JOHN P. COX

# Theory of Stellar Pulsation. (PSA-2)

PRINCETON LEGACY LIBRARY

# Theory of
# Stellar Pulsation

PRINCETON SERIES IN ASTROPHYSICS

Edited by Jeremiah P. Ostriker

Theory of Rotating Stars, *by J. L. Tassoul*
Theory of Stellar Pulsation, *by J. P. Cox*

John P. Cox

# THEORY OF STELLAR PULSATION

PRINCETON UNIVERSITY PRESS
Princeton, New Jersey

Published by Princeton University Press, Princeton, New Jersey
In the United Kingdom: Princeton University Press, Guildford, Surrey



Library of Congress Cataloging in Publication Data will be found on the last printed page of this book

This book has been composed in Times Roman

Clothbound editions of Princeton University Press books are printed on acid-free paper, and binding materials are chosen for strength and durability

Printed in the United States of America by Princeton
University Press, Princeton, New Jersey

Princeton Legacy Library edition 2017
Paperback ISBN: 978-0-691-61597-4
Hardcover ISBN: 978-0-691-62996-4

To Richard N. Thomas

who provided me with the opportunity
to develop extensive lecture notes,
on which this book is partly based.

# Contents

# Preface

It is now about thirty years since a new text or monograph dealing mainly with the theory of stellar pulsation has been published in the English language. The last such book, to the best of my knowledge, was S. Rosseland's classic, *The Pulsation Theory of Variable Stars* (Oxford University Press, 1949), which dealt almost exclusively with the theory of purely radial oscillations.* Even the monumental and remarkably comprehensive encyclopedia article by P. Ledoux and Th. Walraven *(Handb. d. Phys.,* Vol. **51,** 1958) is by now more than twenty years out of date. This article has been the standard reference in the field for many years and will no doubt continue in this role; indeed, the work is referred to in almost every section of this book. However, some of the most important astrophysical problems having to do with variable stars have been solved since the Ledoux-Walraven article was published. Also, much of the basic theory of nonradial oscillations was not developed until the mid and late 1960's, and new developments are still occasionally coming in. Some of these recent developments have been described in various review articles (see the references given in Chapter 1). These articles, however, along with most of the background material needed for a detailed understanding of the theory of stellar pulsation, are scattered throughout the physical and astrophysical literature. In this book I have collected much of this material into one place, and attempted to fill the need in the astrophysical literature for an up-to-date, reasonably comprehensive, and sufficiently detailed treatment of these matters.

The theory of both radial and nonradial oscillations is discussed in this book. However, the recent (mid 1960's and later) extensions of the theory into general relativity are considered for the most part outside the scope of the present work. Thus, except as mentioned otherwise, the treatment throughout most of this book is based on nonrelativistic, Newtonian physics.

The book is divided into three main parts. Part I, consisting of five chapters, is devoted to fundamentals. It contains a brief summary of the main observations (Chap. 3); a brief summary of the basic equations of hydrodynamics and heat flow, couched in forms suitable for later astrophysical applications (Chap. 4); and a fairly thorough discussion of the linear theory (Chap. 5). I have taken considerable pains to elucidate the

*Recently, a monograph dealing with nonradial stellar oscillations has been published by Unno, Osaki, Ando, and Shibahashi (1979).

differences between Eulerian and Lagrangian variations, and to write the linearized equations for the more general case in which there is a velocity field present. Much of this material is based on a paper published in 1967. I also refer to an unpublished proof worked out as recently as 1974.

Part II, consisting of eight chapters, is concerned with purely radial oscillations. The theory of linear, adiabatic, radial oscillations is presented in considerable detail (Chapter 8), as is the theory and calculations of nonadiabatic and nonlinear radial oscillations (Chapters 9 through 12). Some simple models of stellar pulsation (essentially radial) are described in Chapter 13. Some of the important recent developments in our understanding of variable stars are also summarized.

Part III, made up of the final six chapters, is concerned primarily with the theory of nonradial stellar oscillations. Most of the conventional notation and terminology associated with nonradial oscillations is contained in Chapter 17. The bulk of this chapter was written in 1976–1977; therefore, the level of sophistication contained in, for example, the papers of Christensen-Dalsgaard (1979), Shibahashi (1979), and Wolff (1979), and in the monograph by Unno, Osaki, Shibahashi, and Ando (1979), is not reflected in this chapter. This fact, though regrettable, is also inevitable in a rapidly developing field, especially when delays in publication are taken into account. In this Part I have included not only the topics that might logically be considered a part of this subject, but also some of the newer developments referred to above. Chapter 19, in particular, is devoted to "miscellaneous" topics. These are primarily characterized by a relaxation of one or more of the assumptions usually adopted—assumptions which are held through most of the rest of the book.

The book is aimed at about the level of the first-year graduate student. A knowledge of calculus, differential equations, vector analysis, and matrix algebra is assumed. Because of space limitations, I have not been able to include detailed proofs and derivations in most cases. Therefore, considerable demands may be made on the reader.

Although the book was not written as a text (for example, no problems are included), it may nevertheless be useful in that capacity. There is probably more than enough material to comprise a one- or two-semester graduate course in variable stars. The book will probably be of greatest use to students and research workers in this and related fields.

*Theory of Stellar Pulsation* has grown, for the most part, out of lecture notes developed for a graduate course in variable stars that the author has given at the University of Colorado several times over the past fifteen years. Some of the contents of several of the author's review papers on variable stars have also been incorporated (of course, some of the contents of some of these review papers had their first origin in these same lecture notes!).

I take great pleasure in acknowledging the many people who have contributed, directly or indirectly, to this project. I would particularly like to express my gratitude to the University of Colorado for granting me a sabbatical leave during the Spring semester of 1975, when the actual transformation from lecture notes into a book was begun and partially completed. I am also grateful to Professor V. H. Regener for making available the facilities of the Department of Physics and Astronomy of the University of New Mexico during part of this period, and to Professor D. S. King of that Department for much moral support and for many useful discussions.

Particular thanks go to Dr. Morris Aizenman for suggestions regarding publication and for many discussions, and to Professor S. Chandrasekhar for reading and making useful comments on some of the chapters, as well as for much general assistance and advice. I am grateful to Professor Carl Hansen, who read and commented on many chapters, and offered useful general suggestions; to Professor N. Baker, who also read and commented on several chapters; and to Professor M. Smith, who made helpful comments and suggestions, mostly about Chapter 17. The efforts and interest of Dr. C. G. Davis, who kindly brought to the writer's attention a number of references and conference proceedings which might otherwise have been missed, are also much appreciated. On the other hand, I offer sincere apologies in advance to those persons whose work has been insufficiently cited or referenced.

Useful discussions with Professor A. Weigert, Dr. M. Aizenman, and Professor P. Smeyers have resulted in a clarification of many concepts for the writer. I am especially grateful to Professor Smeyers for some expert help on some difficult points. I also appreciate the very proficient assistance of D. Schwank, R. Gross, and B. Carroll in attending to the seemingly infinite amount of work required in assembling a book of this kind. My gratitude also to Allen Wynne and Myrle Crouch for their kind assistance with some of the bibliographical material. Professor R. McCray was also most helpful, particularly with respect to some of the references on X-ray bursters.

I wish to thank my wife, Dr. J. B. Blizard-Cox, for much patience, encouragement, and moral support over the nearly four years and many absences required to write this book.

Discussions with numerous persons, in addition to those mentioned above, have influenced, either directly or indirectly, the final form of the book. Among these persons are T. Adams, G. L. Berggren, W. Brittin, T. Brown, R. Buchler, J. Castor, J. Christensen-Daalsgard, R. Christy, L. Cloutman, B. Cogan, A. Cox, E. F. Cox, W. R. Davey, D. Davison, R. Deupree, B. L. Dickerson, W. Dziembowski, D. Eilers, J. Faller, D. Fischel, W. Fitch, P. Flower, R. H. Garstang, M. Goossens, D. Gough, H.

