FOREIGN TECHNOLOGY DIVISION

AIR FORCE SYSTEMS COMMAND

APPLIED METHODS OF CALCULATION OF SHELLS AND THIN-WALLED CONSTRUCTIONS

by

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FOREIGN TECHNOLOGY DIVISION

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THE RIGIDITY OF SHELLS

U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* <u>ye</u> initially, after vowels, and after ъ, ь; <u>e</u> elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FTD-MT

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}
rot	curl
lg	log

In the book is examined a wide circle of problems on calculation of shells of revolution and elements of thin-walled constructions for strength, rigidity and stability under various forms of force action. Many of these problems appeared in recent years in connection with the development of new technology. Such problems include, for example, calculations of all sorts of doughnut-shaped shells, loaded by internal pressure, spherical shells, loaded by local loads, and so forth.

Problems of stability of shells are given in the book in a new formulation, the basis of which is formed by the fact that on the contour of pits and bulges, forming as a result of loss of stability, there take place inherent boundary conditions. These conditions on the contour of half-waves are determined by loading conditions and the proposed form of loss of stability.

The new approach to these problems refines and expands the concept of stability of shells and gives the possibility of solving practically important problems.

The book is designed for scientific workers and engineers of aviation and other branches of industry and can be useful to college students.

Tables 14. Illustrations 233. Bibliography 30 names.

PREFACE

At present at the disposal of the engineer-designer there is a number of fundamental works on the theory of plates, thin elastic shells and thin-walled three-dimensional systems, the authors of which are known Soviet Scientists (V. Z. Vlasov, V. V. Novozhilov, I. F. Obraztsov, S. N. Kan, V. I. Feodos'yev, A. S. Vol'mir and others).

In these works are given not only general principles and methods of calculation of plates, shells and thin-walled constructions, but there are also provided solutions of many practically important problems, which the engineer encounters in the process of designing and calculation of a flight vehicle.

This book contains solutions of numerous new problems, having appeared in recent years in connection with the development of constructions of flight vehicles. For some of them, basically problems on the strength and stability of shells of revolution, it is very difficult to find rational solutions within the framework of existing theory of shells. To get an effective engineering solution some additional simplifications must be used, ensuing from analysis of actual work of the construction. Of course, the introduction of such simplifications makes the solution of a complex problem less strict, but it gives the possibility of using it directly in the process of designing. This namely is the basic goal that the author pursued. Solutions given in the book of certain more complex problems are obtained without estimation of the accuracy of quantities of components of stressed and deformed state of constructions.

Questions of theory in the book are touched upon only in the limited volume in which they are necessary for explanation of solutions of some problems.

To get numerical results in the book the energy method is widely used, the effectiveness of which has been shown on numerous examples.

The book consists of three sections. In the small section "Strength and stability of rods and plates" the reader becomes acquainted with the most widespread applied methods of solution of typical problems of structural mechanics, which are illustrated by examples of calculation of rods and plates. Subsequently these methods are used in the remaining sections of the book with solution of problems of strength and stability of shells. Furthermore, this section contains some new results on the calculation of round and square membranes.

The section "Strength of shells" is dedicated to stress and rigidity analysis of smooth and reinforced shells of revolution with various forms of force effects. This section contains solutions of some problems of strength calculation of thin-walled three-dimensional systems.

The last section is dedicated to questions of stability of shells. During examination of these questions the author used his treatment of the problem of stability of shells as a basis. To some readers, who were accustomed to traditional methods of investigation of problems of stability, this treatment can seem questionable. However, numerous experimental results, obtained recently during tests of models of cylindrical shells by Russian and foreign researchers, satisfactorily agree with results of calculations obtained on the basis of theoretical reasons of the author.

The author accepts with gratitude all remarks of the readers, directed toward improvement of the book. Direct them to the address of the publishing house (Moscow, K-51, Petrovka, 24).

The author expresses gratitude to A. N. Yelpat'yevskiy and A. I. Sverdlov for the valuable remarks made by them while examining and editing the manuscript.

P A R T I

STRENGTH AND STABILITY OF RODS AND PLATES

CHAPTER I

STRENGTH OF RECTANGULAR PLATES AND MEMBRANES

§ 1. Basic Information from the Theory of Rectangular Plates of Small Deflection

By plates of small deflection we mean such plates, deflections of which, computed from the middle plane,[1] are small in comparison with the thickness of plate. In such plates membrane stresses in the middle surface[2] can be disregarded in comparison with bending stresses.

The basis of investigation of these plates within limits of elastic deformations is formed by the following differential equation:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\,\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q, \tag{1.1}$$

where w – the sought deflection at any point of the plate; q – intensity of external load; $D=\frac{E\delta^3}{12(1-\mu^2)}$ – cylindrical rigidity of plate; δ – thickness of plate; μ – Poisson's ratio.

[1]The plane, dividing the thickness of the plate in half, is called the middle.

[2]The middle plane after bending of plate is called the middle surface.

Bending and twisting moments are determined by formulas[1]

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu \frac{\partial^2 w}{\partial x^2}\right),$$

$$M_{xy} = -(1-\mu) D \frac{\partial^2 w}{\partial x^2 \partial y^2}.$$

The greatest magnitudes of normal stresses take place at the surface of the plate and are equal to

$$\sigma_{x\,max} = \pm \frac{6M_x}{\delta^2}, \quad \sigma_{y\,max} = \pm \frac{6M_y}{\delta^2}, \quad \sigma_{xy\,max} = \pm \frac{6M_{xy}}{\delta^2}.$$

With solution of equation (1.1) it is necessary to know four boundary conditions in the direction of each axis of coordinates. These conditions can be the following.

1. Edge is freely supported (Fig. 1).

Fig. 1.

In this instance on the edge of the plate deflection w and bending moment M_x (or M_y)

$$w = 0, \quad M_x = 0 \quad (\text{or} \quad M_y = 0).$$

2. Edge is rigidly fixed (Fig. 2).

Fig. 2.

In this instance deflection and the angle of rotation should be equal to zero:

$$w = 0, \quad \frac{\partial w}{\partial x} = 0 \left(\text{or} \quad \frac{\partial w}{\partial y} = 0\right).$$

[1]In these formulas deflection is considered positive if it is directed toward concavity of the middle surface of the plate.

3. Edge is free (Fig. 3).

Fig. 3.

In this instance on the edge of the bending moment and shearing forces should become zero:

$$M_x = 0 \quad (\text{or } M_y = 0),$$
$$V_x = 0 \quad (\text{or } V_y = 0).$$

Shearing forces through deflection are expressed by the following formulas:

$$V_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right],$$
$$V_y = -D\left[\frac{\partial^3 w}{\partial y^3} + (2-\mu)\frac{\partial^3 w}{\partial y\,\partial x^2}\right].$$

4. Edge is elastically supported on a beam with flexural rigidity EJ and torsional rigidity C. In this instance the boundary conditions on edge $x = a$ will be (Fig. 4):

$$EJ\left(\frac{\partial^4 w}{\partial y^4}\right)_{x=a} = (V_x)_{x=a} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right].$$

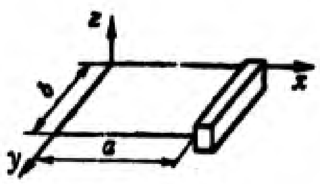

Fig. 4.

The second condition will consist of equality of bending moment of the plate and twisting moment of the beam:

$$C\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_{x=a} = (M_x)_{x=a} = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)_{x=a}.$$

Analogously, if edge $x = 0$ is elastically supported,

$$EJ\left(\frac{\partial^4 w}{\partial y^4}\right)_{x=0} = (V_x)_{x=0} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2}\right]_{x=0},$$
$$C\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)_{x=0} = (M_x)_{x=0} = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right)_{x=0}.$$

§ 2. Methods of Solution of Equations of Bending of Plates

Method of Double Trigonometric Series

This method is especially convenient for plates freely supported along the contour. The desired deflection of plate is sought in the standard of double trigonometric series, each term of which satisfies boundary conditions of the problem and is provided with an indeterminate coefficient.

In an analogous series there is laid out the load affecting the plate, where in practically encountered cases no limitations on the character of load are imposed, i.e., it can be both distributed, and in the form of concentrated forces.

After substitution of the accepted expression for deflection w and load affecting the plate represented by double series in equation (1.1), we can determine all the coefficients in w.

For example, for a plate freely supported along the entire contour with sides a and b, being under the action uniform load q (Fig. 5), we take

$$w=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

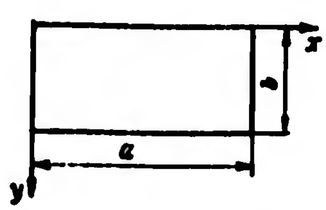

Fig. 5.

Let us formulate the necessary derivatives from this expression and substitute them in equation (1.1). Then

$$D\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}=q. \quad (1.2)$$

Now let us expand the right side of this equation into series in terms of sines

$$q=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

For determination of coefficients B_{mn} let us multiply the right and left sides of the last expression by $\sin\frac{m'\pi x}{a}\sin\frac{n'\pi y}{b}\,dx\,dy$ and integrate with respect to the entire area of the plate. Considering in this case the orthogonality of trigonometric functions on the interval of integration

$$\int_0^a \sin\frac{m\pi x}{a}\sin\frac{m'\pi x}{a}\,dx=\begin{cases}\frac{a}{2} & \text{when } m=m'\\ 0 & \text{when } m\neq m'\end{cases}$$

$$\int \sin\frac{n\pi y}{b}\sin\frac{n'\pi y}{b}\,dy=\begin{cases}\frac{b}{2} & \text{when } n=n'\\ 0 & \text{when } n\neq n',\end{cases}$$

we obtain

$$B_{mn}=\frac{16q}{\pi^2 mn}.$$

Then

$$q=\frac{16q}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

If the load was applied on a limited section of the plate surface, then integration in the left side of expansion q would be necessary only within this section, since everywhere outside it $q \equiv 0$.

From equation (1.2) we obtain the expression for coefficients A_{mn}:

$$A_{mn}=\frac{16q}{\pi^2 Dmn\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}.$$

Then the expression for deflection takes the form

$$w=\frac{16q}{\pi^2D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}.$$

By substituting the corresponding derivatives from deflection in expressions for moments, we can determine stresses σ_x and σ_y at any point of the plate. In this case the obtained series will converge slower than the original series. For example, the series for stresses at the point of application of concentrated force will even be divergent.

The double series are summed up for assigned value of $x=a_1, y=b_1$ in the following manner:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\frac{m\pi a_1}{a}\sin\frac{n\pi b_1}{b}}{mn\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}U_{mn}=$$
$$=U_{11}+U_{12}+U_{13}+\ldots+U_{21}+U_{22}+\ldots+U_{31}+U_{32}+U_{33}+\ldots$$

Bubnov-Galerkin Method.

For solution of equation (1.1) by Bubnov-Galerkin method we assigned the suitable expression for deflection in the form of a series of functions with indeterminate coefficients

$$w=A_1\varphi_1(x, y)+A_2\varphi_2(x, y)+\ldots, \qquad (1.3)$$

where $\varphi_1(x, y), \varphi_2(x, y), \ldots$ — linear-independent functions, which satisfy all boundary conditions of the problem and more or less correctly reflect the shape of the deformed surface of the plate.

Indeterminate coefficients $A_1, A_2, \ldots$ are determined from equation

$$\iint_s\left[D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)-q\right]\varphi_i\,dx\,dy=0 \qquad (1.4)$$
$$i=1, 2, \ldots$$

Integration in equation (1.4) is performed with respect to the entire area of plate. In this case there is obtained as many equations as indeterminate coefficients A_i. The result of the solution of the problem will be more accurate, the more terms that are in expression (1.3).

Example. Let us examine bending of a rigidly fixed rectangular plate with constant load q (Fig. 6).

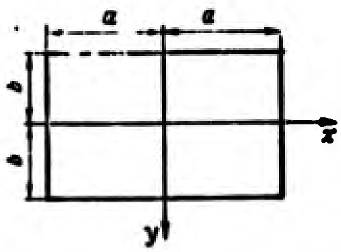

Fig. 6.

Boundary conditions of the problem can be satisfied if for deflection w we take expression

$$w = A_1\varphi_1 + A_2\varphi_2 + \ldots,$$

where

$$\varphi_1 = (x^2 - a^2)^2 (y^2 - b^2)^2,$$
$$\varphi_2 = (x^2 - a^2)^2 (y^2 - b^2)^3$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$

Being limited by the first approach, we take

$$w = A_1 (x^2 - a^2)^2 (y^2 - b^2)^2.$$

From this expression let us formulate the corresponding derivatives and substitute them in Bubnov-Galerkin equation (1.4):

$$\int_{-a}^{a}\int_{-b}^{b} \{8DA_1 [3(x^2 - a^2)^2 + 4(3x^2 - a^2)(3y^2 - b^2) + 3(y^2 - b^2)^2] - q\} \times$$
$$\times (x^2 - a^2)^2 (y^2 - b^2)^2 \, dx\, dy = 0.$$

After integration we obtain the following expression for A_1:

$$A_1 = \frac{7q}{128D\left(a^4 + \frac{4}{7}a^2b^2 + b^4\right)}.$$

Then for deflection w we will have

$$w=\frac{7q\,(x^2-a^2)^2\,(y^2-b^2)^2}{128D\left(a^4+\frac{4}{7}a^2b^2+b^4\right)}.$$

In the examined problem function w must be satisfied only by geometric boundary conditions. Therefore, it was easily selected. In case of mixed conditions at the boundary it is difficult to select such a function, therefore it is expedient to use generalized equation of the Bubnov-Galerkin method in the following form:

$$\iint_F\left(D\nabla^2\nabla^2w-q\right)\delta w\,dx\,dy+\int\left[M_x\delta\left(\frac{\partial w}{\partial x}\right)\right]dy+$$
$$+\int\left[M_y\delta\left(\frac{\partial w}{\partial y}\right)\right]dx+\int(V_x\delta w)\,dy+\int(V_y\delta w)\,dx=0,$$

where unary integrals are taken along the boundary of the region. With use of this equation it is necessary to satisfy all geometric and optionally all power boundary conditions of the problem, since the given equation expresses its equality to zero of the first variation of total energy, and on the basis of origin of possible displacements this is entirely sufficient for equilibrium of any mechanical system. There is more on this in § 3.

Kantorovich-Vlasov Method.

This method is more accurate than the Bubnov-Galerkin method, and it involves the following. In order to avoid integration of equations in partial derivatives, the solution is sought in the form of the product of two functions

$$w=X(x)\,Y(y),$$

when one of them is selected earlier so that boundary conditions of the problem would be satisfied. In this case for determination of the second function we substitute the accepted expression for w in equation (1.1). We multiply it by selected function and ntegrate within the limits of change of the given function. For example, with selected function $Y(y)$ the shown equation takes the following form, analogous in form to equations of the Bubnov-Galerkin method:

$$\int^{b}\{D[X^{IV}(x)Y(y)+2X^{II}(x)Y^{II}(y)+X(x)Y^{IV}(y)]-q\}Y(y)\,dy=0. \qquad (1.5)$$

Hence after integration we obtain the usual differential equation for function $X(x)$.

The accepted expression for sought function w can be substituted even in the functional of total potential energy of the given problem. After integration of this functional with respect to variable selected function and application of known rules of calculus of variations to it after this, the equation can be obtained for determination of unknown function $X(x)$.

Example. Let us assume there is a plate loaded by uniformly distributed pressure with two rigidly fixed opposite sides, and two other sides fixed in any manner.

For the solution of this problem let us take the first approximation, used by us during illustration of the Bubnov-Galerkin method. In the Kantorovich-Vlasov method the expression for deflection can be taken in two forms:

$$w=W(x)(y^2-b^2)^2$$

or

$$w=(x^2-a^2)^2W(y).$$

Let us take the first of these experssions, which satisfies the condition of rigid framing on the ends $y=\pm b$.

Let us formulate the necessary derivatives of this expression and substitute them in equation (1.5):

$$\int_{-b}^{b}\{D[(y^2-b^2)^2W^{IV}(x)+8(3y^2-b^2)W^{II}(x)+24W(x)]-q\}(y^2-b^2)\,dy=0.$$

After integration of this equation within the shown limits we obtain

$$\frac{16b^4}{63}W^{IV}(x)-\frac{32}{63}b^2W^{II}(x)+8W(x)=\frac{q}{3D}.$$

This equation is already solved accurately.

In theory of differential equations it is proven that the general integral of a heterogeneous equation consists of a particular solution, corresponding to the right side W_0, and general integral $\overline{W}$ of homogeneous equation. In our case this will be

$$\frac{16b^4}{63}W^{IV}(x)-\frac{32b^2}{63}W^{II}(x)+8W(x)=0,$$

i.e.,

$$W=W_0+\overline{W}.$$

If q = const, then W_0 = A and

$$A=\frac{q}{24D}.$$

Then

$$W=\frac{q}{24D}+A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}+A_3e^{\lambda_3 x}+A_4e^{\lambda_4 x},$$

where λ_i — roots of characteristic equation

$$\frac{16b^4}{63}\lambda^4-\frac{32b^2}{63}\lambda^2+8=0.$$

Arbitrary constants of integrations will be determined from boundary conditions at ends $x=\pm a$.

Thus, we find the final expression for deflection w:

$$w=\left(\frac{q}{24D}+A_1e^{\lambda_1 x}+A_2e^{\lambda_2 x}+A_3e^{\lambda_3 x}+A_4e^{\lambda_4 x}\right)(y^2-b^2)^2.$$

Method of Finite Differences.

The method of finite differences is based on replacement of the original differential equation by an equation in finite differences. For this purpose it is necessary to change from differential operations in the original equation to operations in finite differences. For derivation of these relationships we will basically proceed from the possibility of expansion of the sought function into Taylor series.

If the quantity of some function at point x is known, then the quantity of this function at point $(x + h)$ will be

$$f(x+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+\ldots$$

Let us assume we know the quantity of function f at point k (Fig. 7) on straight line Ox.

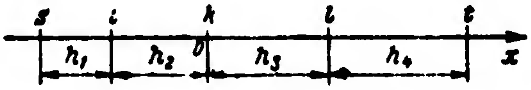

Fig. 7.

The quantity of the function at point l will be

$$f_l=f_k+\frac{h_3}{1!}f'_k+\frac{h_3^2}{2!}f''_k+\frac{h_3^3}{3!}f'''_k+\ldots$$

From this expansion the expression can be obtained for the first derivative at point k:

$$f'_k=\frac{f_l-f_k}{h_3}-\left(\frac{h_3}{2!}f''_k+\frac{h_3^2}{3!}f'''_k+\frac{h_3^3}{4!}f_k^{IV}+\ldots\right). \qquad (1.6)$$

The first term in the right side of this expression is the tangent of angle of slope of chord AB to axis x (Fig. 8).

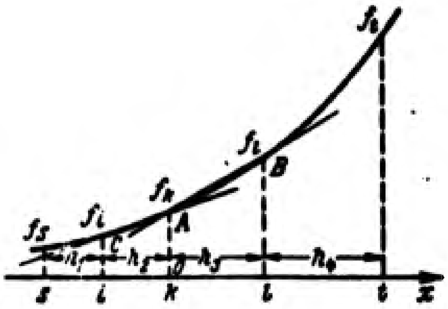

Fig. 8.

Analogously the quantity of function f at point i can be obtained, for which Taylor series it is necessary to put $(-h_2)$. In this instance for derivative of function f at point k we obtain expression

$$f'_k=\frac{f_k-f_i}{h_2}+\left(\frac{h_2}{2!}f''_k-\frac{h_2^2}{3!}f'''_k+\frac{h_2^3}{4!}f_k^{IV}-\ldots\right). \qquad (1.7)$$

In this expansion the first term of the right side is the tangent of angle of slope of chord CA to axis x.

Thus, for the derivative at point k we obtain two expressions - (1.6) and (1.7).

By dropping in these expressions the terms contained in parentheses, we obtain approximate expressions for the derivative of function f at point k on the right and on the left:

$$f'_k = \frac{f_l - f_k}{h_3} + O(h),$$

$$f'_k + \frac{f_k - f_i}{h_2} + O(h).$$

The accuracy of these formulas will be evaluated by the first large dropped term.

The more accurate expression for the first derivative at point k will be equal to the arithmetic mean of expressions (1.6) and (1.7):

$$f'_k = \frac{1}{2}\left(\frac{f_l - f_k}{h_3} + \frac{f_k - f_i}{h_2}\right) + \frac{1}{2}\left(\frac{h_2 - h_3}{2!} f''_k - \frac{h_3^2 + h_2^2}{3!} f'''_k + \dots\right). \qquad (1.8)$$

By dropping in this expression the terms in the second parentheses on the right, we obtain the approximate expression of averaged first derivative at point k:

$$f'_k = \frac{1}{2}\left(\frac{f_l - f_k}{h_3} + \frac{f_k - f_i}{h_2}\right) + O(h). \qquad (1.9)$$

The accuracy of this formula will be of order h. If we assume that $h_1 = h_2 = h_3 = h_4 = h$, then for the first derivative we obtain expression

$$f'_k = \frac{f_l - f_i}{2h} + O(h^2).$$

The accuracy of this formula will be higher. The largest of the dropped terms here has order h^2.

To get the expression of the second derivative at point k let us exclude f'_k from expansions (1.6) and (1.7):

$$f''_k = \frac{2}{h_2 + h_3}\left(\frac{f_l - f_k}{h_3} - \frac{f_k - f_i}{h_2}\right) - 2\left[\frac{h_3^2 - h_2^2}{3!(h_3 + h_2)} f'''_k + \frac{h_3^3 + h_2^3}{4!(h_3 + h_2)} f^{IV}_k + \dots\right]$$

By dropping the terms contained in brackets here, we obtain the approximate expression for the second derivative at point k with precision of order h:

$$f''_k = \frac{2}{h_2+h_3}\left(\frac{f_i-f_k}{h_3}-\frac{f_k-f_l}{h_2}\right)+O(h).$$

If $h_1 = h_2 = h_3 = h_4 = h$, we obtain a more accurate expression for the second derivative

$$f''_k = \frac{2}{h^2}\left(\frac{f_i+f_l}{2}-f_k\right)+O(h^2).$$

Thus, we obtained expressions for the first and second derivatives of f at point k through values of this function at adjacent points on the right and left.

Let us use these formulas to get higher order derivatives at point k:

$$f'''_k = \frac{d}{dx}(f''_k) = \frac{d}{dx}\left\{\frac{2}{h_2+h_3}\left(\frac{f_i-f_k}{h_3}-\frac{f_k-f_l}{h_2}\right)-\right.$$
$$\left.-2\left[\frac{h_3^2-h_2^2}{3!(h_3+h_2)}f'''_k+\frac{h_3^3+h_2^3}{4!(h_3+h_2)}f_k^{IV}+\dots\right]\right\}=$$
$$=\frac{2}{(h_2+h_3)h_3}\left(\frac{df}{dx}\right)_i+\frac{2}{(h_2+h_3)h_2}\left(\frac{df}{dx}\right)_l-\frac{2}{h_2h_3}\left(\frac{df}{dx}\right)_k-$$
$$-2\frac{d}{dx}\left[\frac{h_3^2-h_2^2}{3!(h_3+h_2)}f'''_k+\frac{h_3^3+h_2^3}{4!(h_3+h_2)}f_k^{IV}+\dots\right].$$

By applying the difference operation to the first derivatives at points l, i and k according to formula (1.9), we obtain to h accuracy the expression for the third derivative

$$f'''_k = f_k\left[\frac{1}{h_2+h_3}\left(\frac{1}{h_2^2}-\frac{1}{h_3^2}\right)-\frac{1}{h_2h_3}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\right]+$$
$$+f_l\left[\frac{1}{h_2(h_2+h_3)}\left(\frac{1}{h_1}-\frac{1}{h_2}\right)+\frac{1}{h_2^2h_3}\right]+$$
$$+f_i\left[\frac{1}{h_3(h_2+h_3)}\left(\frac{1}{h_3}-\frac{1}{h_4}\right)-\frac{1}{h_2h_3^2}\right]-$$
$$-\frac{f_s}{h_1h_2(h_2+h_3)}+\frac{f_t}{h_3h_4(h_2+h_3)}+O(h).$$

Analogously we can obtain expressions for derivatives with respect to variable y, having substituted in the obtained formulas l by n, i by m, s by u, t by v and distances h with appropriate interlinear number by distance b (Fig. 9):

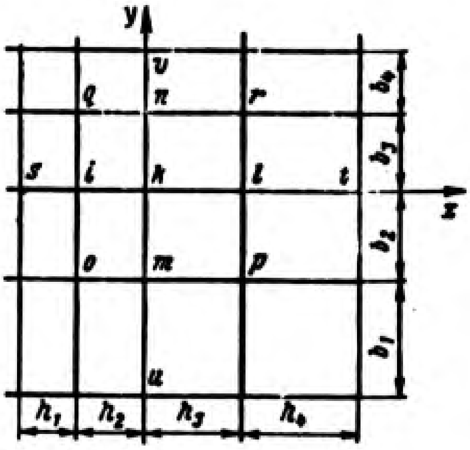

Fig. 9.

$$f'_k = \frac{1}{2}\left(\frac{f_n - f_k}{b_3} + \frac{f_k - f_m}{b_2}\right) + O(b),$$

$$f'_k = \frac{f_n - f_m}{2b} + O(b^2),$$

$$f''_k = \frac{2}{b_2 + b_3}\left(\frac{f_n - f_k}{b_3} - \frac{f_k - f_m}{b_2}\right) + O(b),$$

$$f''_k = \frac{2}{b^2}\left(\frac{f_m + f_n}{2} - f_k\right) + O(b^2). \qquad (1.10)$$

Let us show how mixed derivatives can be obtained at point k. For example

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right)_k = \frac{d}{dy}\left(\frac{df}{dx}\right) = \frac{d}{dy}\left[\frac{1}{2}\left(\frac{f_l - f_k}{h_3} + \frac{f_k - f_i}{h_2}\right) + \right.$$
$$\left. + \frac{1}{2}\left(\frac{h_2 - h_3}{2!} f''_k - \frac{h_2^2 + h_3^2}{3!} f'''_k + \ldots\right)\right] = \frac{1}{2h_3}\left(\frac{df}{dy}\right)_l - \frac{1}{2h_2}\left(\frac{df}{dy}\right)_i +$$
$$+ \frac{1}{2}\left(\frac{1}{h_2} - \frac{1}{h_3}\right)\left(\frac{df}{dy}\right)_k + \frac{1}{2}\frac{d}{dy}\left(\frac{h_2 - h_3}{2!} f''_k - \frac{h_2^2 + h_3^2}{3!} f'''_k + \ldots\right).$$

By applying the difference operation to derivatives at points l, i and k according to formula (1.10), we obtain

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_k = \frac{f_k}{4}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\left(\frac{1}{b_2}-\frac{1}{b_3}\right)+\frac{f_l}{4h_3}\left(\frac{1}{b_2}-\frac{1}{b_3}\right)-$$

$$-\frac{f_n}{4b_3}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)-\frac{f_m}{4b_2}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)+\frac{f_0}{4h_2b_2}-\frac{f_q}{4h_2b_3}+\frac{f_r}{4h_3b_3}-\frac{f_p}{4h_3b_2}.$$

The accuracy of this formula can be of order $O(h)$ or $O(b)$ depending on what order the derivatives were taken.

