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THE FIXED POINT PROPERTY OF THE PRODUCTS AND HYPERSPACES OF ARBOROIDS

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Abstract. This paper is motivated by the Cauty's result [1] which states that if X is a dendroid, then 2^X and $C(X)$ have the fixed point property. The main purpose of this paper is to study the fixed point property of the hyperspaces of arboroids. It is proved, using the inverse systems method and Cauty's result, that if X is an arboroid and $f : 2^X \to 2^X$ is a mapping, then f has the fixed point property. Similar theorem it is proved for $f : C(X) \to C(X)$.

1. Introduction

All spaces in this paper are compact Hausdorff and all mappings are continuous. The weight of a space X is denoted by $\mathrm{w}(X)$. The cardinality of a set A is denoted by $\mathrm{card}(A)$. We shall use the notion of inverse system as in [5, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

Let A be a partially ordered directed set. We say that a subset $A_1 \subset A$ *majorates* [3, p. 9] another subset $A_2 \subset A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subset A$, denotes the lower upper bound of B (if such an element exists in A). Let $\tau \geq \aleph_0$ be a cardinal number. A subset B of A is said to be *τ-closed* in A if for each chain $C \subset B$, with $\mathrm{card}(B) \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be *τ-complete* if for each chain C of A of elements of A with $\mathrm{card}(C) \leq \tau$, there exists an element $\sup C$ in A.

Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system X into the system* $\mathbf{Y}$ [3, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array}$$

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diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [3, p. 17] if for each real-valued mapping $f : \lim \mathbf{X} \to \mathbb{R}$ there exist an $a \in A$ and a mapping $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ*-directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of the members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

Lemma 1.1. [3, Corollary 1.3.2, p. 18]. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-directed inverse system of compact spaces with surjective bonding mappings, then it is factorizing.*

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be τ*-continuous* [3, p. 19] if for each chain B in A with $\mathrm{card}(B) < \tau$ and $\sup B = b$, the diagonal product $\Delta\{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be a τ*-system* [3, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

A σ*-system* is τ-system, where $\tau = \aleph_0$. The following theorem is called the *Spectral Theorem* [3, p. 19].

Theorem 1.2. [3, Theorem 1.3.4, p. 19]. *If a τ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a,\ q_{ab},\ A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of the limit projections of systems in Theorem 1.2 is essential [3, p. 21].

A *fixed point of a function* $f : X \to X$ is a point $p \in X$ such that $f(p) = p$. A space X is said to have the *fixed point property* provided that every mapping $f : X \to X$ has a fixed point.

In the sequel we shall use the following result.

Theorem 1.3. *Let $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ be a σ-system of compact spaces with limit X and onto projections $p_a : X \to X_a$. Let $\{f_a : X_a \to X_a\} : \mathbf{X} \to \mathbf{X}$ be a morphism. Then the induced mapping $f = \lim\{f_a\} : X \to X$ has a fixed point if and only if each mapping $f_a : X_a \to X_a$, $a \in A$, has a fixed point.*

As an immediate consequence of this theorem and the Spectral theorem 1.2 we have the following result.

Theorem 1.4. *Let a non-metric continuum X be the inverse limit of an inverse σ-system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ such that each X_a has the fixed point property and each bonding mapping p_{ab} is onto. Then X has the fixed point property.*

The following result enables to prove Theorem 1.6 as the application of Theorem 1.4.

Theorem 1.5. [7, Theorem 1.6, p. 402]. *If X is the Cartesian product $X = \prod\{X_s : s \in S\}$, where $\operatorname{card}(S) > \aleph_0$ and each X_s is compact, then there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\operatorname{card}(a) = \aleph_0$, such that X is homeomorphic to $\lim \mathbf{X}$.*

Theorem 1.6. *Let S be an infinite set and $Q = \Pi\{X_s : s \in S\}$ Cartesian product of compact spaces. If each product $X_{s_1} \times X_{s_2} \times ... \times X_{s_n}$ of finitely many spaces X_s has the fixed point property, then Q has the fixed point property.*

Proof. We shall consider the following cases.