Hill, S. Hill, N. Hoffman, D. Hummer, C. Keller, G. E. Langer, J. Latour, P. Ledoux, J. Lesh, D. Lind, J. M. Malville, J. McGraw, P. Melvin, B. Mihalis, D. Mihalis, G. Nelson, Y. Osaki, A. Phelps, R. Ross, H. Saio, E. Schmidt, M. Schwarzschild, R. Scuflaire, N. Simon, A. Skumanich, W. Spangenburg, W. Sparks, S. Starrfield, R. Stellingwerf, C. Sterken, R. Stobie, P. Stry, J. Toomre, H. Van Horn, G. Wallerstein, B. Warner, J. C. Wheeler, C. Whitney, D. Winget, C. Wolff, C. Zafiratos, and K. Ziebarth.

I am also especially grateful to the many students, too numerous to mention by name, who have suffered through my lectures, and whose questions have resulted in a sharpening of my understanding of certain points. This enhanced degree of understanding has presumably resulted in a clarification of a number of discussions in the book.

Sincere thanks are also extended to the numerous persons who have sent me preprints of their work over the past few years.

For their skillful preparation of the typescript and assistance with numerous other editorial matters, I am greatly indebted to Lorraine Volsky, Leslie Haas, and Gwendy Romey.

For their kind permission to reproduce material extensively, I am grateful to The Institute of Physics, The Astronomical Society of Japan, Astronomy and Astrophysics, The Astrophysical Journal, and Annual Reviews, Inc. I would also like to thank the Princeton University Press and its editorial staff for much patience and forbearance. Particular thanks are extended to Mr. Edward Tenner.

Much of the support for the time required to write this book has been provided by National Science Foundation Grants MPS72-05309, AST72-05039 A04, AST77-23183, AST76-01586, and AST78-42115, all through the University of Colorado. I am grateful also to Dr. Charles F. Keller, who made the facilities of the Los Alamos Scientific Laboratory available to the writer (support provided by the Energy Research and Development Administration), and I appreciate the interest of and many discussions with Dr. Arthur N. Cox, of that Laboratory, over a number of years.

*August 15, 1979*

# I

# PRELIMINARIES

# 1

# Introduction

Pulsating stars are stars in which large-scale dynamical motions, usually including the entire star, and usually more or less rhythmic, are present. The simplest kind of such motion is a purely radial pulsation, in which the star maintains a spherical shape at all times, but changes its volume, as if it were breathing.

The study of pulsating stars constitutes a relatively small, but highly important, area of modern stellar astrophysics. The idea that certain types of variable stars owe their variability to periodic or cyclic expansions and contractions dates from the work of Shapley (1914), and was given a firm mathematical foundation by Eddington (1918a,b). Since then the "pulsation hypothesis" has gained wide acceptance. The study of pulsating stars, both theoretical and observational, has proved a powerful tool in the study of stellar structure and in other aspects of modern astrophysics. (Summaries of some of the early history of the pulsation theory have been given by Rosseland 1949, Chap. 1; Eddington 1926, Chap. 8; and Ledoux and Walraven 1958.) One of the more spectacular and far-reaching fruits of the observational study of one of the best-known types of pulsating stars, the *classical Cepheids,* is the famous *period-luminosity* relation (see §3.1). This relation provides the astronomer with one of the most basic "yardsticks" for the measurement of truly great astronomical distances, of the order of the mean separation between galaxies, and has played a crucial role in the establishment of the basic distance scale of the universe. In addition, the attempts to understand the cause and nature of stellar pulsations have served as a challenge to the theorist, and have provided some fascinating and, in some ways, unique applications of physical theory. Further discussion of the importance and significance of the study of pulsating stars will be found in §19.7.

Pulsating stars comprise only a subset of the wider class of *intrinsic variable stars.* These are stars whose variability arises from causes entirely *within* themselves, and not from geometric effects such as eclipses in binary stars; or to some external agency such as interaction with the interstellar medium or with circumstellar matter. The whole class of intrinsic variables includes many different kinds of objects, some of which, such as the *quasi-stellar objects,* are probably not stars in the usual sense of the term. (For recent reviews of these objects, see, for example, Burbidge and Burbidge 1967; Perry, Burbidge, and Burbidge 1978; Schmidt 1969; an updated, semi-popular account is given in H. Smith

1978). The intrinsic variables are usually divided into two broad groups, the *pulsating variables* and the *eruptive variables*. This monograph is concerned primarily with the former group: a brief survey of the types of stars included therein will be presented in Chapter 3. Among the eruptive variables are the spectacular novae and supernovae, which will not be discussed in detail in this book. Recent reviews of certain types of eruptive variables may be found, however, in Shklovsky (1968) and in Oke and Searle (1974) (supernovae); Payne-Gaposchkin (1957) (novae); Kukarkin and Parenago (1963), Payne-Gaposchkin (1954), Ledoux and Walraven (1958) (the whole class of intrinsic variables, including recurrent novae and nova-like stars); Mumford (1967) (dwarf novae); Robinson (1976), Warner (1976a) (cataclysmic variables in general); and Herbig (1962) (T Tauri stars). See also many of the papers in Kippenhahn, Rahe, and Strohmeier (1977). Unless we explicitly state otherwise, we shall always in this book mean "intrinsic variable star" when we use the term "variable star."

The most general definition of a variable star is that it is a star whose physical properties change with time. However, a more restricted definition is implied in normal usage: by variable stars is usually meant stars whose properties change appreciably at a rate fairly easily detectable by astronomers—during, say, a few seconds or fractions of seconds to a few years or decades.

The most obvious and most easily detectable distinguishing feature of a variable star is its apparent brightness: most such stars, in fact, are detected by their light variations. Other observable properties, such as spectral type or color, and radial velocity, usually also vary during the light variations. In the case of *pulsars* (for recent reviews, see, for example, Hewish 1970; Ruderman 1972, 1975; Ginzburg and Zheleznyakov 1975; F. G. Smith 1977; Taylor and Manchester 1977), it is the variable radio radiation, on time scales of a few seconds to a few hundredths of a second, by which these objects are generally detected. However, light variations, synchronized with the radio variations, have been detected in the Crab pulsar, NP-0532 (Cocke, Disney, and Taylor 1969). The light variations are synchronized with the X-ray pulses in some X-ray "pulsars" (e.g., Hiltner and Mook 1970; Lamb and Sorvari 1972; Davidson, Henry, Middleditch, and Smith 1972; Forman, Jones, and Liller 1972). On the other hand, in some cases, such as in spectrum or magnetic variables or in the "line profile variable B stars" (the "53 Persei stars") (M. Smith 1977; Smith and McCall 1978; M. Smith 1978, 1979a,b), the brightness may be almost constant in time, and some other property, such as spectral details or magnetic field strength, may betray the variability of the star (for example, Deutsch 1958; Sargent 1964; Ledoux and Renson 1966). For example, the eleven-year solar cycle makes the sun, strictly speaking, a variable star (not to mention the small-scale oscillations recently reported by Hill and collaborators [Hill, Stebbins, and Brown 1975; Brown, Stebbins, and Hill 1976; Hill 1978; and numerous papers in Hill and Dziembowski 1979]; see also the numerous references in Gough 1977c and, in relation to the whole question of the possible variability of the sun, White 1977 and Eddy 1978). Also, the "X-ray bursters" exhibit variations on scales of minutes (see, e.g., Gursky 1977; Lewin 1977; Lewin *et al.* 1977; Lewis and van Paradijs 1979), while "$\gamma$-ray bursts" are characterized by time scales for variability of ~0.1–100s (e.g., Strong, Klebesadel, and Evans 1975; Klebesadel and Strong 1976; Cline and Desai 1976; Fishman, Watts, and Derrickson 1978). (For a recent review of X-ray sources in general, see Ostriker 1977.)

The variations associated with the pulsating variables may be periodic or cyclic, semi-regular, or irregular. The corresponding time scales range all the way from a few tens of seconds to a few years. It is, of course, possible that time scales lying outside this range exist; but then the problem of detection might become somewhat difficult.

