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DIMENSIONS, EMBEDDINGS, AND ATTRACTORS

JAMES C. ROBINSON

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To my family: Tania, Joseph, & Kate.

Contents

Preface

The main purpose of this book is to bring together a number of results concerning the embedding of 'finite-dimensional' compact sets into Euclidean spaces, where an 'embedding' of a metric space (X, ϱ) into $\mathbb{R}^n$ is to be understood as a homeomorphism from X onto its image. A secondary aim is to present, alongside such 'abstract' embedding theorems, more concrete embedding results for the finite-dimensional attractors that have been shown to exist in many infinite-dimensional dynamical systems.

In addition to its summary of embedding results, the book also gives a unified survey of four major definitions of dimension (Lebesgue covering dimension, Hausdorff dimension, upper box-counting dimension, and Assouad dimension). In particular, it provides a more sustained exposition of the properties of the box-counting dimension than can be found elsewhere; indeed, the abstract results for sets with finite box-counting dimension are those that are taken further in the second part of the book, which treats finite-dimensional attractors.

While the various measures of dimension discussed here find a natural application in the theory of fractals, this is not a book about fractals. An example to which we will return continually is an orthogonal sequence in an infinite-dimensional Hilbert space, which is very far from being a 'fractal'. In particular, this class of examples can be used to show the sharpness of three of the embedding theorems that are proved here.

My models have been the classic text of Hurewicz & Wallman (1941) on the topological dimension, and of course Falconer's elegant 1985 tract which concentrates on the Hausdorff dimension (and Hausdorff measure). It is a pleasure to acknowledge formally my indebtedness to Hunt & Kaloshin's 1999 paper 'Regularity of embeddings of infinite-dimensional fractal sets into finite-dimensional spaces'. It has had a major influence on my own research over the last ten years, and one could view this book as an extended exploration of the ramifications of the approach that they adopted there.

My interest in abstract embedding results is related to the question of whether one can reproduce the dynamics on a finite-dimensional attractor using a finite-dimensional system of ordinary differential equations (see Chapter 10 of Eden, Foias, Nicolaenko, & Temam (1994), or Chapter 16 of Robinson (2001), for example). However, there are still only partial results in this direction, so this potential application is not treated here; for an up-to-date discussion see the paper by Pinto de Moura, Robinson, & Sánchez-Gabites (2010).

I started writing this book while I was a Royal Society University Research Fellow, and many of the results here derive from work done during that time. I am currently supported by an EPSRC Leadership Fellowship, Grant EP/G007470/1. I am extremely grateful to both the Royal Society and to the EPSRC for their support.

I would like to thank Alexandre Carvalho, Peter Friz, Igor Kukavica, José Langa, Eric Olson, Eleonora Pinto de Moura, and Alejandro Vidal López, all of whom have had a hand in material that is presented here. In particular, Eleonora was working on closely-related problems for her doctoral thesis during most of the time that I was writing this book, and our frequent discussions have shaped much of the content and my approach to the material. I had comments on a draft version of the manuscript from Witold Sadowski, Jaime Sánchez-Gabites, and Nicholas Sharples: I am extremely grateful for their helpful and perceptive comments. David Tranah, Clare Dennison, and Emma Walker at Cambridge University Press have been most patient as one deadline after another was missed and extended; that one was finally met (nearly) is due in large part to a kind invitation from Marco Sammartino to Palermo, where I gave a series of lectures on some of the material in this book in November 2009.

Many thanks to my parents and to my mother-in-law; in addition to all their other support, their many days with the children have made this work possible. Finally, of course, thanks to Tania, my wife, and our children Joseph and Kate, who make it all worthwhile; this book is dedicated to them.

Introduction

Part I of this book treats four different definitions of dimension, and investigates what being 'finite dimensional' implies in terms of embeddings into Euclidean spaces for each of these definitions.

Whitney (1936) showed that any abstract n-dimensional C^r manifold is C^r-homeomorphic to an analytic submanifold in $\mathbb{R}^{2n+1}$. This book treats embeddings for much more general sets that need not have such a smooth structure; one might say 'fractals', but we will not be concerned with the fractal nature of these sets (whatever one takes that to mean).