Case 1 $\operatorname{card}(S) = \aleph_0$. We may assume that $S = \mathbb{N}$. The proof is a straightforward modification of the proof of [10, Corollary 3.5.3, pp. 106-107]. Let $f : Q \to Q$ be continuous. For every $n \in \mathbb{N}$ define

$$K_n = \{x \in Q : (x_1, ..., x_n) = (f(x)_1, ..., f(x)_n)\}.$$

It is clear that for every n the set K_n is closed in Q and that $K_{n+1} \subset K_n$. For every $n \in \mathbb{N}$, let o_n be a given point of X_n and $p_n : Q \to X_1 \times ...\times X_n$ be the projection. Define continuous function $f_n : X_1 \times ... \times X_n \to X_1 \times ... \times X_n$ by

$$f_n(x_1, ..., x_n) = (p_n f)(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...).$$

By assumption of Theorem f_n has the fixed point property, say $(x_1, ..., x_n)$. It follows that

$$(x_1, ..., x_n, o_{n+1}, o_{n+2}, ...) \in K_n.$$

We conclude that $\{K_n : n \in \mathbb{N}\}$ is a decreasing collection of nonempty closed subsets of Q. By compactness of Q we have that

$$K = \cap\{K_n : n \in \mathbb{N}\}$$

is nonempty. It is clear that every point in K is a fixed point of f.

Case 2 $\operatorname{card}(A) \geq \aleph_1$. By Theorem 1.5 there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\operatorname{card}(a) = \aleph_0$, such that Q is homeomorphic to $\lim \mathbf{X}$. By Case 1 each Y_a has the fixed point property. Finally, by Theorem 1.4 we infer that Q has the fixed point property. □

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $\operatorname{Bd}(U)$ is metrizable for each $U \in \mathcal{B}$.

Theorem 1.7. [8, Theorem 9, p. 205]. *Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces and surjective bonding mappings p_{ab}. Then:*

- **1):** *There exists an inverse system $M(\mathbf{X}) = \{M_a, m_{ab}, A\}$ of compact spaces such that m_{ab} are monotone surjections and $\lim \mathbf{X}$ is homeomorphic to $\lim M(\mathbf{X})$,*
- **2):** *If $\mathbf{X}$ is σ-directed, then $M(\mathbf{X})$ is σ-directed,*
- **3):** *If $\mathbf{X}$ is σ-complete, then $M(\mathbf{X})$ is σ-complete,*
- **4):** *If every X_a is a metric space and $\lim \mathbf{X}$ is locally connected (a rim-metrizable continuum), then every M_a is metrizable.*

REMARK. Let us observe that the projections $m_a : \lim M(\mathbf{X}) \to M_a, a \in A$, are monotone. In the case of the locally connected spaces or the rim-metrizable continua, we have the following result.

Theorem 1.8. [8, Theorem 10, p. 207]. *Let* $\mathbf{X} = \{X_a, p_{ab}, A\}$ *be a* σ*-system of compact spaces and surjective bonding mappings* p_{ab}. *If* $\lim \mathbf{X}$ *is a locally connected space (a rim-metrizable continuum), then there exists an* $a \in A$ *such that the projection* p_b *is monotone, for every* $b \geq a$.

For a compact space X we denote by 2^X the hyperspace of all nonempty closed subsets of X equipped with the Vietoris topology. $C(X)$ and $X(n)$, where n is a positive integer, stand for the sets of all connected members of 2^X and of all nonempty subsets consisting of at most n points, respectively, both considered as subspaces of 2^X.

For a mapping $f : X \to Y$ define $2^f : 2^X \to 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. It is known that 2^f is continuous, $2^f(C(X)) \subset C(Y)$ and $2^f(X(n)) \subset Y(n)$. The restriction $2^f|C(X)$ is denoted by $C(f)$.

An element $\{x_a\}$ of the Cartesian product $\prod\{X_a : a \in A\}$ is called a *thread* of $\mathbf{X}$ if $p_{ab}(x_b) = x_a$ for any a, $b \in A$ satisfying $a \leq b$. The subspace of $\prod\{X_a : a \in A\}$ consisting of all threads of $\mathbf{X}$ is called the limit of the inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ and is denoted by $\lim \mathbf{X}$ or by $\lim\{X_a, p_{ab}, A\}$ [5, p. 135].