The discovery of periodic or cyclic variables came relatively late in the whole history of astronomy. Apparently, the first authenticated discovery of such a variable star was that of $o$ Ceti (Mira), a Long Period Variable (see Chap. 3), by Fabricius in 1596 (Ledoux and Walraven 1958). A few supernovae, such as the Crab supernova of 1054, Tycho's supernova of 1572, and Kepler's supernova of 1604, had been recorded, but these belong to the class of eruptive variables. Before the end of the eighteenth century, only sixteen variable stars had been discovered, two of which were later found to be eclipsing binaries and five of which were novae (Campbell and Jacchia 1941). Two of these were classical Cepheids: $\delta$ Cephei, the prototype of this kind of star (see Chap. 3), discovered by John Goodricke in 1784; and $\eta$ Aquilae, discovered by Edward Pigott also in 1784 (Campbell and Jacchia 1941). The total number of intrinsic variable stars now known in the Galaxy is some 25,000, of which over 20,000 are listed in the catalog of Kukarkin *et al.* (1969). Over ninety per cent of these are pulsating variables. The total number of such variables in the entire Galaxy is estimated to be $\sim 2 \times 10^6$ (Kukarkin and Parenago 1963). However, since the total number of stars in the whole Galaxy is some $10^{11}$–$10^{12}$, it follows that only about one star in $10^5$–$10^6$ is a pulsating star. Stellar pulsation is therefore quite rare, on the whole, among stars. Nevertheless, it is highly important in astrophysics, as will be seen in later portions of this book. (The recent discovery of the variable white dwarfs, or "ZZ Ceti stars" [McGraw 1977; Robinson and McGraw 1976a,b; Robinson, Nather, and McGraw 1976; Nather 1978], may cause the above numbers to be revised somewhat.)

In Chapter 2 we introduce, primarily for orientation, some important time scales for stars. In Chapter 3 we present a brief survey of empirical information on pulsating variables. Since the observational literature on variable stars is quite extensive, and since a number of good reviews of this subject exist (to be referenced there), we shall only mention the particularly important points.

In Chapter 4 we shall summarize some basic theoretical information which will frequently be referred to in later parts of the book. We shall here and throughout, except when explicitly stated otherwise, employ nonrelativistic mechanics and Newtonian gravitation theory. The neglect of special relativity in the consideration of pulsating stars is well justified in most cases because the relevant velocities are generally small compared to the speed of light. The neglect of general relativistic gravitation theory is also well justified for most pulsating stars because the gravitational fields are usually very weak; equivalently, the mean radii of most kinds of pulsating stars are much larger than their Schwarzschild radii $R_s = 2GM/c^2$, where $G$ is the gravitation constant, $M$ is the mass of the star, and $c$ is the velocity of light. Examples of stellar objects in which these approximations are not justified are dense white dwarfs (see, e.g., Misner, Thorne, and Wheeler 1973); neutron stars; "supermassive stars," if they exist (see, e.g., Wagoner 1969); and collapsed stars, or "black holes" (see, e.g., Ruffini and Wheeler 1971; Penrose 1972; Zeldovitch and Novikov 1971, Chap. 11; Thorne 1967a; Eardley and Press 1975). These developments in the general relativistic theory of pulsating stars will not be considered in detail in this book (however, see §19.5 and Cox 1974a).

The linear theory of stellar oscillations is discussed in Chapter 5. This theory has played a vital role in the development of our present understanding of pulsating stars. Until recent years this theory formed the basis of nearly all theoretical discussions of pulsating stars, even though it was well known that pulsations of actual stars are generally of a large enough amplitude that nonlinear effects are certainly important. The linear theory is nevertheless extremely useful, in part because its relative mathematical simplicity facilitates understanding in physical terms of some of the complicated phenomena involved. This theory is also useful if we believe that at least some types of actual stellar oscillations arose because the star was at one time unstable against infinitely small oscillations. The fact that most of the recognized types of pulsating stars occupy more-or-less well defined regions on the Hertzsprung-Russell (H-R) diagram (see Fig. 3.1 below) suggests a relation between linear instability, which depends (presumably) on the "static" characteristics of a star, and actual stellar oscillations.

Part II will be devoted exclusively to purely radial motion, which will receive a fair amount of emphasis in this book. There are two main reasons for this relatively heavy emphasis. First, this is the simplest kind of motion for spherical stars. It is therefore relatively tractable mathematically, and many of its aspects can be understood physically. Second, most actual pulsating stars appear, fortunately, to be undergoing predominantly just this simple kind of motion.

Part III will be devoted primarily to the theory of nonradial stellar oscillations.

In Chapter 19, certain complicating factors, such as rotation, viscosity, magnetic fields, thermal imbalance, and general relativity in stellar pulsations (both radial and nonradial) will either be dealt with briefly, or at least mentioned, with appropriate references to the literature. A few other miscellaneous topics, such as secular stability of stars, will be referred to, and some comments will be made about the significance of stellar oscillation theory to other areas of astrophysics.

Other recent reviews of pulsating stars and pulsation theory have been provided by Payne-Gaposchkin (1951, 1954); Ledoux and Walraven (1958); Ledoux and Whitney (1961); Ledoux 1963, 1965, 1974, 1978); Zhevakin (1963); Christy (1966a, 1967, 1968, 1969a,b, 1970); J. P. Cox (1967, 1974a, 1975, 1976a, 1979); A. N. Cox and J. P. Cox (1967); King and Cox (1968); J. P. Cox and Giuli (1968; Chap. 27); Iben (1971a); Hoffmeister (1971); Percy (1975); Glasby (1975); and Kukarkin (1976). Earlier reviews are those of Eddington (1926, Chap. 8); and Rosseland (1949). Other useful recent collections of papers on pulsating stars are the proceedings of the Third I.A.U. Colloquium on Variable Stars (Bamberg, Germany, 1965), *The Position of Variable Stars on the* H-R *Diagram;* the proceedings of the Fifth I.A.U. Colloquium on Variable Stars (Bamberg, Germany, 1971), *New Directions and New Frontiers in Variable Star Research;* Detre (1968); Philip (1972); Strohmeier and Knigge (1972); Demarque (1973); Ledoux, Noels, and Rodgers (1974); Fischel and Sparks (1975); Fitch (1976a); A. N. Cox and Deupree (1976); Kippenhahn, Rahe, and Strohmeier (1977); Fischel, Lesh, and Sparks (1978); and Hill and Dziembowski (1979). A monograph on nonradial stellar oscillations has recently been published by Unno, Osaki, Ando, and Shibahashi (1979).

Some works dealing with wave phenomena in general have been found very helpful to the author, and may also be helpful to the reader. Among these are Morse (1936), Greenspan (1968), Tolstoy (1973), Lighthill (1978), and Main (1978).

# 2

## Some Important Time Scales

In this chapter we shall consider, for orientation, some important stellar time scales and their rough orders of magnitude. Time scales lying in the general range of a few seconds to a few years will be of particular interest in connection with pulsating stars.

### 2.1. THE PULSATION PERIOD

The first and most relevant of these time scales for pulsating stars is the pulsation period $\Pi$ of the fundamental mode of purely radial oscillations. While accurate methods of calculating $\Pi$ will be considered in later chapters, it is instructive to consider first some simple, approximate methods of estimating its value that give nearly the same results as do the more elaborate methods. To order of magnitude, this value of $\Pi$ also applies to the lower pressure and gravity modes of nonradial oscillations of somewhat realistic stellar models (see Chapter 17).

Perhaps the most general of these simple methods is that described by Cox (1967). This method uses the fact that stellar pulsations (at least those of low modes) can be regarded, approximately, as a kind of "long-wave" acoustics (wavelength of the "sound wave" of the order of or larger than the dimensions of the system), as is shown by Ledoux and Walraven (1968, Sect. 60; see also §8.9 of this book). The pulsation period $\Pi$ then ought to be of the order of the time required for a sound wave to propagate through the mean or equilibrium diameter of the star. A general expression for the Laplacian (adiabatic) sound speed (see, e.g., §5.5), averaged in some suitable manner over the entire star in its equilibrium state, can be obtained from the virial theorem (see, e.g., Cox and Giuli 1968, Chap. 17). This expression is essentially independent of the material properties of the star.

A crude but roughly equivalent method of obtaining an expression for the mean sound speed is the following. The equation of hydrostatic equilibrium is used (see, e.g., Chapter 4), and all quantities therein are regarded as average, or representative, values throughout the star. Substituting these values into the expression (eq. [5.38]) for the sound speed then yields the desired result.