Analogously any other derivative can be obtained. All subsequently derivatives are written out bel w.

$$\left(\frac{df}{dx}\right)_k = \frac{1}{2}\left(-\frac{f_l}{h_2}+\frac{f_l}{h_3}\right)+\frac{1}{2}f_k\left(\frac{1}{h_2}-\frac{1}{h_3}\right).$$

$$\left(\frac{df}{dy}\right)_k = \frac{1}{2}\left(-\frac{f_m}{b_2}+\frac{f_n}{b_3}\right)+\frac{1}{2}f_k\left(\frac{1}{b_2}-\frac{1}{b^3}\right).$$

$$\left(\frac{d^2f}{dx^2}\right)_k = \frac{2}{h_2+h_3}\left(\frac{f_l-f_k}{h_3}-\frac{f_k-f_l}{h_2}\right).$$

$$\left(\frac{d^2f}{dy^2}\right)_k = \frac{2}{b_2+b_3}\left(\frac{f_n-f_k}{b_3}-\frac{f_k-f_m}{b_2}\right).$$

$$4\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_k = \left(\frac{1}{h_2}-\frac{1}{h_3}\right)\left(\frac{1}{b_2}-\frac{1}{b_3}\right)f_k-\left(\frac{1}{b_2}-\frac{1}{b_3}\right)\left(\frac{f_l}{h_2}-\frac{f_l}{h_3}\right)-$$

$$-\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\left(\frac{f_m}{b_2}-\frac{f_n}{b_3}\right)+\frac{1}{h_2}\left(\frac{f_0}{b_2}-\frac{f_q}{b_3}\right)-\frac{1}{h_3}\left(\frac{f_p}{b_2}-\frac{f_r}{b_3}\right).$$

$$\left(\frac{d_3f}{dx^3}\right)_k = \left[\frac{1}{h_2+h_3}\left(\frac{1}{h_2^2}-\frac{1}{h_3^2}\right)-\frac{1}{2h_2h_3}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\right]f_k+$$

10) $$+\left[\frac{1}{h_3(h_2+h_3)}\left(\frac{1}{h_3}-\frac{1}{h_4}\right)-\frac{1}{h_2h_3^3}\right]f_l+\left[\frac{1}{h_2(h_2+h_3)}\left(\frac{1}{h_1}-\frac{1}{h_2}\right)+\right.$$

$$\left.+\frac{1}{h_2^2h_3}\right]f_i-\frac{1}{h_2+h_3}\left(\frac{f_s}{h_1h_2}-\frac{f_t}{h_3h_4}\right).$$

$$\left(\frac{d^3f}{dy^3}\right)_k = \left[\frac{1}{b_2+b_3}\left(\frac{1}{b_2^2}-\frac{1}{b_3^2}\right)-\frac{1}{2b_2b_3}\left(\frac{1}{b_2}-\frac{1}{b_3}\right)\right]f_k+$$

$$+\left[\frac{1}{b_3(b_2+b_3)}\left(\frac{1}{b_3}-\frac{1}{b_4}\right)-\frac{1}{b_2b_3^2}\right]f_n+\left[\frac{1}{b_2(b_2+b_3)}\left(\frac{1}{b_1}-\frac{1}{b_2}\right)+\right.$$

$$\left.+\frac{1}{b_2^2b_3}\right]f_m-\frac{1}{b_2+b_3}\left(\frac{f_u}{b_1b_2}-\frac{f_v}{b_3b_4}\right).$$

$$\left(\frac{\partial^3 f}{\partial x\,\partial y^2}\right)_k = \left[-\frac{1}{b_2b_3}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\right]f_k+\left[\frac{1}{b_2(b_2+b_3)}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\right]f_m+$$

$$+\left[\frac{1}{b_3(b_2+b_3)}\left(\frac{1}{h_2}-\frac{1}{h_3}\right)\right]f_n+\left[\frac{1}{b_2h_3(b_2+b_3)}\right]f_p+\left[\frac{1}{b_2h_3(b_2+b_3)}\right]f_r+$$

$$+\left[\frac{1}{h_2b_2b_3}\right]f_l-\left[\frac{1}{h_3b_2b_3}\right]f_l-\left[\frac{1}{h_2b_3(b_2+b_3)}\right]f_q-\left[\frac{1}{h_2b_2(b_2+b_3)}\right]f_0.$$

$$\left(\frac{\partial^3 f}{\partial x^2 \partial y}\right)_k = -\left[\frac{1}{h_2 h_3}\left(\frac{1}{b_2}-\frac{1}{b_3}\right)\right] f_k + \left[\frac{1}{h_2(h_2+h_3)}\left(\frac{1}{b_2}-\frac{1}{b_3}\right)\right] f_i +$$

$$+\left[\frac{1}{h_3(h_2+h_3)}\left(\frac{1}{b_2}-\frac{1}{b_3}\right)\right] f_l + \left[\frac{1}{h_2 b_3(h_2+h_3)}\right] f_q +$$

$$+\left[\frac{1}{h_3 b_3(h_2+h_3)}\right] f_r + \left[\frac{1}{b_2 h_2 h_3}\right] f_m - \left[\frac{1}{b_3 h_2 h_3}\right] f_n -$$

$$-\left[\frac{1}{b_2 h_3(h_2+h_3)}\right] f_p - \left[\frac{1}{b_2 h_2(h_2+h_3)}\right] f_o.$$

$$\frac{1}{4}(h_2+h_3)(b_2+b_3)\left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right)_k = \left(\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\frac{1}{b_2}+\frac{1}{b_3}\right) f_k -$$

$$-\frac{1}{h_2}\left(\frac{1}{b_2}+\frac{1}{b_3}\right) f_i - \frac{1}{h_3}\left(\frac{1}{b_2}+\frac{1}{b_3}\right) f_l - \frac{1}{b_2}\left(\frac{1}{h_2}+\frac{1}{h_3}\right) f_m -$$

$$-\frac{1}{b_3}\left(\frac{1}{h_2}+\frac{1}{h_3}\right) f_n + \frac{f_o}{h_2 b_2} + \frac{f_q}{h_2 b_3} + \frac{f_p}{h_3 b_2} + \frac{f_r}{h_3 b_3}.$$

$$\frac{1}{4}(h_2+h_3)\left(\frac{d^4 f}{dx^4}\right)_k = \left[\frac{1}{h_2^2}\left(\frac{1}{h_1+h_2}+\frac{1}{h_3}\right) + \frac{1}{h_3^2}\left(\frac{1}{h_2}+\frac{1}{h_3+h_4}\right)\right] f_k -$$

$$-\left[\frac{1}{h_1+h_2}\left(\frac{1}{h_1 h_2}+\frac{1}{h_2^2}\right) + \frac{1}{h_2^2 h_3}\right] f_i - \left[\frac{1}{h_3+h_4}\left(\frac{1}{h_3 h_4}+\frac{1}{h_3^2}\right) + \frac{1}{h_2 h_3^2}\right] f_l +$$

$$+\left[\frac{1}{h_1 h_2(h_1+h_2)}\right] f_s + \left[\frac{1}{h_3 h_4(h_3+h_4)}\right] f_t.$$

$$\frac{1}{4}(b_2+b_3)\left(\frac{d^4 f}{dy^4}\right)_k = \left[\frac{1}{b_2^2}\left(\frac{1}{b_1+b_2}+\frac{1}{b_3}\right) + \frac{1}{b_3^2}\left(\frac{1}{b_2}+\frac{1}{b_3+b_4}\right)\right] f_k -$$

$$-\left[\frac{1}{b_1+b_2}\left(\frac{1}{b_1 b_2}+\frac{1}{b_2^2}\right) + \frac{1}{b_2^2 b_3}\right] f_m - \left[\frac{1}{b_3+b_4}\left(\frac{1}{b_3 b_4}+\frac{1}{b_3^2}\right) + \frac{1}{b_2 b_3^2}\right] f_n +$$

$$+\left[\frac{1}{b_1 b_2(b_1+b_2)}\right] f_u + \left[\frac{1}{b_3 b_4(b_3+b_4)}\right] f_v.$$

Assuming in these expressions $h_1 = h_2 = h_3 = h_4 = h$, $b_1 = b_2 = b_3 = b_4 = b$, we obtain

$$\left.\begin{aligned}
\left(\frac{df}{dx}\right)_k &= \frac{1}{2h}(f_l - f_i),\\
\left(\frac{df}{dy}\right)_k &= \frac{1}{2b}(f_n - f_m),\\
\left(\frac{d^2 f}{dx^2}\right)_k &= \frac{1}{h^2}(f_l + f_i - 2f_k),\\
\left(\frac{d^2 f}{dy^2}\right)_k &= \frac{1}{b^2}(f_n + f_m - 2f_k),\\
\left(\frac{d^3 f}{dx^3}\right)_k &= \frac{1}{2h^3}(f_t - f_s + 2f_i - 2f_l),\\
\left(\frac{d^3 f}{dy^3}\right)_k &= \frac{1}{2b^3}(f_v - f_u + 2f_m - 2f_n),\\
\left(\frac{d^4 f}{dx^4}\right)_k &= \frac{1}{h^4}(6f_k - 4f_l - 4f_i + f_t + f_s).
\end{aligned}\right\} \qquad (1.11)$$

$$\left.\begin{aligned}
\left(\frac{d^4 f}{dy^4}\right)_k &= \frac{1}{b^4}\,(6f_k - 4f_n - 4f_m + f_v + f_u),\\
\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_k &= \frac{1}{4hb}\,(f_0 - f_p + f_r - f_q),\\
\left(\frac{\partial^3 f}{\partial x^2\,\partial y}\right)_k &= \frac{1}{2bh^2}\,(2f_m - 2f_n + f_q - f_0 + f_r - f_p),\\
\left(\frac{\partial^3 f}{\partial x\,\partial y^2}\right)_k &= \frac{1}{2hb^2}\,(2f_i - 2f_l + f_p - f_0 + f_r - f_q),\\
\left(\frac{\partial^4 f}{\partial x^2\partial y^2}\right)_k &= \frac{1}{h^2b^2}\,(4f_k - 2f_i - 2f_l - 2f_n - 2f_m + f_0 + f_p + f_r + f_q).
\end{aligned}\right\} \qquad (1.11)$$

By substituting the differential operations in equation (1.1) by the operations in finite differences for formula (1.11), we obtain

$$6W_k\left(1 + \frac{4}{3}\xi + \xi^2\right) - 4(1 + \xi)(W_i + W_l + \xi W_n + \xi W_m) +$$

$$+ W_t + W_s + \xi^2(W_u + W_v) + 2\xi(W_0 + W_p + W_r + W_q) = \frac{qh^4}{D}. \qquad (1.12)$$

where

$$\xi = \left(\frac{h}{b}\right)^2.$$

The possible types of boundary conditions enumerated above should also be represented through finite differences.

1. Edge is freely supported:

$$W_k = 0,$$

$$M_x = -D\left[\frac{W_i + W_l - 2W_k}{h^2} + \mu\,\frac{W_n + W_m - 2W_k}{b^2}\right] = 0.$$

2. Edge is rigidly fixed:

$$W_k = 0, \quad \left(\frac{\partial W}{\partial x}\right)_k = \frac{W_i - W_l}{2h} = 0.$$

3. Edge is free:

(1.11)

$$M_x = -D\left[\frac{W_i + W_l - 2W_k}{h^2} + \mu\,\frac{W_n + W_m - 2W_k}{h^2}\right] = 0,$$

$$V_x = -D\left[\frac{W_t - W_l + 2W_l - 2W_i}{2h^3} + \right.$$

$$\left. + \frac{2-\mu}{2hb^2}\,(2W_i - 2W_l - W_0 + W_p + W_r - W_q)\right] = 0.$$

With solution of the problems by the method of finite differences on the surface of plates we apply a grid with sides parallel to axes of coordinates. Points of intersection of the lines of this grid are numbered. For each of these nodal points we then formulate equation (1.12). There will be as many equations as numbers on the grid. In order to reduce the amount of these numbers, where this is possible we use conditions of symmetry of the problem. Having solved the obtained system of equations, we find the amount of deflection at each nodal point of the applied grid. By using the found magnitude of deflections, we can determine the bending moments at these points. The accuracy of solution of problems by this method will be higher, the more closely-spaced the grid that is applied.

Simultaneously with this, of course, the difficulty of solution of a large number of simultaneous equations increases. It is possible to considerably reduce this laboriousness if we use extrapolation formulas, making it possible by the first two-three approximations to obtain a refined following approximation, while not solving the problem itself. The simplest in a practical respect are Richardson formulas extrapolation, which are based on the following reasonings. With calculation of derivatives through averaged differences in expansion of Taylor series we were limited by the first two terms and they allowed the largest error of order h^2. Dropped terms had the form

$$\varepsilon = f_1(x)\,h^2 + f_2(x)\,h^4 + \dots$$

Consequently, if we solved some problem with the aid of approximate finite-difference equations and found the result equal to A, this result could have been refined by having added dropped error ε to it. In view of the act that this error with such formulation of the problem remains unknown, it is possible to proceed in the following manner. Let us suppose we solved the given problem at two different grid spacings h_1 and h_2 and obtained the magnitudes of the sought function at the given value of x, equal to A_1 and A_2 respectively. Then more accurate quantity of the sought function at point x will be

$$A = A_1 + \varepsilon_1 = A_1 + f_1(x)\,h_1^2 + f_2(x)\,h_1^4 + \ldots$$
$$A = A_2 + \varepsilon_2 = A_2 + f_1(x)\,h_2^2 + f_2(x)\,h_4^2 + \ldots$$

Being limited in the right side of these expressions by the largest components, it is possible to write

$$A \approx A_1 + f_1(x)\,h_1^2, \quad A \approx A_2 + f_1(x)\,h_2^2.$$

By excluding hence $f_1(x)$, we obtain the refined quantity of sought function, which we designate through $A_{\text{экстр}}$ (extrapolated):

$$A_{\text{экстр}} = \frac{h_1^2}{h_1^2 - h_2^2} A_2 - \frac{h_2^2}{h_1^2 - h_2^2} A_1.$$

Refined quantity of the sought function according to the first three approximations can be obtained analogously:

$$A_{\text{экстр}} = \frac{h_2^2 h_3^2}{(h_1^2 - h_2^2)(h_1^2 - h_3^2)} A_1 - \frac{h_1^2 h_3^2}{(h_1^2 - h_2^2)(h_2^2 - h_3^2)} A_2 + \frac{h_1^2 h_2^2}{(h_1^2 - h_3^2)(h_2^2 - h_3^2)} A_3.$$

By assuming in these formulas

$$h_1 = \frac{a_2 - a_1}{n_1}, \quad h_2 = \frac{a_2 - a_1}{n_2}, \quad h_3 = \frac{a_2 - a_1}{n_3},$$

where a_1, a_2 - limits of change of variable of integration x; n_1, n_2, n_3 - numbers of divisions of length $(a_2 - a_1)$, we obtain the following Richardson formulas:

$$A_{n_1, n_2} = A_2 \frac{n_2^2}{n_2^2 - n_1^2} - A_1 \frac{n_1^2}{n_2^2 - n_1^2},$$

$$A_{n_1, n_2, n_3} = A_1 \frac{n_1^4}{(n_2^2 - n_1^2)(n_3^2 - n_1^2)} - A_2 \frac{n_2^4}{(n_2^2 - n_1^2)(n_3^2 - n_2^2)} + A_3 \frac{n_3^4}{(n_3^2 - n_1^2)(n_3^2 - n_2^2)}.$$

With solution of problems by the method of finite differences we have the possibility of determining the quantities of the sought function at earlier marked points of the applied grid. For determination of the quantity of this function at intermediate points let us use Lagrange formulas, which are derived in the following manner. Let us assume we found quantities of the sought function in three consecutively arranged points *k*, *l*, *i* (Fig. 10).

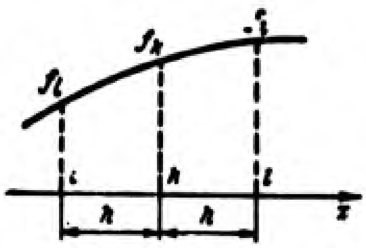

Fig. 10.

With sufficient accuracy the equation of the curve passing through three points can be substituted by a parabola of the second power

$$f(x)=a_0+a_1x+a_2x^2.$$

Let us find the coefficients of this parabola from conditions

$$x=0\quad f(0)=f_k,\quad x=h\quad f(h)=f_i,$$
$$x=-h\quad f(-h)=f_l.$$

Hence we find

$$a_0=f_k,\quad a_1=\frac{f_i-f_l}{2h},\quad a_2=\frac{f_i+f_l-2f_k}{2h^2}.$$

Then we obtain the following Lagrange interpolation formula with respect to three points:

$$f(x)=f_k+\frac{f_i-f_l}{2h}x+\frac{f_i+f_l-2f_k}{2h^2}x^2.$$

Analogous formulas can be obtained even for a large number of points. According to these formulas one can determine the quantites of sought function at any intermediate point x according to quantities of this function at adjacent points.

Let us show the utilization of the method of finite differences for a problem about bending of a rectangular plate by uniform load at various, but symmetrically arranged boundary conditions (Fig. 11).

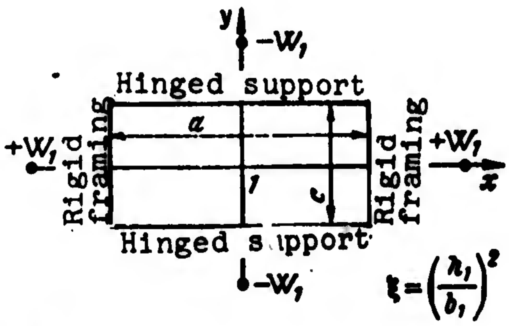

Fig. 11.

First approximation:

$$h_1 = \frac{a}{2}, \quad b_1 = \frac{c}{2},$$

$$\xi = \left(\frac{h_1}{b_1}\right)^2 = \left(\frac{a}{c}\right)^2.$$

Let us formulate equation (1.12) for point 1:

$$6W_1\left(1 + \frac{4}{3}\xi + \xi^2\right) - 4(1+\xi)(0+0+0+0) + W_1 + W_1 +$$

$$+ \xi^2(-W_1 - W_1) + 2\xi(0+0+0+0) = \frac{qh_1^4}{D}.$$

Hence we obtain the magnitude of deflection at point 1:

$$W_1 = \frac{qa^4}{64(2 + 2\xi + \xi^2)D}.$$

If the plate is square, $\xi = 1$, then

$$W_1 = 0{,}0341\frac{qa^4}{E\delta^3}.$$

The second approximation (Fig. 12):

$$h_2 = \frac{a}{3}, \quad b_2 = \frac{c}{3}, \quad \xi = \left(\frac{a}{c}\right)^2.$$

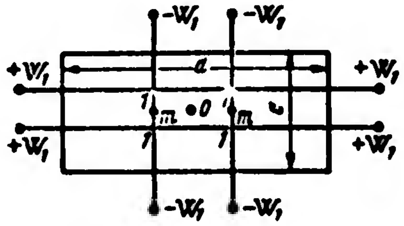

Fig. 12.

In this instance because of the symmetry of deflection relative to the middle of the plate all nodal points are marked by the same figure 1. Let us formulate equation (1.12) for any of these points:

$$6W_1\left(1+\frac{4}{3}\xi+\xi^2\right)-4(1+\xi)(0+W_1+0+\xi W_1)+W_1+0+$$

$$+\xi^2(-W_1+0)+2\xi(0+W_1+0+0)=\frac{qh_2^4}{D}.$$

Hence we obtain

$$W_1=\frac{qa^4}{81(3+2\xi+\xi^2)D}.$$

When $\xi = 1$ we will have

$$W_1=0{,}0224\,\frac{qa^4}{E\delta^3}.$$

By using the Lagrange interpolation formula, we find the magnitude of deflection at point m:

$$W_m=W_1+\frac{(W_1-0)\,0{,}5\,b}{2b}+\frac{(W_1+0-2W_1)\,0{,}25b^2}{2b^2}=1{,}125W_1.$$

In order to determine the magnitude of deflection in the center of the plate at point O, it is necessary once again to use interpolation with respect to points m. With very high requirements for the accuracy of solution of the problem such double interpolation can lead to errors. In this case we are not aiming at obtaining an accurate solution, but illustrate the method, therefore let us allow double interpolation:

$$W_0=W_m+\frac{(W_m-0)\,0{,}5h}{2h}+\frac{(W_m+0-2W_m)\,0{,}25h^2}{2h^2}=1{,}125W_m$$

or

$$W_0=1{,}125\cdot 1{,}125W_1=0{,}0284\,\frac{qa^4}{E\delta^3}.$$

Let us determine the refined magnitude of deflection in the center of the plate by the first two obtained approximations, using extrapolation:

$$W_{экстр} = (-0.8\cdot 0.0341 + 1.8\cdot 0.0281)\frac{qa^4}{Eh^3} = 0.0239\frac{qa^4}{Eh^3}.$$

The accurate solution of this problem gives coefficient 0.0209 for deflection in the center of the plate. Thus, the error of approximate solution is equal to 14%.

§ 3. Application of the Origin of Virtual Displacements for Investigation of Bending of Plates. Ritz Method

The origin of virtual displacements is one of the basic principles of mechanics, which asserts that if the body is in a state of equilibrium, then the sum of work of all forces applied to this body at any permitted connections of very small (virtual) displacements is equal to zero.

During examination of deformable bodies we apply this origin to the total expression of potential energy of the system. By potential energy of the system we mean the work that forces of the system both internal and external accomplish during transition of the system from deformed state to nondeformed.

By internal forces of the system we mean those forces which appear between particles of the body with respect to its deformation. These are forces of elasticity.

The sum of these works is numerically equal to potential energy of the system:

$$Э = V + T,$$

where V - energy of deformation or work of internal forces; T - work or potential of external forces.

Inasmuch as the considered system is in equilibrium, accordingly to the origin of virtual displacements

$$\delta Э=\delta V+\delta T=0,$$

where δ indicates the possible change in coordinates of points of the body from the position of equilibrium.

This equation is used during solution of many practically important problems during strength calculation of various constructions.

Total potential energy for plates, expressed through bending w, has the form

$$Э=\frac{D}{2}\iint_{S}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\right.$$
$$\left.+2(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy+T,$$

where $D=\frac{Eh^3}{12(1-\mu^2)}$ — flexural rigidity of the plate.

For solution of a particular problem a suitable expression for w must be selected, which satisfies the assigned boundary conditions, with indeterminate parameters,

$$w=a_1\varphi_1(x,y)+a_2\varphi_2(x,y)+\ldots$$

and it must be substituted in Э. Indeterminate parameters a_i are determined from condition

$$\delta Э=\frac{\partial Э}{\partial a_1}\delta a_1+\frac{\partial Э}{\partial a_2}\delta a_2+\ldots=0,$$

which is the total differential of the function of many variables. In this case, proceeding from the beginning of virtual displacements, this differential must be equal to zero.

Since variations δa_1, δa_2, ... are arbitrary and nonzero, for fulfillment of condition $\delta Э = 0$ it is necessary to assume

$$\frac{\partial Э}{\partial a_1}=0,\quad \frac{\partial Э}{\partial a_2}=0,\ldots$$

Each of these equations is nothing else but equality of the sum of works with variation of some parameter to zero.

These equations give the possiblity of determining all unknown parameters entering the expression of deflection.

<u>Example</u>. Bending of hinge-supported rectangular plate by a uniform load (Fig. 13). A suitable expression for deflection, which satisfies assigned boundary conditions, in this case can be taken in the form of double series in cosines:

$$w = \sum_{m=1,3,5...}^{\infty} \sum_{n=1,3,5...}^{\infty} A_{mn} \cos\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b}.$$

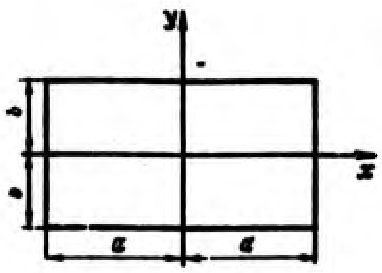

Fig. 13.

By substituting the accepted expression for w in the expression for Э and integrating within $(-a, +a)$ and $(-b, +b)$, we obtain

$$Э = \frac{Dab}{2} A_{mn}^2 \left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2 + T.$$

Work of load q will be

$$T = -q \int_{-a}^{a} \int_{-b}^{b} w\,dx\,dy = \frac{16abq}{\pi^2} \frac{A_{mn}}{mn} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}.$$

Then

$$Э = \frac{Dab}{2} A_{mn}^2 \left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2 - \frac{16abq}{\pi^2} \frac{A_{mn}}{mn} \sin\frac{m\pi}{2} \sin\frac{n\pi}{2}.$$

Unknown parameters A_{mn} are found from equations

$$\frac{\partial Э}{\partial A_{mn}} = DabA_{mn}\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2 - \frac{16abq}{\pi^2}\,\frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{mn} = 0,$$

whence

$$A_{mn} = \frac{16q\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}}{D\pi^2 mn\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2}.$$

Then

$$w = \frac{16q}{D\pi^2}\sum_{m=1,3,5\ldots}^{\infty}\sum_{n=1,3,5\ldots}^{\infty}\frac{\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b}}{mn\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2}.$$

If the plate was affected by concentrated force P at point $x=a_1,\ y=b_1$ (Fig. 14), then work of external forces would be

$$T = P(w)_{\substack{x=a_1\\ y=b_1}} = P\sum_{m=1,3,5\ldots}^{\infty}\sum_{n=1,3,5\ldots}^{\infty} A_{mn}\cos\frac{m\pi a_1}{2a}\cos\frac{n\pi b_1}{2b}.$$

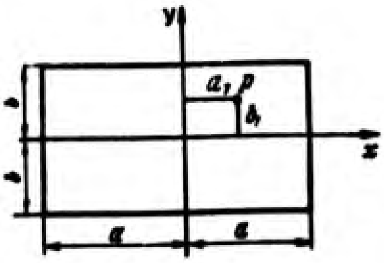

Fig. 14.

Then for w we obtain expression

$$w = \frac{P}{Dab}\sum_{m=1,3\ldots}^{\infty}\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos\frac{m\pi a_1}{2a}\cos\frac{n\pi b_1}{2b}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b}}{\left[\left(\frac{m\pi}{2a}\right)^2 + \left(\frac{n\pi}{2b}\right)^2\right]^2}.$$

The obtained series converge rather rapidly, and two-three terms in the expansion give a good result.

§ 4. Basic Information from the Theory of Rectangular Plates of Large Deflection

If deflection of a plate is commensurate with its thickness, then we cannot disregard stresses in its middle surface, as took place with derivation of equation (1.1). These stresses will be commensurate with bending stresses. Differential equation of equilibrium of such a plate has the form

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q+N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x \partial y}. \qquad (1.13)$$

In the right side of this equation along with lateral load q there appeared vertical components of forces affecting the middle surface. For determination of these forces it is necessary to have additional equations, which can be obtained from condition of equilibrium of the element of the plate in tangential plane.

From condition of equilibrium of forces in the direction of axes x and y (Fig. 15) we obtain

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=0, \quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0.$$

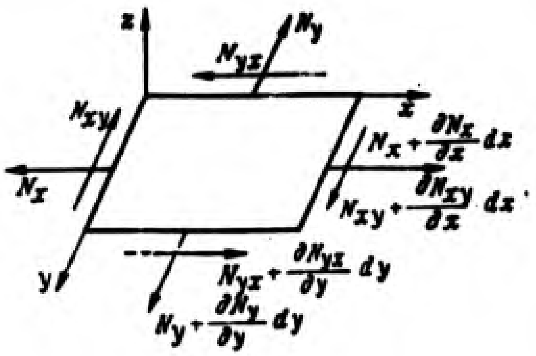

Fig. 15.

From the condition of equality of the moments of all forces relative to axis z to zero we find that

$$N_{xy}=N_{yx}.$$

Then the equations of equilibrium of forces in tangential plane will finally take the form

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0. \tag{1.14}$$

Thus for determination of four unknown functions (w, N_x, N_y, N_{xy}) we have three simultaneous equations. The fourth, lacking equation can be obtained on the basis of the following considerations.

For biaxial stressed state Hooke law is written in the form

$$\varepsilon_x=\frac{1}{E\delta}(N_x-\mu N_y),\quad \varepsilon_y=\frac{1}{E\delta}(N_y-\mu N_x),\quad \varepsilon_{xy}=\frac{2(1+\mu)}{E\delta}N_{xy}. \tag{1.15}$$

On the other hand, for components of deformation $\varepsilon_x, \varepsilon_y, \varepsilon_{xy}$ expressions can be obtained through components of displacement u, v and w of points of the middle surface of the plate. From Fig. 16 it is evident that points A, B and C after deformation transferred to position A_1, B_1 and C_1. The sides of element dx, dy changed their length and became equal $(1+\varepsilon_x)dx$ and $(1+\varepsilon_y)dy$. With axes of coordinates x, y, z they formed angles, cosines of which are equal to l_1, m_1, n_1 and l_2, m_2, n_2 respectively.

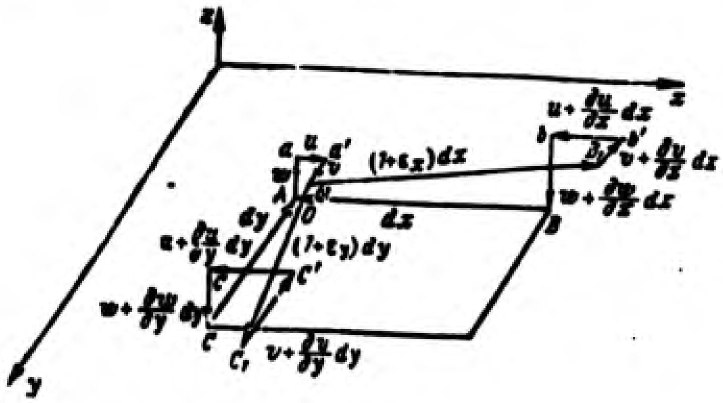

Fig. 16.

Let us project closed three-dimensional polygons $Oaa'A_1B_1b'bB$ and $Oaa'A_1C_1C'cC$ to axes x, y, z. Inasmuch as these polygons are closed, the projections of their sides to these axes will be equal to zero. Let us furnish the sides of these polygons with pointers in the direction of revolution. Projections of sides of polygon $Oaa'A_1B_1b'bB$ to axes x, y, z will be

$$-dx + u + (1+\varepsilon_x) l_1 dx - u - \frac{\partial u}{\partial x} dx = 0,$$

$$v + (1+\varepsilon_x) m_1 dx - v - \frac{\partial v}{\partial x} dx = 0,$$

$$w + (1+\varepsilon_x) n_1 dx - w - \frac{\partial w}{\partial x} dx = 0.$$

Hence we obtain

$$(1+\varepsilon_x) l_1 = 1 + \frac{\partial u}{\partial x}, \quad (1+\varepsilon_x) m_1 = \frac{\partial v}{\partial x}, \quad (1+\varepsilon_x) n_1 = \frac{\partial w}{\partial x}.$$

Let us square the right and left sides of these equalities and sum up. Then when $l_1^2 + m_1^2 + n_1^2 = 1$ we obtain

$$2\varepsilon_x + \varepsilon_x^2 = 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2.$$

By disregarding quantity ε_x^2 in comparison with ε_x, we will have

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right].$$

Usually to get approximate expressions for components of deformation the squares of derivatives of functions u and v are disregarded in comparison with $\left(\frac{\partial w}{\partial x}\right)^2$. Then

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \qquad (1.16)$$

Analogously we can obtain

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2. \qquad (1.17)$$

To get deformation ε_{xy}, characterizing the change of right angle between segments dx and dy, we use the following formula, known from analytical geometry for cosine of the angle between two straight lines:

$$\cos\varphi = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

Let us substitute here

$$l_1 = \frac{1 + \frac{\partial u}{\partial x}}{1+\varepsilon_x}, \quad m_1 = \frac{\frac{\partial v}{\partial x}}{1+\varepsilon_x}, \quad n_1 = \frac{\frac{\partial w}{\partial x}}{1+\varepsilon_x}$$

and analogous expressions for l_2, m_2, n_2:

$$l_2=\frac{\frac{\partial u}{\partial y}}{1+\varepsilon_y},\quad m_2=\frac{1+\frac{\partial v}{\partial y}}{1+\varepsilon_y},\quad n_2=\frac{\frac{\partial w}{\partial y}}{1+\varepsilon_y}.$$

Then we will have

$$\cos\varphi=\cos(90-\varepsilon_{xy})=\sin\varepsilon_{xy}\approx\varepsilon_{xy}\approx\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}. \qquad (1.18)$$

By excluding the derivatives of functions u and v from the obtained expressions for $\varepsilon_x, \varepsilon_y$ and ε_{xy}, we obtain

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}-\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y}=\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}.$$

Here instead of components of deformation let us substitute their expressions according to Hooke law. Then

$$\frac{1}{E\delta}\left[\left(\frac{\partial^2 N_x}{\partial y^2}-\mu\frac{\partial^2 N_y}{\partial y^2}\right)+\left(\frac{\partial^2 N_y}{\partial x^2}-\mu\frac{\partial^2 N_x}{\partial x^2}\right)-2(1+\mu)\frac{\partial^2 N_{xy}}{\partial x\,\partial y}\right]=$$
$$=\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}.$$

Thus, we obtain the fourth lacking equation, which is called the equation of compatibility of deformations.

For solution of the obtained system of equations with four unknown functions it is possible to proceed in the following manner. Let us introduce a new as yet unknown function ϕ, so that it would satisfy equations (1.14). For this it is sufficient to assume

$$N_x=\frac{\partial^2\varphi}{\partial y^2},\quad N_y=\frac{\partial^2\varphi}{\partial x^2},\quad N_{xy}=-\frac{\partial^2\varphi}{\partial x\,\partial y}.$$

By substitution of these expressions in equations (1.14) we can be certain that they are identically satisfied. Thus, from the system of four equations only the two following simultaneous equations relative to functions ϕ and w will remain:

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q+\frac{\partial^2\varphi}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2\varphi}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\,\frac{\partial^2 w}{\partial x\partial y},$$
$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=E\delta\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}\right].$$

If the flexural rigidity of the plate is small, such a plate is called a membrane. In this instance equations take the form $D=0$)

$$\frac{\partial^2\varphi}{\partial y^2}\frac{\partial^2 w}{\partial x^2}-2\frac{\partial^2\varphi}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+q=0,$$

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=E\delta\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]. \qquad (1.19)$$

Despite the fact that the written equations have been known more than 50 years, accurate methods of their solution have barely been developed. Therefore, in practical cases for solution of problems we more frequently use the origin of virtual displacements, applying it to the total expression of potential energy of the plate or membrane. Sometimes for integration of these equations it is possible to successfully use the Bubnov-Galerkin method or method of finite differences.

§ 5. Application of Origin of Virtual Displacements for Investigation of Rectangular Membranes

Let us assume the membrane is under the action of distributed load q. Then the total potential energy

$$Э=\frac{1}{2}\iint_s (N_x\varepsilon_x+N_y\varepsilon_y+N_{xy}\varepsilon_{xy})\,dx\,dy+T,$$

where through T there is designated the potential of load q and integration is performed with respect to the entire area of membrane. The coefficient 1/2 considers the circumstance that forces in the membrane increase from zero to their finite quantity according to linear law (Hooke law) and the work of these forces on corresponding displacements will be equal to the area of a triangle.

In the expression under the integral sign by replacing forces N_x, N_y, N_{xy} by their expressions through components of deformation according to formulas of Hooke law (1.15), and components of deformation according to formulas (1.16)-(1.18) through components of displacement, we obtain

$$\text{Э} = \frac{E\delta}{2(1-\mu^2)} \iint\limits_s \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial v}{\partial y}\left(\frac{\partial w}{\partial y}\right)^2 + \right.$$

$$+ \frac{1}{4}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]^2 + 2\mu\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial y}\left(\frac{\partial w}{\partial x}\right)^2 + \right.$$

$$\left. + \frac{1}{2}\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{1-\mu}{2}\left[\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + \right.$$

$$\left.\left. + 2\frac{\partial u}{\partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + 2\frac{\partial v}{\partial x}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right\} dx\,dy - \iint\limits_s q w\,dx\,dy.$$

Let us apply this expression for the problem of a square membrane (Fig. 17) with side $2a$ loaded with constant pressure q.