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with the natural projections $p_a : \lim \mathbf{X} \to X_a$, for $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $\mathbf{X}(n) = \{X_a(n), 2^{p_{ab}}|\ X_b(n), A\}$ form inverse systems. For each $F \in 2^{\lim \mathbf{X}}$, i.e., for each closed $F \subseteq \lim \mathbf{X}$ the set $p_a(F) \subseteq X_a$ is closed and compact. Thus, we have a mapping $2^{p_a} : 2^{\lim \mathbf{X}} \to 2^{X_a}$ induced by p_a for each $a \in A$. Define a mapping $M : 2^{\lim \mathbf{X}} \to \lim 2^{\mathbf{X}}$ by $M(F) = \{p_a(F) : a \in A\}$. Since $\{p_a(F) : a \in A\}$ is a thread of the system $2^{\mathbf{X}}$, the mapping M is continuous and one-to-one. It is also onto since for each thread $\{F_a : a \in A\}$ of the system $2^{\mathbf{X}}$ the set $F' = \bigcap\{p_a^{-1}(F_a) : a \in A\}$ is non-empty and $p_a(F') = F_a$. Thus, M is a homeomorphism. If $P_a : \lim 2^{\mathbf{X}} \to 2^{X_a}, a \in A$, are the projections, then $P_a M = 2^{p_a}$. Identifying F with $M(F)$ we have $P_a = 2^{p_a}$.

Lemma 1.9. . *Let* $X = \lim \mathbf{X}$. *Then* $2^X = \lim 2^{\mathbf{X}}$, $C(X) = \lim C(\mathbf{X})$ *and* $X(n) = \lim \mathbf{X}(n)$.

2. The arboroids as the inverse limit space of dendroids

A continuum X with precisely two non-separating points is called a *generalized arc.*

A *simple n-od* is the union of n generalized arcs $A_1O, A_2O, ..., A_\alpha O$, each two of which have only the point O in common. The point O is called the *vertex* or the *top* of the n-od.

By a *branch point* of a compact space X we mean a point p of X which is the vertex of a simple triod lying in X. A point $x \in X$ is said to be *end point* of X if for each neighborhood U of x there exists a neighborhood V of x such that $V \subset U$ and $\mathrm{card}(Bd(V)) = 1$.

Let S be the set of all end points and of all branch points of a continuum X. An arc pq in X is called a *free arc* in X if $pq \cap S = \{p, q\}$.

A continuum is a *graph* if it is the union of a finite number of metric free arcs. A *tree* is an acyclic graph.

A continuum X is *tree-like* (*arc-like*) if for each open cover $\mathcal{U}$ of X, there is a tree (arc) $X_{\mathcal{U}}$ and a $\mathcal{U}$-mapping $f_{\mathcal{U}} : X \to X_{\mathcal{U}}$ (the inverse image of each point is contained in a member of $\mathcal{U}$).

Every tree-like continuum is hereditarily unicoherent.

A non-metric hereditarily unicoherent continuum which is arcwise connected by generalized arcs is said to be an *arboroid*. A metrizable hereditarily unicoherent continuum which is arcwise connected is said to be a *dendroid*. Every arboroid is tree-like [4, Corollary, p. 20]. If X is an arboroid and $x, y \in X$, then there exists a unique arc $[x, y]$ in X with endpoints x and y. If $[x, y]$ is an arc, then $[x, y] \diagdown \{x, y\}$ is denoted by (x, y).

A point t of an arboroid X is said to be a *ramification point* of X if t is the only common point of some three arcs such that it is the only common point of any two, and an end point of each of them.

A point e of an arboroid X is said to be *end point* of X if there exists no arc $[a, b]$ in X such that $x \in [a, b] \diagdown \{a, b\}$.