We will consider four major definitions of dimension:

(i) The (Lebesgue) covering dimension $\dim(X)$, based on the maximum number of simultaneously intersecting sets in refinements of open covers of X (Chapter 1). This definition is topologically invariant, and is primarily used in the classical and abstract 'Dimension Theory', elegantly developed in Hurewicz & Wallman's 1941 text, and subsequently by Engelking (1978), who updates and extends their treatment.

(ii) The Hausdorff dimension $d_{\mathrm{H}}(X)$, the value of d where the 'd-dimensional Hausdorff measure' of X switches from ∞ to zero (Chapter 2). Hausdorff measures (and hence the Hausdorff dimension) play a large role in geometric measure theory (Federer, 1969), and in the theory of dynamical systems (see Pesin (1997)); the standard reference is Falconer's 1985 tract, and subsequent volumes (Falconer, 1990, 1997).

(iii) The (upper) box-counting dimension $d_{\mathrm{B}}(X)$, essentially the scaling as $\epsilon \to 0$ of $N(X, \epsilon)$, the number of ϵ-balls required to cover X, i.e. $N(X, \epsilon) \sim \epsilon^{-d_{\mathrm{B}}(X)}$ (Chapter 3). This dimension has mainly found application in the field of dynamical systems, see for example Falconer (1990), Eden *et al.* (1994), C. Robinson (1995), and Robinson (2001).

(iv) The Assouad dimension $d_{\mathrm{A}}(X)$, a 'uniform localised' version of the box-counting dimension: if $B(x,\rho)$ denotes the ball of radius ρ centred at $x \in X$, then $N(X \cap B(x,\rho), r) \sim (\rho/r)^{d_{\mathrm{A}}(X)}$ for every $x \in X$ and every $0 < r < \rho$ (Chapter 9). This definition appears unfamiliar outside the area of metric spaces and most results are confined to research papers (e.g. Assouad (1983), Luukkainen (1998), Olson (2002); but see also Heinonen (2001, 2003)).

For any compact metric space (X, ϱ) we will see that

$$\dim(X) \le d_{\mathrm{H}}(X) \le d_{\mathrm{B}}(X) \le d_{\mathrm{A}}(X),$$

and there are examples showing that each of these inequalities can be strict. We will check that each definition satisfies the natural properties of a dimension: monotonicity ($X \subseteq Y$ implies that $d(X) \le d(Y)$); stability under finite unions ($d(X \cup Y) = \max(d(X), d(Y))$); and the dimension of $\mathbb{R}^n$ is n (a consistent way to interpret this so that it makes sense for all the definitions above is that $d(K) = n$ if K is a compact subset of $\mathbb{R}^n$ that contains an open set). We will also consider how each definition behaves for product sets.

Our main concern will be with the embedding results that are available for each class of 'finite-dimensional' set. The embedding result for sets with finite covering dimension, due to Menger (1926) and Nöbeling (1931) (given as Theorem 1.12 here), is in a class of its own. The result guarantees that when $\dim(X) \le d$, a generic set of continuous maps from a compact metric space (X, ϱ) into $\mathbb{R}^{2d+1}$ are embeddings.

The results for sets with finite Hausdorff, upper box-counting, and Assouad dimension are of a different cast. They are expressed in terms of 'prevalence' (a version of 'almost every' that is applicable to subsets of infinite-dimensional spaces, introduced independently by Christensen (1973) and Hunt, Sauer, & Yorke (1992), and the subject of Chapter 5), and treat compact subsets of Hilbert and Banach spaces. Using techniques introduced by Hunt & Kaloshin (1999), we show that a 'prevalent' set of continuous linear maps $L : \mathscr{B} \to \mathbb{R}^k$ provide embeddings of X when $d(X - X) < k$, where

$$X - X = \{x_1 - x_2 : \; x_1, x_2 \in X\}$$

and d is one of the above three dimensions (see Figure 1). Note that if one wishes to show that a linear map provides an embedding, i.e. that $Lx = Ly$ implies that $x = y$, this is equivalent to showing that $Lz = 0$ implies that $z = 0$ for $z \in X - X$. This is why the natural condition for such results is one on the 'difference' set $X - X$; but while $d_{\mathrm{B}}(X - X) \le 2d_{\mathrm{B}}(X)$, there are examples of