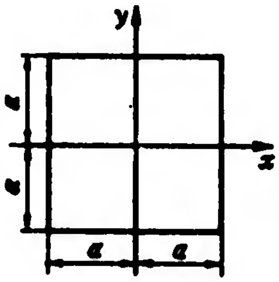

Fig. 17.

Proceeding from the fact that deflection w should be an even function relative to the center of the membrane, let us take the following expression for it, which becomes zero on the contour:

$$w = C\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2a}.$$

For selection of suitable expressions for displacements u and v we will follow the following considerations. If the accepted expression for w is substituted in the right side of equation (1.19), we obtain an equation for function ϕ with a known right side. The particular solution of such an equation can be sought in the form corresponding to its right side. Having determined the structure of function in this way by formulas of Hook law we can determine the structure of functions u and v. The just described method of determining the structure of function u and v is extremely bulky, and it can be bypassed in the following manner. In the expression for Э there are the following products of functions u and w:

$$\frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2, \quad \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial y}\right)^2, \quad \frac{\partial u}{\partial y}\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}. \tag{1.20}$$

It turns out that the structure of function u will coincide with this result if we determine it from conditions

$$u_1=\int\left(\frac{\partial w}{\partial x}\right)^2 dx, \quad u_2=\int\left(\frac{\partial w}{\partial y}\right)^2 dx, \quad u_3=\int\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\,dy,$$

which are obtained from products of (1.20). Constants of integration are dropped here. By summing up the values of u_1, u_2, u_3, obtained after integration, and having furnished each term with an as yet interterminate coefficient, we obtain the general structure for function u. Thus we will have

$$u=A_1\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}+A_2\sin\frac{\pi x}{a}.$$

Analogously

$$v=B_1\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}+B_2\sin\frac{\pi y}{a}.$$

Boundary conditions for these displacements are the following. $x=\pm a$, $y=\pm a$ there should be $u=0$, $v=0$. From these conditions we obtain

$$A_1=A_2=A, \quad B_1=B_2=B.$$

Then finally

$$u=A\left(1+\cos\frac{\pi y}{a}\right)\sin\frac{\pi x}{a},$$

$$v=B\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{a}.$$

Now the total potential energy of the membrane will have the form

$$\frac{2(1-\mu^2)}{\pi^2 E\delta}\,\mathcal{Э}=\frac{7-\mu}{2}(A^2+B^2)+(1+\mu)AB-$$
$$-\frac{\pi(2-\mu)}{8a}(A+B)C^2+\frac{5\pi^2C^4}{256a^2}-q_0aC.$$

Here is designated

$$q_0 = \frac{32(1-\mu^2)\,qa}{\pi^4 E\delta}.$$

For determination of parameters *A*, *B* and *C* for the obtained expression of total energy let us apply the origin of virtual displacements. This will give us the following equations:

$$\frac{\partial \text{Э}}{\partial A} = 0, \quad \frac{\partial \text{Э}}{\partial B} = 0, \quad \frac{\partial \text{Э}}{\partial C} = 0.$$

From the solution of these equations we find

$$A = B = \frac{\pi(2-\mu)}{64a}C^2, \qquad C = \frac{16a}{\pi^2}\sqrt[3]{\frac{(1-\mu^2)\,qa}{[10-(2-\mu)^2]\,E\delta}}.$$

Having expressions for *A*, *B* and *C*, by formulas (1.15)-(1.17) we can obtain expressions for stresses:

$$\sigma_x = \frac{N_x}{\delta} = \frac{8}{\pi^2}\sqrt[3]{\frac{Eq^2a^2}{(1-\mu^2)\,[10-(2-\mu)^2]^2\delta^2}} \times$$

$$\times\left[(2+\mu)\left(\cos^2\frac{\pi y}{2a} + \mu\cos^2\frac{\pi x}{2a}\right) - 2\mu(1+\mu)\cos^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2a}\right],$$

$$\sigma_y = \frac{N_y}{\delta} = \frac{8}{\pi^2}\sqrt[3]{\frac{Eq^2a^2}{(1-\mu^2)\,[10-(2-\mu)^2]^2\delta^2}} \times$$

$$\times\left[(2+\mu)\left(\cos^2\frac{\pi x}{2a} + \mu\cos^2\frac{\pi y}{2a}\right) - 2\mu(1+\mu)\cos^2\frac{\pi x}{2a}\cos^2\frac{\pi y}{2a}\right].$$

For displacements we will have

$$u = \frac{8(2-\mu)a}{\pi^3}\sqrt[3]{\frac{(1-\mu^2)^2q^2a^2}{[10-(2-\mu)^2]^2E^2\delta^2}}\cos^2\frac{\pi y}{2a}\sin\frac{\pi x}{a},$$

$$v = \frac{8(2-\mu)a}{\pi^3}\sqrt[3]{\frac{(1-\mu^2)^2q^2a^2}{[10-(2-\mu)^2]^2E^2\delta^2}}\cos^2\frac{\pi x}{2a}\sin\frac{\pi y}{a},$$

$$w = \frac{16a}{\pi^2}\sqrt[3]{\frac{(1-\mu^2)\,qa}{[10-(2-\mu)^2]\,E\delta}}\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2a}.$$

The greatest stresses and greatest deflection will be in the center of the membrane. When $\mu = 0.3$

$$\sigma_x = \sigma_y = 0{,}499\sqrt[3]{\frac{Eq^2a^2}{\delta^2}}, \quad w = 0{,}82a\sqrt[3]{\frac{qa}{E\delta}}.$$

C H A P T E R II

STRENGTH OF ROUND PLATES AND MEMBRANES

§ 6. Basic Information from the Theory of Round Plates of Small Deflection

We obtai the differential equation of equilibrium of round plates from equation (1.1) by its conversion into polar coordinates.

Let us assume on the plate we have point *M* (Fig. 18). Rectangular coordinates x, y of this point is connected with polar coordinates α, r by the following equalities:

$$x = r\cos\alpha = \varphi(\alpha, r), \quad y = r\sin\alpha = \psi(\alpha, r).$$

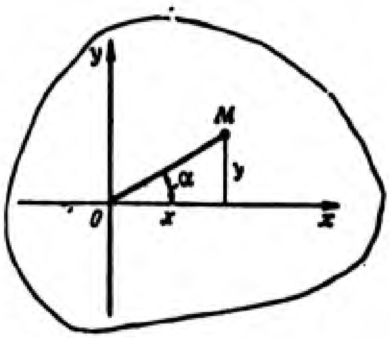

Fig. 18.

Having connection of old coordinates with new, by known formulas of differential calculus it is possible to change from differentiation with respect to variable x, y to differentiation with respect to variable α, r by formulas

$$\frac{\partial w}{\partial \alpha}=\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial \alpha}+\frac{\partial w}{\partial y}\frac{\partial \psi}{\partial \alpha},$$

$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial r}+\frac{\partial w}{\partial y}\frac{\partial \psi}{\partial r}.$$

But

$$\frac{\partial \varphi}{\partial \alpha}=-r\sin\alpha,\quad \frac{\partial \psi}{\partial \alpha}=r\cos\alpha,$$

$$\frac{\partial \varphi}{\partial r}=\cos\alpha,\quad \frac{\partial \psi}{\partial r}=\sin\alpha.$$

Then

$$\frac{\partial w}{\partial \alpha}=-r\sin\alpha\frac{\partial w}{\partial r}+r\cos\alpha\frac{\partial w}{\partial y},$$

$$\frac{\partial w}{\partial r}=\cos\alpha\frac{\partial w}{\partial x}+\sin\alpha\frac{\partial w}{\partial y}.$$

From these expressions we find

$$\frac{\partial w}{\partial x}=\cos\alpha\frac{\partial w}{\partial r}-\frac{1}{r}\sin\alpha\frac{\partial w}{\partial \alpha},$$

$$\frac{\partial w}{\partial y}=\sin\alpha\frac{\partial w}{\partial r}+\frac{1}{r}\cos\alpha\frac{\partial w}{\partial \alpha}.$$

The second and mixed derivatives can be found in the following manner:

$$\begin{aligned}\frac{\partial^2 w}{\partial x^2}&=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial}{\partial x}\left(\cos\alpha\frac{\partial w}{\partial r}-\frac{1}{r}\sin\alpha\frac{\partial w}{\partial x}\right)=\\&=\frac{\partial}{\partial r}\cos\alpha\frac{\partial w}{\partial x}-\frac{\partial}{r\partial \alpha}\sin\alpha\frac{\partial w}{\partial x}=\\&=\cos\alpha\frac{\partial}{\partial r}\left(\cos\alpha\frac{\partial w}{\partial r}-\frac{1}{r}\sin\alpha\frac{\partial w}{\partial \alpha}\right)-\\&-\frac{1}{r}\sin\alpha\frac{\partial}{\partial \alpha}\left(\cos\alpha\frac{\partial w}{\partial r}-\frac{1}{r}\sin\alpha\frac{\partial w}{\partial \alpha}\right)=\\&=\cos^2\alpha\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\sin 2\alpha\frac{\partial w}{\partial \alpha}-\frac{1}{r}\sin 2\alpha\frac{\partial^2 w}{\partial \alpha\,\partial r}+\\&+\frac{1}{r}\sin^2\alpha\frac{\partial w}{\partial r}+\frac{1}{r^2}\sin^2\alpha\frac{\partial^2 w}{\partial \alpha^2}.\end{aligned}$$

Analogously

$$\frac{\partial^2 w}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right)=\sin^2\alpha\frac{\partial^2 w}{\partial r^2}+\frac{1}{r^2}\sin 2\alpha\frac{\partial^2 w}{\partial\alpha\,\partial r}+$$

$$+\frac{1}{r}\cos^2\alpha\frac{\partial w}{\partial r}-\frac{1}{r^2}\sin 2\alpha\frac{\partial w}{\partial\alpha}+\frac{1}{r^2}\cos^2\alpha\frac{\partial^2 w}{\partial\alpha^2},$$

$$\frac{\partial^2 w}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)=\frac{1}{2}\sin 2\alpha\frac{\partial^2 w}{\partial r^2}-$$

$$-\frac{1}{r^2}\cos^2\alpha\frac{\partial w}{\partial\alpha}+\frac{1}{r}\cos^2\alpha\frac{\partial^2 w}{\partial\alpha\,\partial r}-\frac{1}{r}\sin^2\alpha\frac{\partial^2 w}{\partial\alpha\,\partial r}-$$

$$-\frac{1}{2r}\sin 2\alpha\frac{\partial w}{\partial r}-\frac{1}{2r^2}\sin 2\alpha\frac{\partial^2 w}{\partial\alpha^2}+\frac{1}{r^2}\sin^2\alpha\frac{\partial w}{\partial\alpha}.$$

Subsequently we will be limited by examination of symmetric bending of round plates. In this instance all the derivatives according to angle α must become zero. In this case we obtain

$$\left.\begin{aligned}\frac{\partial^2 w}{\partial x^2}&=\cos^2\alpha\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\sin^2\alpha\frac{\partial w}{\partial r},\\ \frac{\partial^2 w}{\partial y^2}&=\sin^2\alpha\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\cos^2\alpha\frac{\partial w}{\partial r},\\ \frac{\partial^2 w}{\partial x\,\partial y}&=\frac{1}{2}\sin 2\alpha\frac{\partial^2 w}{\partial r^2}-\frac{1}{2r}\sin 2\alpha\frac{\partial w}{\partial r}.\end{aligned}\right\}\qquad(2.1)$$

Since angle α is selected arbitrarily, if we change the designation of old axes, i.e., designate axis x through y, and axis y through x, the right sides of expressions (2.1) should not be changed. The structure of the right sides of these expressions will not be changed only when we assume $\alpha = 0$. Then finally we obtain

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial^2 w}{\partial r^2},\qquad \frac{\partial^2 w}{\partial y^2}=\frac{1}{r}\frac{\partial w}{\partial r},\qquad \frac{\partial^2 w}{\partial x\,\partial y}=0.$$

Now equation (1.1) for the round plate can be written so:

$$\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)=\frac{q}{D},$$

or after fulfillment of the shown operations of differentiation

$$\frac{d^4 w}{dr^4}+\frac{2}{r}\frac{d^3 w}{dr^3}-\frac{1}{r^2}\frac{d^2 w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}=\frac{q}{D}.$$

This equation for convenience of integration is sometimes written in the form:

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}=\frac{q}{D}. \qquad (2.2)$$

Expressions for bending moments will have the form

$$M_x\sim M_r=-D\left(\frac{d^2w}{dr^2}+\frac{\mu}{r}\frac{dw}{dr}\right),$$
$$M_y\sim M_t=-D\left(\frac{1}{r}\frac{dw}{dr}+\mu\frac{d^2w}{dr^2}\right).$$

The expression for shearing force can be found from equation (2.2):

$$q=D\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}.$$

Shearing force Q_r per unit of length

$$Q_r=\frac{1}{2\pi r}\int_0^{2\pi}\int_0^{r} qr\,d\alpha\,dr=$$
$$=\frac{D}{2\pi r}\int_0^{2\pi}\int_0^{r}\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}rd\,r\,d\alpha=$$
$$=\frac{D}{2\pi r}2\pi\int_0^{r} d\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}=$$
$$=\frac{D}{r}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]\right\}=D\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right]=$$
$$=D\left(\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr}\right).$$

Equation (2.2) can be integrated in such a sequence.

Let us preliminarily introduce new dimensionless quantity ρ by formula

$$\rho=\frac{r}{R},$$

where R – external radius of the plate.

Consequently, $0 \leq \rho \leq 1$ and $dr = R d\rho$.

Then let us rewrite equation (2.2) in the form

$$\frac{d}{d\rho}\left\{\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\frac{dw}{d\rho}\right)\right]\right\} = \frac{qR^4\rho}{D}.$$

By integrating, we obtain

$$\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\frac{dw}{d\rho}\right)\right] = \frac{R^4}{D}\int q\rho\,d\rho + C_1',$$

$$\frac{d}{d\rho}\left[\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\frac{dw}{d\rho}\right)\right] = \frac{R^4}{D}\,\frac{1}{\rho}\int q\rho\,d\rho + \frac{C_1'}{\rho},$$

$$\frac{1}{\rho}\,\frac{d}{d\rho}\left(\rho\frac{dw}{d\rho}\right) = \frac{R^4}{D}\int\left(\frac{1}{\rho}\int q\rho\,d\rho\right)d\rho + C_1'\ln\rho + C_2',$$

$$\frac{d}{d\rho}\left(\rho\frac{dw}{d\rho}\right) = \frac{R^4}{D}\rho\int\left(\frac{1}{\rho}\int q\rho\,d\rho\right)d\rho + C_1'\rho\ln\rho + C_2'\rho,$$

$$\rho\frac{dw}{d\rho} = \frac{R^4}{D}\int\left[\rho\int\left(\frac{1}{\rho}\int q\rho\,d\rho\right)d\rho\right]d\rho + C_1'\int\rho\ln\rho\,d\rho + \frac{1}{2}C_2'\rho^2 + C_3',$$

$$\frac{dw}{d\rho} = \frac{R^4}{D}\,\frac{1}{\rho}\int\left[\rho\int\left(\frac{1}{\rho}\int q\rho\,d\rho\right)d\rho\right]d\rho +$$
$$+ C_1'\frac{1}{\rho}\int\rho\ln\rho\,d\rho + \frac{1}{2}C_2'\rho + \frac{C_3'}{\rho},$$

$$w = \frac{R^4}{D}\int\left\{\frac{1}{\rho}\int\left[\rho\int\left(\frac{1}{\rho}\int q\rho\,d\rho\right)d\rho\right]d\rho\right\}d\rho +$$
$$+ C_1'\int\left(\frac{1}{\rho}\int\rho\ln\rho\,d\rho\right)d\rho + \frac{1}{4}C_2'\rho^2 + C_3'\ln\rho + C_4',$$

or finally

$$w = \frac{R^4}{D}\int\left\{\frac{1}{\rho}\int\left[\rho\int\left(\frac{1}{\rho}\int q\rho\,d\rho\right)d\rho\right]d\rho\right\}d\rho +$$
$$+ C_1\rho^2\ln\rho + C_2\rho^2 + C_3\ln\rho + C_4.$$

Constants of integration in this expression are determined in each concrete case of calculation from boundary conditions; two boundary conditions will be on the external contour and two – on the inside, if the plate has opening, or from conditions at the center of the plate, ensuing from the physical essence of the problem, if the plate has no opening in the center.

As an example let us examine bending of a hinged-supported plate by uniform load q = const. In this instance the expression for w takes the form

$$w=\frac{qR^4\rho^4}{64D}+C_1\rho^2\ln\rho+C_2\rho^2+C_3\ln\rho+C_4.$$

For determination of constants of integration we have the following boundary conditions at $\rho=1$:

$$w=0,\ M_r=0,$$

$$\frac{dw}{d\rho}=\frac{qR^4\rho^3}{16D}+C_1(2\rho\ln\rho+\rho)+2C_2\rho+\frac{C_3}{\rho},$$

$$\frac{d^2w}{d\rho^2}=\frac{3qR^4\rho^2}{16D}+C_1(2\ln\rho+3)+2C_2-\frac{C_3}{\rho^2},$$

$$M_r=-D\left\{\frac{(3+\mu)qR^2\rho^2}{16D}+\frac{C_1}{R^2}[(3+\mu)+2(1+\mu)\ln\rho]+\frac{2(1+\mu)}{R^2}C_2-\frac{(1-\mu)C_3}{R^2\rho^2}\right\}.$$

When $\rho=1$ we will have (on the contour)

$$\frac{qR^4}{64D}+C_2+C_4=0,$$

$$\frac{(3+\mu)qR^2}{16D}+\frac{3+\mu}{R^2}C_1+\frac{2(1+\mu)}{R^2}C_2-\frac{(1-\mu)}{R^2}C_3=0.$$

To these two equations, ensuing from conditions of the problem when $\rho=1$, one should join two more from conditions in the center of the plate ($\rho=0$). In the center of the plate deflection and moment must have finite quantities. For this it is necessary to assume $C_1 = 0$, $C_3 = 0$. Then finally

$$C_2=-\frac{(3+\mu)qR^4}{32(1+\mu)D},\qquad C_4=\frac{(5+\mu)qR^4}{64(1+\mu)D}.$$

Expressions for deflection and moment M_r take the form

$$w=\frac{qR^4}{64D}\left[\rho^4-\frac{2(3+\mu)}{1+\mu}\rho^2+\frac{5+\mu}{1+\mu}\right],$$

$$M_r=-\frac{(3+\mu)qR^2}{16}(\rho^2-1).$$

For stresses σ_r we obtain

$$\sigma_r = \pm \frac{6M_r}{\delta^2} = \pm \frac{6(3+\mu)qR^2}{16\delta^2}(\varrho^2 - 1).$$

The greatest stresses and deflection will be in the center of the plate $(\varrho = 0)$:

$$w_{max} = \frac{(5+\mu)\,qR^4}{64D}, \quad \sigma_{r\,max} = \pm \frac{3(3+\mu)\,qR^2}{8\delta^2}.$$

§ 7. Application of the Origin of Virtual Displacements for Symmetrically Loaded Round Plates and Membranes

The expression for potential energy of a symmetrically loaded round plate can be obtained from the corresponding expression for a rectangular plate (§ 3), if in the latter we change from variables x, y to new variable r according to formulas of the previous paragraph. In this case we obtain

$$Э = \frac{D}{2}\iint_{s}\left[\left(\frac{d^2w}{dr^2}\right)^2 + \left(\frac{1}{r}\frac{dw}{dr}\right)^2 + 2\mu\frac{1}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right] r\,dr\,d\alpha + T.$$

After integration with respect to α

$$Э = \pi D\int_{R_B}^{R_H}\left[\left(\frac{d^2w}{dr^2}\right)^2 + \left(\frac{1}{r}\frac{dw}{dr}\right)^2 + \frac{2\mu}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}\right] r\,dr + T,$$

where R_B, R_H – internal and external radii of the plate; T – work of external forces.

As an example let us consider bending of a rigidly fixed round plate, loaded by concentrated force in the center. In this instance $R_B = 0$.

As usual, for solution of the problem by this method it is necessary to assign a suitable expression for w, satisfying prescribed boundary conditions:

$$w = A_0\left(1 - \frac{r^2}{R_н^2}\right)^2 + A_1\left(1 - \frac{r^4}{R_н^4}\right)^2 + \ldots$$

Being limited by the first term of this series, we take

$$w = A_0\left(1 - \frac{r^2}{R_н^2}\right)^2. \qquad (2.3)$$

By substituting the necessary derivatives from this expression in Э and having integrated within limits of $0-R_H$, we find

$$Э = \frac{32\pi D}{3R_н^2}A_0^2 - PA_0.$$

The origin of virtual displacements gives the following equation for determination of parameter A_0:

$$\frac{dЭ}{dA_0} = 0.$$

From this condition we find

$$A_0 = \frac{3PR_н^2}{64\pi D}.$$

Then

$$w = \frac{3PR_н^2}{64\pi D}\left(1 - \frac{r^2}{R_н^2}\right)^2.$$

Deflection in the center

$$w = 0{,}015\,\frac{PR_н^2}{D}.$$

Accurate solution of the given problem for deflection in the center gives the following quantity:

$$w = 0{,}02\,\frac{PR_н^2}{D}.$$

For improvement of the approximate result the number of terms in the expression for w must be increased.

During investigation of symmetrically loaded round membranes it is also suitable to use the origin of virtual displacements.

The expression for potential energy of a symmetrically loaded round membrane can be obtained from corresponding expression of § 5.

By virtue of symmetry of loading the component of shearing strain in this case will be equal to zero, and the expression for total energy will obtain the form

$$Э=\frac{1}{2}\iint_{s}(N_r\varepsilon_r+N_\theta\varepsilon_\theta)r\,dr\,d\alpha+T.$$

For connection of stresses with components of deformation we have formulas of Hooke law:

$$\varepsilon_r=\frac{1}{E\delta}(N_r-\mu N_\theta),$$
$$\varepsilon_\theta=\frac{1}{E\delta}(N_\theta-\mu N_r).$$

Expressions for components of deformation can be obtained from Fig. 19 (see Fig. 16).

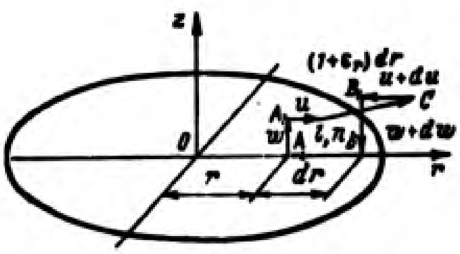

Fig. 19.

Let us project closed polygon AA_1B_1CB to axes r, z and let us equate the sums of projections to zero. From the equations obtained in this case we find

$$(1+\varepsilon_r)n = \frac{dw}{dr}, \quad (1+\varepsilon_r)l = 1+\frac{du}{dr}.$$

If the left and right sides of these equalities are squared and summed up, then, by omitting the reasonings analogous to those made in § 4 with deviation of ε_x, ε_y in rectangular axes, we obtain the following expression for ε_r:

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2.$$

The component of deformation in circumferential direction is determined by expression

$$\varepsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r}.$$

Then, considering formulas of Hooke law and the just obtained relationships for ε_r and ε_θ from displacements u and w, we obtain the following expression for total energy of a symmetrically loaded round membrane at constant pressure q:

$$Э = \frac{\pi E h}{1-\mu^2}\int_0^a \left[\left(\frac{du}{dr}\right)^2 + \frac{du}{dr}\left(\frac{dw}{dr}\right)^2 + \frac{1}{4}\left(\frac{dw}{dr}\right)^4 + \frac{u^2}{r^2} + \right.$$
$$\left. + 2\mu\frac{u}{r}\frac{du}{dr} + \mu\frac{u}{r}\left(\frac{dw}{dr}\right)^2\right] r\,dr - 2\pi q\int_0^a wr\,dr.$$

The last term in this expression represents work of normal pressure q. Having the expression for total energy, it is possible to change to selection of suitable expressions for u and w.

For deflection w let us take expression (2.3), which we used during investigation of bending of a round plate

$$w = A_0\left(1 - \frac{r^2}{a^2}\right)^2.$$

The structure of function u is determined as was stipulated in § 5, proceeding from the expression of potential energy:

$$u_1 = \int \left(\frac{dw}{dr}\right)^2 dr, \quad u_2 = \frac{1}{r}\left(\frac{dw}{dr}\right)^2.$$

By substituting here the derivatives of w, we obtain

$$u = A_1'r + A_3'r^3 + A_5'r^5 + A_7'r^7.$$

In order that the given expression would satisfy conditions on the contour $u = 0$ when $r = a$, it is necessary to assume

$$A_1' = -a^2A_3' - a^4A_5' - a^6A_7'.$$

Then finally

$$u = A_1\left(1 - \frac{r^2}{a^2}\right)r + A_2\left(1 - \frac{r^4}{a^4}\right)r + A_3\left(1 - \frac{r^6}{a^6}\right)r.$$

After selection of functions w and u it is possible to calculate the potential energy of membrane Э and, by making use of the origin of virtual displacements, to determine constants A_0, A_1, A_2, A_3 from equations

$$\frac{\partial Э}{\partial A_0} = 0, \quad \frac{\partial Э}{\partial A_1} = 0, \quad \frac{\partial Э}{\partial A_2} = 0, \quad \frac{\partial Э}{\partial A_3} = 0.$$

By solving the equations obtained in this case, we find

$$A_0 = a\sqrt[3]{\frac{21(1-\mu^2)qa}{2(23 + 14\mu - 9\mu^2)E\delta}},$$

$$A_1 = (3-\mu)\frac{A_0^2}{a^2},$$

$$A_2 = -\frac{2(5-\mu)A_0^2}{3a^2},$$

$$A_3 = \frac{(7-\mu)A_0^2}{6a^2}.$$

Stresses in the membrane σ_r and σ_θ are determined by formulas of Hooke law:

$$\sigma_r = \frac{E}{1-\mu^2}(\varepsilon_r + \mu\varepsilon_\theta), \quad \sigma_\theta = \frac{E}{1-\mu^2}(\varepsilon_\theta + \mu\varepsilon_r).$$

In expanded form these stresses will be

$$\sigma_r = E\sqrt[3]{\frac{21^2(1-\mu^2)^2 q^2 a^2}{4(23+14\mu-9\mu^2)^2 E^2 h^2}}\left[\frac{5+2\mu-3\mu^2}{6(1-\mu^2)} - \frac{r^2}{a^2} + \right.$$
$$\left. + \frac{2}{3}\frac{r^4}{a^4} - \frac{r^6}{6a^6}\right],$$
$$\sigma_\theta = E\sqrt[3]{\frac{21^2(1-\mu^2)^2 q^2 a^2}{4(23+14\mu-9\mu^2)^2 E^2 h^2}}\left[\frac{5-3\mu}{6(1-\mu^2)} - \frac{3r^2}{a^2} + \right.$$
$$\left. + \frac{10r^4}{3a^4} - \frac{7r^6}{6a^6}\right].$$

By assuming $\mu = 0.3$, we find stresses and deflection w in the center $r = 0$ and on contour $r = a$ of the membrane[1]:

$$(\sigma_r)_{r=0} = (\sigma_\theta)_{r=0} = 0{,}495E\sqrt[3]{\frac{q^2a^2}{E^2h^2}};$$
$$(\sigma_r)_{r=a} = 0{,}242E\sqrt[3]{\frac{q^2a^2}{E^2h^2}};$$
$$(\sigma_\theta)_{r=a} = 0{,}073E\sqrt[3]{\frac{q^2a^2}{E^2h^2}},$$
$$(w)_{r=0} = 0{,}71a\sqrt[3]{\frac{qa}{Eh}}.$$

[1]Numerical coefficients in the obtained formulas for stresses are somewhat different from coefficients in analogous formulas given in the book of S. P. Timoshenko "Plates and shells." This difference is apparently a consequence, in the first place, of different methods of solution of the given problem,and, secondly, a different approach to selection of approximating functions for displacements in the tangential plane for the deformed membrane.

The second observation pertains to formulas obtained for a square membrane in § 5.

CHAPTER III

STABILITY OF PLATES

Phenomenon of loss of stability has been connected with the possibility of appearance of elastic systems that are different from original forms of equilibrium. If, for example, the plate is loaded with compressive forces, acting in its middle plane, then for a certain magnitude of these forces the original rectilinear surface of the plate can cease being uniquely possible. Along with this form there can exist other forms of equilibrium, connected with warpage of the surface of the plate. The load at which this phenomenon occurs is called critical.

Subsequently we will examine only such cases when the load does not change its direction after loss of stability of the plate. Such loads are called conservative.

Below are examined only two methods of solution of problems of stability, which are widely applied in calculation practice: static method and energy method. The first method is based on the use of equation (1.1), the second – expressions of total energy (§ 3).

Let us begin with examination of the first method.

§ 8. Static Method of Investigation of Stability of Rectangular Plates

In this case the differential equation of equilibrium has the form

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q.$$

By lateral load q here we should mean that fictitious distributed load, which distributed forces give in the middle surface with their projection to a nondeformed plane of the plate. Thus, we can obtain (see § 4)

$$q=N_x\frac{\partial^2 w}{\partial x^2}+N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}.$$

Then the equation for investigation of stability of plates takes the form

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=-N_x\frac{\partial^2 w}{\partial x^2}-N_y\frac{\partial^2 w}{\partial y^2}+2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y}.$$

Here signs for N_x and N_y are changed to opposite – compression. Expressions for moments and shearing forces, and also formulation of boundary conditions in this case remain the same as during investigation of the strength of plates.

Forces N_x, N_y, N_{xy}, connected with effective contour forces, in the most general case can be quantities variable at each point of the plate. In this instance at first it is necessary to solve the two-dimensional problem of theory of elasticity about the distribution of these forces along the plane of the plate, and then it is even possible to solve the problem of stability of this plate, being under the action of a specified system of external forces.

But in the most important cases and in the calculations most frequently encountered in practice these forces can be considered uniformly distributed along the plane of the plate and coinciding with their distribution along the contour.

The given equation of stability cannot be accurately solved, and therefore it is necessary to use approximate methods, which was indicated above (§ 2).

Let us give some examples. In the beginning let us examine stability of an evenly compressed plate, hinge-supported along the contour (Fig. 20).

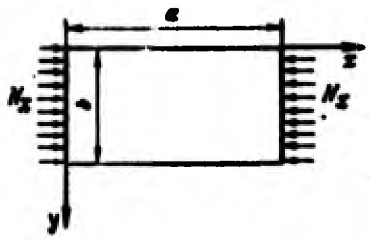

Fig. 20.

In this instance N_x = const along the entire plane of the plate; $N_y = 0$, $N_{xy} = 0$. Then

$$D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=-N_x\frac{\partial^2 w}{\partial x^2}.$$

With hinge-support there should be:

$$M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\mu\frac{\partial^2 w}{\partial y^2}\right)=0,\quad w=0 \text{ when } x=0,\ x=a,$$

$$M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)=0,\quad w=0 \text{ when } y=0,\ y=b.$$

We will satisfy these boundary conditions if for deflection we take

$$w=A\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \qquad (3.1)$$

Of course, it would have been possible to take another expression for deflection, which satisfies the same boundary conditions.

The expression for deflection (3.1) shows that after the loss of stability the surface of the plate in the direction of axis x was bent along m half-waves, and along axis y – along n half-waves.