Let Y^X be the set of all mappings of X to Y. If Y is a metric space with a metric d, then on the set Y^X one can define a metric $\widehat{d}$ by letting

$$\widehat{d}(f, g) = \sup_{x \in X} d\left(f(x), g(x)\right).$$

Proposition 1. *Let X be any tree-like continuum, let P be a polyhedron with a given metric d, $r > 0$ a real number and $f : X \to P$ a mapping. Then there exist a tree Q, a mapping $g : X \to Q$ and a mapping $p : Q \to P$ such that $g(X) = Q$ and $\widehat{d}(f, pg) \leq r$.*

Proof. Let K be a triangulation of P of mesh not greater than $r/2$. Let a_i be the vertices of K, and let St a_i be the open star of K around the vertex a_i. Hence, $\{\text{St } a_i\}$ is an open covering for P, and so is $\mathcal{U} = \{f^{-1}(\text{St } a_i)\}$ for X. There exist a tree Q and a mapping $g : X \to Q$ such that g is an $\mathcal{U}$-mapping and $g(X) = Q$. There exists a triangulation L of Q with vertices b_j such that the cover $\mathcal{V} = \{g^{-1}(\text{St } b_j)\}$ refines the cover $\mathcal{U}$. Let x be a point of X and let s be a simplex of Q with vertices $b_{j_1}, ..., b_{j_k}$ containing $g(x)$. This means that $\left\{g^{-1}(\text{St } b_{j_1}), ..., g^{-1}(\text{St } b_{j_k})\right\}$ is a collection of some $g^{-1}(\text{St } b_j)$ containing x. It follows that $g^{-1}(\text{St } b_{j_1}) \cap ... \cap g^{-1}(\text{St } b_{j_k}) \neq \emptyset$. We infer that St $b_{j_1} \cap ... \cap$ St $b_{j_k} \neq \emptyset$. Let $p : Q \to P$ be a simplicial mapping sending each vertex b_j of Q into a vertex a_i having the property that $g^{-1}(\text{St } b_i) \subset f^{-1}(\text{St } a_i)$. It remains to prove that $d(f, pg) \leq r$. Now, for each $g^{-1}(\text{St } b_{i_j})$ we have some $f^{-1}(\text{St } a_{i_j})$ with $g^{-1}(\text{St } b_{i_j}) \subset f^{-1}(\text{St } a_{i_j})$. From $g^{-1}(\text{St } b_{j_1}) \cap ... \cap g^{-1}(\text{St } b_{j_k}) \neq \emptyset$ it follows that $f^{-1}(\text{St } b_{j_1}) \cap ... \cap f^{-1}(\text{St } b_{j_k}) \neq \emptyset$, i.e., that there exists a simplex σ of K with vertices $b_{j_1}, ..., b_{j_k}$ such that $f(x) \in$ St σ. Clearly, $pg(x) \in$ St σ. Finally, $\widehat{d}(f, pg) \leq r$. □

Proposition 2. *If $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is an inverse system of tree-like continua and if p_{ab} are onto mappings, then the limit $X = \lim \mathbf{X}$ is a tree-like continuum.*

Proof. Let $\mathcal{U} = \{U_1, ..., U_n\}$ be an open covering of X. There exist an $a \in A$ and an open covering $\mathcal{U}_a = \{U_{1a}, ..., U_{ka}\}$ such that $\left\{p_a^{-1}(U_{1a}), ..., p_a^{-1}(U_{ka})\right\}$ refines the covering $\mathcal{U}$. There exist a tree T_a and a $\mathcal{U}_a$-mapping $f_{\mathcal{U}_a} : X_a \to T_a$ since X_a is tree-like. It is clear that $f_{\mathcal{U}_a} p_a : X \to T_a$ is a $\mathcal{U}$-mapping. Hence, X is tree-like. □

Proposition 3. *If X is a tree-like continuum, Q a tree and $f : X \to Q$ is a mapping, then $f(X)$ also is a tree.*

Proof. This follows from the fact that a subcontinuum of a tree is a tree. □

The following result is an expanding theorem of tree-like continua into inverse σ-systems of metric tree-like continua.

Theorem 2.1. *If X is a tree-like non-metric continuum, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric tree-like continua X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$.*

Proof. Let us observe that Propositions 1-3 are the conditions (A)-(C) in [9, p. 220]. Then from Mardešić's General Expansion Theorem [9, Theorem 2] it follows that there exists an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and onto bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. It remains to prove that there exists such σ-system. The proof is broken into several steps.