This procedure shows that, very nearly, $\Pi(\bar{\rho})^{1/2}$ = *constant*. This is the famous period-mean density relation, which seems to be satisfied by most types of pulsating stars (see, e.g., Chapter 3). According to this relation, a large, tenuous star will have a longer period than will a small, compact star.

The details of the above considerations will reveal that the *constant* in the above expression contains the factor $\Gamma_1^{-1/2}$, where $\Gamma_1$ is one of the adiabatic exponents (see Chapter 4), here assumed constant. A more careful derivation of the above expression shows that this factor should actually be replaced by the factor $(3\Gamma_1 - 4)^{-1/2}$, which arises from the spherical symmetry and the variations of gravity which are not fully taken into account in the above considerations. This latter factor may cause $\Pi$ to be considerably larger than the value given in the above expression if $\Gamma_1$ is close to 4/3; this will be the case in relativistic white dwarfs or neutron stars, or in very massive stars where radiation pressure is more important than gas pressure. For $\Gamma_1 = 4/3$, $\Pi = \infty$; and the star is dynamically unstable if $\Gamma_1 < 4/3$.

Another simple, approximate expression for the pulsation period of a star, which, moreover, yields the correct factor $(3\Gamma_1 - 4)^{-1/2}$, is the following. Suppose that the entire mass $M$ of the star is concentrated in a point at the center, and that the stellar surface, lying at a mean distance $R$ from the center, is represented by a thin, spherical shell of this radius, having a mass $m$ small compared with $M$, and offering no resistance, other than inertia, to changes in its radius (that is, the shell is completely compressible, inviscid, and has zero surface tension). The entire volume within the shell is filled with a uniform, massless gas whose only function is to supply pressure to support the shell against gravity, and the shell is surrounded by vacuum (pressure $P = 0$). If $r$ is the instantaneous radius of the membrane, its equation of motion is

$$m\ddot{r} = 4\pi r^2 P - \frac{GMm}{r^2}, \tag{2.1}$$

where a dot denotes the time derivative, $P$ denotes the (spatially constant) gas pressure inside the membrane, and $G$ is the constant of gravitation. We now assume small, adiabatic oscillations about the hydrostatic equilibrium state ($\ddot{r} = 0$); that is, $\delta P/P = \Gamma_1\, \delta\rho/\rho$, where $\delta P$, for example, denotes the departure of the pressure from its equilibrium value. Linearizing eq. (2.1) (further details regarding linearization may be found in Chap. 5) and assuming a time dependence of the form $e^{i\sigma t}$, it is a simple matter to show that the angular pulsation frequency $\sigma$ is given by the relation

$$\sigma^2 = (3\Gamma_1 - 4) \cdot \frac{GM}{R^3} = (3\Gamma_1 - 4) \cdot \tfrac{4}{3}\pi G\bar{\rho}, \tag{2.2}$$

which defines the mean density $\bar{\rho}$.We then obtain the following expression for the pulsation period $\Pi = 2\pi/\sigma$:

$$\Pi = 2\pi/[(3\Gamma_1 - 4) \cdot 4/3\, \pi G\bar{\rho}]^{1/2} . \tag{2.3}$$

Interestingly enough, this is precisely the expression for the fundamental pulsation period of purely radial pulsations of the homogeneous (constant-density) model of given $\Gamma_1$ and $\bar{\rho}$.

By carefully following through the derivation of eq. (2.2), it is easy to discover the origin of the "magic" critical number 4/3 (see above remarks concerning dynamical instability). Write the number as (2 + 2)/3. One of the 2's comes from the inverse square character of Newtonian gravitation (which is the only kind we consider in this book unless we specifically state otherwise); the other 2 comes from the fact that the total pressure force on a sphere of radius $r$ varies as $r^2$. The 3 comes from the three-dimensionality of physical space: the volume of a sphere of radius $r$ varies as $r^3$.

It is customary to write the period-mean density relation in the form

$$\Pi(\bar{\rho}/\bar{\rho}_\odot)^{1/2} = Q, \tag{2.4}$$

where $Q \sim (G\bar{\rho}_\odot)^{-1/2}$ ($\bar{\rho}_\odot = 1.41$ gm cm$^{-3}$ = mean density of the sun) is the "pulsation constant." It is not actually a constant, as its value depends, generally only weakly, on $\Gamma_1$ and on the structure of the star. Accurate calculations show that, for the fundamental radial mode and for $\Gamma_1 = 5/3$,

$$0\overset{d}{.}03 \lesssim Q \lesssim 0\overset{d}{.}12, \tag{2.5}$$

while a representative value is $Q \approx 0\overset{d}{.}04$. Fitting formulae, giving $Q$ as a function of stellar parameters (mostly mass and equilibrium radius), have been provided by Cox, King, and Stellingwerf (1972) and Faulkner (1977b). Since $Q$ is the period that the sun would have if it were pulsating, we see that its period would be of the order of an hour. Observations (uncertain as they are!) of many variable stars yield values in the general range

$$0\overset{d}{.}02 \lesssim Q \lesssim 0\overset{d}{.}11, \tag{2.6}$$

in reasonable agreement with theory.

The pulsation periods to be expected of known kinds of stars can be estimated on the basis of the period-mean density relation (2.4). Considering stars of mean densities lying between those of moderately dense white dwarfs, $\bar{\rho} \sim 10^6$ gm cm$^{-3}$, and those of tenuous red supergiants, $\bar{\rho} \sim 10^{-9}$ gm cm$^{-3}$, we obtain periods lying in the approximate range

$$3 \text{ seconds} \lesssim \Pi \lesssim 1000 \text{ days}, \tag{2.7}$$

which nicely spans the range of periods observed for most types of periodic or cyclic intrinsic variables (see Chapter 3). This rough agreement provides good general support for the pulsation theory of variable stars. There are stronger and more specific arguments in favor of this theory that are, however, outside the scope of this book (see, e.g., Eddington 1926, Chap. 8). We may note that, had neutron stars (which are probably represented by the pulsars), with mean densities $\bar{\rho} \sim 10^{15}$ gm cm$^{-3}$, been included in the above selection of stars, the lower limit of the above period range would have been a few milliseconds. While there is as yet no direct evidence that neutron stars are pulsating, some of the finer details, with time scales of a few milliseconds, observed in the pulsed radio radiation from pulsars (e.g., Taylor and Huguenin 1971), could well be a result of pulsations.

## 2.2 THE "FREE-FALL" TIME

The "free-fall," or "dynamical," time scale, $t_{ff}$, is the characteristic time associated with dynamical collapse, or with the orbital motion of a satellite circling the parent body very close to its surface; $t_{ff}$ is also the characteristic time for a significant departure from hydrostatic equilibrium to alter the state of a star appreciably.

A simple estimate of the order of magnitude of $t_{ff}$ can be obtained by calculating the time required for a unit mass to fall freely through a distance of the order of $R$ (stellar radius) under the influence of a (constant) gravitational acceleration equal to the surface gravity $GM/R^2$ of a star of mass $M$. This procedure yields

$$t_{ff} \sim (G\bar{\rho})^{-1/2} \tag{2.8}$$

(other approximate methods of obtaining eq. [2.8] are presented in, e.g., Cox and Giuli 1968, Chap. 1). Equation (2.8) shows that, aside from numerical factors generally of order unity, $t_{ff}$ is of the order of the pulsation period $\Pi$. This well-known result is a consequence of the fact that the characteristic velocities associated with low-order, largely radial pulsations (the sound speed) and with dynamical processes (e.g., free fall or orbital speeds) or low-order, nonradial gravity oscillations are all determined, via the virial theorem, essentially by the gravitational energy of the star.