By substituting the accepted expression for w in the equation of equilibrium, we obtain when $A \neq 0$

$$D\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]^2 = N_x\left(\frac{m\pi}{a}\right)^2.$$

Hence we find

$$N_x=\pi^2 D\frac{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}{\left(\frac{m}{a}\right)^2}.$$

For purposes of calculation the smallest magnitude of compressive force N_x must be found. From the structure of the last expression it is evident that minimum N_x will be at $n = 1$, i.e., when in the direction of axis y of the plate there will be only one half-wave. Then

$$N_x=\frac{\pi^2 D}{a^2}\left[m+\frac{1}{m}\left(\frac{a}{b}\right)^2\right]^2.$$

As can be seen from this expression, at specified sizes of the plate the magnitude of compressive force N_x depends on the number of half-waves m. In each concrete case it would be necessary to assume $m = 1, 2, 3 \ldots$ and to take the smallest of all values of N_x obtained in this case for calculation. However, to get a visible finite formula we will consider that the plate is rather long and the quantity of all types of values of numbers m is also rather large. Under such an assumption it can be considered that force N_x is a continuous function of parameter m. On the basis of such assumption the given expression can be differentiated. Then for finding the minimum load we have equation

$$\frac{dN_x}{dm}=2\pi^2\frac{D}{b^2}\left[1-\frac{1}{m^2}\left(\frac{a}{b}\right)^2\right]\left[m+\frac{1}{m}\left(\frac{a}{b}\right)^2\right]=0.$$

Hence we find

$$m=\frac{a}{b}.$$

Since numbers m must be whole natural numbers, the least quantities for force N_x will be when

$$m = \frac{a}{b} = 1; \quad m = \frac{a}{b} = 2; \quad m = \frac{a}{b} = 3\ldots$$

and all these least quantities will be equal to each other. Therefore,

$$N_{x\,\min} = \frac{\pi^2 D}{a^2}\left(\frac{a}{b} + \frac{a}{b}\right)^2 = \frac{4\pi^2 D}{b^2}.$$

Let us construct curves of the relationship of force N_x to ratio a/b at equal quantities of m (Fig. 21).

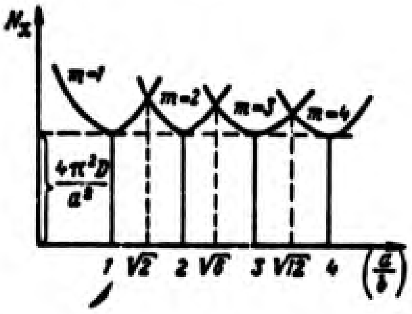

Fig. 21.

Let us find the points of intersection of curves during the transition from m to $m + 1$ half-waves. From the condition of equality of critical forces at points of intersection we find

$$m + \frac{1}{m}\left(\frac{a}{b}\right)^2 = m + 1 + \frac{1}{m+1}\left(\frac{a}{b}\right)^2.$$

From this equation we obtain

$$\frac{a}{b} = \sqrt{m(m+1)}.$$

Having given different values to parameter m, let us find the points of intersection of the curves in Fig. 21. When $m=1$ $\frac{a}{b} = \sqrt{2}$, when $m=2$ $\frac{a}{b} = \sqrt{6}$, etc. At very large m we obtain $a/b \approx m$, i.e., rather long plates after loss of stability are approximately divided into square half-waves. In this case for critical magnitude of force N_x we obtain

$$N_{кр} = \frac{4\pi^2 D}{b^2}.$$

If for a specified plate ratio a/b is not very great, the, having substituted it into the expression for N_x instead of m, we can determine critical magnitude of compressive force for the given plate. We examined this case of hinge-support of plate along the contour. With other boundary conditions it would be necessary to take another expression for deflection, which satisfies these boundary conditions, and to use other approximate methods, described in § 2.

Let us show how it is possible to rather simply obtain the solution of the problem of stability, using the method of finite differences for a plate hinge-supported on three sides and with the fourth side free with uniform compression (Fig. 22).

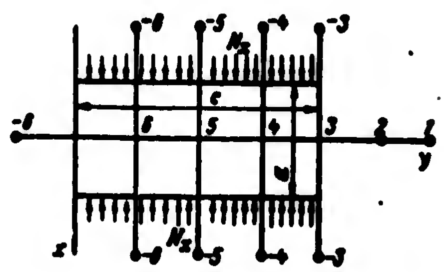

Fig. 22.

Equation of stability in finite differences in this case assumes the form

$$\frac{D}{h^4}[20W_k - 8(W_i + W_j + W_m + W_n) + 2(W_p + W_q + W_r + W_s) + W_t + W_u + W_v + W_w] = -\frac{N_x}{h^2}(W_i + W_j - 2W_k).$$

We will satisfy boundary conditions if we assume

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2}\right) = 0 \text{ when } y = 0 \text{ and } y = c,$$

$$V_y = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x^2\,\partial y}\right] = 0 \text{ when } y = c,$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2}\right) = 0 \text{ when } x = \pm\frac{a}{2}$$

and on the contour deflection is equal to zero everywhere, except side $y = 0$.

In finite differences these boundary conditions have the form

$$M_y = -D\left(\frac{W_n + W_n - 2W_k}{b^2} + \mu\,\frac{W_l + W_l - 2W_k}{h^2}\right) = 0,$$

$$V_y = -D\left[\frac{W_v - W_u + 2W_m - 2W_n}{2b^3} + (2-\mu)\,\frac{2W_m - 2W_n + W_q - W_0 + W_r - W_p}{2bh^2}\right] = 0,$$

$$M_x = -D\left(\frac{W_l + W_i - 2W_k}{h^2} + \mu\,\frac{W_m + W_n - 2W_k}{b^2}\right) = 0.$$

From these equations for the accepted numeration of grid nodes on Fig. 22 we obtain

$$\left.\begin{array}{l} W_k = 0 \\ W_l = -W_l \end{array}\right\}\text{ long side} \qquad \left.\begin{array}{l} W_k = 0 \\ W_n = -W_m \end{array}\right\}\text{ short side on the left}$$

For the short side on the right when $\mu = 0.3$ we obtain the following two equations, which connect outer contour points 1 and 2 with internal contour points 3, 4, 5:

$$\left.\begin{array}{l} V_y = -D\left(\dfrac{W_1 - W_5 + 2W_4 - 2W_2}{2b^3} + 1{,}7\,\dfrac{2W_1 - 2W_2 + 0 - 0 + 0 - 0}{2bh^2}\right) = 0, \\ M_y = -D\left(\dfrac{W_2 + W_4 - 2W_3}{b^2} + 0{,}3\,\dfrac{0 + 0 - 2W_3}{h^2}\right) = 0. \end{array}\right\} \qquad (3.2)$$

From the given equations we obtain the connection of outer contour points with internal. For internal contour points we obtain the following four equations of equilibrium.

Point 3:

$$\frac{D}{b^4}\left[20W_3 - 8(0 + 0 + W_4 + W_2) + 2(0 + 0 + 0 + 0) - W_3 - W_3 + W_1 + W_5\right] = -\frac{N_x}{h^2}(0 + 0 - 2W_3).$$

Point 4:

$$\frac{D}{b^4}[20W_4-8(0+0+W_3+W_5)+2(0+0+0+0)-W_4-W_4+W_2+W_6]=$$
$$=-\frac{N_x}{h^2}(0+0-2W_4).$$

Point 5:

$$\frac{D}{b^4}[20W_5-8(0+0+W_4+W_6)+2(0+0+0+0)-W_5-W_5+W_3+0]=$$
$$=-\frac{N_x}{h^2}(0+\;\;-2W_5).$$

Point 6:

$$\frac{D}{b^4}[20W_6-8(0+0+W_5+0)+2(0+0+0+0)-W_6-W_6+W_4-W_6]=$$
$$=-\frac{N_x}{h^2}(0+0-2W_6).$$

By excluding W_1 and W_2 from these equations with the aid of equations (3.2), we obtain the following homogeneous system of four equations:

$$\left.\begin{aligned} -8W_5+(17-2k)W_6=0,\\ W_3-8W_4+(18-2k)W_5-8W_6=0,\\ -5{,}4W_3+(17-2k)W_4-8W_5+W_6=0,\\ (11{,}24-k)W_3-10{,}8W_4+2W_5=0, \end{aligned}\right\} \qquad (3.3)$$

where

$$k=\frac{N_x}{D}\left(\frac{b}{4}\right)^2.$$

In this system of equations there is a total of five unknown quantities: at points 3, 4, 5 and 6 deflections W_3, W_4, W_5, W_6 and load parameter k.

Of all these unknowns only parameter k interests us. For its determination from the second equation it is possible to determine, for example, W_3 and to substitute its value in the remaining three. Then from one of these equations we determine W_4 and substitute its value in the remaining two, which will already contain only W_5 and W_6. By excluding from these equations, for example, W_5, we obtain one equation

$$W_6(8k^4 - 297{,}9k^3 + 3641k^2 - 1662k + 23\,320) = 0.$$

Since $W_6 \neq 0$, there should be

$$8k^4 - 297{,}9k^3 + 3641k^2 - 1662k + 23\,320 = 0. \tag{3.4}$$

We will arrive at exactly the same result if we equate the determinant of equations (3.3) to zero.

Inasmuch as the smallest magnitude of load N_x interests us, we must determine the smallest root of equation (3.4). Then by trial and error we can be certain that $k_{\min} = 2.55$. Then

$$\frac{N_x}{D}\left(\frac{b}{4}\right)^2 \approx 2{,}55$$

or

$$N_{кр} = 40{,}8\,\frac{D}{b^2}.$$

Accurate solution of the problem in this formula leads to coefficient 43.4.

From the provided examples of problems on the stability of rectangular plates it is evident that the formula for critical load can always be represented in the form

$$N_{кр} = k\,\frac{\pi^2 D}{b^2}$$

or when $\mu = 0.3$

$$\sigma_{кр} = k\,\frac{0{,}9E}{\left(\frac{b}{\delta}\right)^2}. \tag{3.5}$$

Numerical value of coefficient k in this formula will depend both on the character of load affecting the plate and on boundary conditions of the problem. In reference books on structural mechanics there is a sufficient quantity of these coefficients for different cases of loading and fixing of the plate contour.

As can be seen from the structure of formula (3.5) itself, it is valid only within limits of elasticity $\sigma_{кр} \leq \sigma_p$. If only $\sigma_{кр} > \sigma_p$ (and practically $\sigma_{кр} > \sigma_s$), then the given formula is impossible to use. In these cases we use Tetmayer-Yasinskiy formula, which is well recommended in practice. This formula is illustrated by Fig. 23, whence

$$\sigma_{кр} = \sigma_p + mn \approx \sigma_s + mn$$

or

$$\frac{\sigma_в - \sigma_s}{mn} = \frac{\left(\frac{b}{\delta}\right)_{кр}}{\left(\frac{b}{\delta}\right)_{кр} - \left(\frac{b}{\delta}\right)}.$$

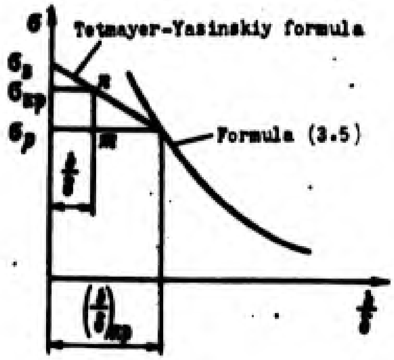

Fig. 23.

Here we obtain Tetmayer-Yasinskiy formula

$$\sigma_{кр} = \sigma_в - (\sigma_в - \sigma_s)\frac{\left(\frac{b}{\delta}\right)}{\left(\frac{b}{\delta}\right)_{кр}},$$

where

$$\left(\frac{b}{\delta}\right)_{кр} = \sqrt{k\frac{0,9E}{\sigma_s}}.$$

If $\left(\frac{b}{\delta}\right) > \left(\frac{b}{\delta}\right)_{кр}$, then we should use formula (3.5), if however $\left(\frac{b}{\delta}\right) < \left(\frac{b}{\delta}\right)_{кр}$, then we should use Tetmayer-Yasinskiy formula.

ble

§ 9. Application of Origin of Virtual Displacements for Investigation of Stability of Plates

In more complex cases of loading of plates for solution of questions of stability the energy method can be successfully used, based on the origin of virtual displacements.

Let us examine a plate in deformed state after loss of stability. Total energy of the plate in this state will be equal to the work of internal forces of elasticity and to work of external forces. Work of forces of elasticity is numerically equal to energy of bending and twisting of a plate and is expressed by formula

$$Э_{изг}=\frac{D}{2}\iint_{s}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\right.$$
$$\left.+2(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy,$$

[изг = bending]

where the integral is extended along the entire area of the plate.

Elastic energy, accumulated in the plate before buckling, does not play a role in questions of investigation of stability, and we do not consider it. Work of external forces, acting in the middle plane of the plate:

$$T=-\frac{1}{2}\iint\left[N_x\left(\frac{\partial w}{\partial x}\right)^2+N_y\left(\frac{\partial w}{\partial y}\right)^2+2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy.$$

Total energy of a buckled plate after loss of stability

$$Э=Э_{изг}+T=\frac{D}{2}\iint\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\right.$$
$$\left.+2(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy-\frac{1}{2}\iint\left[N_x\left(\frac{\partial w}{\partial x}\right)^2+N_y\left(\frac{\partial w}{\partial y}\right)^2+\right.$$
$$\left.+2N_{xy}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]dx\,dy.$$

$\left.\frac{}{}\right)_{кр}.$

Inasmuch as the plate in the considered state is in equilibrium under the action of specified external loads, by applying the origin of virtual displacements to this state it can be said that the sum of work of all forces on virtual displacements will be equal to zero, i.e.,

$$\delta Э = 0.$$

For all practical purposes with use of this condition it is necessary to preliminarily assign a suitable expression for deflection w, certainly satisfying all the geometric boundary conditions of the problem and optionally (but desirably) force conditions:

$$w = a_1 f_1(x, y) + a_2 f_2(x, y) + \ldots,$$

where f_1, f_2, ... – known functions, which satisfy boundary conditions; a_1, a_2 – unknown coefficients (parameters).

Having substituted this function of deflection in the expression of energy and integrated it with respect to the area of the entire plate, we obtain that total energy will be expressed in the function of unknown parameters a_1, a_2, ...:

$$Э = Э(a_1, a_2, \ldots).$$

The origin of virtual displacements leads to equations (§ 3)

$$\frac{\partial Э}{\partial a_1} = 0, \quad \frac{\partial Э}{\partial a_2} = 0, \ldots \ldots \tag{3.6}$$

There will be as many such equations as unknown parameters a_i.

Equations obtained in this case will be homogeneous. The solution of such a system of equations will be the equality of determinant to zero. By equating the determinant to zero, we obtain one equation, from which the critical value of external force will be determined.

As an example let us take the problem about stability of a rectangular hinge-supported plate, being under the action of compressive load N_x, evenly distributed along sides $x = 0$ and $x = a$ (see Fig. 20).

As the expression for deflection w we take

$$w = a_1 xy\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right) + a_2 x^2 y^2 \left(1 - \frac{x}{a}\right)^2\left(1 - \frac{y}{b}\right)^2.$$

For simplicity of computations we will be limited by the first term

$$w = a_1 xy\left(1 - \frac{x}{a}\right)\left(1 - \frac{y}{b}\right).$$

The given expression satisfies only one condition $w = 0$ along the contour and does not satisfy force boundary conditions, nevertheless this expression can be used for approximate solution of the problem. The total expression for potential energy will have the form

$$Э = \frac{Daba_1^2}{15}\left(\frac{8}{3} + \frac{b^2}{a^2}\right) - \frac{N_x ab^3 a_1^2}{180}.$$

Conditions (3.6) in this case lead to equation

$$\frac{\partial Э}{\partial a_1} = \frac{2Daba_1}{15}\left(\frac{8}{3} + \frac{b^2}{a^2}\right) - \frac{2N_x ab^3 a_1}{180} = 0.$$

Hence we obtain

$$N_{кр} = \frac{12D}{b^2}\left(\frac{8}{3} + \frac{b^2}{a^2}\right).$$

This formula can be written in such form:

$$N_{кр} = \frac{k_1 \pi^2 D}{b^2},$$

where

$$k_1 = 1{,}22\left(\frac{8}{3} + \frac{b^2}{a^2}\right).$$

Exact solution of the given problem leads to the following value of this coefficient:

$$k_{точн}=\left(\frac{a}{b}\right)^2\left(1+\frac{b^2}{a^2}\right)^2.$$

[точн = exact]

Table 1 for comparison contains values of these coefficients for some ratios *a/b*.

Table 1.

$\frac{a}{b}$	0,2	1	1,41
k_1	33,7	4,48	3,87
$k_{точн}$	27	4	4,49

From this table it is evident that the greatest error of approximate solution is obtained for small values of *a/b*. It is sufficient to increase the number of terms in the expression for deflection, as the error of approximate solution is reduced. However, a somewhat better result can be obtained if for deflection we take an expression, which will satisfy force boundary conditions as much as possible. For example, let us take

$$w=a_1x\left(1-\frac{x}{a}\right)\sin\frac{\pi y}{b}.$$

By omitting all intermediate computations, let us give the finished result for coefficient k_1 (Table 2).

Table 2.

$\frac{a}{b}$	0,2	1,0	1,4
k_1	32,44	4,2	4,56
$k_{точн}$	27	4	4,49

In all cases a reliable result can be obtained if a suitable expression satisfies all boundary conditions of the problem. For example, expression

$$w = A(x^4 - 2ax^3 + a^3x)\sin\frac{\pi y}{b}$$

satisfies all conditions of hinged support. In this instance for coefficient k_1 we obtain the following values (Table 3).

Table 3.

$\frac{a}{b}$	0,2	1,0	1,4
k_1	27,2	3,98	4,43
$k_{точн}$	27	4	4,49

C H A P T E R IV

STABILITY OF RODS

§ 10. Euler Form of Loss of Stability of Rods

Critical loads for rods, just as for plates, can be determined both from solution of differential equation of the problem and by the energy method.

Differential equation of equilibrium of a bent rod, as it is known, has the form

$$EJy'' = M.$$

In the case of loading of rod in Fig. 24 the expression for bending moment in an arbitrary section of the rod will be

$$M = P(\delta - y).$$

Fig. 24.

Then

$$EJy'' = P(\delta - y)$$

or

$$y'' + k^2 y = \frac{P\delta}{EJ},$$

where

$$k^2 = \frac{P}{EJ}.$$

The solution of this differential equation will consist of the solution of homogeneous equation (without the right side) and solution of equation with the right side (particular solution):

$$y = A\cos kx + B\sin kx + \delta.$$

For determination of constants of integration we have the following boundary conditions:

$$\text{when } x=0 \quad y=0,\ y'=0,$$
$$\text{when } x=l \quad y=\delta.$$

From these boundary conditions we find

$$B=0,$$
$$A=-\delta.$$

Then $y = \delta(1 - \cos kx)$.

This expression for deflection should be valid on the entire length of the rod. In order to obtain the magnitude of deflection on the free end $x = l$, equal to $y = \delta$, it is necessary to assume $(\cos kx)_{x = l} = 0$, i.e., should be

$$kl = \frac{(2n+1)\pi}{2}$$

or

$$k = \frac{(2n+1)\pi}{2l} = \sqrt{\frac{P}{EJ}},$$

whence

$$P = \frac{\pi^2 EJ(2n+1)^2}{4l^2}.$$

The smallest magnitude for force P will be $n = 0$:

$$P_{кр} = \frac{\pi^2 EJ}{4l^2}.$$

This is Euler critical force for the rod.

With other boundary conditions the numerical coefficient in the given formula will be different. Generally it is possible to write

$$P_{кр} = C\frac{\pi^2 EJ}{l^2}, \tag{4.1}$$

where C – numerical coefficient depending on boundary conditions (Table 4).

Table 4.

Character of boundary conditions	One rigidly fixed, the other free	Both ends hinge supported	One rigidly fixed, the other hinged	Both ends rigidly fixed
C	$\frac{1}{4}$	1	$\frac{1}{2}$	4

Let us set the limits of applicability of formula (4.1).

Having divided $P_{кр}$ by the area of cross section of the rod, we obtain critical stress

$$\sigma_{кр} = \frac{C\pi^2 EJ}{l^2 F}$$

or

$$\sigma_{кр} = \frac{C\pi^2 E}{\left(\frac{l}{i}\right)^2}, \tag{4.2}$$

where

$$i = \sqrt{\frac{J}{F}}.$$

Formula (4.2) is applicable if

$$\sigma_{кр} < \sigma_p.$$

In limiting case

$$\frac{C\pi^2 E}{\left(\frac{l}{i}\right)^2} = \sigma_p,$$

whence

$$\left(\frac{l}{i}\right)_{\kappa p} = \sqrt{\frac{C\pi^2 E}{\sigma_p}}.$$

If $\left(\frac{l}{i}\right) < \left(\frac{l}{i}\right)_{\kappa p}$, then formula (4.2) becomes inapplicable, since critical stress is obtained higher than σ_p. In this instance Tetmayer-Yasinskiy formula should be used (Fig. 25).

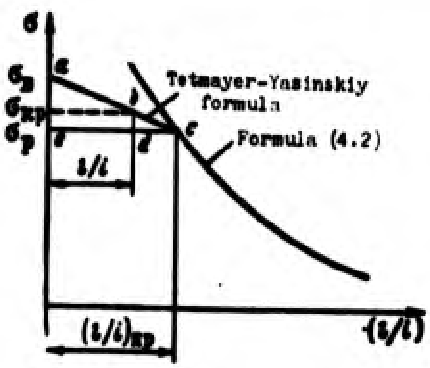

Fig. 25.

From similarity of triangles *bod* and *aoe* we find

$$\sigma_{\kappa p} = \sigma_b - (\sigma_b - \sigma_p)\frac{\frac{l}{i}}{\left(\frac{l}{i}\right)_{\kappa p}}.$$

After multiplication of $\sigma_{\kappa p}$ by the area of cross section of the rod we obtain the expression for critical force beyond the limit of proportionality.

§ 11. Local Stability of Compressed Rods[1]

If the rod consists of separate plate elements (Fig. 26), then, besides the forms of loss of stability examined in § 10, accompanied

[1]Here, just as in § 10, it is assumed that the rod under compression is not twisted. Such a calculation scheme does not contradict experimental results for rolled aircraft profiles.

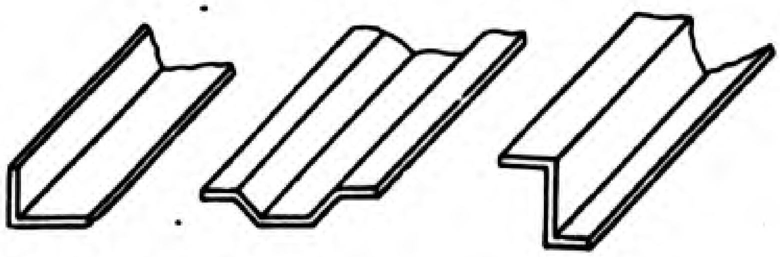

Fig. 26.

by general bending of the axis, there can still be a local loss of stability of the web or the wall as plates. The magnitude of critical stresses for the web and the wall of the profile can be determined by formula (3.5). This formula was obtained for a long rectangular plate, compressed in the direction of the long side and losing stability with the formation of square half-waves, moreover along the width of such a plate only one half-wave is generated.

The magnitude of coefficient k in (3.5) should be taken depending on the boundary conditions on the contour of half-wave. For example, there is examined a profile of constant thickness, shown in Fig. 27 (the upper web is shown after loss of stability; broken lines show boundaries of half-waves). Let us consider each half-wave of the upper web as a square plate with side equal to the width of web. Boundary conditions for it will be the following: side ab – hinge supported, sides ad and bc – hinge supported, side dc – free.

In this instance coefficient k = 1.44.

When determining the critical stresses for the wall coefficient k = 4 as for a hinge-supported square plate with side equal to the width of the wall.

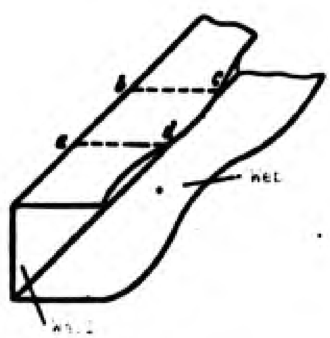

Fig. 27.

In order that the considered profile would be of equal strength both in terns of overall and local stability, it is necessary that critical stresses in both cases be equal to each other:

for a plate

$$\sigma_{кр}=k\frac{0,9E}{\left(\frac{b}{\delta}\right)^2}; \tag{4.3}$$

for a rod

$$\sigma_{кр}=\frac{C\pi^2E}{\left(\frac{l}{i}\right)^2}.$$

From condition of equality of critical stresses we obtain the connection of all geometric parameters of the profile with coefficients k and C:

$$\frac{\frac{b}{\delta}}{\frac{l}{i}}=\frac{1}{\pi}\sqrt{\frac{0,9k}{C}}. \tag{4.4}$$

The profile, which satisfies condition (4.4), will be the most advantageous.

Formulas (4.3) and (4.4) will be valid if critical stresses in the profile do not exceed the elastic limit. If critical stresses in the profile, calculated by formulas (4.3), (4.4), are higher than the limit of elasticity, then corresponding Tetmayer-Yasinskiy formulas should be used.

For determination of critical stresses in the wall and web we have Tetmayer-Yasinskiy formula

$$\sigma_{кр}=\sigma_в-(\sigma_в-\sigma_s)\frac{\frac{b}{\delta}}{\left(\frac{b}{\delta}\right)_{пр}},$$

where

$$\left(\frac{b}{\delta}\right)_{пр}=\sqrt{k\frac{0,9E}{\sigma_s}}.$$

From condition of uniform strength of the profile in terms of local and overall stability we obtain

$$\sigma_{\text{в}}-(\sigma_{\text{в}}-\sigma_{\text{п}})\frac{\frac{b}{\delta}}{\left(\frac{b}{\delta}\right)_{\text{кр}}}=\sigma_{\text{в}}-(\sigma_{\text{в}}-\sigma_{\text{п}})\frac{\frac{l}{i}}{\left(\frac{l}{i}\right)_{\text{кр}}},$$

whence

$$\frac{\left(\frac{b}{\delta}\right)_{\text{кр}}}{\left(\frac{l}{i}\right)_{\text{кр}}}=\frac{\left(\frac{b}{\delta}\right)}{\left(\frac{l}{i}\right)}$$

or

$$\frac{\frac{b}{\delta}}{\frac{l}{i}}=\frac{1}{\pi}\sqrt{\frac{0,9k}{C}}.$$

Thus, conditions of uniform strength of the profile both in elastic, and nonelastic ranges coincide, i.e., the profile, most advantageous in the elastic zone, remains the most advantageous also in the nonelastic zone of work of the material.

§ 12. Effective Width of Covering, Working Together with Rod Bracing

If a plate, fastened along the perimeter to the profile, lost stability from compression, then it is excluded from further work and only small narrow strips, adjacent to the profiles, continue to absorb the additional load.

Such phenomenon will be observed when critical stress of the plate is less than the critical stress of the profile. Figure 28 shows a plate fastened on the contour after loss of stability. The central part of the plate after loss of stability is conditionally surrounded by a wavy line. The strips of the plate, adjacent to profiles, with width c on each side, which continue to absorb the load, can be approximately determined from the following reasonings.

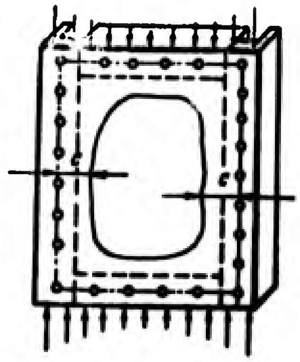

Fig. 28.

Let us consider a plate with width $2c$ as freely supported and for determination of its critical stress let us apply formula (3.5) when $k = 4$:

$$\sigma_{кр} = 3{,}6 \frac{E}{\left(\frac{b}{\delta}\right)^2}.$$

Having substituted here the dimension of $2c$ instead of b, we obtain

$$\sigma_{кр} = 3{,}6 \frac{E}{\left(\frac{2c}{\delta}\right)^2},$$

whence we find

$$2c = 19\delta \sqrt{\frac{E}{\sigma_{кр}}}. \qquad (4.5)$$

This is the so-named Karman formula. A strip of plate with dimension c, adjacent to the profile, is called the effective width.

From the condition of compatibility of deformations of adjacent strip and profile

$$\frac{3{,}6}{\left(\frac{2c}{\delta}\right)^2} = \frac{\sigma_{кр.стр}}{E}$$

we obtain

$$2c = 1{,}9\delta \sqrt{\frac{E}{\sigma_{кр.стр}}}, \qquad (4.6)$$

where $\sigma_{\text{кр.стр}}$ – critical stress of stringer.

Dimension c is usually read from the place of attachment of the profile to the plate on the side of the plate (Fig. 29).

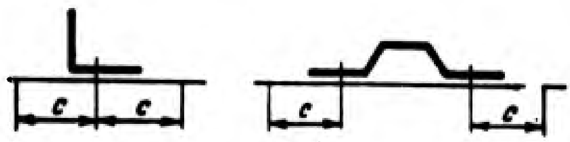

Fig. 29.

The effective width c, calculated by formula (4.6), is then considered during determination of the cross-sectional area of the profile during its calculation for compression. Formula (4.6) is used even during calculation of warped panels.

P A R T II

STRENGTH OF SHELLS

C H A P T E R V

MOMENTLESS SHELLS OF REVOLUTION

In this part we examined some questions of calculation of shells, widely applied in various constructions. In Chapter V let us pause on calculation of momentless shells of revolution. By momentless shells it has been accepted to mean a shell, the stressed state of which is determined basically by membrane ("chain") stresses. Bending stresses in such shells are usually small in comparison with membrane. Formulas ensuing from the momentless theory play a basic role in strength calculations of thin-walled vessels and capacities with internal pressure. Momentless stressed state in such constructions is usually disturbed either at places of attachment of edges of shells, or at places of uneven change in the thickness, at places of joining of shells of various geometric shape, and also at places of uneven change of load. This type of problems is examined in Chapter VI.

§ 13. Some Information from the Geometry of Shells

Subsequently we will consider only shells of revolution. By shell of revolution we mean a body formed by the revolution of two flat curves around an axis, lying in their plane and not intersecting these curves. The distance between these curves forms the thickness of the shell.

The surface, dividing the thickness of the shell in half, is called the middle.

By prescribing the shape of middle surface and shell thickness, we comprehensively determine the shell in a geometric relationship. Subsequently we will consider only covers of constant thickness.

The track of intersection of shell with the plane passing through the axis of revolution, is called the meridian (Fig. 30). Meridians coincide with generatrices of the shell. The track of intersection of the shell with the plane perpendicular to the axis of shell is called the parallel or parallel circle.

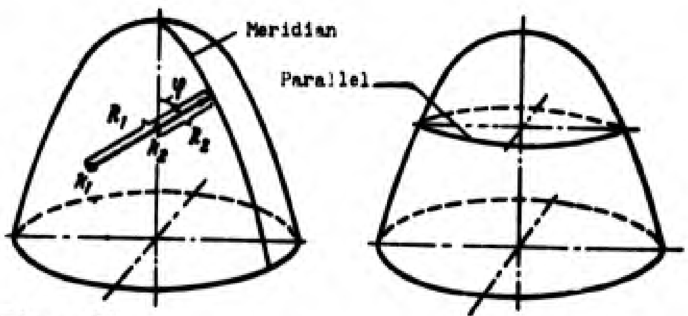

Fig. 30.

The radius of the meridian is called the first principal radius of curvature and is designated R_1.

Radius of curvature of the curve, obtained from intersection of the meridian by a plane perpendicular to this meridian, is called the second principal radius and is designated R_2.

Both radii lie on one straight line, being the normal to the meridian. The point of intersection of this normal with axis of shell k_2 and point k_1 of the end of the normal are centers of curvature of the surface at the given point. The angle between the normal to meridian and axis of shell ϕ is called the angle of latitude of the considered point.

§ 14. Equilibrium Equations of the Shell with Axisymmetrical Load. Laplace Equation

Let us examine the equilibrium conditions of a shell, loaded symmetrically relative to its axis.