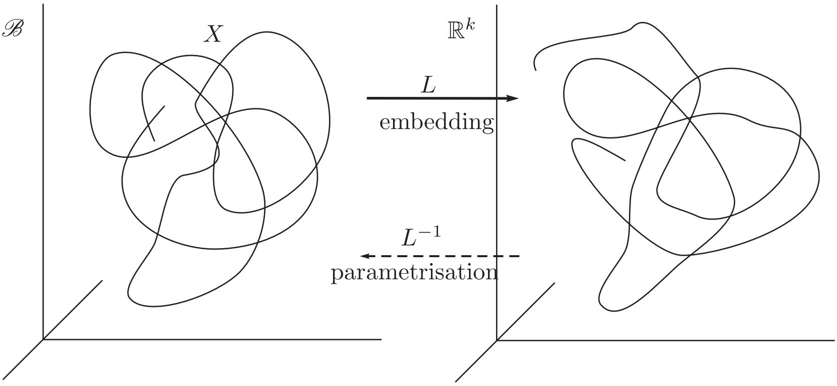

Figure 1 The linear map $L : \mathscr{B} \to \mathbb{R}^k$ embeds X into $\mathbb{R}^k$. The inverse mapping L^{-1} provides a parametrisation of X using k parameters.

sets for which $d_{\rm H}(X) = 0$ but $d_{\rm H}(X - X) = \infty$ (and similarly for the Assouad dimension).

Where the embedding results for these three dimensions differ from one another is in the smoothness of the parametrisation of X provided by L^{-1}. In the Hausdorff case this inverse can only be guaranteed to be continuous (Chapter 6); in the upper box-counting case it will be Hölder (Chapter 8); and in the Assouad case it will be Lipschitz to within logarithmic corrections (Chapter 9). Simple examples of orthogonal sequences in ℓ^2 (or related examples in c_0, the space of sequences that tend to zero) show that the results we give cannot be improved when the embedding map L is linear.

Chapter 4 presents an embedding result for subsets X of $\mathbb{R}^N$ with box-counting dimension $d < (N - 1)/2$. The ideas here form the basis of the results for subsets of Hilbert and Banach spaces that follow, and justify the development of the theory of prevalence in Chapter 5 and the definition of various 'thickness exponents' (the thickness exponent itself, the Lipschitz deviation, and the dual thickness) in Chapter 7.

Part II discusses the attractors that arise in certain infinite-dimensional dynamical systems, and the implications of the results of Part I for this class of finite-dimensional sets. In particular, the embedding result for sets with finite box-counting dimension is used toward a proof of an infinite-dimensional version of the Takens time-delay embedding theorem (Chapter 14) and it is shown that a finite-dimensional set of real analytic functions can be parametrised using a finite number of point values (Chapter 15).

Chapter 10 gives a very cursory summary of some elements of the theory of Sobolev spaces and fractional power spaces of linear operators, which are required in order to discuss the applications to partial differential equations. It is shown how the solutions of an abstract semilinear parabolic equation, and of the two-dimensional Navier–Stokes equations, can be used to generate an infinite-dimensional dynamical system whose evolution is described by a nonlinear semigroup.

The global attractor of such a nonlinear semigroup is a compact invariant set that attracts all bounded subsets of the phase space. A sharp condition guaranteeing the existence this global attractor is given in Chapter 11, and it is shown that such an object exists for the semilinear parabolic equation and the Navier–Stokes equations that were treated in the previous chapter.

Chapter 12 provides a method for bounding the upper box-counting dimension of attractors in Banach spaces. While there are powerful techniques available for attractors in Hilbert spaces, these are already presented in a number of other texts, and outlining the more general Banach space technique is more in keeping with the overall approach of this book (the Hilbert space method is covered here in an extended series of exercises). In particular, we show that any attractor of the abstract semilinear parabolic equation introduced in Chapter 10 will be finite-dimensional.