## 2.3 THE KELVIN TIME

The "Kelvin time," $t_K$, is essentially the "relaxation time" for departures of a star from *thermal* equilibrium, that is from balance between energy generated by thermonuclear reactions in the stellar interior and energy lost by radiation, both photonic and neutrinic, through the stellar surface. The order of magnitude of $t_K$ can be estimated as follows. Let $E_{th}$ be the total internal (thermal) energy of a star and $L$ the *luminosity* (net rate of loss of energy through the surface) of the star. Then we have, to order of magnitude,

$$t_K \sim E_{th}/L. \tag{2.9}$$

However, $E_{th}$ can be related to $\Omega$, the gravitational energy of the star, by the virial theorem. This theorem can be written in the following general form, for a self-gravitating system in hydrostatic equilibrium that possesses no mass motions (for example, turbulence, rotation, pulsation) and no magnetic fields, and for which the pressure vanishes on the surface:

$$3\int_V PdV = -\Omega, \tag{2.10}$$

where $P$ is the total pressure and the integration is extended over the entire volume $V$ of the star, and

$$\Omega \equiv -\int_M \frac{Gmdm}{r} \equiv -q\frac{GM^2}{R} \tag{2.11}$$

is the gravitational potential energy of a spherical star. Here $q$ is a dimensionless constant whose value depends on the mass concentration of the star but is of order unity for chemically homogeneous stars, and the integration is extended over the entire stellar mass $M$. If we assume that the pressure is supplied by a simple, perfect, nonrelativistic gas, then we have $E_{th} = (3/2)\int_V PdV$, which yields the simple form $E_{th} = -(1/2)\,\Omega$ of the virial theorem. Using this last result in eqs. (2.9) and (2.11) for $\Omega$, and taking $q \approx 3/2$, we obtain

$$t_k \sim 3/4\,\frac{GM^2}{LR} \sim 2\times 10^7\,\frac{M^2}{LR}\text{ years}, \tag{2.12}$$

where $L$, $M$, and $R$ are in solar units. The Kelvin time $t_K$ is also the time that would be required for a star to contract from infinite dispersion to its present radius if $L$ were to remain constant during the entire contraction.

The Kelvin time is normally not of immediate concern as far as the *periods* of pulsating stars are concerned. However, as we shall see (Chapter 9), it is relevant in connection with growth rates, or $e$-folding times, for the growth or decay of pulsations.

A useful dimensionless quantity is the ratio of the free-fall time (~ pulsation period) to the Kelvin time:

$$\frac{t_{ff}}{t_k} \sim \frac{\Pi}{t_K} \sim \frac{LR^{5/2}}{G^{3/2}M^{5/2}} \sim 10^{-12}\frac{LR^{5/2}}{M^{5/2}}, \tag{2.13}$$

if $L$, $M$, and $R$ are in solar units. It is thus seen that for stars not differing greatly from the sun, the pulsation period is many orders of magnitude smaller than the Kelvin time.

## 2.4 THE "NUCLEAR" TIME

The "nuclear" time scale, $t_{\text{nuc}}$, is only of indirect interest in connection with pulsating stars, but knowledge of its value is useful for orientation. This time scale is, loosely speaking, the time required for the properties of a star to change appreciably as a result of nuclear evolution (changes in internal chemical composition due to nuclear transmutations). For a hydrogen-burning star we may make use of the fact that an amount of energy $\sim 0.007\, c^2 \sim 6 \times 10^{18}$ ergs ($c$ = light velocity) is released per gram of hydrogen that is fused into helium. Assuming that $\sim 10\%$ of the mass of the star is available for this fusion, we obtain

$$t_{\text{nuc}} \sim 10^{10}\, M/L \text{ years}, \tag{2.14}$$

where again $M$ and $L$ are in solar units. It is seen that, normally,

$$t_{\text{nuc}} \sim 10^3\, t_K.$$

# 3

## Some Observational Considerations

In this chapter we shall discuss briefly certain topics related to variable stars that are of primarily observational interest. Further details of the observed characteristics of the numerous types of variable stars can be found in the comprehensive and detailed discussions of, for example, Ledoux and Walraven (1958), Payne-Gaposchkin (1951, 1954), Payne-Gaposchkin and Gaposchkin (1963), Kukarkin and Parenago (1963), Hoffmeister (1971), Strohmeier and Knigge (1972), Kukarkin (1976), Pel (1978), and in certain of the references given in Chapter 1. See also the summary in J. P. Cox (1974a). Because of the vastness of the literature on this subject and the rate at which discoveries are being made, no claim is made for completeness for the most up-to-date observational results. Interested persons are advised to check the current astronomical and astrophysical literature. We have not included the *spectrum and magnetic variables* (see, e.g., Ledoux and Renson 1966), the *flare (UV Ceti)* stars (see, e.g., Lovell 1971), nor the *T Tauri* stars (see, e.g., Herbig 1962, 1978) among the pulsating variables, because it is not clear that their characteristics are necessarily directly related to pulsations. We have also not included the quasi-stellar objects, for reasons given earlier (Chapter 1); nor the pulsars, as their main observed characteristics are generally believed to be a result of *rotation* rather than pulsation (see, e.g., Hewish 1970; Cameron 1970; Ruderman 1972; Canuto 1977). We have also not included the recently observed oscillations of some of the cataclysmic variables (see, e.g., Warner and Robinson 1972; Patterson, Robinson, and Nather 1976; Warner 1976a,b; Robinson 1976; Stiening, Hildebrand, and Spillar 1979), as the nature and cause of these oscillations are unknown. Some of the material in this chapter has been borrowed from J. P. Cox (1974a).

We have summarized in Table 3.1 some of the properties of most of the recognized types of pulsating variables. In Figure 3.1 are shown the locations of some of these various types, as well as some others, on a Hertzsprung-Russell (H-R) diagram.[1]

The classical Cepheids and W Virginis stars are sometimes called

[1]This, as well as other items of astronomical nomenclature and general information, may be found in any text on introductory astrophysics, for example, Aller (1963), Swihart (1968), Unsöld (1977), Smith and Jacobs (1973), Rose (1973), or Harwit (1973).

## TABLE 3.1

### The Pulsating Variables*

| Kind of Star | Range of Periods | Charac-teristic Period | Population Type | Range of Spectral Types | Absolute Magnitude ($M_V$) |
|---|---|---|---|---|---|
| RR Lyrae | 1.5–24 h | 0.5 d | II | A2–F2 | 0.0 to +1.0 |
| Classical Cepheids | 1–50 d | 5–10 d | I | F6–K2 | −0.5 to −6 |
| W Virginis Stars | 2–45 d | 12–20 d | II | F2–G6(?) | 0 to −3 |
| RV Tauri Stars | 20–150 d | 75 d | II | G, K | ~−3 |
| Red Semi-Regular Variables | 100–200 d | 100 d | I and II | (K), M, R, N, S | −1 to −3 |
| Long Period Variables | 100–700 d | 270 d | I and II | $M_e$, $R_e$, $N_e$, $S_e$ | +1 to −2 |
| $\beta$ Cephei Stars | | | | | |
| ("$\beta$ Canis Majoris Stars") | 4–6 h | 5 h | I | B1–B2 | −3.5 to −4.5 |
| Dwarf Cepheids and $\delta$ Scuti Stars | 1–3 h | 2 h | I | A2–F5 | +2 to +3 |
| Beat Cepheids | 1–7 d | 2 d | I(?) | F0–G0(?) | −1 to −3(?) |
| Variable White Dwarfs | | | | | |
| ("ZZ Ceti Stars") | 200–1000 s | 500 s(?) | I(?) | A5–F5(?) | +10 to +15(?) |

*Adapted from Table 1 of Cox (1974a), courtesy of the Institute of Physics.