Figure 31 shows a shell, from which element s_1s_2 has been mentally cut off by two meridion sections at angle $d\theta$ to each other and by two tapered sections at angle $d\phi$, to which for equilibrium there are applied as yet unknown internal forces N_ϕ and N_θ and also external load with intensity q. Let us formulate the equation of equilibrium of the separated element in the direction of radius of curvature of the meridian (Fig. 32):

$$N_\theta R_1\, d\varphi\, d\theta \sin\varphi + (N_\varphi + dN_\varphi)(r + dr)\, d\theta\, d\varphi + qR_1 r\, d\theta\, d\varphi = 0.$$

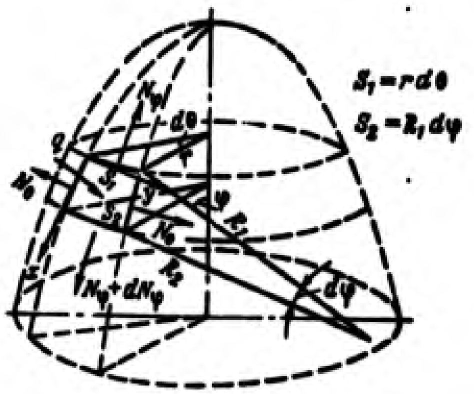

Fig. 31.

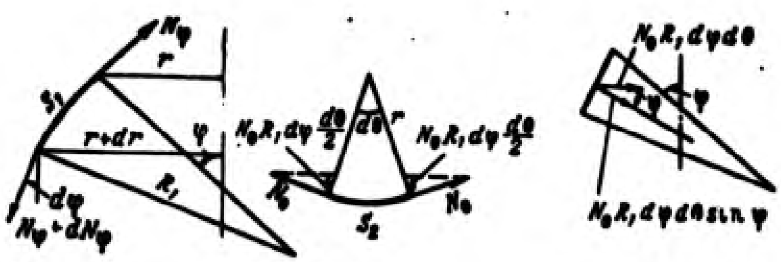

Fig. 32.

By disregarding infinitely small quantities dr and dN_ϕ in this equation and cancelling by $d\theta d\phi$, we obtain

$$N_\theta R_1 \sin\varphi + N_\varphi r + qR_1 r = 0.$$

According to Fig. 31 we have

$$r = R_2 \sin\varphi.$$

Then

$$N_\theta R_1 + N_\varphi R_2 + q R_1 R_2 = 0.$$

Let us divide this equality by $R_1 R_2$. Then

$$\frac{N_\varphi}{R_1} + \frac{N_\theta}{R_2} = -q. \qquad (5.1)$$

This is the Laplace equation.

In equation (5.1) the two unknowns are N_ϕ and N_θ. To get the second equation let us examine the equilibrium parts of the shell, located over the parallel circle, determined by angle ϕ (Fig. 33):

$$2\pi r N_\varphi \sin\varphi + Q = 0,$$

whence

$$N_\varphi = -\frac{Q}{2\pi r \sin\varphi} = -\frac{Q}{2\pi R_2 \sin^2\varphi}, \qquad (5.2)$$

where Q – resultant of the entire load, located above the considered section.

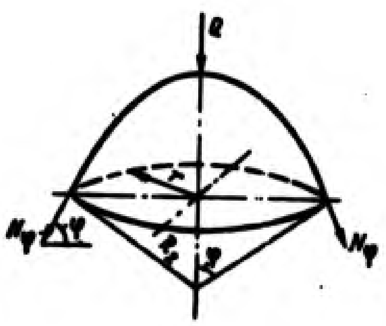

Fig. 33.

From equations (5.1) and (5.2) we can determine both membrane forces with axisymmetrical loading of the shell of revolution. Let us examine particular cases of application of these equations.

Figure 34a shows a spherical shell, loaded by internal uniform pressure q

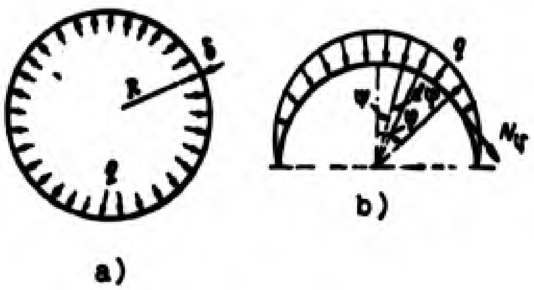

Fig. 34.

In this instance

$$N_\varphi = N_\theta = N, \quad R_1 = R_2 = R,$$

$$\frac{2N}{R} - q = 0, \quad N = \frac{qR}{2}, \quad \sigma = \frac{qR}{2\delta},$$

where δ – thickness of shell.

The spherical shell under load, changing according to law $q = q_0 \cos^2 \phi$, is shown in Fig. 34b.

In this case for the resultant of force Q we have expression

$$Q = \int_0^{2\pi}\int_0^{\varphi} qR\, d\psi\, r\, d\theta \cos\psi.$$

Then

$$N_\varphi = -\frac{Q}{2\pi R \sin^2\varphi} =$$

$$= -\frac{q_0 R}{\sin^2\varphi}\int_0^{\varphi} \cos^3\psi \sin\psi\, d\psi = -\frac{q_0 R}{4}(1 + \cos^2\varphi).$$

From Laplace equation we obtain

$$N_\theta = -\frac{q_0 R}{4}(3\cos^2\varphi - 1).$$

From these expressions it is evident that force N_ϕ is compressive everywhere, and force N_θ changes its sign. At angle $\phi \approx 55°$ it becomes it becomes zero, and $\phi > 55°$ it will be tensile.

In the case of a cylindrical shell, loaded by internal uniform pressure (Fig. 35), we have $R_1 = \infty$, $R_2 = R$, where R – radius of cylinder. Then

$$\frac{N_\theta}{R} - q = 0, \quad N_\theta = qR, \quad \sigma_\theta = \frac{qR}{\delta}.$$

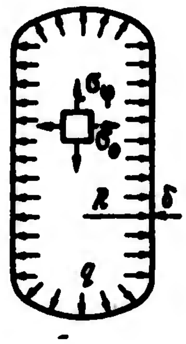

Fig. 35.

Meridion stresses are determined from equation (5.2):

$$N_\varphi = \frac{\pi R^2 q}{2\pi R} = \frac{qR}{2}, \quad \sigma_\varphi = \frac{qR}{2\delta}.$$

In the case of a tapered shell under internal uniform pressure q (Fig. 36)

$$R_1 = \infty, \quad R_2 = x\,\mathrm{tg}\,\alpha,$$
$$N_\theta = qR_2 = qx\,\mathrm{tg}\,\alpha, \quad \sigma_\theta = \frac{qx\,\mathrm{tg}\,\alpha}{\delta}.$$

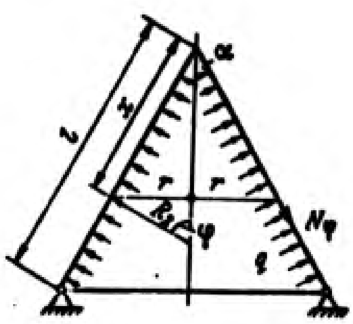

Fig. 36.

Let us determine meridion stresses from equation (5.2):

$$N_\varphi = \frac{\pi r^2 q}{2\pi r \cos\alpha}$$

or

$$\sigma_\varphi = \frac{q x \operatorname{tg}\alpha}{2\delta}.$$

Figure 37 shows an ellipsoidal doughnut-shaped shell, loaded by internal uniform pressure. In this case from the condition of equilibrium of section of shell AB in the direction of axis y we obtain

$$2\pi r N_\varphi \sin\varphi = \pi (r^2 - r_0^2) q,$$

whence for meridion force

$$N_\varphi = \frac{(r^2 - r_0^2) q}{2r \sin\varphi}. \qquad (5.3)$$

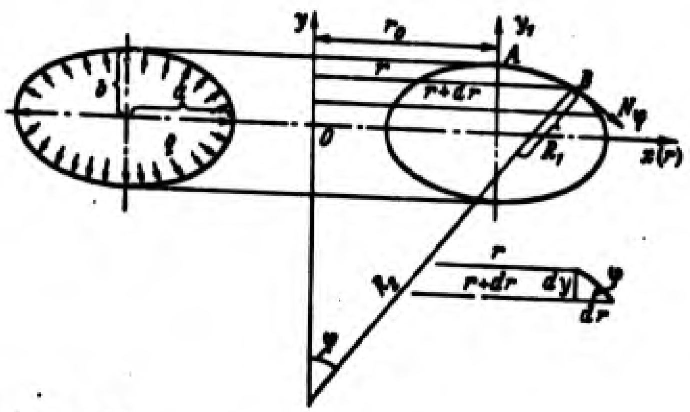

Fig. 37.

The expression for circumferential section N_θ is obtained from Laplace equation, having substituted the found value of N_ϕ in it:

$$N_\theta = q R_2 \left(1 - \frac{r^2 - r_0^2}{2 R_1 r \sin\varphi}\right). \qquad (5.4)$$

In formulas (5.3) and (5.4) it is most convenient to exp ess principal radii of curvature R_1 and R_2, and also sin ϕ as a function of r.

For R_1 we have the following expression from analytical geometry:

$$R_1 = \frac{\left[1 + \left(\frac{dr}{dy}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2r}{dy^2}}.$$

According to Fig. 37

$$R_2 = \frac{r}{\sin\varphi}.$$

According to this figure we find

$$\frac{dy}{dr} = -\operatorname{tg}\varphi$$

or

$$\frac{dr}{dy} = -\operatorname{ctg}\varphi.$$

Let us assume we have assigned the equation of ellipse in axes x, y_1:

$$\frac{x^2}{a^2} + \frac{y_1^2}{b} - 1 = 0.$$

This equation in axes r, y will have the form

$$\frac{(r-r_0)^2}{a^2} + \frac{y^2}{b^2} - 1 = 0,$$

whence

$$r = r_0 + \sqrt{a^2\left(1 - \frac{y^2}{b^2}\right)}.$$

From this expression let us find the first and second derivatives:

$$\frac{dr}{dy} = -\frac{a\sqrt{a^2 - (r-r_0)^2}}{b(r-r_0)},$$

$$\frac{d^2r}{dy^2} = -\frac{a^4}{b^2(r-r_0)^3}.$$

Thus, we expressed the first and second derivatives in the function of radius r. Now for principal radii of curvature R_1 and R_2, and also for $\sin\phi$ we obtain the following expressions:

$$\left.\begin{aligned} R_1 &= \frac{[(b^2-a^2)(r-r_0)^2+a^4]^{\frac{3}{2}}}{a^4 b}, \\ R_2 &= \frac{r[(b^2-a^2)(r-r_0)^2+a^4]^{\frac{1}{2}}}{b(r-r_0)}, \\ \sin\varphi &= \frac{b(r-r_0)}{[(b^2-a^2)(r-r_0)^2+a^4]^{\frac{1}{2}}}. \end{aligned}\right\} \qquad (5.5)$$

If we substitute expressions (5.5) in (5.3) and (5.4), we obtain the formulas for stresses:

$$\left.\begin{aligned} \sigma_\varphi &= \frac{q(r+r_0)}{2rb\delta}\sqrt{(b^2-a^2)(r-r_0)^2+a^4}, \\ \sigma_\theta &= \frac{q}{2b\delta}\,\frac{2r(b^2-a^2)(r-r_0)+a^4}{\sqrt{(b^2-a^2)(r-r_0)^2+a^4}}. \end{aligned}\right\} \qquad (5.6)$$

From these formulas it is evident that stress σ_ϕ will always be positive, if the shell is affected by internal pressure. Regarding stresses σ_θ, at a certain value of r it passes through zero and changes its sign.

For example, if internal pressure acts, then when $r < r_0$ the annular stress σ_θ will be tensile, when $r > r_0$ – compressive. Therefore, this part of the doughnut-shaped shell at some magnitude of such stresses can lose stability.

For a round torus (Fig. 38), loaded by constant internal pressure q, when $a = b$ the expressions (5.6) for stresses take the form

$$\sigma_\varphi = \frac{q(r+r_0)a}{2r\delta}, \quad \sigma_\theta = \frac{qa}{2\delta}. \qquad (5.7)$$

From (5.7) for a round torus the corresponding formulas can be obtained for cylindrical and spherical shells. The second of the

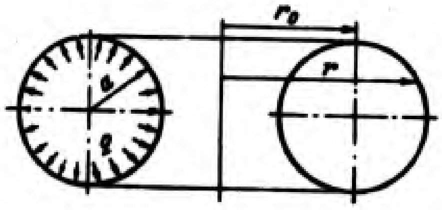

Fig. 38.

given formulas is not connected to the characteristic parameter of torus with radius r_0 and therefore can be immediately used if computing the axial stresses in a cylindrical shell (Fig. 39).

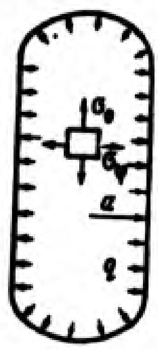

Fig. 39.

Let us rewrite the formula for stresses σ_ϕ in the following form:

$$\sigma_\varphi = \frac{qa}{2\delta}\left(1 + \frac{r_0}{r}\right).$$

With increase in radius r_0 the current radius r increases simultaneously with it. Then in the limit

$$\lim\left(\frac{r_0}{r}\right)_{r_0\to\infty} = 1.$$

Therefore, it is possible to write

$$\sigma_\varphi = \frac{qa}{2\delta}\left[1 + \lim\left(\frac{r_0}{r}\right)_{r_0\to\infty}\right] = \frac{qa}{\delta}.$$

By this formula we determine circumferential stresses in a cylindrical shell with constant internal pressure q.

In the case of a sphere in formulas (5.7) it is necessary to place $r_0 = 0$.

In the case of an ellipsoidal shell, loaded by uniform internal pressure, in formulas (5.6) it is necessary to place $r_0 = 0$. Then the expressions for stresses in this shell (Fig. 40) will have the form

$$\sigma_\varphi = \frac{q}{2b\delta}\sqrt{r^2(b^2-a^2)+a^4},$$

$$\sigma_\theta = \frac{q}{2b\delta}\,\frac{2r^2(b^2-a^2)+a^4}{\sqrt{r^2(b^2-a^2)+a^4}}. \qquad (5.8)$$

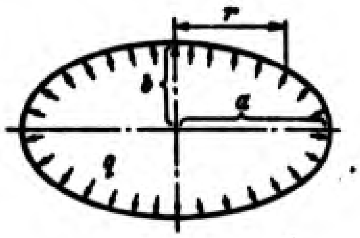

Fig. 40.

From these formulas it is also evident that stress σ_ϕ will be of one sign everywhere at any value of r; stress σ_θ at some value of r passes through zero and changes its sign at a point which can be found from equation

$$2r^2(b^2-a^2)+a^4=0,$$

whence

$$r=\sqrt{-\frac{a^4}{2(b^2-a^2)}}, \qquad (a>b).$$

§ 15. Stresses in Shells from Hydrostatic Pressure

All liquid bodies in contrast to solid take the form of the container in which they are located.

Pressure inside the liquid is composed of pressure of the free surface (q_H) and pressure of the liquid column with height from the considered point to the free surface. In Fig. 41 the force of pressure on area F in side the liquid

$$P=\gamma hF+q_{n}F,$$

where γ – specific weight of liquid.

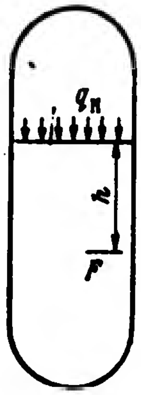

Fig. 41.

Having divided this expression by F, we obtain the total specific pressure at the given point:

$$q=\frac{P}{F}=\gamma h+q_{n}.$$

Pressure of liquid on area F is called hydrostatic pressure. Hydrostatic pressure of liquid at the given point coincides with normal stress inside the liquid. Normal stress of liquid at the given point is identical in all directions (Pascal law).

Hy‿ static pressure is always directed normal to the surface on which it acts. The magnitude of hydrostatic pressure at the given point is measured by the height of liquid column above this point and depends on the shape of the container in which this liquid is located.

Figure 42 shows examples of pressure diagrams for certain containers.

For determination of membrane stresses in shells of revolution from hydrostatic pressure equations (5.1) and (5.2) are used. In these equations the magnitude of hydrostatic pressure, expressed depending on the height of the liquid column must be substituted.

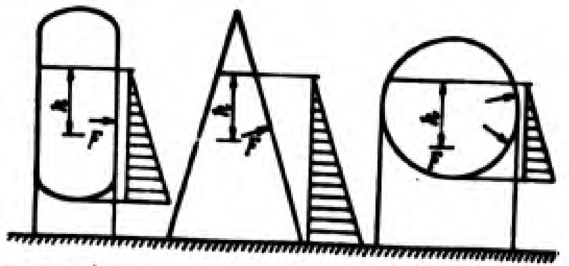

Fig. 42.

In the case of cylindricll shell (Fig. 43)

$$R_1=\infty,\quad R_2=R,\quad q=-(H-x)\gamma.$$

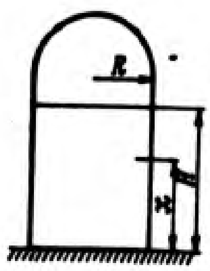

Fig. 43.

From Laplace equation it follows that

$$\frac{N_\theta}{R}=(H-x)\gamma$$

or

$$\sigma_\theta=\frac{\gamma(H-x)R}{\delta}.$$

In this case $N_\phi = 0$.

In the case of tapered shell (Fig. 44a) we have $R_1 = \infty$, $R_2 = x\ \mathrm{tg}\ \alpha$:

$$q=-\gamma[x\cos\alpha-(l-H)],$$

$$N_\theta=-qR_2=\gamma x[x\cos\alpha-(l-H)]\,\mathrm{tg}\,\alpha.$$

For determination of forces N_ϕ let us examine Fig. 44b.

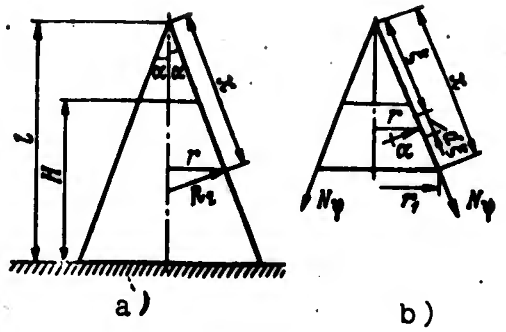

Fig. 44.

From the condition of equilibrium of this part of the cone in the direction of its axis we obtain

$$\int_0^{2\pi}\int_{\frac{l-H}{\cos\alpha}}^{x} q\,d\xi\,\sin\alpha\, r\, d\theta = 2\pi r_1 N_\varphi \cos\alpha.$$

Having substituted here

$$q=\gamma[\xi\cos\alpha-(l-H)],$$
$$r=\xi\sin\alpha,\quad r_1=x\sin\alpha,$$

after integration we obtain

$$N_\varphi=\frac{\gamma\,\mathrm{tg}\,\alpha}{6x}\left[\frac{(l-H)^3}{\cos^2\alpha}+2x^3\cos\alpha-3(l-H)x^2\right].$$

In the given formulas the magnitudes of x lie within

$$\frac{l}{\cos\alpha}\geqslant x\geqslant\frac{l-H}{\cos\alpha}.$$

The greatest magnitudes of force N_θ and N_ϕ are obtained when $x=\frac{l}{\cos\alpha}$:

$$N_{\theta\,max}=\frac{\gamma l H\,\mathrm{tg}\,\alpha}{\cos\alpha},$$
$$N_{\varphi\,max}=\frac{\gamma H^2(3l-H)\,\mathrm{tg}\,\alpha}{6l\cos\alpha}.$$

If the cone is completely filled with liquid, then $H = l$. Then the expressions for forces will have the following form:

$$N_\varphi = \frac{1}{3}\gamma x^2 \sin\alpha, \quad N_\theta = \gamma x^2 \sin\alpha.$$

Let us examine loading cases of a tapered shell, shown in Fig. 45:

$$R_1 = \infty, \quad R_2 = x \operatorname{tg} \alpha,$$
$$q = -\gamma (H - x\cos\alpha),$$
$$N_\theta = -qR_2 = \gamma (H - x\cos\alpha) x \operatorname{tg}\alpha.$$

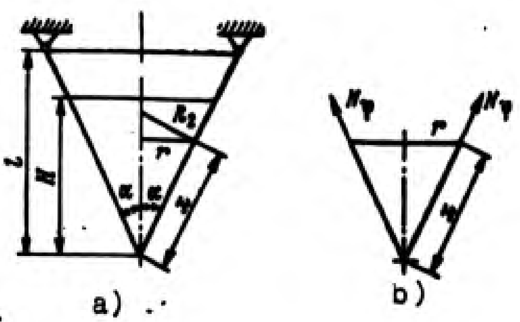

Fig. 45.

For determination of meridion forces N_ϕ let us examine the equilibrium of the lower part of the cone, determined by distance x from its vertex (Fig. 45b). From condition of equilibrium of all forces, acting on this part of the shell in the direction to its axis, we obtain

$$2\pi r N_\varphi \cos\alpha = V\gamma,$$

where V - volume of the balanced part of liquid; $V\gamma$ - weight of this liquid.

Volume of liquid

$$V = \frac{1}{3}\pi r^2 x \cos\alpha + \pi r^2 (H - x\cos\alpha).$$

Taking into account that $r = x \sin\alpha$, and using the given quantities for V, we finally obtain the formula for force N_ϕ:

$$N_\varphi = \gamma x \left(\frac{1}{2} H - \frac{1}{3} x \cos\alpha\right) \operatorname{tg}\alpha.$$

This force reaches the greatest magnitude when $x=\frac{3H}{4\cos\alpha}$:

45: $$(N_\varphi)_{max}=\frac{3\gamma H^2 \operatorname{tg}\alpha}{16\cos\alpha}.$$

As is easily checked, the greatest magnitude of annular force N_θ will be when $x=\frac{H}{2\cos\alpha}$:

$$(N_\theta)_{max}=\frac{\gamma H^2 \operatorname{tg}\alpha}{4\cos\alpha}.$$

If a tapered shell in the shape of the bottom of a tank will be loaded according to Fig. 46, for forces N_ϕ and N_θ we have the following expressions:

$$N_\varphi=\frac{1}{2}\gamma x\left(h+l-\frac{2}{3}x\cos\alpha\right)\operatorname{tg}\alpha,$$
$$N_\theta=\gamma x(h+l-x\cos\alpha)\operatorname{tg}\alpha.$$

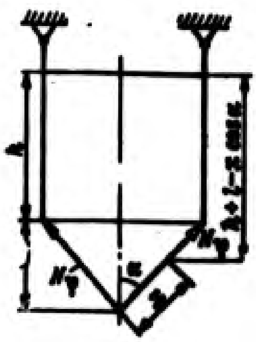

Fig. 46.

Let us examine the lower bottom, made in the shape of doughnut-shaped shell of ellipsoidal section, loaded by hydrostatic pressure. Weight of the shaded part of liquid (Fig. 47) will be

$$G=\gamma\int_y^b(\pi r^2-\pi r_0^2)dy+\gamma(\pi r^2-\pi r_0^2)(H+y)=$$
$$=\pi\gamma\int_y^b(r^2-r_0^2)\,dy+\pi\gamma(r^2-r_0^2)(H+y).$$

By substituting the quantity r^2 in the expression under the integral sign by formula $r=r_0+a\sqrt{1-\frac{y^2}{b^2}}$ and integrating within the shown limits, we obtain

$$G=\pi\gamma\left[\frac{1}{2}\pi r_0 ab-r_0 ay\sqrt{1-\frac{y^2}{b^2}}-r_0 ab\arcsin\frac{y}{b}+\right.$$
$$\left.+\frac{2}{3}a^2b-a^2y+\frac{1}{3}\frac{a^2y^3}{b^2}+(r^2-r_0^2)(H+y)\right].$$

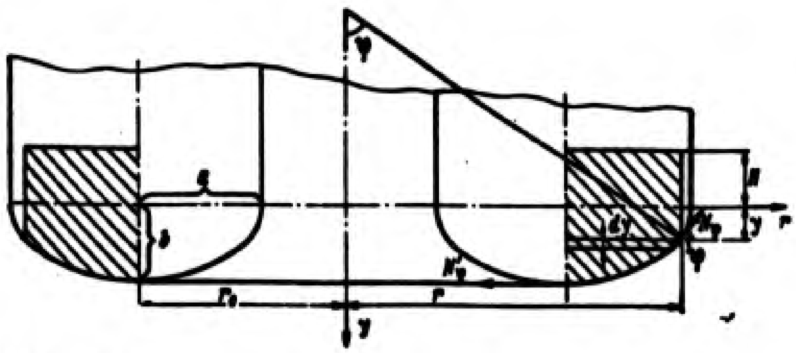

Fig. 47.

Let us replace y here by its expression through r according to formula

$$y=b\sqrt{1-\frac{(r-r_0)^2}{a^2}}.$$

Finally

$$G=\pi\gamma\left\{\frac{1}{2}\pi r_0 ab+\frac{2}{3}a^2b+H(r^2-r_0^2)-r_0 ab\arcsin\times\right.$$
$$\left.\times\sqrt{1-\frac{(r-r_0)^2}{a^2}}-\frac{1}{3}b\sqrt{1-\frac{(r-r_0)^2}{a^2}}[2(a^2-r^2)+r_0(r_0+r)]\right\}.$$

Now let us formulate the condition of equilibrium of the weight of liquid in the shaded volume and meridion forces N_ϕ on the axes of shell. We have

$$2\pi rN_\varphi\sin\varphi=G,$$

whence

$$N_\varphi=\frac{G}{2\pi r\sin\varphi}.$$

After replacement of sin ϕ in the last expression by its quantity according to formula (5.5) we obtain

$$N_\varphi = \frac{\gamma}{2br(r-r_0)}\left\{\frac{1}{2}\pi r_0 ab + \frac{2}{3}a^2 b + H(r^2 - r_0^2) - \right.$$
$$- r_0 ab \arcsin\sqrt{1 - \frac{(r-r_0)^2}{a^2}} -$$
$$\left. - \frac{1}{3} b \sqrt{1 - \frac{(r-r_0)^2}{a^2}}\left[2(a^2 - r^2) + r_0(r_0 + r)\right]\right\} \times$$
$$\times \sqrt{(b^2 - a^2)(r - r_0)^2 + a^4}. \qquad (5.9)$$

The expression for forces N_θ is found from Laplace equation

$$N_\theta = -\left(q + \frac{N_\varphi}{R_1}\right)R_2,$$

where

$$q = -\gamma(H + y) = -\gamma\left[H + b\sqrt{1 - \frac{(r-r_0)^2}{a^2}}\right].$$

By substituting the quantities of R_1 and R_2 according to formula (5.5) and the quantities of N_ϕ found above in the formula for N_θ, we obtain

$$N_\theta = \frac{\gamma H\left[2r(b^2 - a^2)(r - r_0) + a^4\right]}{2b\sqrt{(b^2 - a^2)(r - r_0)^2 + a^4}} +$$
$$+ \frac{\gamma\sqrt{1 - \frac{(r_0 - r)^2}{a^2}}}{2(r - r_0)^2\sqrt{(b^2 - a^2)(r - r_0)^2 + a^4}}\left\{2r(r - r_0)\left[(b^2 - a^2)(r - r_0)^2 + a^4\right] + \right.$$
$$\left. + \frac{1}{3}a^4\left[2(a^2 - b^2) + r_0(r_0 + r)\right]\right\} -$$
$$- \frac{\gamma a^5}{2(r - r_0)^2\sqrt{(b^2 - a^2)(r - r_0)^2 + a^4}} \times$$
$$\times \left[\frac{1}{2}\pi r_0 + \frac{2}{3}a - r_0 \arcsin\sqrt{1 - \frac{(r_0 - r)^2}{a^2}}\right]. \qquad (5.10)$$

The given expressions for N_ϕ and N_θ when $r = r_0$ gives indeterminacy of type 0/0. By applying the L'Hôptial rule to them, when $r = r_0$ we obtain

$$N_\varphi = \frac{\gamma H a^2}{b}, \quad N_\theta = \frac{1}{2}\gamma a^2\left(1 + \frac{H}{b}\right).$$

Let us examine the lower bottom, made in the form of an ellipsoid of revolution, loaded by hydrostatic pressure. By substituting the quantity of $r = 0$, in the formula we obtain (Fig. 48)

$$N_\varphi=\frac{\gamma}{2br^2}\left[\frac{2}{3}a^2b+Hr^2-\frac{2}{3}b(a^2-r^2)\times\right.$$
$$\left.\times\sqrt{1-\frac{r^2}{a^2}}\,\right]\sqrt{(b^2-a^2)r^2+a^4},$$
$$N_\theta=\frac{\gamma H[2(b^2-a^2)r^2+a^4]}{2b\sqrt{(b^2-a^2)r^2+a^4}}+ \qquad (5.11)$$
$$+\gamma\frac{\left\{r^2[(b^2-a^2)r^2+a^4]+\frac{1}{3}a^4(a^2-r^2)\right\}\sqrt{1-\frac{r^2}{a^2}}-\frac{1}{3}a^6}{r^2\sqrt{(b^2-a^2)r^2+a^4}}.$$

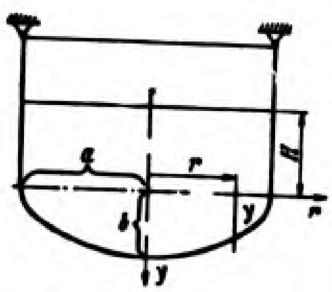

Fig. 48.

By fulfilling limiting passage when in these formulas $r = 0$, we find

$$N_\varphi=\frac{1}{2}\gamma a^2\left(1+\frac{H}{b}\right),\quad N_\theta=\frac{1}{2}\gamma a^2\left(1+\frac{H}{b}\right).$$

From formula (5.11) when $a = b$ the formula for a hemisphere can be obtained (Fig. 49)

$$N_\varphi=\frac{\gamma a}{2r^2}\left[Hr^2+\frac{2}{3}a^3-\frac{2}{3}a(a^2-r^2)\sqrt{1-\frac{r^2}{a^2}}\right],$$
$$N_\theta=\frac{\gamma aH}{2}+\gamma\frac{(2a^2r^2+a^4)\sqrt{1-\frac{r^2}{a^2}}-a^4}{3r^2}.$$

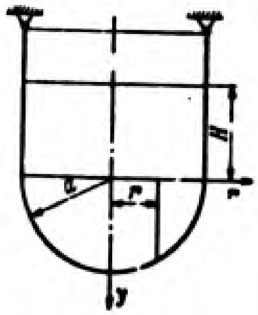

Fig. 49.

When $r = 0$ we obtain

$$N_\varphi = \frac{\gamma a(a+H)}{2}, \quad N_\theta = \frac{\gamma a(a+H)}{2}.$$

When $r = a$ we obtain

$$N_\varphi = \frac{1}{2}\gamma a\left(H + \frac{2}{3}a\right), \quad N_\theta = \frac{1}{2}\gamma a\left(H - \frac{2}{3}a\right).$$

From the last expression it is evident that when $H = \frac{2}{3}a$ the annular forces will be equal to zero. When $H < \frac{2}{3}a$ this force becomes compressive, and when $H = 0$ it reaches its maximum

$$N_{\theta\,max} = -\frac{1}{3}\gamma a^2.$$

If the lower bottom is made in the form of a circular torus, then, by assuming $a = b$ (Fig. 50) in formulas (5.9) and (5.10), we obtain

$$\left.\begin{aligned} N_\varphi = {} & \frac{\gamma a}{2r(r-r_0)}\left\{\frac{1}{2}\pi r_0 a^2 + \frac{2}{3}a^3 + H(r^2 - r_0^2) - \right. \\ & - r_0 a^2 \arcsin\sqrt{1 - \frac{(r_0-r)^2}{a^2}} - \\ & \left. - \frac{1}{3}a\left[2(a^2 - r^2) + r_0(r_0 + r)\right]\sqrt{1 - \frac{(r_0-r)^2}{a^2}}\right\}, \\ N_\theta = {} & \frac{\gamma H a}{2} + \frac{\gamma}{2(r_0-r)^2}\left\{\frac{1}{3}a^2(4r^2 - 5r_0 r + \right. \\ & + 2a^2 + r_0^2)\sqrt{1 - \frac{(r_0-r)^2}{a^2}} - a^3\left[\frac{1}{2}\pi r_0 + \right. \\ & \left.\left. + \frac{2}{3}a - r_0 \arcsin\sqrt{1 - \frac{(r_0-r)^2}{a^2}}\right]\right\}. \end{aligned}\right\} \quad (5.12)$$

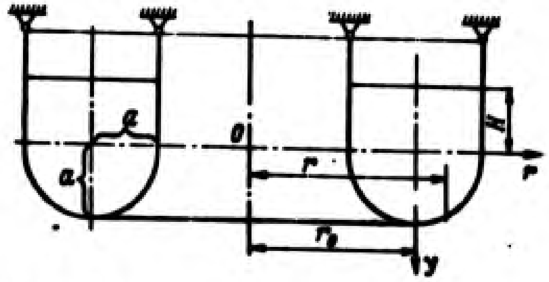

Fig. 50.