Before proving the final two 'concrete' embedding theorems in Chapters 14 and 15, Chapter 13 provides two results that guarantee that an attractor has zero 'thickness': we show first that if the attractor consists of smooth functions then its thickness exponent is zero, and then that the attractors of a wide variety of models (which can be written in the abstract semilinear parabolic form) have zero Lipschitz deviation. This, in part, answers a conjecture of Ott, Hunt, & Kaloshin (2006).

Most of the chapters end with a number of exercises. Many of these carry forward portions of the argument that would break the flow of the main text, or discuss related approaches. Full solutions of the exercises are given at the end of the book.

All Hilbert and Banach spaces are real, throughout.

PART I

Finite-dimensional sets

1

Lebesgue covering dimension

There are a number of definitions of dimension that are invariant under homeomorphisms, i.e. that are topological invariants – in particular, the large and small inductive dimensions, and the Lebesgue covering dimension. Although different a priori, the large inductive dimension and the Lebesgue covering dimension are equal in any metric space (Katětov, 1952; Morita, 1954; Chapter 4 of Engelking, 1978), and all three definitions coincide for separable metric spaces (Proposition III.5 A and Theorem V.8 in Hurewicz & Wallman (1941)). A beautiful exposition of the theory of 'topological dimension' is given in the classic text by Hurewicz & Wallman (1941), which treats separable spaces throughout and makes much capital out of the equivalence of these definitions. Chapter 1 of Engelking (1978) recapitulates these results, while the rest of his book discusses dimension theory in more general spaces in some detail.

This chapter concentrates on one of these definitions, the Lebesgue covering dimension, which we will denote by $\dim(X)$, and refer to simply as the covering dimension. Among the three definitions mentioned above, it is the covering dimension that is most suitable for proving an embedding result: we will show in Theorem 1.12, the central result of this chapter, that if $\dim(X) \leq n$ then a generic set of continuous maps from X into $\mathbb{R}^{2n+1}$ are homeomorphisms, i.e. provide an embedding of X into $\mathbb{R}^{2n+1}$.

There is, unsurprisingly, a topological flavour to the arguments involved here, and consequently they are very different from those in the rest of this book. However, any survey of embedding results for finite-dimensional sets would be incomplete without including the 'fundamental' embedding theorem that is available for sets with finite covering dimension.

1.1 Covering dimension

Let (X, ϱ) be a metric space, and A a subset[1] of X. A *covering* of $A \subseteq X$ is a finite collection $\{U_j\}_{j=1}^r$ of open subsets of X such that

$$A \subseteq \bigcup_{j=1}^{r} U_j.$$

The *order* of a covering is the largest integer n such that there are $n+1$ members of the covering that have a nonempty intersection. A covering β is a *refinement* of a covering α if every member of β is contained in some member of α.

Definition 1.1 A set $A \subseteq X$ has $\dim(A) \le n$ if every covering has a refinement of order $\le n$. A set A has $\dim(A) = n$ if $\dim(A) \le n$ but it is not true that $\dim(A) \le n - 1$.

Clearly dim is a topological invariant. We now prove some elementary properties of the covering dimension, following Munkres (2000) and Edgar (2008).

Proposition 1.2 *Let* $B \subseteq A \subseteq X$, *with* B *closed. If* $\dim(A) = n$ *then* $\dim(B) \le n$.

Proof Let α be a covering of B by open subsets $\{U_j\}$ of X. Cover A by the sets $\{U_j\}$, along with the open set $X \setminus B$. Let β be a refinement of this covering that has order at most n. Then the collection

$$\beta' := \{U \in \beta : U \cap B \ne \emptyset\}$$

is a refinement of α that covers B and has order at most n. □

The assumption that B is closed makes the proof significantly simpler, but the result remains true for an arbitrary subset of A, see Theorem 3.2.13 in Edgar (2008), or Theorem III.1 in Hurewicz & Wallman (1941). However, the following 'sum theorem' is not true unless one of the spaces is closed: in fact, $\dim(X) = n$ if and only if X can be written as the union of $n + 1$ subsets all of which have dimension zero (see Theorem III.3 in Hurewicz & Wallman (1941)).

[1] In the context of metric spaces it is somewhat artificial to make the definition in this form, since (A, ϱ) is a metric space in its own right. But our main focus in what follows will be on subsets of Hilbert and Banach spaces, where the underlying linear structure of the ambient space will be significant.