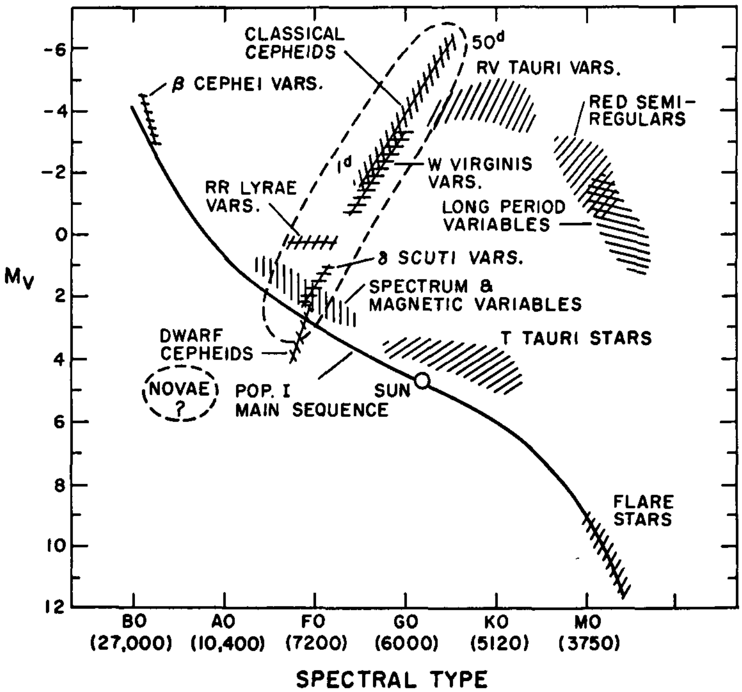


Figure 3.1. Location of a number of various types of intrinsic variables on the Hertzsprung-Russell diagram. From Figure 1 of J. P. Cox (1974a), courtesy of the Institute of Physics.

collectively "Cepheids," being regarded as counterparts, distinguished by Population type, of a single kind of star. The red semi-regular variables and the Long Period Variables are sometimes collectively referred to as the "red variables."

The group of stars in the upper part of Table 3.1 (the RR Lyrae variables, Cepheids, RV Tauri stars, and the red variables) is sometimes referred to as the *Great Sequence.* Note that, as one descends this part of the table, the characteristic periods become progressively longer and the stars become progressively redder (cooler).

Such a general correlation between period and spectral type (or color) can easily be shown to be just what would be expected for a case of radial pulsations. We may say, alternatively, that large stars have relatively long periods (long sound travel times through their diameters): such stars are, for given luminosity, relatively cool. Therefore, increasing periods and increasing coolness tend to go together.

The significance of the nearly vertical oval region shown by dashed lines in Figure 3.1 is that most pulsating stars lying in this region (the RR Lyrae variables, classical Cepheids, W Virginis variables, and dwarf Cepheids and $\delta$ Scuti variables) are thought to owe their instability to a common physical mechanism (second ionization of helium in the envelope), the details of which will be discussed in some detail in Chapter 10. This oval region is sometimes referred to loosely as the "instability strip" or "instability region." It has been suggested by Van Horn (1978), Nather (1978), and Hansen (1979) that this instability region might even include the variable white dwarfs (or "ZZ Ceti" stars, see below). However, see J. P. Cox and Hansen (1979).

For recent reviews of some of the short-period variables, see Petersen (1976), McNamara and Feltz (1978), and Breger (1979).

If one examines the frequency distribution of periods for the pulsating variables in the Galaxy, corrected for selection effects, one finds more or less well-defined peaks at the characteristic periods for the various kinds of pulsators listed in Table 3.1 (e.g., Payne-Gaposchkin 1954, p. 17). This fact suggests that the classification of pulsating stars into distinct types has some basis in reality.

The most common kind of pulsating variable, in terms of numbers per unit volume of space, is found to be, at least in the part of the Galaxy in the vicinity of the sun, the recently discovered variable white dwarfs (the "ZZ Ceti stars," McGraw 1977, Nather 1978). They appear to outnumber all other types of variables stars by a considerable factor ($>10^2$?).

## 3.1 CLASSICAL CEPHEIDS AND THE PERIOD-LUMINOSITY RELATION

Because of the importance of the classical Cepheids and their role in establishing the basic distance scale of the universe, through the famous period-luminosity relation, we devote here a special section to this type of variable star.

The prototype of this kind of star is $\delta$ Cephei, with a period of $5^d\!.366$ (Kukarkin *et al.* 1969, 1974, 1976). Polaris is another classical Cepheid, although the light variations are small ($<0^m\!.1$). Classical Cepheids are yellow giants and supergiants, and are therefore highly luminous (see Table 3.2 below) and visible, if not dimmed by interstellar extinction, at great distances. Classical Cepheids have been observed in about thirty external galaxies.

The periods of classical Cepheids are nearly all confined to the range $1^d$–$50^d$, but a few Cepheids in the Large Magellanic Cloud have periods approaching $100^d$; periods in the Small Magellanic Cloud extend up to about $200^d$ (Payne-Gaposchkin and Gaposchkin 1965). (The classical

Cepheid in the Galaxy with the longest known period is BP Her, with a period of $83^{d}.1$, according to Makarenko 1972.)

About 700 classical Cepheids are known in the Galaxy (Payne-Gaposchkin and Haramundanis 1970), and they are all closely confined to the Galactic plane and partake of the rotation of the Galaxy. They are extreme Population I objects. Because of their confinement to the Galactic plane, they are heavily obscured and reddened by interstellar dust. They are all too distant for their distances to be measured by the usual direct methods (for example, by triangulation). Hence, until recently, the only way to determine the distances of Cepheids was to use statistical methods based on the solar motion relative to the nearby stars. These methods do not always yield very accurate or reliable results. Since the mid-1950's, however, some thirteen classical Cepheids have been discovered in galactic (open) clusters (for the history of these findings, see Fernie 1969). These discoveries have made possible more accurate determinations of the distances of Cepheids (see, e.g., Kraft 1961; Sandage and Tammann 1968, 1969, 1976a,b and references therein; Geyer 1970; Schaltenbrand and Tammann 1970; Pel 1978), and hence of the zero point of the period-luminosity relation (see below).

Some properties of classical Cepheids in the Galaxy are summarized in Table 3.2 (from Cox 1974a). The masses given in this table are only estimates, based on recent stellar evolution calculations with conventional masses. These masses may be only upper limits since actual Cepheid masses may be somewhat smaller than is indicated by the evolutionary calculations (see §19.7). Unfortunately, reliable empirical masses are not available for any Cepheids, since most Cepheids are either single or members of such widely separated binaries that reliable orbital elements, and hence masses, cannot be obtained (see, e.g., Latyshev 1969; Abt 1959).

The light curves of classical Cepheids are skew symmetric and highly

TABLE 3.2

Properties of Galactic Classical Cepheids*

| | Range | |
|---|---|---|
| Property | From | To |
| Period (Π) | $1^{d}$ | $50^{d}$ |
| Mean Luminosity ($L$) | 300 $L_{\odot}$ | 26,000 $L_{\odot}$ |
| Median Spectral Type | F5 | G5 |
| Mean Radius ($R$) | 14 $R_{\odot}$ | 200 $R_{\odot}$ |
| Mass ($M$) | $\lesssim$3.7 $M_{\odot}$ | $\lesssim$14 $M_{\odot}$ |

*Subscript ⊙ denotes solar values.

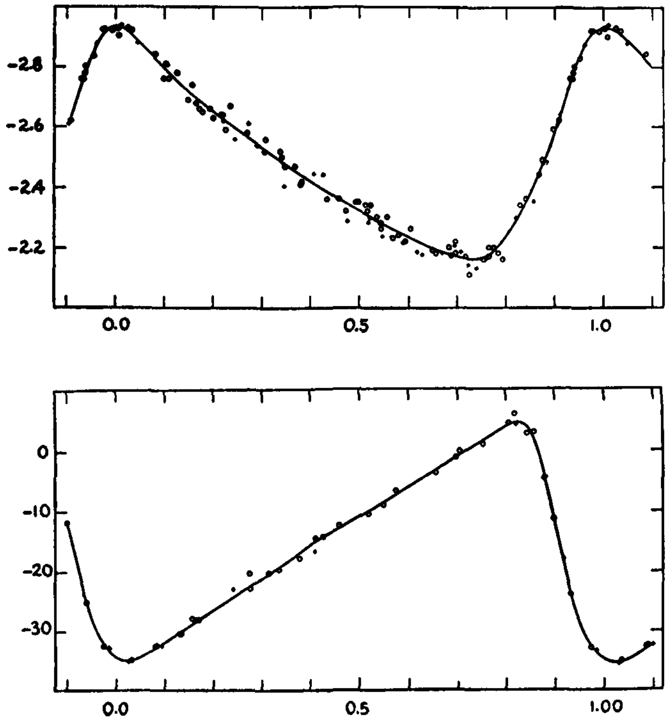


Figure 3.2. Light curve (upper figure) and radial velocity curve (lower figure) (astronomical sign convention) for δ Cephei. Abscissae are phase, and the ordinate of the light curve is apparent magnitude (arbitrary zero point). The ordinate of the velocity curve is in units of km $s^{-1}$ and the zero-point is not corrected for the velocity of approach of the center of mass of the star, relative to the sun, of 16 km $s^{-1}$ (from Goldberg and Aller 1943).

periodic, repeating faithfully over many periods (see Fig. 3.2). The total magnitude range (visual) is about $1^m$; this increases slowly and somewhat erratically with increasing period.