Formulas (5.12) for forces when $r = r_0$ give indetermining 0/0. By means of limiting passage there was obtained:

$$N_\varphi = \gamma Ha, \quad N_\theta = \frac{1}{2}\gamma a^2\left(1+\frac{H}{a}\right).$$

§ 16. Calculation of Upper End Plates of Tanks from the Action of Internal Hydrostatic Pressure

Above we considered the problems of calculation of the lower bottom of containers, loaded by internal hydrostatic pressure.

In this case let us examine the calculation of the upper end plates, when they are subjected to the action of internal hydrostatic pressure. Just as earlier, solution of the posed problem is started with examination of an ellipsoidal doughnut-shaped end plate (Fig. 51).

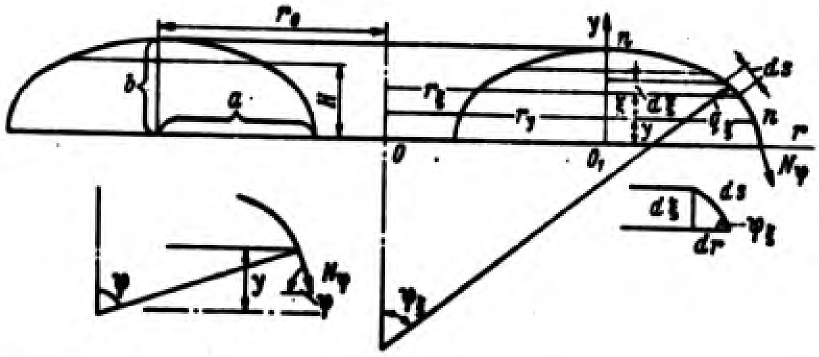

Fig. 51.

Equation of equilibrium in the direction of axis y for the upper part of torus n-n will have the form

$$\int_0^{2\pi}\int_0^{H-y} q_\xi r_\xi \cos\varphi_\xi \, d\theta \, ds = \int_0^{2\pi} r_y N_\varphi \sin\varphi_y \, d\theta, \qquad (5.13)$$

where $q_\xi = \gamma(H-(y+\xi))$ – hydrostatic pressure in section $(y + \xi)$, read from point O_1; H – height of liquid level, read from base of torus; γ – specific weight of liquid.

The remaining designations are clear from Fig. 51.

For running radii r_ξ and r_y there are equations:

$$\frac{(r_y-r_0)^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{(r_\xi-r_0)^2}{a^2}+\frac{(y+\xi)^2}{b^2}=1,$$

and also the following obvious relationships:

$$ds=\frac{d\xi}{\sin\varphi_\xi},$$

$$\mathrm{tg}\,\varphi_y=-\frac{dy}{dr_y}, \qquad \mathrm{ctg}\,\varphi_y=-\frac{dr_y}{dy},$$

$$\mathrm{tg}\,\varphi_\xi=-\frac{d\xi}{dr_\xi}, \qquad \mathrm{ctg}\,\varphi_\xi=-\frac{dr_\xi}{d\xi}.$$

By using the given relationships during calculation of integrals in equation (5.13), we obtain the following expression for force N^{Φ}:

$$\begin{aligned} N_\varphi = {} & \frac{\gamma a\sqrt{(b^2-a^2)(r-r_0)^2+a^4}}{b^3 r(r-r_0)}\left\{\frac{aH^3}{6}+r_0 b^3\times\right. \\ & \times\left[H-\frac{1}{2}b\sqrt{1-\frac{(r_0-r)^2}{a^2}}\right]\frac{r-r_0}{a}-ab^2\times \\ & \times\left[\frac{1}{2}H-\frac{1}{3}b\sqrt{1-\frac{(r_0-r)^2}{a^2}}\right]\left[1-\frac{(r_0-r)^2}{a^2}\right]- \\ & -\frac{1}{2}Hr_0b^2\sqrt{1-\frac{H^2}{b^2}}-\frac{1}{2}r_0b^3\left[\arcsin\frac{H}{b}-\right. \\ & \left.\left.-\arcsin\sqrt{1-\frac{(r_0-r)^2}{a^2}}\right]\right\}. \end{aligned} \qquad (5.14)$$

Force N_θ is determined from Laplace equation

$$\frac{N_\varphi}{R_1}+\frac{N_\theta}{R_2}=\gamma(H-y),$$

where R_1, R_2 are given by formulas (5.5).

In expanded form for N_θ we obtain formula

$$N_\theta=\frac{\gamma r}{b(r-r_0)}\left[H-b\sqrt{1-\frac{(r_0-r)^2}{a^2}}\right]\sqrt{(b^2-a^2)(r-r_0)^2+a^4}+$$
$$+\frac{\gamma a^5}{b^3(r-r_0)^2\sqrt{(b^2-a^2)(r-r_0)^2+a^4}}\left\{\frac{1}{2}r_0b^3\left[\arcsin\frac{H}{b}-\right.\right.$$
$$\left.-\arcsin\sqrt{1-\frac{(r-r_0)^2}{a^2}}\right]+\frac{1}{2}Hr_0b^2\sqrt{1-\frac{H^2}{b^2}}+ \tag{5.15}$$
$$+ab^2\left[\frac{1}{2}H-\frac{1}{3}b\sqrt{1-\frac{(r-r_0)^2}{a^2}}\right]\left[1-\frac{(r-r_0)^2}{a^2}\right]-$$
$$\left.-r_0b^2\left[H-\frac{1}{2}b\sqrt{1-\frac{(r-r_0)^2}{a^2}}\right]\frac{r-r_0}{a}-\frac{aH^3}{6}\right\}.$$

From formulas (5.14) and (5.15) various particular formulas can be obtained for other shapes of shells, which are under hydrostatic pressure. For example, when $r_0 = 0$ we obtain formulas for ellipsoid of revolution

$$N_\varphi=\frac{\gamma a^2}{b^3r^2}\left[\frac{1}{6}H^3-b^2\left(\frac{1}{2}H-\frac{1}{3}b\sqrt{1-\frac{r^2}{a^2}}\right)\left(1-\frac{r^2}{a^2}\right)\right]\times$$
$$\times\sqrt{(b^2-a^2)r^2+a^4}, \tag{5.16}$$
$$N_\theta=\frac{\gamma}{6}\left(H-b\sqrt{1-\frac{r^2}{a^2}}\right)\sqrt{(b^2-a^2)r^2+a^4}+$$
$$+\frac{\gamma a^6}{b^3r^2\sqrt{(b^2-a^2)r^2+a^4}}\times$$
$$\times\left[b^2\left(\frac{1}{2}H-\frac{1}{3}b\sqrt{1-\frac{r^2}{a^2}}\right)\left(1-\frac{r^2}{a^2}\right)-\frac{1}{6}H^3\right]. \tag{5.17}$$

By assuming $r = a$ in these formulas, we obtain expressions for forces at the equator of ellipsoid of revolution

$$N_\varphi=\frac{\gamma aH^3}{6b^2},\quad N_\theta=\gamma aH\left(1-\frac{a^2H^2}{6b^4}\right).$$

When $H=\frac{b^2}{a}\sqrt{2}$ this force obtains the greatest magnitude

$$N_{\theta\,\max}=\frac{2}{3}\sqrt{2}\gamma b^2.$$

In the case of a spherical shell in formulas (5.16) and (5.17) one should assume $a = b = R$. Then

$$N_\varphi = \frac{\gamma R}{r^2}\left[\frac{1}{6}H^3 - R^2\left(\frac{1}{2}H - \frac{1}{3}R\sqrt{1-\frac{r^2}{R^2}}\right)\left(1-\frac{r^2}{R^2}\right)\right],$$

$$N_\theta = \gamma R\left(H - R\sqrt{1-\frac{r^2}{R^2}}\right) +$$
$$+\frac{\gamma R}{r^2}\left[R^2\left(\frac{1}{2}H - \frac{1}{3}R\sqrt{1-\frac{r^2}{R^2}}\right)\left(1-\frac{r^2}{R^2}\right) - \frac{H^3}{6}\right].$$

When $r = R$

$$N_\varphi = \frac{\gamma H^3}{6R},\quad N_\theta = \gamma R H\left(1 - \frac{H^2}{6R^2}\right).$$

If the level of liquid will be equal to the radius of sphere $H = R$, then for forces at the equator we obtain the following quantities:

$$N_\varphi = \frac{1}{6}\gamma R^2,\quad N_\theta = \frac{5}{6}\gamma R^2.$$

In the case of a round torus in formulas (5.14) and (5.15) one should assume $a = b$. Then when $r = r_0 + a$ we obtain **magnitudes** of forces at the equator

$$N_\varphi = \frac{\gamma}{r_0 + a}\left[\frac{1}{6}H^3 + r_0 a H\left(1 - \frac{1}{2}\sqrt{1-\frac{H^2}{a^2}}\right) - \right.$$
$$\left. - \frac{1}{2}r_0 a^2\left(\arcsin\frac{H}{a} - \pi\right)\right],$$
$$N_\theta = \gamma H(r_0 + a) + \frac{\gamma}{a}\left[\frac{1}{2}r_0 a^2\left(\arcsin\frac{H}{a} - \pi\right) - \right.$$
$$\left. - r_0 a H\left(1 - \frac{1}{2}\sqrt{1-\frac{H^2}{a^2}}\right) - \frac{1}{6}H^3\right].$$

When $r = r_0 - a$ we obtain formulas for forces on the inside of the torus

$$N_\varphi = \frac{\gamma}{r_0 - a}\left[-\frac{1}{6}H^3 + r_0 a H\left(1 + \frac{1}{2}\sqrt{1-\frac{H^2}{a^2}}\right) + \right.$$
$$\left. + \frac{1}{2}r_0 a^2\left(\arcsin\frac{H}{a} - \pi\right)\right],$$
$$N_\theta = -\gamma H(r_0 - a) + \frac{\gamma}{a}\left[-\frac{1}{6}H^3 + r_0 a H\left(1 + \frac{1}{2}\sqrt{1-\frac{H^2}{a^2}}\right) + \right.$$

$$+\frac{1}{2}r_0a^2\left(\arcsin\frac{H}{a}-\pi\right)\Bigg].$$

§ 17. Stresses in a Spherical Shell, Supported on Some Cross Section and Loaded by Hydrostatic Pressure

From Fig. 52 it is evident that the intensity of hydrostatic pressure

$$q=-\gamma R(\cos\alpha_0-\cos\alpha).$$

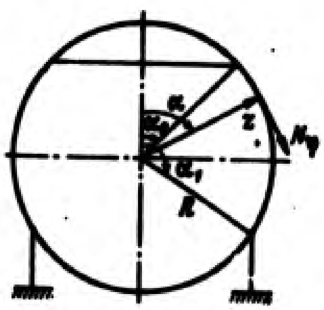

Fig. 52.

From condition of equilibrium of the part of the sphere, determined by angle α in the direction to the axis of the shell, we obtain

$$\int_0^{2\pi}\int_{\alpha_0}^{\alpha} qR^2\sin\alpha\cos\alpha\,d\alpha\,d\theta+2\pi RN_\varphi\sin^2\alpha=0.$$

Let us substitute here the expression for q and integrate. Then

$$N_\varphi=\frac{\gamma R^2}{6\sin^2\alpha}(\cos\alpha-\cos\alpha_0)[2\cos^2\alpha-(\cos\alpha_0+\cos\alpha)\cos\alpha_0]. \qquad (5.18)$$

The expression for force N_θ we obtain from Laplace equation

$$N_\theta=-qR-N_\varphi$$

or

$$N_\theta=\gamma R^2(\cos\alpha_0-\cos\alpha)\left[1-\frac{2\cos^2\alpha-(\cos\alpha_0+\cos\alpha)\cos\alpha_0}{6(1-\cos^2\alpha)}\right]. \qquad (5.19)$$

Expressions (5.18), (5.19) are valid when $\alpha_0 \leqslant \alpha \leqslant \alpha_1$. Now we obtain the expressions for forces when $\alpha > \alpha_1$ (Fig. 53). The expression for pressure remains as before.

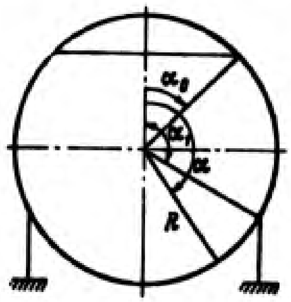

Fig. 53.

From condition of equilibrium of part of the sphere at angle α we find

$$\int_0^{2\pi}\int_\alpha^{\pi} qR^2 \sin\alpha \cos\alpha \, d\alpha \, d\theta + 2\pi R N_\varphi \sin^2\alpha = 0.$$

After substitution of quantity q in the expression under the integral sign and after integration we obtain

$$N_\varphi = \frac{1}{6}\gamma R^2\left(3\cos\alpha_0 + 2 + \frac{2\cos^2\alpha}{1-\cos\alpha}\right). \qquad (5.20)$$

The expression for force N_θ is determined from Laplace equation

$$N_\theta = -qR - N_\varphi$$

or

$$N_\theta = \frac{1}{6}\gamma R^2\left(3\cos\alpha_0 - 2 - 6\cos\alpha - \frac{2\cos^2\alpha}{1-\cos\alpha}\right). \qquad (5.21)$$

When $\alpha_0 = 0$ formulas (5.20), (5.21) are somewhat simplified and they take the form convenient for investigation:

$$\left.\begin{aligned} N_\varphi &= \frac{1}{6}\gamma R^2\left(1 - \frac{2\cos^2\alpha}{1+\cos\alpha}\right), \\ N_\theta &= \frac{1}{6}\gamma R^2\left(5 - 6\cos\alpha + \frac{2\cos^2\alpha}{1+\cos\alpha}\right), \end{aligned}\right\} 0 \leqslant \alpha \leqslant \alpha_1 \qquad (5.22)$$

$$N_\varphi=\frac{1}{6}\gamma R^2\left(5+\frac{2\cos^2\alpha}{1-\cos\alpha}\right),$$
$$N_\theta=\frac{1}{6}\gamma R^2\left(1-6\cos\alpha-\frac{2\cos^2\alpha}{1-\cos\alpha}\right).\quad \alpha_1\leqslant\alpha\leqslant\pi \qquad (5.23)$$

If in (5.22) and (5.23) we assume $\alpha=\alpha_1=\frac{\pi}{2}$, we obtain the following expressions for forces:

over the support (top)

$$N_\varphi^{\text{в}}=\frac{1}{6}\gamma R^2,\quad N_\theta^{\text{в}}=\frac{5}{6}\gamma R^2,$$

under the support (bottom)

$$N_\varphi^{\text{н}}=\frac{5}{6}\gamma R^2,\quad N_\theta^{\text{н}}=\frac{1}{6}\gamma R^2.$$

From comparison of the obtained expressions for forces N_ϕ over the support and under the support it is evident that at the place of support the shells are changed unevenly by the magnitude of vertical reaction along the support section. From comparison of expressions for N_θ it is evident that they are also changed unevenly. Thus, the membrane theory does not satisfy requirements of continuity of forces on the support section, and bending of the shell should appear in this zone.

Let us examine further the case when $\alpha_0=\frac{\pi}{2}$. We will have the following expressions for forces:

$$N_\varphi=\frac{1}{3}\gamma R^2\left(1+\frac{\cos^2\alpha}{1-\cos\alpha}\right),$$
$$N_\theta=-\frac{1}{3}\gamma R^2\left(1+3\cos\alpha+\frac{\cos^2\alpha}{1-\cos\alpha}\right).$$

From these expressions it is evident that force N_ϕ will be tensile everywhere, force N_θ – compressive. Its value will be greatest when $\alpha=\frac{\pi}{2}$:

$$N_{\theta\,max}=-\frac{1}{3}\gamma R^2.$$

§ 18. Stresses in a Hemispherical Shell Located Under Axisymmetrical Aerodynamic Load

Figure 54 shows a diagram of loading on a hemispherical shell by aerodynamic load, changing according to law

$$q = q_0 \frac{(\cos\varphi - \cos\varphi_0)^2}{(1 - \cos\varphi_0)^2}.$$

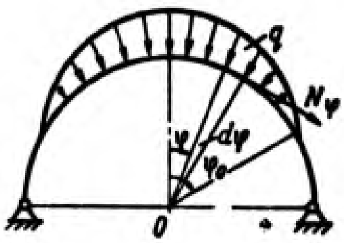

Fig. 54.

In order to obtain expressions for internal forces N_ϕ, let us examine the equilibrium of part of the sphere, determined by angle ϕ. Let us formulate the equilibrium conditions of all forces on the axis of shell:

$$\int_0^{2\pi}\int_0^{\varphi} qRr\cos\varphi\, d\varphi\, d\theta + \int_0^{2\pi} N_\varphi r \sin\varphi\, d\theta = 0,$$

where $r = R\sin\phi$.

After substitution of the expression for q and integration we obtain

$$N_\varphi = -\frac{q_0 R}{(1-\cos\varphi_0)^2 \sin^2\varphi}\left[\frac{1}{4} + \frac{3}{4}\cos^2\varphi - \cos^4\varphi - \frac{2}{3}(1-\cos^3\varphi)\cos\varphi_0 + \frac{1}{2}\cos^2\varphi_0 \sin^2\varphi\right]. \qquad (5.24)$$

The expression for annular force N_θ is determined from Laplace equation

$$N_\theta = \frac{q_0 R}{(1-\cos\varphi_0)^2}\left\{-(\cos\varphi-\cos\varphi_0)^2 + \frac{1}{\sin^2\varphi}\left[\frac{1}{4}+\frac{3}{4}\cos^2\varphi - \right.\right.$$
$$\left.\left. -\cos^4\varphi - \frac{2}{3}(1-\cos^3\varphi)\cos\varphi_0 + \frac{1}{2}\cos^2\varphi_0\sin^2\varphi\right]\right\}. \qquad (5.25)$$

Formulas (5.24), (5.25) are valid if $0<\varphi<\varphi_0$. To get formulas that are valid when $\phi > \phi_0$, equilibrium of the unloaded part of the shell must be examined. In this case we found

$$N_\varphi = -\frac{N_\varphi^0 \sin^2\varphi_0}{\sin^2\varphi}, \quad N_\theta = -N_\varphi.$$

Here $N_\varphi^0 = (N_\varphi)_{\varphi=\varphi_0}$. N_φ is given by formula (5.24).

§ 19. Displacements In a Symmetrically Loaded Cylindrical Shell

The problem about determination of displacements in momentless shells of revolution is expediently started with examination of a circular cylindrical shell. Figure 55 shows an element of this shell with dimensions dx, dy in the position before and after application of axisymmetrical load to the cylinder. Here u and w are components of total displacement in the direction of axes x and z respectively; l_1, l_2 – quantities of cosines of angles of element A_1B_1 after deformation with axes x and z; ε_x – relative elongation of element AB after deformation; ε_y – the same of element AD.

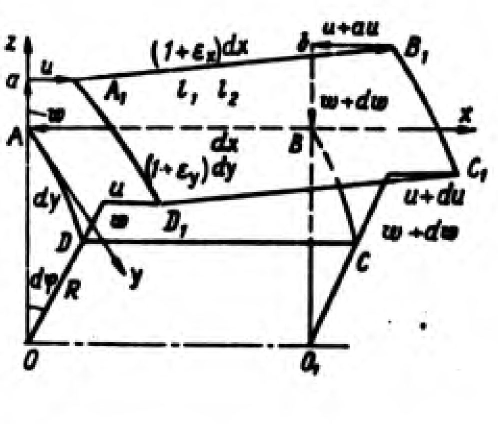

Fig. 55.

For determination of ε_x it is possible to project a closed polygon AaA_1B_1bB and x and z. Pointers on the ends of the sides of this polygon indicate the sequence of revolution:

$$u+dx(1+\varepsilon_x)l_1-u-du-dx=0,$$
$$w+dx(1+\varepsilon_x)l_2-w-dw=0.$$

Hence we have

$$(1+\varepsilon_x)l_1=1+\frac{du}{dx}, \qquad (1+\varepsilon_x)l_2=\frac{dw}{dx}.$$

By squaring these expressions and summing up, we obtain

$$(1+\varepsilon_x)^2(l_1^2+l_2^2)=\left(1+\frac{du}{dx}\right)^2+\left(\frac{dw}{dx}\right)^2,$$

or when $l_1^2+l_2^2=1$

$$1+2\varepsilon_x+\varepsilon_x^2=1+2\frac{du}{dx}+\left(\frac{du}{dx}\right)^2+\left(\frac{dw}{dx}\right)^2.$$

By disregarding here small quantity ε_x (in comparison with 2), we obtain

$$\varepsilon_x=\frac{du}{dx}+\frac{1}{2}\left[\left(\frac{du}{dx}\right)^2+\left(\frac{dw}{dx}\right)^2\right].$$

We are limited in this expression by the first linear term

$$\varepsilon_x=\frac{du}{dx}.$$

Component of deformation in circumferential direction

$$\varepsilon_y=\frac{2\pi(R+w)-2\pi R}{2\pi R}=\frac{w}{R}.$$

Furthermore, components of deformations can be expressed through stresses according to Hooke law for biaxial stressed state:

$$\varepsilon_x=\frac{1}{E}(\sigma_x-\mu\sigma_y), \qquad \varepsilon_y=\frac{1}{E}(\sigma_y-\mu\sigma_x).$$

Then

$$\frac{du}{dx}=\frac{1}{E}(\sigma_x-\mu\sigma_y), \qquad \frac{w}{R}=\frac{1}{E}(\sigma_y-\mu\sigma_x).$$

For example, in the case of loading a tank by boost pressure, hydrostatic pressure and axial force of compression (Fig. 56), for stresses we will have the following expressions:

$$\sigma_x = \frac{q_H R}{2\delta} - \frac{P}{2\pi R\delta}, \qquad \sigma_y = \frac{q_H R}{\delta} + \frac{\gamma(H-x)R}{\delta}.$$

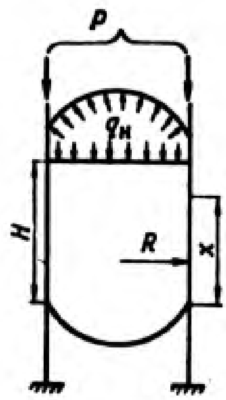

Fig. 56.

For determination of displacement u we will have equation

$$\frac{du}{dx} = \frac{1}{E\delta}\left\{\frac{q_H R}{2} - \frac{P}{2\pi R} - \mu\left[\gamma R(H-x) + q_H R\right]\right\}.$$

After integration of this equation we obtain

$$u = \frac{x}{E\delta}\left\{\left(\frac{q_H R}{2} - \frac{P}{2\pi R}\right) - \mu\left[\gamma R\left(H - \frac{1}{2}x\right) + q_H R\right]\right\} + C. \qquad (5.26)$$

Constant of integration C is determined from condition $u = 0$ when $x = 0$ and must be equal to zero.

Expression for w obtains the form

$$w = \frac{R}{E\delta}\left[\gamma R(H-x) + q_H R - \frac{\mu}{2}\left(q_H R - \frac{P}{\pi R}\right)\right]. \qquad (5.27)$$

Let us apply expressions (5.26), (5.27) for determination of lowering of the level of liquid as a result of deformation of the tank from the effect of loads shown in Fig. 56.

The expression for increase in the tank volume can be obtained from Fig. 57. Increase of volume dV of the considered element will be equal to the difference of its volume before and after deformation. Thus, with accuracy to small terms of the first order we obtain

$$dV=\frac{1}{2}(1+\varepsilon_y)dy(R+w)(1+\varepsilon_x)dx-\frac{1}{2}dyRdx=$$
$$=\left(w+\frac{1}{2}R\frac{du}{dx}\right)Rdx\,d\theta.$$

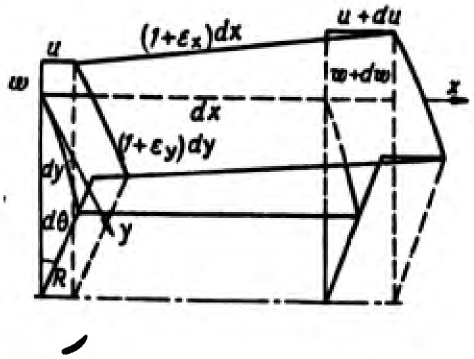

Fig. 57.

Total increase in the volume for the filled part of the tank

$$\Delta V=\int_0^{2\pi}\int_0^H\left(w+\frac{1}{2}R\frac{du}{dx}\right)Rdx\,d\theta=2\pi R\int_0^H\left(w+\frac{1}{2}R\frac{du}{dx}\right)dx.$$

After substitution here of expressions u and w and integration we obtain

$$\Delta V=\frac{2\pi R^2H}{E\delta}\left[\left(\frac{5}{4}-\mu\right)q_nR+\frac{\left(1-\frac{1}{2}\mu\right)}{2}\gamma RH-\frac{1-2\mu}{4\pi}\frac{P}{R}\right].$$

Lowering of the level of liquid

$$\Delta H=\frac{\Delta V}{\pi R^2}=\frac{2H}{E\delta}\left[\left(\frac{5}{4}-\mu\right)q_nR+\frac{\left(1-\frac{1}{2}\mu\right)}{2}\gamma RH-\frac{1-2\mu}{4\pi}\frac{P}{R}\right]. \quad (5.28)$$

If the thickness of the shell will be variable, it is possible to proceed in the following manner. Divide the length of the shell into sections with identical thickness and use formula (5.28) for each of these sections. For the height of liquid column for each section we should take the distance from the considered section to the mirror of liquid. Total lowering of the level of liquid will be equal to the sum of ΔH, obtained for the shown sections.

§ 20. Displacements In a Symmetrically Loaded Tapered Shell

Expressions for components of deformation of a tapered shell can be obtained from Fig. 58, where l_1, l_2 – direction cosines of element dx in deformed state with axes x and z.

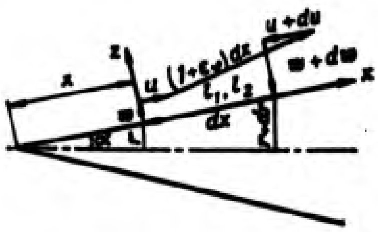

Fig. 58.

By proceeding just as in the case of a cylindrical shell, we obtain

$$u+(1+\varepsilon_\varphi)dx l_1-u-$$
$$-du-dx=0,$$
$$w+(1+\varepsilon_\varphi)dx l_2-w-dw=0.$$

From these equations let us find quantity ε_ϕ with accuracy to linear terms:

$$\varepsilon_\varphi=\frac{du}{dx}.$$

The deformation component in circumferential direction

$$\varepsilon_\theta=\frac{2\pi(r+w\cos\alpha+u\sin\alpha)-2\pi r}{2\pi r}=\frac{w}{r}\cos\alpha+\frac{u}{r}\sin\alpha.$$

But since $r = x\sin\alpha$

$$\varepsilon_\theta=\frac{w}{x}\operatorname{ctg}\alpha+\frac{u}{r}.$$

Let us exclude displacement u from the obtained expressions. Then

$$\frac{dw}{dx}=\left[\frac{d(x\varepsilon_\theta)}{dx}-\varepsilon_\varphi\right]\operatorname{tg}\alpha.$$

By integrating this expression, we find that

$$w = \operatorname{tg}\alpha \int \left[\frac{d(x\varepsilon_\theta)}{dx} - \varepsilon_\varphi \right] dx + C. \tag{5.29}$$

According to Hooke law the deformation components can be expressed through force in the form

$$\varepsilon_\varphi = \frac{1}{E\delta}(N_\varphi - \mu N_\theta), \qquad \varepsilon_\theta = \frac{1}{E\delta}(N_\theta - \mu N_\varphi).$$

Having expressions in each concrete case for forces N_ϕ and N_θ, we can determine displacements u and w.

Let us apply the given theory for determination of increase in the volume of tapered bottom of the tank (Fig. 59). Expressions for forces N_ϕ and N_θ in this case have the form

$$N_\varphi = \frac{q_н x \operatorname{tg}\alpha}{2} + \frac{\gamma x}{2}\left[(h+l) - \frac{2}{3}x\cos\alpha\right]\operatorname{tg}\alpha,$$
$$N_\theta = q_н x \operatorname{tg}\alpha + \gamma x[(h+l) - x\cos\alpha]\operatorname{tg}\alpha.$$

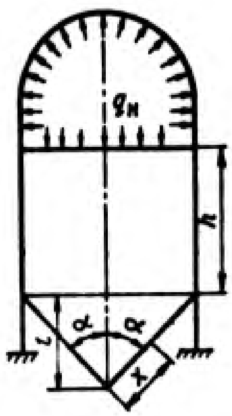

Fig. 59.

In this case for deformation components we obtained expressions

$$\varepsilon_\varphi = \frac{x\operatorname{tg}\alpha}{E\delta}\left\{(0,5-\mu)q_н + \gamma\left[(0,5-\mu)(h+l) + \left(\mu - \frac{1}{3}\right)x\cos\alpha\right]\right\},$$
$$\varepsilon_\theta = \frac{x\operatorname{tg}\alpha}{E\delta}\left\{\left(1-\frac{\mu}{2}\right)q_н + \gamma\left[\left(1-\frac{\mu}{2}\right)(h+l) - \left(1-\frac{\mu}{3}\right)x\cos\alpha\right]\right\}.$$

After substitution of these expressions in (5.29) and integration for w we obtain

$$w = \frac{x^2 \operatorname{tg}^2 \alpha}{E\delta}\left\{\frac{3}{4} q_н + \gamma\left[\frac{3}{4}(h+l) - \frac{8}{9} x \cos\alpha\right]\right\} + C.$$

Let us find displacement u from equation

$$u = x\varepsilon_1 - w \operatorname{ctg} \alpha =$$
$$= \frac{x^2 \operatorname{tg} \alpha}{E\delta}\left\{\frac{1-2\mu}{4} q_н + \gamma\left[\frac{1-2\mu}{4}(h+l) - \frac{1-3\mu}{9} x \cos\alpha\right]\right\} - C \operatorname{ctg} \alpha.$$

For determination of constant of integration C we have condition $u = 0$ when $x = \frac{l}{\cos\alpha}$, from which

$$C = \frac{\operatorname{tg}^2 \alpha}{E\delta}\left(\frac{l}{\cos\alpha}\right)^2\left[\frac{1-2\mu}{4} q_н + \gamma\left(\frac{1-2\mu}{4} h + \frac{5-6\mu}{36} l\right)\right].$$

Then

$$u = \frac{\operatorname{tg} \alpha}{E\delta}\left\{\frac{1-2\mu}{4}\left(x^2 - \frac{l^2}{\cos^2\alpha}\right)[q_н - (h+l)\gamma] - \frac{1-3\mu}{9}\gamma\left(x^3 \cos\alpha - \frac{l^3}{\cos^2\alpha}\right)\right\}. \quad (5.30)$$

For displacement w we obtain

$$w = \frac{\operatorname{tg}^2 \alpha}{E\delta}\left\{\left[\frac{3}{4} x^2 + \frac{1-2\mu}{4}\left(\frac{l}{\cos\alpha}\right)^2\right][q_н + (h+l)\gamma] - \gamma\left[x^3 \cos\alpha + \frac{1-3\mu}{9} l \left(\frac{l}{\cos\alpha}\right)^2\right]\right\}. \quad (5.31)$$

From expression (5.30) it is evident that when $x = 0$ and $\gamma = 0$ the lowest point of the tapered shell with loading only by internal pressure $q_н$ tries to raise upward. Formulas (5.30), (5.31) do not correspond to all boundary conditions of the problem, connected with restraining of cone, which is a consequence of the momentless theory, which we used in obtaining these formulas. Therefore, it is necessary to look on them as approximate.