Step 1. For each subset Δ_0 of $(A, \leq)$ we define sets Δ_n, $n = 0, 1, ...,$ by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y) : x, y \in \Delta_n\}$, where $m(x,y)$ is a member of A such that $x, y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n : n \in \mathbb{N}\}$. It is clear that $\operatorname{card}(\Delta) = \operatorname{card}(\Delta_0)$. Moreover, Δ is directed by $\leq$. For each directed set $(A, \leq)$ we define

$$A_\sigma = \{\Delta : \emptyset \neq \Delta \subset A, \operatorname{card}(\Delta) \leq \aleph_0 \quad \text{and } \Delta \text{ is directed by } \leq\}.$$

Step 2. *If A is a directed set, then A_σ is σ-directed and σ-complete.* Let $\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ be a countable subset of A_σ. Then $\Delta_0 = \cup\{\Delta^1, \Delta^2, ..., \Delta^n, ...\}$ is a countable subset of A_σ. Define sets Δ_n, $n = 0, 1, ...,$ by the inductive rule $\Delta_{n+1} = \Delta_n \bigcup \{m(x,y) : x, y \in \Delta_n\}$, where $m(x, y)$ is a member of A such that $x, y \leq m(x,y)$. Let $\Delta = \bigcup\{\Delta_n\colon n \in \mathbb{N}\}$. It is clear that $\operatorname{card}(\Delta) = \operatorname{card}(\Delta_0)$. This means that Δ is countable. Moreover $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Hence A_σ is σ-directed. Let us prove that A_σ is σ-complete. Let $\Delta^1 \subset \Delta^2 \subset ... \subset \Delta^n \subset ...$ be a countable chain in A_σ. Then $\Delta = \cup\{\Delta^i : i \in \mathbb{N}\}$ is countable and directed subset of A, i.e., $\Delta \in A_\sigma$. It is clear that $\Delta \supseteq \Delta^i$, $i \in \mathbb{N}$. Moreover, for each $\Gamma \in A_\sigma$ with property $\Gamma \supseteq \Delta^i$, $i \in \mathbb{N}$, we have $\Gamma \supseteq \Delta$. Hence $\Delta = \sup\{\Delta^i : i \in \mathbb{N}\}$. This means that A_σ is σ-complete.

Step 3. If $\Delta \in A_\sigma$, let $\mathbf{X}^\Delta = \{X_b, p_{bb'}, \Delta\}$ and $X_\Delta = \lim \mathbf{X}^\Delta$. If $\Delta, \Gamma \in A_\sigma$ and $\Delta \subseteq \Gamma$, let $P_{\Delta\Gamma}\colon X_\Gamma \to X_\Delta$ denote the map induced by the projections $p^\Gamma_\delta : X_\Gamma \to X_\delta$, $\delta \in \Delta$, of the inverse system $\mathbf{X}^\Gamma$.

Step 4. *If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is an inverse system, then $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ is a σ-directed and σ-complete inverse system such that $\lim \mathbf{X}$ and $\lim \mathbf{X}_\sigma$ are homeomorphic.* Each thread $x = (x_a : a \in A)$ induces the thread $(x_a : a \in \Delta)$ for each $\Delta \in A_\sigma$, i.e., the point $x_\Delta \in X_\Delta$. This means that we have a mapping $H : \lim \mathbf{X} \to \lim \mathbf{X}_\sigma$ such that $H(x) = (x_\Delta : \Delta \in A_\sigma)$. It is obvious that H is continuous and 1-1. The mapping H is onto since the collections of the threads $\{x_\Delta : \Delta \in A_\sigma\}$ induces the thread in $\mathbf{X}$. We infer that H is a homeomorphism since $\lim \mathbf{X}$ is compact.

Step 5. *Every X_Δ is a metric tree-like continuum.* Apply Proposition 2.

Step 6. *Every projection $P_\Delta : \lim \mathbf{X}_\sigma \to X_\Delta$ is onto.* This follows from the assumption that the bonding mappings p_{ab} are surjective.

Finally, $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ is a desired σ-system. □

Now we shall prove an expanding theorem of arboroids into inverse σ-systems of dendroids.