The shapes of the light curves are correlated with the periods. This correlation is known as the *Hertzsprung relation,* and is illustrated, e.g., in Figure 4 of Cox (1974a) (this figure may also be found in Payne-Gaposchkin 1951). (For further discussion of the Hertzsprung relation, see, e.g., Payne-Gaposchkin 1961; Payne-Gaposchkin and Gaposchkin 1966). Note that a secondary hump often appears on the descending branch at periods between $7^d$ and $9^d$. It should be noted, however, that the Hertzsprung relation is statistical in character, as there are many individual exceptions to it (see, e.g., Figure 5 in Cox 1974a).

The spectra and colors of Cepheids also change during the light variation. The spectra are earliest (closest to the O end of the spectral sequence) at maximum brightness, and the spectral changes are consistent with the changes in color. For $\delta$ Cephei, for example, the spectrum varies between F5 and G2 during the cycle (Kukarkin *et al.* 1969); this variation corresponds to a total change of about 1500°C in effective temperature. Most of the variation in brightness arises from the temperature variations; the radius variations are relatively small (fractional semi-amplitude around 0.05–0.10; see, e.g., Nikolov and Tsvetko 1972), and have only a minor effect on the light curves.

The radial velocity curves of classical Cepheids tend to be roughly mirror images of the light curves when the astronomical sign convention regarding radial velocities is used, as shown in Figure 3.2 (from Cox 1974a). If the velocity curve represents the motion of the stellar surface, then the phase relation between the light and velocity curves implies that the star is brightest when it is expanding through its equilibrium radius, and not when its radius is smallest, as might be expected from naive considerations. This retardation of maximum brightness behind minimum radius has been called the "phase lag discrepancy." The phase lag of maximum luminosity behind minimum radius would be about 90° if the light and velocity curves were sinusoidal. However, because of the skewness of the curves, the phase lag is actually considerably smaller than this, perhaps 0.1–0.2 periods. The physical cause of the phase lag has been clarified in recent years, and will be discussed further in Chapter 11.

The total velocity amplitude typically lies in the range 30–40 km $s^{-1}$, but increases slowly and erratically with increasing period $\Pi$, up to some 50–60 km $s^{-1}$ for $\Pi \approx 30^d - 40^d$. Note that, as a result of foreshortening and limb darkening, the true velocity amplitudes are larger than the above values by a factor which is customarily taken to be 24/17 (see §3.4).

Perhaps the most important function of classical Cepheids for the astronomer is their use as powerful distance indicators; they are still the most important tool for establishing the basic distance scale of the universe (Sandage and Tammann 1971; Sandage 1972). This use is based on the well-known period-luminosity relation, which was discovered in 1912 by Leavitt of Harvard on the basis of Cepheids in the Small Magellanic Cloud (Pickering 1912). She found that the mean luminosity increases monotonically with increasing period, but she was unable to specify the zero point of the relation. The history of the determination of this zero point makes a fascinating chapter in the history of astronomy, and has been described by Baade (1956, 1963) and Fernie (1969). Suffice it to say here that the "doubling" of the size of the universe in the early 1950's was the result of the discovery by Baade, using the then newly operative 200 inch Palomar telescope, of an error in the earlier determinations of the zero point: this error had gone undetected for approximately forty years! The question of this zero point is certainly one of the most basic problems of observational astrophysics, because of its importance in the establishment of the distance scale for truly large astronomical distances.

Recent discussions of the empirical period-luminosity relation of classical Cepheids are due to Fernie (1967), Sandage and Tammann (1968, 1969, 1974, 1976a,b), Geyer (1970), van Genderin (1970), Schaltenbrand and Tammann (1970), Gaposchkin (1972), and Pel (1978). The Sandage and Tammann (1968) period-luminosity relation is shown in Figure 3.3 (from Cox 1974a). This is a composite relation, containing Galactic Cepheids as well as Cepheids found in other galaxies. These authors conclude that there is no reason to doubt that a "universal" period-luminosity relation exists for at least all the galaxies included in their study. However, the question of the universality of the period-luminosity relation is apparently not yet entirely settled (see, e.g., Fernie 1969; Gascoigne 1969).

Although the Sandage and Tammann period-luminosity relation is nonlinear, the departures from linearity are rather small. The central line

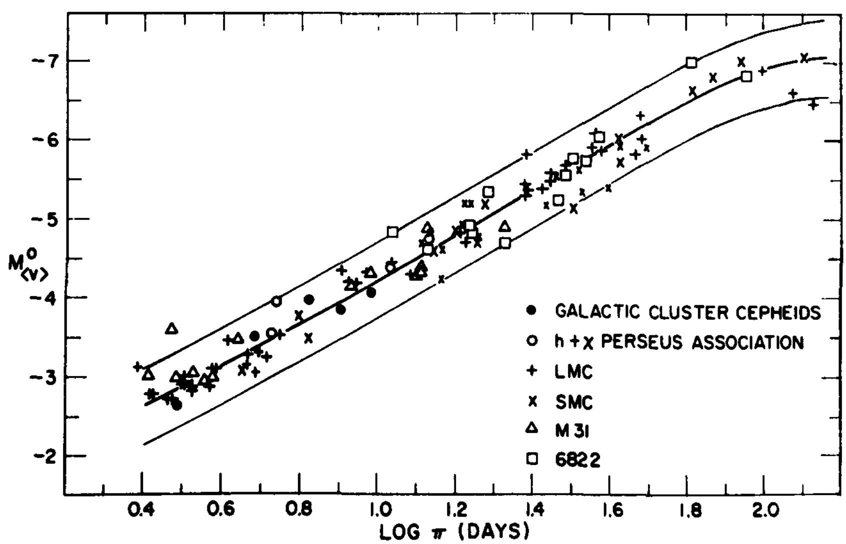


Figure 3.3. The composite period-luminosity relation of Sandage and Tammann (1968). The relation is based on Cepheids in our own Galaxy as well as in others, identified in the figure. The ordinate is absolute visual magnitude, and the superscript 0 means that the absolute magnitudes have been corrected for the effects of interstellar reddening and extinction (after Sandage and Tammann 1968). (Courtesy of *The Astrophysical Journal*, published by the University of Chicago Press, and of the authors.)

of the band shown in Figure 3.3 can be represented adequately over most of its length by the following relation:

$$M_{\langle V \rangle}{}^0 = -2.80 \log \Pi_d - 1.43 \qquad (0.4 \lesssim \log \Pi_d \lesssim 1.7), \qquad (3.1)$$

where the subscript $\langle V \rangle$ denotes an average over period, the superscript 0 means that the absolute magnitudes have been corrected for interstellar reddening and extinction, and the subscripts $d$ mean that the periods are in days. Using the relations among $M_{\langle V \rangle}$, color $(B\text{-}V)$, $T_e$, and $M_{\text{bol}}$ given by Kraft (1961), we may also write eq. (3.1) as

$$\log\left(\frac{L}{L_\odot}\right) = 1.15 \log \Pi_d + 2.47 \qquad (0.4 \lesssim \log \Pi_d \lesssim 1.7), \qquad (3.2)$$

where $L_\odot$ denotes the solar luminosity.

The scatter shown in Figure 3.3 about the central line is thought to be mostly intrinsic and a result of the finite width of the region of instability, and possibly of the presence of stars pulsating in different modes (see Chapter 10 and §19.7). The total intrinsinc width of the period-luminosity relation is approximately $1^m$ at a given period.

There appear to be certain differences between Cepheids in the Galaxy and in the Magellanic Clouds. Perhaps most striking are the differences in the period distributions of Cepheids in these systems. Thus, for example, in the Small Magellanic Cloud there are a great many Cepheids having $\Pi \lesssim 3^d$, whereas in the Galaxy very few Cepheids have periods as short as this. The Cepheids in the Large Magellanic Cloud are intermediate in this respect between those in the Galaxy and in the Small Magellanic Cloud (see, e.g., data summarized by Hofmeister 1967).