Let us apply these formulas for determination of lowering of the level of liquid in tapered shell.

First of all expressions must be obtained for increase in the elementary volume. In Fig. 60 element $dxdy$ is represented in the position before and after deformation. For calculation of the volume of prism $OABCDO_1$ let us apply the formula for a truncated cone

$$V=\frac{\pi h}{3}(R^2+r^2+Rr).$$

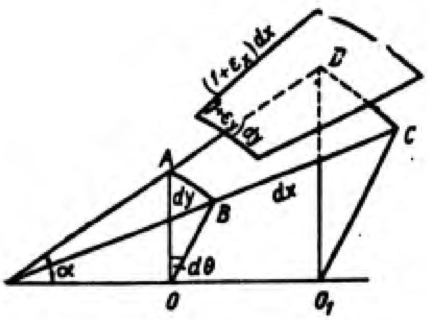

Fig. 60.

After division of the right side of this formula by 2π and multiplication by $d\theta$ it can be used for determination of the volume of a wedge-shaped tapered prism.

Volume before deformation (Fig. 61)

$$r=x\sin\alpha,\quad R=(x+dx)\sin\alpha,\quad h=dx\cos\alpha,$$

$$V_0=\frac{dx\cos\alpha}{6}\left[(x+dx)^2\sin^2\alpha+x^2\sin^2\alpha+x\,(x+dx)\sin^2\alpha\right].$$

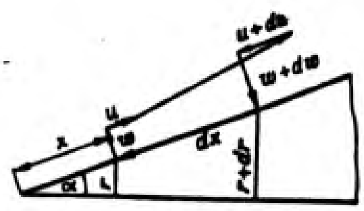

Fig. 61.

Volume after deformation (see Fig. 61)

$$r=x\sin\alpha+w\cos\alpha+u\sin\alpha,$$

$$R=(x+dx)\sin\alpha+(w+dw)\cos\alpha+(u+du)\sin\alpha,$$
$$h=(dx+du)\cos\alpha,$$
$$V=\frac{(dx+du)\cos\alpha}{6}\{[(x+dx)\sin\alpha+(w+dw)\cos\alpha+$$
$$+(u+du)\sin\alpha]^2+(x\sin\alpha+w\cos\alpha+u\sin\alpha)^2+$$
$$+[(x+dx)\sin\alpha+(w+dw)\cos\alpha+(u+du)\sin\alpha]\times$$
$$\times(x\sin\alpha+w\cos\alpha+u\sin\alpha)\}.$$

The difference of volumes with accuracy to quantities of the second order of smallness

$$dV=V-V_0=\frac{x\cos\alpha}{2}(w\sin 2\alpha+2u\sin^2\alpha)\,dx\,d\theta.$$

Total increase in the volume of cone

$$\Delta V_{\kappa}=\frac{1}{2}\cos\alpha\int_0^{2\pi}\int_0^{l/\cos\alpha}(w\sin 2\alpha+2u\sin^2\alpha)\,x\,dx\,d\theta.$$

After substitution here of expressions (5.30), (5.31) and integration we obtain

$$\Delta V_{\kappa}=\frac{2\pi\sin^3\alpha}{E\delta}\left\{\frac{1}{4}\left(\frac{l}{\cos\alpha}\right)^4\left[\left(1-\frac{\mu}{2}\right)q_{\kappa}-\frac{1+\mu}{2}(h+l)\gamma\right]+\right.$$
$$\left.+\frac{1-2\mu}{4}\left(\frac{l}{\cos\alpha}\right)^4(h+l)\gamma-\frac{10-3\mu}{45}\gamma\left(\frac{l}{\cos\alpha}\right)^5\cos\alpha\right\}.$$

Let us use the obtained value of ΔV_{κ} for determination of lowering of the level of liquid, being in the cone.

Since the volume of the cone was increased by ΔV_{κ} the level of liquid will drop by quantity $(l-l')$ (Fig. 62). In order to preserve the previous level in the cone, a quantity of liquid equal to ΔV_{κ} should be added in it. Then according to the formula for truncated cone

$$\frac{\pi h}{3}(R_{\kappa}^2+R_{\kappa}R_{\kappa}'+R_{\kappa}'^2)=\Delta V_{\kappa},$$

where

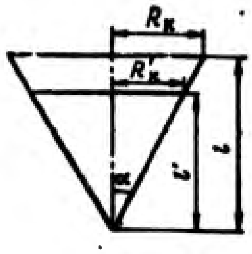

Fig. 62.

$$h = l - l',$$
$$R'_K = R_K - h \operatorname{tg} \alpha.$$

Then

$$\frac{\pi h}{3}\left[R_K^2 + R_K(R_K - h \operatorname{tg} \alpha) + (R_K - h \operatorname{tg} \alpha)^2\right] = \Delta V_K.$$

Since the amount of lowering of level h is always substantially less than R_K it is possible to write

$$\frac{\pi h}{3} 3R_K^2 \approx \Delta V_K.$$

Hence we obtain the following expression for lowering the level of liquid in tapered shell:

$$h = \frac{\Delta V_K}{\pi R_K^2}.$$

§ 21. Displacements In Shells of Arbitrary Shape with Axisymmetrical Load

In the shell of revolution is described by an arbitrary surface of revolution (Fig. 63), then expressions for components of deformation can be obtained exactly as was done earlier (see § 20).

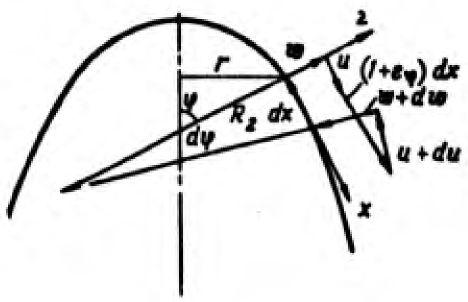

Fig. 63.

By projecting a closed polygon to axes x and z, we obtain

$$u+(1+\varepsilon_\varphi)\,dxl_1-u-du-(w+dw)\frac{dx}{R_1}-dx=0,$$

$$w+(1+\varepsilon_\varphi)\,dxl_2+(u+du)\frac{dx}{R_1}-w-dw=0.$$

Hence

$$(1+\varepsilon_\varphi)^2(l_1^2+l_2^2)=\left(1+\frac{du}{dx}+\frac{w}{R_1}\right)^2+\left(\frac{dw}{dx}-\frac{u}{R_1}\right)^2.$$

Taking into account that $l_1^2 + l_2^2 = 1$, and disregarding small quantity $\varepsilon\phi$ in comparison with 2, we obtain

$$\varepsilon_\varphi=\frac{du}{dx}+\frac{w}{R_1}+\frac{1}{2}\left[\left(\frac{du}{dx}\right)^2+\left(\frac{dw}{dx}\right)^2+\left(\frac{u}{R_1}\right)^2+\right.$$
$$\left.+\left(\frac{w}{R_1}\right)^2+2\frac{w}{R_1}\frac{du}{dx}-2\frac{u}{R_1}\frac{dw}{dx}\right].$$

Subsequently we will be limited only by the linear part of this expression

$$\varepsilon_\varphi=\frac{du}{dx}+\frac{w}{R_1}. \qquad (5.32)$$

The expression for deformation component in circumferential direction we obtain from relationship

$$\varepsilon_\theta=\frac{2\pi(r+w\sin\varphi+u\cos\varphi)-2\pi r}{2\pi r}=\frac{w}{r}\sin\varphi+\frac{u}{r}\cos\varphi.$$

Taking into account that $r = R_2 \sin\phi$, we finally find

$$\varepsilon_\theta=\frac{w}{R_2}+\frac{u}{R_2}\operatorname{ctg}\varphi. \qquad (5.33)$$

By having expressions for deformation components (5.32), (5.33), we can determine displacements u and w.

From (5.32)

$$w=R_1\varepsilon_\varphi-R_1\frac{du}{dx}.$$

Then for ε_θ we find

$$\varepsilon_\theta = \frac{R_1}{R_2}\varepsilon_\varphi - \frac{R_1}{R_2}\frac{du}{dx} + \frac{u}{R_2}\operatorname{ctg}\varphi.$$

Since $dx = R_1 d_\phi$,

$$\varepsilon_\theta = \frac{R_1}{R_2}\varepsilon_\varphi - \frac{du}{R_1 d\varphi} + \frac{u}{R_2}\operatorname{ctg}\varphi,$$

or

$$\frac{du}{d\varphi} - u\operatorname{ctg}\varphi = R_1\varepsilon_\varphi - R_2\varepsilon_\theta,$$

or

$$\sin\varphi\,\frac{d}{d\varphi}\left(\frac{u}{\sin\varphi}\right) = R_1\varepsilon_\varphi - R_2\varepsilon_\theta.$$

By integrating the last expression, we obtain

$$u = \sin\varphi\left[\int\frac{R_1\varepsilon_\varphi - R_2\varepsilon_\theta}{\sin\varphi}\,d\varphi + C\right]. \qquad (5.34)$$

Here deformation components ε_ϕ and ε_θ must be expressed through stresses σ_ϕ and σ_θ according to Hooke law for a case of biaxial stressed state:

$$\varepsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \mu\sigma_\theta), \qquad \varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_\varphi).$$

As an example let us determine displacements u and w for an ellipsoid of revolution, being under internal uniform pressure.

In this instance for stresses we obtained formulas

$$\sigma_\varphi = \frac{q}{2b\delta}\sqrt{(b^2-a^2)r^2+a^4},$$

$$\sigma_\theta = \frac{q}{2b\delta}\,\frac{2r^2(b^2-a^2)+a^4}{\sqrt{(b^2-a^2)r^2+a^4}}.$$

Furthermore, expressions for R_1, R_2 and $\sin\phi$ are given by formula (5.5).

These formula for an ellipsoid are expressed through variable r. Therefore, in expression (5.34) it is necessary to change from integration with respect to ϕ to integration with respect to r. From Fig. 64 we have

$$(R_1 d\varphi)^2 = (dy)^2 + (dr)^2, \quad R_1 d\varphi = \sqrt{1 + \left(\frac{dy}{dr}\right)^2}\, dr,$$

$$d\varphi = \frac{1}{R_1}\sqrt{1 + \left(\frac{dy}{dr}\right)^2}\, dr.$$

Furthermore, earlier we obtained

$$\frac{dy}{dr} = -\frac{br}{a\sqrt{a^2 - r^2}}.$$

By considering all the cited formulas and substituting them in expression (5.34), we will have

$$u = \frac{br}{\sqrt{(b^2 - a^2) r^2 + a^4}} \left\{ \frac{q(b^2 - a^2)}{2E\delta b^2 a} \int \left[\frac{(1 - 2\mu) r \sqrt{(b^2 - a^2) r^2 + a^4}}{\sqrt{a^2 - r^2}} - \right.\right.$$
$$\left.\left. - \frac{a^4 r}{\sqrt{(b^2 - a^2) r^2 + a^4}\sqrt{a^2 - r^2}} \right] dr + C \right\}.$$

After integration of this expression with the aid of substitution of $r = a \sin \xi$ we obtain

$$u = \frac{q a^2 b (b^2 - a^2) r}{2E\delta b^2 \sqrt{(b^2 - a^2) r^2 + a^4}} \left\{ \left[1 - \frac{1 - 2\mu}{2}\left(\frac{b}{a}\right)^2 \right] \times \right.$$
$$\times \ln\left[\sqrt{1 - \frac{b^2}{a^2}} \sqrt{1 - \left(\frac{r}{a}\right)^2} + \sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\left(\frac{r}{a}\right)^2} \right] -$$
$$\left. - \frac{1}{2}(1 - 2\mu)\sqrt{1 - \left(\frac{r}{a}\right)^2}\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\left(\frac{r}{a}\right)^2} \right\} +$$
$$+ \frac{brC}{\sqrt{(b^2 - a^2) r^2 + a^4}}.$$

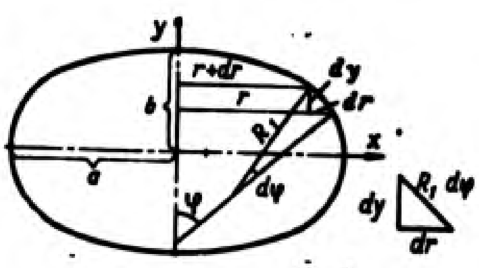

Fig. 64.

Let us find the constant of integration from condition $u = C$ when $r = a$

$$C = -\frac{qa^2(b^2-a^2)}{2E\delta b^2}\left[1-\frac{1-2\mu}{2}\left(\frac{b}{a}\right)^2\right]\ln\frac{b}{a}.$$

Then

$$u = \frac{qa^2(b^2-a^2)r}{2E\delta b\sqrt{(b^2-a^2)r^2+a^4}}\left\{\left[1-\frac{1-2\mu}{2}\left(\frac{b}{a}\right)^2\right]\times\right.$$
$$\times\left(\ln\left[\sqrt{1-\left(\frac{b}{a}\right)^2}\sqrt{1-\left(\frac{r}{a}\right)^2}+\right.\right.$$
$$\left.\left.+\sqrt{1-\left[1-\left(\frac{b}{a}\right)^2\right]\left(\frac{r}{a}\right)^2}\right]-\ln\frac{b}{a}\right)-$$
$$\left.-\frac{1-2\mu}{2}\sqrt{1-\left(\frac{r}{a}\right)^2}\sqrt{1-\left[1-\left(\frac{b}{a}\right)^2\right]\left(\frac{r}{a}\right)^2}\right\}.$$

Let us find expressions for w from relationship (5.33)

$$w = R_2\varepsilon_\theta - u\,\mathrm{ctg}\,\varphi,$$

where

$$\mathrm{ctg}\,\varphi = \frac{a\sqrt{a^2-r^2}}{br},$$

$$\frac{2E\delta b^2}{q}w = (2-\mu)(b^2-a^2)r^2+(1-\mu)a^4-$$
$$-\frac{a^3(b^2-a^2)\sqrt{a^2-r^2}}{\sqrt{(b^2-a^2)r^2+a^4}}\left\{\left[1-\frac{1-2\mu}{2}\left(\frac{b}{a}\right)^2\right]\times\right.$$
$$\times\left(\ln\left[\sqrt{1-\left(\frac{b}{a}\right)^2}\sqrt{1-\left(\frac{r}{a}\right)^2}+\sqrt{1-\left[1-\left(\frac{b}{a}\right)^2\right]\left(\frac{r}{a}\right)^2}\right]-\right.$$
$$\left.\left.-\ln\frac{b}{a}\right)-\frac{1-2\mu}{2}\sqrt{1-\left(\frac{r}{a}\right)^2}\sqrt{1-\left[1-\left(\frac{b}{a}\right)^2\right]\left(\frac{r}{a}\right)^2}\right\}.$$

Displacements at the pole and at the equator of the ellipsoid will be accordingly:

at the pole when $r = 0$, $u_n = 0$,

$$\frac{2E\delta b^2 w_n}{q} = (1-\mu)a^4 - a^2(b^2-a^2)\left\{\left[1-\frac{1-2\mu}{2}\left(\frac{b}{a}\right)^2\right]\times\right.$$
$$\left.\times\left(\ln\left[\sqrt{1-\left(\frac{b}{a}\right)^2}+1\right]-\ln\frac{b}{a}\right)-\frac{1-2\mu}{2}\right\};$$

at the equator when $r = a$

$$u_0 = 0, \quad w_0 = \frac{qa^2}{2E\delta}\left(2 - \mu - \frac{a^2}{b^2}\right).$$

When $\frac{a}{b} = 2, \quad \mu = 0,3$ we obtain

$$w_a = 3\frac{qa^2}{E\delta}, \quad w_0 = -1,15\frac{qa^2}{E\delta},$$

i.e., ellipsoid is elongated in the direction of the axis of rotation and is contracted at the equator.

In the case of a sphere when $a = b$ for the equator and for the pole we obtain the same value of w, equal to

$$w = \frac{(1-\mu)qa^2}{2E\delta}.$$

§ 22. Determination of Reduction of the Level of Liquid in Spherical Tanks from Internal Pressure

The problem of determination of lowering of the level of liquid in spherical tanks in accurate formation (from positions of momentless theory) leads to very complex formulas, inconvenient for practical utilization. The approximate solution of this problem is given here.

Let us assume there is a spherical tank, liquid level in which is defined by angle α_0. The tank is supported along the circumference, defined by angle α_1 (Fig. 65). For determination of the increase in volume the tank is divided into zones. For each zone stresses σ_f and σ_θ (on the upper and lower sections of each zone) are calculated by the appropriate formulas of § 17.

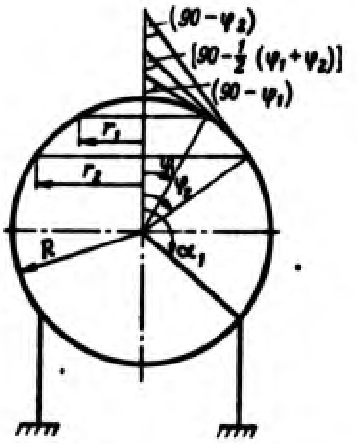

Fig. 65.

Then deformation components are determined in meridion and circumferential directions:

$$\varepsilon_\varphi = \frac{1}{E}(\sigma_\varphi - \mu\sigma_\theta)_{\substack{\varphi=\varphi_1,\\ \varphi=\varphi_2}}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \mu\sigma_\varphi)_{\substack{\varphi=\varphi_1,\\ \varphi=\varphi_2}}.$$

After this the increase in volume of truncated cone can be determined by expression

$$\Delta V = \frac{\pi(h+\Delta h)}{3}\left[(r_2+\Delta r_2)^2+(r_1+\Delta r_1)^2+(r_1+\Delta r_1)(r_2+\Delta r_2)\right] - \frac{\pi h}{3}(r_2^2+r_1^2+r_1 r_2) = \frac{\pi}{3}\left[r_1 h(2\Delta r_1+\Delta r_2) + r_2 h(2\Delta r_2+\Delta r_1) + \Delta h(r_1^2+r_2^2+r_1 r_2)\right].$$

Here the following designations are introduced:

$$r_1 = R\sin\varphi_1,\quad r_2 = R\sin\varphi_2,\quad \Delta r_1 = r_1(\varepsilon_\theta)_{\varphi=\varphi_1} = R(\varepsilon_\theta)_{\varphi=\varphi_1}\cdot\sin\varphi_1,$$

$$\Delta r_2 = r_2(\varepsilon_\theta)_{\varphi=\varphi_2} = R(\varepsilon_\theta)_{\varphi=\varphi_2}\cdot\sin\varphi_2,$$

$$h = l\cos\left(90^\circ - \frac{\varphi_1+\varphi_2}{2}\right),\quad \Delta h = h\varepsilon_\varphi^{cp},$$

$$\varepsilon_\varphi^{cp} = \frac{1}{2}\left[(\varepsilon_\varphi)_{\varphi=\varphi_1} + (\varepsilon_\varphi)_{\varphi=\varphi_2}\right];$$

R – radius of sphere; l – length of chord, determined by angle $(\phi_2-\phi_1)$; h – height of zone.

The remaining designations are clear from Fig. 65.

By having the expression for increase in volume of each zone, we can determine the total increase in volume of the entire tank:

$$\Delta V_c = \sum_{i=1}^{n} \Delta V_i,$$

where n – number of zones.

The amount of lowering of the liquid level can be determined from the condition where the volume of deformed tank is supplemented by the volume of liquid, equal to ΔV_c, and the level of liquid is raised to initial. From this condition by analogy with § 21 we approximately obtain the amount of lowering of the level

$$H = \frac{\Delta V_c}{\pi R^2 \sin^2 u_0}.$$

§ 23. Calculation of Shells for Arbitrary Load. Differential Equations of Equilibrium

In this paragraph are examined some practically important problems, connected with calculation of shells of revolution for arbitrary nonaxisymmetric load. Generally such shells should be examined with allowance for their work on bending. In this case these shells will be examined only from positions of momentless theory. Therefore, the subsequently obtained results should be viewed as approximate.

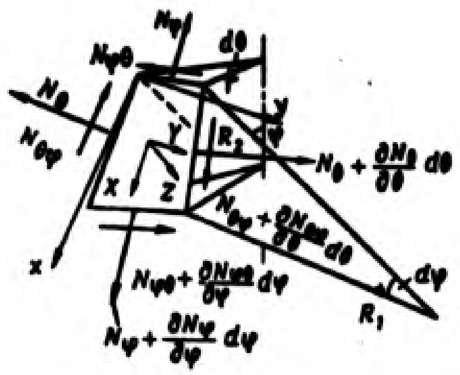

Fig. 66.

Let us turn to Fig. 66, which shows an element of a shell with sides $rd\theta$ and $R_1 d_\phi$ and components of external load X, Y, Z, positive directions of which coincide with the direction of local axes of coordinates x, y, z. On the sides of the chosen element are applied internal forces N_ϕ, N_θ, $N_{\theta\phi}$ and $N_{\phi\theta}$, expressing the effect of the discarded part of the shell. Let us formulate the sums of projections of all forces affecting the chosen element of shell to axes x, y, z of moving coordinate system.

To axis x

$$-N_\varphi r d\theta+\left(N_\varphi+\frac{\partial N_\varphi}{\partial\varphi}d\varphi\right)\left(r+\frac{\partial r}{\partial\varphi}d\varphi\right)d\theta-N_{\theta\varphi}R_1 d\varphi+$$

$$+\left(N_{\theta\varphi}+\frac{\partial N_{\theta\varphi}}{\partial\theta}d\theta\right)R_1 d\varphi-$$

$$-\left(N_\theta+\frac{\partial N_\theta}{\partial\theta}d\theta\right)R_1 d\varphi d\theta\cos\varphi+Xrd\theta R_1 d\varphi=0.$$

To axis y

$$-N_{\varphi\theta} r d\theta+\left(N_{\varphi\theta}+\frac{\partial N_{\varphi\theta}}{\partial\varphi}d\varphi\right)\left(r+\frac{\partial r}{\partial\varphi}d\varphi\right)d\theta-N_\theta R_1 d\varphi+$$

$$+\left(N_\theta+\frac{\partial N_\theta}{\partial\theta}d\theta\right)R_1 d\varphi+$$

$$+\left(N_{\theta\varphi}+\frac{\partial N_{\theta\varphi}}{\partial\theta}d\theta\right)R_1 d\varphi d\theta\cos\varphi+Yrd\theta R_1 d\varphi=0.$$

To axis z

$$\left(N_\varphi+\frac{\partial N_\varphi}{\partial\varphi}d\varphi\right)\left(r+\frac{\partial r}{\partial\varphi}d\varphi\right)d\theta\, d\varphi+$$

$$+\left(N_\theta+\frac{\partial N_\theta}{\partial\theta}d\theta\right)R_1 d\varphi\, d\theta\sin\varphi+Zrd\theta R_1 d\varphi=0.$$

From these equations with accuracy to quantities of the first order of smallness we obtain

$$\frac{\partial}{\partial\varphi}(R_2 N_\varphi\sin\varphi)+R_1\frac{\partial N_{\theta\varphi}}{\partial\theta}-R_1 N_\theta\cos\varphi+XR_1R_2\sin\varphi=0,$$

$$\frac{\partial}{\partial\varphi}(R_1 N_{\varphi\theta}\sin\varphi)+N_{\theta\varphi}R_1\cos\varphi+R_1\frac{\partial N_\theta}{\partial\theta}+YR_1R_2\sin\varphi=0,$$

$$\frac{N_\varphi}{R_1}+\frac{N_\theta}{R_2}+Z=0. \qquad (5.35)$$

Here

$$R_2=\frac{r}{\sin\varphi}.$$

Furthermore, from the law of pairing of tangential stresses it follows that $N_{\theta\phi} = N_{\phi\theta}$.

Equations (5.35) pertain to the type of linear differential first order equations in partial derivatives with variable coefficients. Having excluded force N_θ from these equations, we obtain

$$\frac{\partial(R_2N_\varphi)}{\partial\varphi}+2R_2N_\varphi\operatorname{ctg}\varphi+\frac{R_1}{\sin\varphi}\frac{\partial N_{\theta\varphi}}{\partial\theta}+ZR_1R_2\operatorname{ctg}\varphi+XR_1R_2=0,$$

$$\frac{\partial(R_2N_{\theta\varphi})}{\partial\varphi}+(R_1+R_2)N_{\theta\varphi}\operatorname{ctg}\varphi-\frac{R_2}{\sin\varphi}\frac{\partial N_\varphi}{\partial\theta}-\frac{R_1R_2}{\sin\varphi}\frac{\partial Z}{\partial\theta}+YR_1R_2=0. \quad (5.36)$$

For further simplification of these equations let us introduce new unknown functions by formulas

$$R_2N_\varphi=U,\quad R_2N_{\theta\varphi}=V.$$

Then equations (5.36) will take the following form:

$$\frac{\partial V}{\partial\varphi}+\left(1+\frac{R_1}{R_2}\right)V\operatorname{ctg}\varphi-\frac{1}{\sin\varphi}\frac{\partial U}{\partial\theta}-\frac{R_1R_2}{\sin\varphi}\frac{\partial Z}{\partial\theta}+YR_1R_2=0,$$

$$\frac{\partial U}{\partial\varphi}+2U\operatorname{ctg}\varphi+\frac{R_1}{R_2\sin\varphi}\frac{\partial V}{\partial\theta}+ZR_1R_2\operatorname{ctg}\varphi+XR_1R_2=0. \quad (5.37)$$

One of the general methods of solution of these equations is the method based on expansion of the sought functions into Fourier series. For this it is necessary in the beginning to expand external loads into Fourier series along coordinate θ. Let us assume these expansions have the form

$$X=\sum X_n\cos n\theta,\quad Y=\sum Y_n\sin n\theta,\quad Z=\sum Z_n\cos n\theta.$$

According to this we have the solution of obtained equations in the form

$$V=\sum V_n\sin n\theta,\quad U=\sum U_n\cos n\theta.$$

After substitution of the shown expansions in equations they take a simpler form

$$\frac{dV_n}{d\varphi}+\left(1+\frac{R_1}{R_2}\right)V_n\operatorname{ctg}\varphi+\frac{n}{\sin\varphi}U_n+\frac{nR_1R_2}{\sin\varphi}Z_n+Y_nR_1R_2=0,$$
$$\frac{dU_n}{\partial\varphi}+2U_n\operatorname{ctg}\varphi+\frac{nR_1}{R_2\sin\varphi}V_n+Z_nR_1R_2\operatorname{ctg}\varphi+X_nR_1R_2=0. \qquad (5.38)$$

Despite the apparent simplicity of the structure of these equations, they do not always allow a simple solution. Therefore, let us examine the simplest shapes of shells with the simplest types of loading.

Loading of hemispherical shell by wind load. In this case $R_1 = R_2 = R$. Then equations (5.38) will take the form

$$\frac{dV_n}{d\varphi}+2V_n\operatorname{ctg}\varphi+\frac{n}{\sin\varphi}U_n+\frac{nR^2}{\sin\varphi}Z_n+Y_nR^2=0,$$
$$\frac{dU}{d\varphi}+2U_n\operatorname{ctg}\varphi+\frac{n}{\sin\varphi}V_n+Z_nR^2\operatorname{ctg}\varphi+XR^2=0,$$

After addition and subtraction of these equations and introduction of new unknown functions

$$P_n=V_n-U_n,\quad Q_n=V_n+U_n \qquad (5.39)$$

we obtain the two following independent equations:

$$\frac{dP_n}{d\varphi}+\left(2\operatorname{ctg}\varphi-\frac{n}{\sin\varphi}\right)P_n-Z_nR^2\left(\operatorname{ctg}\varphi-\frac{n}{\sin\varphi}\right)-(X_n-Y_n)R^2=0,$$
$$\frac{dQ_n}{d\varphi}+\left(2\operatorname{ctg}\varphi+\frac{n}{\sin\varphi}\right)Q_n+Z_nR^2\left(\operatorname{ctg}\varphi+\frac{n}{\sin\varphi}\right)+(X_n+Y_n)R^2=0. \qquad (5.40)$$

These equations pertain to first order differential equations with separating variables of the following type:

$$y'+q(x)y+f(x)=0.$$

The solution of this equation, as is known from the theory of differential equations, will be expression

In our case for the first equation (5.40)

$$q(x)=2\operatorname{ctg}\varphi-\frac{n}{\sin\varphi},$$
$$f(x)=-Z_nR^2\left(\operatorname{ctg}\varphi-\frac{n}{\sin\varphi}\right)-(X_n-Y_n)R^2.$$

Furthermore,

$$e^{-\int q(x)dx}=e^{-\int\left(2\operatorname{ctg}\varphi-\frac{n}{\sin\varphi}\right)d\varphi}=e^{\ln\frac{\operatorname{tg}^n 0{,}5\varphi}{\sin^2\varphi}}=\frac{\operatorname{tg}^n 0{,}5\varphi}{\sin^2\varphi},$$
$$e^{\int q(x)dx}=e^{\int\left(2\operatorname{ctg}\varphi-\frac{n}{\sin\varphi}\right)d\varphi}=e^{\ln\frac{\sin^2\varphi}{\operatorname{tg}^n 0{,}5\varphi}}=\frac{\sin^2\varphi}{\operatorname{tg}^n 0{,}5\varphi}.$$

Then from (5.39) for Q_n and P_n we obtain the following expressions:

$$P_n=\frac{\operatorname{tg}^n 0{,}5\varphi}{\sin^2\varphi}\left\{C_1+R^2\int\left[Z_n\left(\operatorname{ctg}\varphi-\frac{n}{\sin\varphi}\right)+(X_n-Y_n)\right]\frac{\sin^2\varphi}{\operatorname{ctg}^n 0{,}5\varphi}d\varphi\right\}.$$

Analogously

$$Q_n=\frac{\operatorname{ctg}^n 0{,}5\varphi}{\sin^2\varphi}\left\{C_2-R^2\int\left[Z_n\left(\operatorname{ctg}\varphi+\frac{n}{\sin\varphi}\right)+(X_n+Y_n)\right]\frac{\sin^2\varphi}{\operatorname{ctg}^n 0{,}5\varphi}d\varphi\right\}.$$

After determination of P_n and Q_n the fought forces are easily determined:

$$N_{\theta\varphi}=\frac{P_n+Q_n}{2R}\sin n\theta,\quad N_\varphi=\frac{Q_n-P_n}{2R}\cos n\theta,$$
$$N_\theta=-RZ_n-N_\varphi.$$

Let us examine the case of the action of wind load on a hemisphere according to law

$$Z=Z_0\sin\varphi\cos\theta,\quad n=1,\quad Z_n=Z_0\sin\varphi,$$
$$X=0,\quad Y=0.$$

For forces N_ϕ, $N_{\theta\phi}$ in this instance we obtain the following expressions:

$$N_\varphi=\frac{\cos\theta}{\sin^3\varphi}\left[\frac{C_2-C_1}{2R}+\frac{C_2+C_1}{2R}\cos\varphi+Z_0R\left(\cos^2\varphi-\frac{1}{3}\cos^4\varphi\right)\right],$$
$$N_{\theta\varphi}=\frac{\sin\theta}{\sin^3\varphi}\left[\frac{C_1+C_2}{2R}+\frac{C_2-C_1}{2R}\cos\varphi+Z_0R\left(\cos\varphi-\frac{1}{3}\cos^3\varphi\right)\right].$$

Constants C_1 and C_2 must be determined from boundary conditions. We will consider that the hemisphere is fastened to a fixed base (Fig. 67). Then the boundary conditions will be the following:

1) projection of tangential forces $N_{\theta\phi}$ when $\phi = \pi/2$ in direction $\theta = 0$ are equal to the projection of wind loading in the same direction:

$$\int_0^{2\pi} (N_{\theta\varphi})_{\varphi=\frac{\pi}{2}} R \sin\theta d\theta + \int_0^{2\pi}\int_0^{\frac{\pi}{2}} Z_0 Rr \sin\varphi \cos\theta \, d\varphi \, d\theta = 0;$$

2) sum of moments of forces N_ϕ when $\phi = \pi/2$ relative to the diameter of hemisphere $\theta = \pi/2$ are equal to the moment of wind load relative to the same diameter:

$$\int (N_\varphi)_{\varphi=\frac{\pi}{2}} R^2 \cos\theta \, d\theta = 0.$$

Moment of wind load is equal to zero, since pressure Z has radial direction.