Theorem 2.2. *If X is an arboroid, then there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendroids X_a and onto mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$.*

Proof. Firstly we recall that each arboroid is tree-like [4, Corollary, p. 20]. Then from Theorem 2.1 it follows that there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and onto bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. It remains to prove that there every X_a is a dendroid. By the fact that X_a is tree-like it follows that X_a is unicoherent. Moreover, it is metric. It remains to prove that X_a is arcwise connected. Let a, b be a pair of points of X_a. There exists a pair x, y of points of X such that $a = P_a(x)$ and $b = P_a(y)$. There exist a unique arc xy in X with end points x and y since X is arcwise connected. Now, $P_a(xy)$ is arcwise connected [13]. This means that there is an arc ab with end points a and b. Thus, X_a is a dendroid. □

A *non-metric or generalized dendrite* is a locally connected arboroid. From Theorem 2.2 we obtain the following result.

Theorem 2.3. *If X is a generalized dendrite, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric dendrites X_Δ and onto monotone projections P_Δ such that X is homeomorphic to* $\lim \mathbf{X}_\sigma$.

Proof. From Theorem 2.2 we have a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of metric dendroids X_Δ and onto mappings $P_{\Delta\Gamma}$ such that X is homeomorphic to $\lim \mathbf{X}_\sigma$. It suffices to prove that every X_Δ is locally connected. This follows from [14, Lemma 1.5, p. 70]. Moreover, by Theorem 1.8 it follows that there exists an $a \in A$ such that the projection P_b is monotone, for every $b \geq a$. □

By similar method of proof we have the following result.

Theorem 2.4. *If X is a rim-metrizable arboroid, then there exists a σ-system $\mathbf{X}_\sigma = \{X_\Delta, P_{\Delta\Gamma}, A_\sigma\}$ of dendroids X_Δ and onto monotone projections P_Δ such that X is homeomorphic to* $\lim \mathbf{X}_\sigma$.

A λ-*dendroid* is an hereditarily decomposable, hereditarily unicoherent continuum. A λ-dendroid is tree-like [4, Corollary, p. 20].

Theorem 2.5. *If X is a non-metric rim-metrizable λ-dendroid, then there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric λ-dendroids X_a and onto mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Proof. From Theorem 2.1 it follows that there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a and onto mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$ since λ-dendroid X is tree-like.We infer that each X_a is unicoherent. It remains to prove that every X_a is hereditarily decomposable. By Theorem 1.8 there exists an $a \in A$ such that the projection p_b is monotone, for every $b \geq a$. Using Theorem 14 of [2, p. 217] (see also [12, p. 297]) we conclude that every X_a is λ-dendroid. □

3. The fixed point property of the hyperspaces of arboroids

Now we shall investigate the fixed point property of the hyperspaces of arboroids. Let us recall the following known results.

Theorem 3.1. [1, Theorem 1, p. 1]. *For every dendroid X, every tree T_0 contained in X and every $\varepsilon > 0$, there exists a tree T contained in X and containing T_0 and an ε-retraction of X onto T.*

Theorem 3.2. [1, Corollaire 1, p. 1]. *If X is a dendroid, then 2^X and $C(X)$ have the fixed point property.*

By this theorem and Theorems 1.4 and 1.9 we shall prove the following result.

Theorem 3.3. *If X is an arboroid, then 2^X has the fixed point property.*

Proof. By Theorem 2.2 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendroids such that all the bonding mappings p_{ab} are surjective and the limit $\lim \mathbf{X}$ is homeomorphic to X. Now we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X (Lemma 1.9). It is clear that the mappings $2^{p_{ab}}$ are onto if the bonding mappings p_{ab} are onto. Now we can apply Theorem 1.4 since, by Theorem 3.2, every 2^{X_a} has the fixed point property. Hence, 2^X has the fixed point property. □

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-system. If we consider the inverse system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$, then $C(p_{ab})$ are not always the surjections. This is the case only if p_{ab} are weakly confluent mappings [11, Theorem (0.49.1), p. 24]. This means that we can apply Theorem 1.4 on the system $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ only if p_{ab} are weakly confluent mappings. Let us recall that a mapping $f : X \to Y$ is *weakly confluent* provided that for each subcontinuum K of Y there exists a component A of $f^{-1}(K)$ such that $f(A) = K$ [11, (0.45.4), p. 22]. Each monotone mapping is weakly confluent. It follows that expanding Theorem 2.2 is not enough for proving the fixed point property of $C(X)$ when X is an arboroid. For this reason we shall consider the fixed point property for 2^X and $C(X)$ if X is a rim-metrizable arboroid.