## 3.2. MORE RECENTLY RECOGNIZED TYPES OF VARIABLE STARS

Besides the types of variable stars referred to above, at least three additional types have recently received considerable attention in the astronomical and astrophysical literature. Moreover, recent discoveries have increased the membership of these types considerably. In view of these considerations, we present here brief descriptions of these types and some references to the literature.

### 3.2a. RAPID BLUE VARIABLES

The rapid blue variables are a somewhat loosely defined class of objects characterized by relatively blue colors and very short-period light variations. The variations are for the most part quite irregular, and sometimes described as "flickering." The periods of light variation range from some tens to some hundreds of seconds. The observational literature on these objects is summarized in Warner and Robinson (1972); Osaki and Hansen (1973a); Warner and Brickhill (1974); Brickhill (1975); Patterson, Robinson, and Nather (1977); Robinson and McGraw (1976a,b); Robinson, Nather, and McGraw (1976); Warner (1976b); Stiening, Hildebrand, and Spillar (1979); and Nather (1978). These objects are probably for the most part dwarf novae (which are thought to be close binaries; see, e.g., Kraft 1962, 1963; Warner 1976a; Patterson, Robinson, and Nather 1977) and variable white dwarfs (Richer and Ulrych 1974; McGraw and Robinson 1975, 1976; Robinson and McGraw 1976a,b; Robinson, Nather, and McGraw 1976; Van Horn 1978; Nather 1978; and Hansen 1979). In fact, it has been pointed out by McGraw (1977) that the variable white dwarfs are the most numerous of the variable stars and that they comprise a new class which he refers to as the "ZZ Ceti stars." The few rapid blue variables that do not belong to one of these two classes are of an uncertain nature (Lamb 1974; Bath, Evans, and Pringle 1974). Rapid rotation, for example, may be involved (e.g., Lamb 1974; Herbst, Hesser, and Ostriker 1974).

According to McGraw (1977) and Nather (1978), there are now known to be twelve apparently otherwise normal DA white dwarfs, with colors in the range $0.16 \leq B\text{-}V \leq 0.20$ (effective temperature $\sim 10^4$°K), which exhibit periodicities mostly in the range 200s–1000s. Further discussion of these stars can be found in McGraw (1977), Van Horn (1978), Nather (1978), Hansen (1979), J. P. Cox and Hansen (1979), and in some of the above references.

### 3.2b. BEAT CEPHEIDS

The beat (or "double-mode") Cepheids (they may not actually be Cepheids at all) consist of a small number of stars (according to Stobie 1977, eleven are known at present) whose light curves are not periodic. Nevertheless, these light curves can be decomposed into essentially only two (and, in one or two cases, three) periodic variations per star. The periodic light curves for each star, when added together, give back the original, observed, nonperiodic light curve. The above periodic variations are assumed to represent distinct pulsation modes, usually assumed to be the radial fundamental, first harmonic, and, when present, second harmonic. These modes are evidently for some reason simultaneously present in these stars; the modes interact with one another and produce "beats." The longest of these periods is normally between two and seven days, the next shortest period is about 70% of the longest period, and the third period, when present, is about 80% of the second period. It may be significant that the ratio of the second longest period to the longest in no case lies outside the range 0.70–0.71 (Stobie 1977, Simon 1979). These stars are located in the H-R diagram near the low-luminosity end of the Cepheid instability strip (the long, nearly vertical, oval region in Fig. 3.1). It is for this reason that they are called "Cepheids." According to Stobie (1977), nearly half the variables in the Galaxy in the appropriate period range are beat Cepheids.

Important information regarding certain aspects of stellar pulsation in general, and of these stars in particular, can be obtained from their multiple periods, largely because the period in a given mode is determined mostly by the mass and radius of the star (see, e.g., Cogan 1970; J. P. Cox, King, and Stellingwerf 1972). Hence, given two periods, both of the above quantities can in principle be determined. Discussions of these stars have been provided by Fitch (1970); Stobie (1970, 1972); Stobie and Hawardin (1972); Rodgers and Gingold (1973); Petersen (1973, 1974, 1978); Schmidt (1974); King, Hansen, Ross, and Cox (1975); Fitch and Szeidl (1976); A. N. Cox and Cox (1976); Cogan (1977, 1978a,b); Faulkner (1977a,b); Saio, Kobayashi, and Takeuti (1977); A. N. Cox, Deupree, King, and Hodson (1977); J. P. Cox (1978a); A. N. Cox, Hodson, and King (1979); see also the review papers by A. N. Cox (1978b) and J. P. Cox (1978, 1979) and the many references therein.

### 3.2c. LINE PROFILE VARIABLE B STARS

These stars, also called "53 Persei stars" by M. A. Smith (1979a,b), are for the most part main sequence or near main sequence stars of spectral classes mainly in the early and mid B's, say from O8 through B5. However, some of these stars are giants or supergiants, and they occupy those parts of the H-R diagram surrounding and in the general vicinity of the $\beta$ Cephei stars. The line profile variable B stars are quite common, and most stars in the appropriate regions of the H–R diagram probably belong to this class (Smith 1979a).

These stars primarily exhibit temporal changes in the shapes of spectrum lines in a more or less periodic fashion, with periods ranging typically from a few hours to about two days (characteristically $\sim\frac{1}{2}$ day). These spectral line shape changes can be interpreted in terms of nonradial oscillations, in particular of *g* modes (M. A. Smith 1977; Smith and McCall 1978; M. Smith 1978, 1979a,b; Smith and Buta 1979; see also Chapter 17 for an explanation of the terminology). There is evidence of rather frequent changes in the character of the oscillations, with a given character persisting for, typically, about a month (Smith 1979b).

Light variations of ~0.1 magnitudes have also been detected in a few stars of this type (Buta and Smith 1979; Smith, Africano, and Worden 1979). Buta and Smith also present a rather nice discussion of the light variations accompanying nonradial stellar oscillations (see also Dziembowski 1977c).

## 3.3 EMPIRICAL DETERMINATION OF RADII OF PULSATING STARS

Most empirical methods of radius determinations for radially pulsating stars are based essentially on a method devised by Baade (1926) and Wesselink (1946, 1947). This method proceeds in principle as follows. If $F_v$ denotes the radiant flux (rate of radiation of energy per unit area) in some spectral band (normally ~700–1000 Å wide), and $L_v$ represents the corresponding luminosity of the star in the spectral band, then there is at each instant a simple relation between $F_v$, $L_v$, and the instantaneous radial distance $R$ to the effective level in the atmosphere where the radiation in the given spectral band originates ($R$ is approximately equal to the instantaneous stellar radius). The basic assumption underlying the Wesselink method is that $F_v$ is (for a given star) a function only of the *color,* measured by the color index $B$-$V$, of the star. Here $B$ and $V$ are apparent magnitudes, corrected for interstellar reddening, in broadband spectral regions centered, respectively, in the blue and visual (yellow-green) regions of the spectrum. If one now selects two phases during the pulsation cycle, say at times $t_1$ and $t_2$, at which the colors are equal, that is $(B\text{-}V)_1 = (B\text{-}V)_2$, then, according to the basic assumption, $F_v(t_1) = F_v(t_2)$. It then follows that

$$\frac{L_1}{L_2} = \left(\frac{R_1}{R_2}\right)^2, \tag{3.3}$$

where subscripts 1 and 2 refer to quantities at times $t_1$ and $t_2$, respectively. Hence, a measurement of the relative brightnesses of the star at two phases of equal color gives a measure of the *ratio* of the radii at these two phases.

On the other hand, if a *velocity curve* is available for the star, then the velocity, say $\dot{R}(t)$, of the stellar surface relative to the center of mass of the star can be obtained once the correction factor, say $p$, for converting from observed radial velocity $V(t)$ (relative to the center of mass) to $\dot{R}(t)$,

$$\dot{R}(t) = -pV(t), \tag{3.4}$$

is chosen. Assuming that the mean level in the atmosphere corresponding to the velocity curve is the same as the mean level referred to in the