Theorem 3.4. *If X is a rim-metrizable arboroid, then $C(X)$ has the fixed point property.*

Proof. By Theorem 2.4 there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendroids such that all the bonding mappings p_{ab} are monotone surjections and the limit $\lim \mathbf{X}$ is homeomorphic to X. It is clear that the mappings $C(p_a)$ are onto if the bonding mappings p_a are monotone. Now we can apply Theorem 1.4 since, by Theorem 3.2, every $C(X_a)$ has the fixed point property. Hence, $C(X)$ has the fixed point property. □

Similarly, by Theorem 2.3, on can prove the following result.

Theorem 3.5. *If X is a generalized dendrite, then 2^X and $C(X)$ have the fixed point property.*

Let Y be a topological space. The *cone* $\text{Cone}(Y)$ over Y is the quotient space obtained from $Y \times [0, 1]$ by shrinking $Y \times \{1\}$ to a point. This point is called the *vertex* of $\text{Cone}(Y)$. The subset $Y \times \{1\}$ of $\text{Cone}(Y)$ is called the *base* of $\text{Cone}(Y)$.

The following result generalize Theorem 22.15 of [6, p. 195].

Theorem 3.6. *Let $X = \text{Cone}(Y)$, where Y is an arboroid. Then 2^X has the fixed point property. Moreover, if Y is rim-metrizable, then $C(X)$ has the fixed point property.*

The *suspension* $\Sigma(Y)$ *over a topological space* Y is the quotient space obtained from $Y \times [-1, 1]$ by shrinking $Y \times \{-1\}$ and $Y \times \{1\}$ to different points point.

By the similar method of proof one can get the following result which generalize Theorem 22.16 of [6, p. 196].

Theorem 3.7. *Let $X = \Sigma(Y)$, where Y is an arboroid. Then 2^X has the fixed point property. Moreover, if Y is rim-metrizable, then $C(X)$ has the fixed point property.*

4. Fixed point property for a product of arboroids

In the sequel we shall use the following result.

Proposition 4. [5, Exercise 2.5.D.(b), p. 143]. *Let* $\mathbf{S}(s) = \{X(s)_\alpha, p(s)_{ab}, A\}$ *be an inverse system for every $s \in S$. Then*

$$\Pi\{\mathbf{S}(s) : s \in S\} = \{\Pi\{X(s)_a : s \in S\}, \Pi\{p(s)_{ab} : s \in S\}, A\}$$

is an inverse system and $\lim(\Pi\{\mathbf{S}(s) : s \in S\})$ *is homeomorphic to* $\Pi\{\lim \mathbf{S}(s) : s \in S\}$.

In this Section we shall generalize the following result in two directions.

Theorem 4.1. [1, Corollaire 2, p. 1]. *Each product of dendroids has the fixed point property.*

Theorem 4.2. *Let $X = \prod\{X(s) : s \in S\}$ be a product of arboroids such that* $\mathrm{w}(X(s)) = \tau$ *for every $s \in S$ and for cardinal number τ. Then X has the fixed point property.*

Proof. If for every $s \in S$ we have an arboroid $X(s)$, then, for every $s \in S$, there exists a σ-directed inverse system $\mathbf{X}(s) = \{X_a(s), p_{ab}(s), A(s)\}$ such that $X(s)$ is homeomorphic to $\lim \mathbf{X}(s)$ and every $X_a(s)$ is a dendroid (Theorem 2.2). If $\mathrm{w}(X(s_1)) = \mathrm{w}(X(s_2)), s_1, s_2 \in S$, then $A(s_1) = A(s_2)$ and we may suppose that $A(s) = A$ for every $s \in S$. By Theorem 4 the family $\Pi\{\mathbf{X}(s) : s \in S\} = \{\Pi\{X_a(s) : s \in S\}, \Pi\{ p_{ab}(s) : s \in S\}, A\}$ is an inverse system and $\lim(\Pi\{\mathbf{X}(s) : s \in S\})$ is homeomorphic to $\Pi\{\lim \mathbf{X}(s) : s \in S\}$. From Theorem 4.1 it follows that each $\Pi\{X_a(s) : s \in S\}$ has the fixed point property. Finally, from Theorem 1.4 it follows that $\Pi\{X(s) : s \in S\}$ has the fixed point property. □

For $\mathrm{card}(A) = 1$ we have the following result.

Corollary 4.3. *Every arboroid has the fixed point property.*

QUESTION. Is it true that the assumption "*of the same weight*" in Theorem 4.2 can be omitted?

We close this section with the result which generalize Theorem 4.1 in the another direction.

Theorem 4.4. *Let X be an arboroid and let $\{D_m : m \in M\}$ be a family of dendroids. The $X \times \Pi\{D_m : m \in M\}$ has the fixed point property.*

Proof. By Theorem 2.2 there exists a σ-directed inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to $\lim \mathbf{X}$ and every X_a is a dendroid. From Theorem 4 it follows that $X \times \Pi\{D_m : m \in M\}$ is homeomorphic to the limit of the system

$$\mathbf{X} \times \Pi\{D_m : m \in M\} = \{X_a \times \Pi\{D_m : m \in M\}, p_{ab} \times identity, A\}.$$

Each $X_a \times \Pi\{D_m : m \in M\}$ has the fixed point property (Theorem 4.1). Finally, by Theorem 1.4, $X \times \Pi\{D_m : m \in M\}$ has the fixed point property □

5. The fixed point property of the hyperspaces of the products of arboroids

In this section we shall use the following result from [6, Exercise 22.20, p. 197].

Proposition 5. *Let $X = \Pi\{X_i : i \leq n \leq \infty\}$ be a product of metric continua. Assume that for each i and each $\varepsilon > 0$ there is a continuous function $f_{i,\varepsilon} : X_i \to f_{i,\varepsilon}(X_i) \subset X_i$, where $f_{i,\varepsilon}(X_i)$ is locally connected and $f_{i,\varepsilon}$ is within ε of the identity map on X. Then 2^X and $C(X)$ have the fixed point property.*

We shall prove that Proposition 5 is true for every product of metric continua.

Theorem 5.1. *Let $X = \Pi\{X_a : a \in A\}$ be a product of metric continua. Assume that for each a and each $\varepsilon > 0$ there is a continuous function $f_{a,\varepsilon} : X_a \to f_{a,\varepsilon}(X_a) \subset X_a$, where $f_{a,\varepsilon}(X_a)$ is locally connected and $f_{a,\varepsilon}$ is within ε of the identity map on X. Then 2^X and $C(X)$ have the fixed point property.*

Proof. By Theorem 1.5 if $X = \prod\{X_a : a \in A\}$, where $\mathrm{card}(A) > \aleph_0$ and each X_a is compact, then there exists a σ-directed inverse system $\mathbf{X} = \{Y_a, P_{ab}, A\}$ of the countable products $Y_a = \prod\{X_\mu : \mu \in a\}$, $\mathrm{card}(a) = \aleph_0$, such that X is homeomorphic to $\lim \mathbf{X}$. Moreover, $P_{ab} : Y_b \to Y_a$ is a projection. This means that if $X_a, a \in A$, are continua, then $P_{ab}, a \leq b$, are monotone. We infer that the systems $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ and $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ have the the surjective bonding mappings. This means that one can apply Theorem 1.4 since each 2^{X_a} and $C(X_a)$ has the fixed point property (Theorem 5) and the projections $P_a : \lim \mathbf{X} \to Y_a$ are surjections. □

Applying Theorems 3.1 and 5.1 one can get the following result which generalize Theorems 3.2 and 3.3.

Theorem 5.2. *Let $X = \Pi\{X_a : a \in A\}$ be a product of dendroids. Then 2^X and $C(X)$ have the fixed point property.*

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