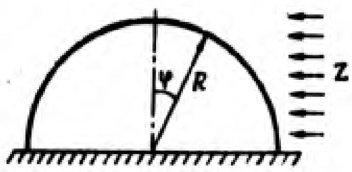

Fig. 67.

From the first condition we obtain

$$\frac{C_1 + C_2}{2} = -\frac{2}{3} Z_0 R^2.$$

The second condition gives

$$C_1 - C_2 = 0.$$

Then for forces N_ϕ, N_θ, $N_{\theta\phi}$ we obtain the following expressions:

$$N_\varphi = -\frac{Z_0 R \cos\varphi \cos\theta}{3\sin^2\varphi}(2 - 3\cos\varphi + \cos^3\varphi),$$

$$N_\theta = \frac{Z_0 R \cos\theta}{3\sin^2\varphi}(2\cos\varphi - 3\sin^2\varphi - 2\cos^3\varphi),$$

$$N_{\theta\varphi} = -\frac{Z_0 R \sin\theta}{3\sin^2\varphi}(2 - 3\cos\varphi + \cos^3\varphi).$$

If we determine constants C_1 and C_2 from the condition that when $\phi = 0$ forces N_ϕ and $N_{\theta\phi}$ should become zero, then we would reveal that they receive the same magnitudes that were given above.

Loading of tapered shell by wind type load. In this instance (Fig. 68)

$$\varphi=\frac{\pi}{2}-\alpha,\quad R_1=\infty,\quad R_1 d\varphi=dx,\quad R_2=x\,\mathrm{tg}\,\alpha.$$

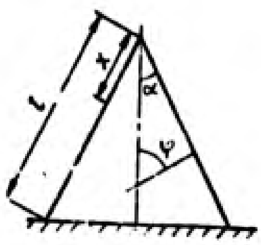

Fig. 68.

Then equations (5.35) will take the form

$$\frac{\partial(xN_\varphi)}{\partial x}+\frac{1}{\sin\alpha}\frac{\partial N_{\theta\varphi}}{\partial\theta}-N_\theta+Xx=0,$$

$$\frac{\partial(xN_{\theta\varphi})}{\partial x}+\frac{1}{\sin\alpha}\frac{\partial N_\theta}{\partial\theta}+N_{\theta\varphi}+Yx=0,$$

$$N_\theta=-Zx\,\mathrm{tg}\,\alpha.$$

After exclusion of force N_θ from these equations and performing analogous conversions with them, as for the case of a sphere, we obtain

$$\frac{dU_n}{dx}+\frac{n}{x\sin\alpha}V_n+Z_n x\,\mathrm{tg}\,\alpha+X_n x=0,$$

$$\frac{dV_n}{dx}+\frac{1}{x}V_n+\frac{nx}{\cos\alpha}Z_n+Y_n x=0, \qquad (5.42)$$

where

$$X=X_n\cos n\theta,\quad Y=Y_n\sin n\theta,\quad Z=Z_n\cos n\theta,$$

$$xN_\varphi=U_n\cos n\theta,\quad xN_{\theta\varphi}=V_n\sin n\theta.$$

Let us apply equationss(5.42) for the case of wind load, changing according to law

$$Z = Z_0\left(\frac{x}{l}\right)^2 \cos\theta,$$
$$X = 0,\ Y = 0,\ n = 1.$$

In this instance equations (5.42) take the form

$$\begin{aligned} &\frac{dU_1}{dx} + \frac{1}{x\sin\alpha}V_1 + Z_0 x\left(\frac{x}{l}\right)^2 \operatorname{tg}\alpha = 0,\\ &\frac{dV_1}{dx} + \frac{1}{x}V_1 + \frac{Z_0 x}{\cos\alpha}\left(\frac{x}{l}\right)^2 = 0. \end{aligned} \qquad (5.43)$$

The second equation is integrated with respect to formula (5.41)

$$V_1 = \frac{1}{x}\left(C_1 - \frac{Z_0 x^5}{5l^2\cos\alpha}\right).$$

After substitution of V_1 in the first equation (5.43) and integration we find

$$U_1 = \frac{Z_0}{4l^2}\left(\frac{2}{5\sin 2\alpha} - \operatorname{tg}\alpha\right)x^4 + \frac{C_1}{x\sin\alpha} + C_2.$$

In this case for forces we obtain the following expressions:

$$N_\varphi = \left[\frac{Z_0}{4l^2}\left(\frac{2}{5\sin 2\alpha} - \operatorname{tg}\alpha\right)x^3 + \frac{C_1}{x^2\sin\alpha} + \frac{C_2}{x}\right]\cos\theta,$$
$$N_{\theta\varphi} = \left(\frac{C_1}{x^2} - \frac{Z_0 x^3}{5l^2\cos\alpha}\right)\sin\theta.$$

Constants C_1 and C_2 in this case must be taken equal to zero, in order to avoid indeterminacy in the magnitudes of forces when $x = 0$. Then finally we obtain

$$N_\varphi = \frac{Z_0}{4l^2}\left(\frac{2}{5\sin 2\alpha} - \operatorname{tg}\alpha\right)x^3\cos\theta,$$
$$N_\theta = -\frac{Z_0\operatorname{tg}\alpha}{l^2}x^3\cos\theta, \qquad N_{\theta\varphi} = -\frac{Z_0 x^3}{5l^2\cos\alpha}\sin\theta.$$

Loading of tapered shell be bending moment and lateral force. First let us examine loading of a tapered shell by bending moment (Fig. 69).

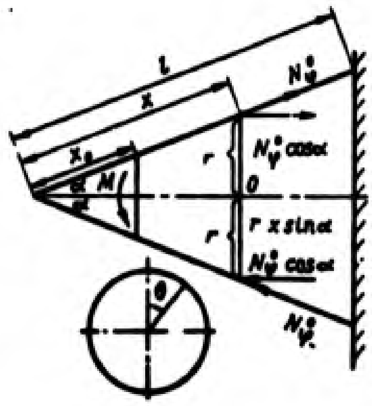

Fig. 69.

In this instance equations (5.35) when $X = 0$, $Y = 0$, $Z = 0$ take the form

$$\frac{\partial N_\varphi}{\partial x}+\frac{1}{x}N_\varphi+\frac{1}{x\sin\alpha}\frac{\partial N_{\theta\varphi}}{\partial\theta}=0,$$
$$\frac{\partial N_{\theta\varphi}}{\partial x}+\frac{2}{x}N_{\theta\varphi}=0, \qquad N_\theta=0. \tag{5.44}$$

Let us integrate the second equation (5.44). After separation of variables we obtain

$$N_{\theta\varphi}=\frac{C(\theta)}{x^2}, \tag{5.45}$$

where $C(\theta)$ – arbitrary function of angle θ.

Before integrating the first equation, let us determine force N_ϕ. For this purpose at distance x from the tip of the cone let us draw a section, along the perimeter of which we apply internal forces N_ϕ, distributed according to cosine law.

From condition of equality of the sum of moments of all forces to zero, applied to the cutoff part of the cone relative to the axis, passing through point O, perpendicular to the drawing, we obtain

$$\int_0^{2\pi} N_\varphi^0 \cos\alpha\cos\theta\, r d\theta\, r\cos\theta = M.$$

Hence

$$N_\varphi^0 = \frac{M}{\pi x^2 \sin^2 \alpha \cos \alpha},$$

where N_ϕ^0 – meridion force in the most remote line.

Meridion force at arbitrary angle θ will be

$$N_\varphi = N_\varphi^0 \cos \theta = \frac{M \cos \theta}{\pi x^2 \sin^2 \alpha \cos \alpha}.$$

Having substituted the obtained expression N_ϕ in the first equation (5.44), we obtain

$$\frac{\partial N_{\theta\varphi}}{\partial \theta} = \frac{M \cos \theta}{\pi x^2 \sin \alpha \cos \alpha}.$$

After integration of this equation we find that

$$N_{\theta\varphi} = \frac{M \sin \theta}{\pi x^2 \sin \alpha \cos \alpha} + C_1(x), \tag{5.46}$$

where $C_1(x)$ – arbitrary function of x.

Thus, for force $N_{\theta\phi}$ we obtain expressions (5.45) and (5.46). From the condition of their equality

$$C(\theta) = \frac{M \sin \theta}{\pi \sin \alpha \cos \alpha} + x^2 C_1(x).$$

Since function $C_1(x)$ does not depend on angle θ, from the last expression it follows that $C_1(x) = 0$. Then for function $C(\theta)$ we obtain

$$C(\theta) = \frac{M \sin \theta}{\pi \sin \alpha \cos \alpha}.$$

Tangential force $N_{\theta\phi}$ is determined from formula (5.46)

$$N_{\theta\varphi} = \frac{M \operatorname{tg} \alpha \sin \theta}{\pi x^2 \sin^2 \alpha}.$$

Distribution of forces $N_{\theta\phi}$ and N_{ϕ} is shown in Fig. 70.

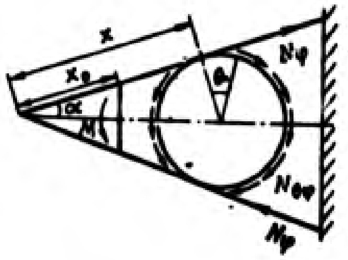

Fig. 70.

As can be seen from this figure, tangential forces $N_{\theta\phi}$ give resultant lateral force

$$Q_1=\int_0^{2\pi} N_{\theta\varphi} r d\theta \sin\theta=\frac{M \operatorname{tg}\alpha}{x \sin\alpha}.$$

Since

$$M=\pi x^2 \sin^2\alpha N_\varphi^0 \cos\alpha,$$

then

$$Q_1=\pi N_\varphi^0 x \sin^2\alpha.$$

This force is equalized in the section of resultant of tangential forces N_{xz} (Fig. 71), equal to

$$Q_2=\int_0^{2\pi} N_\varphi \sin\alpha r d\theta \cos\theta=\pi N_\varphi^0 x \sin^2\alpha.$$

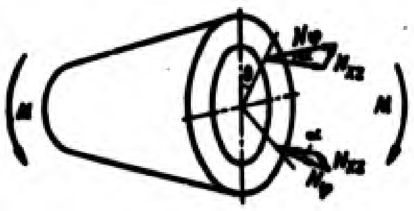

Fig. 71.

From comparison of the right sides of expressions for Q_1 and Q_2 it is evident that they are equal. Consequently, $Q_1 = Q_2$. As follows from Figs. 70 and 71, forces Q_1 and Q_2 are directed along the vertical diameter to directly opposite sides and they lie in the plane of action of the bending moment.

Let us examine further loading of the tapered shell by lateral force Q (Fig. 72).

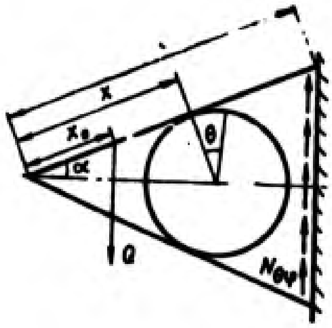

Fig. 72.

Differential equations of the problem in this case remain in the form of (5.44). Function $C(\theta)$ in expression (5.45) for tangential force $N_{\theta\phi}$ we find from condition

$$(N_{\theta\varphi})_{x=x_0}=N^0_{\theta\varphi}\sin\theta$$

or

$$\frac{C(\theta)}{x_0^2}=N^0_{\theta\varphi}\sin\theta,$$

where

$$N^0_{\theta\varphi}=-\frac{Q}{\pi x_0\sin\alpha}.$$

Then

$$N_{\theta\varphi}=\frac{C(\theta)}{x^2}=\frac{N^0_{\theta\varphi}x_0^2\sin\theta}{x^2}=$$
$$=-\frac{Qx_0\sin\theta}{\pi x^2\sin\alpha}.$$

Having substituted the found value of $N_{\theta\phi}$ in the first equation (5.44), we obtain

$$\frac{\partial N_\varphi}{\partial x}+\frac{N_\varphi}{x}=\frac{Qx_0\cos\theta}{\pi x^2\sin\alpha}.$$

As a result of integration of this equation we determine force N_ϕ:

$$N_\varphi=-\frac{Qx_0\cos\theta}{\pi x^2\sin^2\alpha}+\frac{D(\theta)}{x},$$

where $D(\theta)$ – arbitrary function of angle θ.

For determination of this function we have condition

$$(N_\varphi)_{x=x_0}=0,$$

expressing the absence of normal stresses in the section, where force Q is applied.

By using this condition, we find

$$D(\theta)=\frac{Q\cos\theta}{\pi\sin^2\alpha}.$$

Then for N_ϕ finally we obtain expression

$$N_\varphi=\frac{Q(x-x_0)\cos\theta}{\pi x^2\sin^2\alpha}.$$

It is easy to check that the moment of forces N_ϕ in the fixed section of the cone relative to its horizontal diameter is equal to the moment of force Q. Actually,

$$\int_0^{2\pi}(N_\varphi)_{x=l}R^2\cos\alpha\cos\theta\,d\theta=Q(l-x_0)\cos\alpha.$$

Total stresses in the tapered shell with its simultaneous loading by bending moment and lateral force will be

$$N_\varphi=\frac{M\cos\theta}{\pi x^2\sin^2\alpha\cos\alpha}+\frac{Q(x-x_0)\cos\theta}{\pi x^2\sin^2\alpha},$$
$$N_{\theta\varphi}=\frac{Qx_0\sin\theta}{\pi x^2\sin\alpha}-\frac{M\,\mathrm{tg}\,\alpha\sin\theta}{\pi x^2\sin^2\alpha}.$$

From the last expression it is evident that at certain relationships of quantities entering it we can obtain $N_{\theta\phi}=0$. Actually, we have

$$\frac{Qx_0\sin\theta}{\pi x^2\sin\alpha}-\frac{M\,\mathrm{tg}\,\alpha\sin\theta}{\pi x^2\sin^2\alpha}=0.$$

Hence we obtain

$$\frac{Qx_0\cos\alpha}{M}=1.$$

From this condition the conclusion can be made that if in some section of the tapered shell there are applied lateral force and bending moment, where the moment of force Q relative to the tip of the cone is equal to applied moment M, then tangential stresses in the shell will be equal to zero everywhere.

If we change the direction of bending moment, normal stresses from M and Q will be subtracted, and tangential – added.

Loading of cylindrical shell by wind load. In this instance $R_1=\infty$, $R_2=R$, $\varphi=\frac{\pi}{2}$, $R_1 d\varphi=dx$. Then the original equations (5.35) take the form

$$\frac{\partial N_\varphi}{\partial x}+\frac{\partial N_{\theta\varphi}}{R\partial\theta}+X=0,\qquad \frac{\partial N_{\theta\varphi}}{\partial x}+\frac{\partial N_\theta}{R\partial\theta}+Y=0,$$
$$N_\theta=-ZR.$$

Having substituted here $X = 0$, $Y = 0$, $Z = Z_0\cos\theta$, from the third equation we obtain

$$N_\theta=-Z_0R\cos\theta.$$

From the two remaining equations we find

$$N_{\theta\varphi}=-Z_0x\sin\theta+f_1(\theta),$$
$$N_\varphi=\frac{Z_0x^2}{2R}\cos\theta-x\frac{\partial f_1(\theta)}{R\partial\theta}+f_2(\theta).$$

Here $f_1(\theta)$ and $f_2(\theta)$ – arbitrary functions of angle θ, structure of which should be established from the following boundary conditions.

1. The sum of projections of all forces $N_{\theta\phi}$ when $x = 0$ to direction $\theta = 0$ must be equivalent to resultant of force of wind in the same direction (Fig. 73):

$$\int_0^{2\pi}(N_{\theta\varphi})_{x=0}Rd\theta\sin\theta-\int_0^{2\pi}\int_0^{l}ZRd\theta\,dx\cos\theta=0.$$

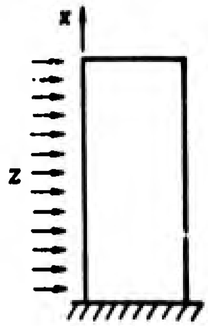

Fig. 73.

From this condition we find

$$f_1(\theta) = Z_0 l \sin\theta.$$

2. Moment of forces N_φ when $x = 0$ relative to diameter $\theta = \pi/2$ must be equal to the moment of forces of wind relative to the same diameter:

$$\int_0^{2\pi} (N_\varphi)_{x=0} R\,d\theta\,R\cos\theta = \int_0^{2\pi}\int_0^{l} ZR\,d\theta\,dx\cos\theta\,x,$$

whence

$$f_2(\theta) = \frac{Z_0 l^2}{2R}\cos\theta.$$

Then for forces N_φ, N_θ, $N_{\theta\varphi}$ we obtain the following expressions:

$$N_\varphi = \frac{Z_0}{2R}(x-l)^2\cos\theta, \qquad N_\theta = -Z_0 R\cos\theta,$$

$$N_{\theta\varphi} = Z_0 l\left(1 - \frac{x}{l}\right)\sin\theta.$$

Bending of cylindrical shell by moment on the end. In this instance differential equations (5.35) will have the form

$$\frac{\partial N_\varphi}{\partial x} + \frac{\partial N_{\theta\varphi}}{R\,\partial\theta} = 0, \qquad \frac{\partial N_{\theta\varphi}}{\partial x} + \frac{\partial N_\theta}{R\,\partial\theta} = 0, \qquad N_\theta = 0. \tag{5.47}$$

From the second equation (5.47)

$$\frac{\partial N_{\theta\varphi}}{\partial x} = 0, \qquad N_{\theta\varphi} = C_1(\theta),$$

where $C_1(\theta)$ – unknown function of angle θ. Having substituted the found value fo $N_{\theta\phi}$ in the first equation, we obtain

$$\frac{\partial N_\varphi}{\partial x}+\frac{\partial C_1(\theta)}{R\partial\theta}=0. \qquad (5.48)$$

Expression for force N_ϕ can be obtained from the formula known from the course of strength of materials

$$N_\varphi=\sigma_\varphi\delta=\frac{M\delta y}{J},$$

where δ – thickness of shell; J – moment of inertia of the cross section of shell; y – distance from horizontal diameter to the considered line.

For moment of inertia J we have expression

$$J=\int_0^{2\pi} dF y^2 = R^3\delta\int_0^{2\pi}\cos^2\theta d\theta=\pi R^3\delta,$$

$$y=R\cos\theta,$$

$$dF=\delta ds=\delta R d\theta.$$

Then

$$N_\varphi=\frac{M\delta R}{\pi R^3\delta}\cos\theta=\frac{M\cos\theta}{\pi R^2}. \qquad (5.49)$$

From this expression it is evident that normal stresses in the cross section of the shell are distributed according to cosine law.

In case of loading of the shell by moment (Fig. 74), the normal stresses do not depend on longitudinal coordinate x. Therefore, in (5.48) one should assume

$$\frac{\partial N_\varphi}{\partial x}=0.$$

Then

$$\frac{\partial C_1(\theta)}{R\partial\theta}=0,$$

whence

$$C_1(\theta)=C_2=\text{const}$$

$$N_{\theta\varphi}=C_2.$$

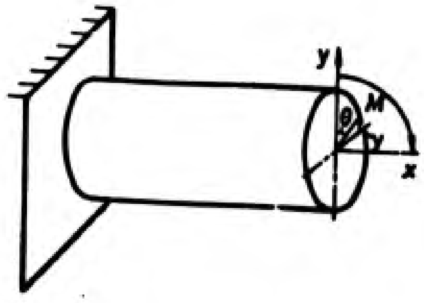

Fig. 74.

Constant of integrations C_2 must be equal to zero, since torque is not applied to the shell.

Consequently, with loading of the shell by pure bending only normal forces N_ϕ appear in it, determined by formula (5.49).

Loading of cylindrical shell by lateral force. In this case we obtain the expressions for internal forces from equations (5.47). From the second equation of this system

$$N_{\theta\varphi}=C_1(\theta).$$

Furthermore, for meridion forces formula (5.49) remains valid, in which the expression of bending moment through force Q should be substituted (Fig. 75)

$$M=Q(l-x).$$

Then

$$N_\varphi=\frac{Q(l-x)\cos\theta}{\pi R^2}.$$

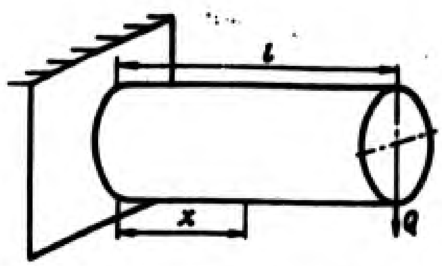

Fig. 75.

In this case equation (5.48) takes the form

$$-\frac{Q\cos\theta}{\pi R^2}+\frac{\partial C_1(\theta)}{R\partial\theta}=0,$$

whence we obtain the expression for $C_1(\theta)$:

$$C_1(\theta)=\frac{Q\sin\theta}{\pi R}+C_3.$$

Consequently

$$N_{\theta\varphi}=C_1(\theta)=\frac{Q\sin\theta}{\pi R}+C_3.$$

Constant C_3 in this case should be assumed equal to zero, since external torsional moment is not applied to the considered shell. Therefore, for tangential forces in a cylindrical shell, loaded by lateral force on the end, we have formula

$$N_{\theta\varphi}=\frac{Q\sin\theta}{\pi R}.$$

Distribution of normal N_ϕ and tangential forces $N_{\theta\phi}$ along the cross section of the shell is shown in Fig. 76.

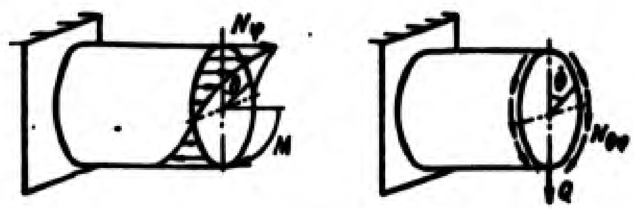

Fig. 76.

Stressed state of a cylindrical shell supported on the ends from the weight of liquid poured in it. The scheme of loading is shown in Fig. 77. Original equations (5.35) in this instance will have the form

$$\frac{\partial N_\varphi}{\partial x}+\frac{\partial N_{\theta\varphi}}{R\partial\theta}=0,\qquad \frac{\partial N_{\theta\varphi}}{\partial x}+\frac{\partial N_\theta}{R\partial\theta}=0,\qquad N_\theta=-ZR, \tag{5.50}$$

where

$$Z=-\gamma h=-\gamma R(\cos\theta-\cos\theta_0).$$

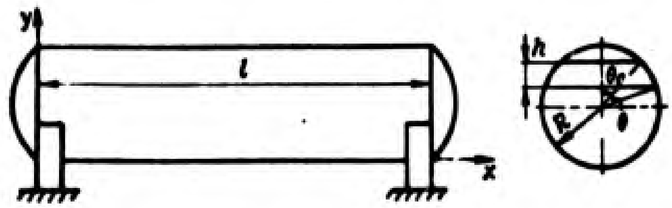

Fig. 77.

From the last equation follows

$$N_\theta = \gamma R^2 (\cos\theta - \cos\theta_0).$$

Then the second equation is reduced to the form

$$\frac{\partial N_{\theta\varphi}}{\partial x} = \gamma R \sin\theta.$$

As a result of integration of this equation we have

$$N_{\theta\varphi} = \gamma R x \sin\theta + C_1(\theta).$$

Let us substitute the found value of $N_{\theta\phi}$ in the first equation (5.50):

$$\frac{\partial N_\varphi}{\partial x} = -\gamma x \cos\theta - \frac{\partial C_1(\theta)}{R\partial\theta}.$$

After integration with respect to x we obtain

$$N_\varphi = -\frac{1}{2}\gamma x^2 \cos\theta - x\frac{\partial C_1(\theta)}{R\partial\theta} + C_2(\theta).$$

Unknown functions $C_1(\theta)$ and $C_2(\theta)$ are found from the following boundary conditions:

$$N_\varphi = 0 \text{ when } \begin{matrix} x=0, \\ x=l. \end{matrix}$$

Then

$$C_2(\theta) = 0, \quad -\frac{1}{2}\gamma l \cos\theta - \frac{\partial C_1(\theta)}{R\partial\theta} = 0,$$

$$\frac{\partial C_1(\theta)}{\partial\theta} = -\frac{1}{2}\gamma Rl\cos\theta.$$

By integrating, we obtain

$$C_1(\theta) = -\frac{1}{2}\gamma Rl\sin\theta + C.$$

Then for tangential force $N_{\theta\phi}$ we will have expression

$$N_{\theta\varphi} = \gamma Rx\sin\theta - \frac{1}{2}\gamma Rl\sin\theta + C.$$

Constant C should be assumed euqal to zero, since tangential forces are not applied to ends of the shell.

Then finally we obtain the following formulas:

$$N_{\varphi} = -\frac{1}{2}\gamma lx\left(1 - \frac{x}{l}\right)\cos\theta,$$
$$N_{\theta} = \gamma R^2(\cos\theta^{*} - \cos\theta_0),$$
$$N_{\theta\varphi} = \gamma Rl\left(\frac{x}{l} - \frac{1}{2}\right)\sin\theta.$$

§ 24. Application of Castigliano Theorem for Problems of Determination of Displacements in Shells

In certain cases for determination of displacements in shells the Castigliano theorem can be useful. According to this theorem displacement, corresponding to the given generalized force factor, is equal to the partial derivative of potential energy in terms of given generalized force:

$$\delta_p = \frac{dЭ}{dP}$$

In this case the potential energy of the given elastic system must be expressed as a function of external forces.

In the case of momentless shell the potential strain energy

$$Э=\frac{1}{2}\iint_F (N_\varphi \varepsilon_\varphi + N_\theta \varepsilon_\theta + N_{\theta\varphi} \varepsilon_{\theta\varphi})\,dF.$$

By substituting here the deformation components through forces according to Hooke law, we obtain

$$Э=\frac{1}{2E\delta}\iint_F [N_\varphi^2 + N_\theta^2 - 2\mu N_\theta N_\varphi + 2(1+\mu) N_{\theta\varphi}^2]\,dF. \tag{5.51}$$

Let us examine some problems, which have practical interest.

Displacement of the point of application of radial concentrated force, acting on the spherical segment. Let us assume the shell in the form of a spherical segment is affected by concentrated force P, applied through a rigid washer. Then from the condition of equilibrium of the upper part of the segment (Fig. 78), determined by angle $\psi + \phi_0$, we obtain

$$N_\varphi = -\frac{P}{2\pi R \sin^2(\varphi_0+\psi)}.$$

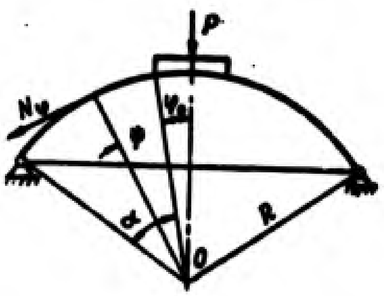

Fig. 78.

The expression for force N_θ we obtain from Laplace equation

$$N_\theta = -N_\varphi = \frac{P}{2\pi R \sin^2(\varphi_0+\psi)}.$$

In view of the symmetry of loading the tangential forces in shell $N_{\theta\phi} = 0$.

For strain energy we obtain expression

$$Э=\frac{(1+\mu)P^2}{4\pi^2E\delta}\int\limits_0^{\alpha}\int\limits_0^{2\pi}\frac{d\psi d\theta}{\sin^3(\varphi_0+\psi)}.$$

After integration we find

$$Э=\frac{(1+\mu)P^2}{4\pi E\delta}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\alpha}{2}}{\operatorname{tg}\frac{\varphi_0}{2}}+\frac{\operatorname{ctg}\varphi_0}{\sin\varphi_0}-\frac{\operatorname{ctg}(\varphi_0+\alpha)}{\sin(\varphi_0+\alpha)}\right].$$

By using the Castigliano theorem, we obtain displacement of the point of application of force P:

$$\delta_p=\frac{dЭ}{dP}=\frac{(1+\mu)P}{2\pi E\delta}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\alpha}{2}}{\operatorname{tg}\frac{\varphi_0}{2}}+\frac{\operatorname{ctg}\varphi_0}{\sin\varphi_0}-\frac{\operatorname{ctg}(\varphi_0+\alpha)}{\sin(\varphi_0+\alpha)}\right].$$

Determination of angle of rotation of a rigid washer, fastened to the spherical segment and loaded by bending moment. For determination of internal elastic forces N_ϕ, N_θ, $N_{\theta\phi}$ we have the following differential equations:

$$\frac{\partial}{\partial\psi}[N_\varphi\sin(\varphi_0+\psi)]+\frac{\partial N_{\theta\varphi}}{\partial\theta}-N_\theta\cos(\varphi_0+\psi)=0,$$
$$\frac{\partial}{\partial\psi}[N_{\theta\varphi}\sin(\varphi_0+\psi)]+\frac{\partial N_\theta}{\partial\theta}+N_{\theta\varphi}\cos(\varphi_0+\psi)=0,$$
$$N_\theta+N_\varphi=0,$$

obtained from equations (5.35) by replacement of ϕ by $\phi_0+\psi$.

After exclusion of force N_θ from these equations we will have

$$\frac{\partial N_\varphi}{\partial\psi}+2N_\varphi\operatorname{ctg}(\varphi_0+\psi)+\frac{1}{\sin(\varphi_0+\psi)}\frac{\partial N_{\theta\varphi}}{\partial\theta}=0,$$
$$\frac{\partial N_{\theta\varphi}}{\partial\psi}+2N_{\theta\varphi}\operatorname{ctg}(\varphi_0+\psi)-\frac{1}{\sin(\varphi_0+\psi)}\frac{\partial N_\varphi}{\partial\theta}=0.$$

Solution of these equations can be sought in the form

$$N_\varphi=S_\varphi\cos\theta,\quad N_{\theta\varphi}=S_{\theta\varphi}\sin\theta.$$

Then

$$\frac{dS_\varphi}{d\psi} + 2S_\varphi \operatorname{ctg}(\varphi_0+\psi) + \frac{1}{\sin(\varphi_0+\psi)} S_{\theta\varphi} = 0,$$

$$\frac{dS_{\theta\varphi}}{d\psi} + 2S_{\theta\varphi} \operatorname{ctg}(\varphi_0+\psi) + \frac{1}{\sin(\varphi_0+\psi)} S_\varphi = 0.$$

First let us sum up both these equations, and then subtract the second from the first. In this case we obtain

$$\frac{dU_1}{d\psi} + \left[2\operatorname{ctg}(\varphi_0+\psi) + \frac{1}{\sin(\varphi_0+\psi)}\right] U_1 = 0,$$

$$\frac{dU_2}{d\psi} + \left[2\operatorname{ctg}(\varphi_0+\psi) - \frac{1}{\sin(\varphi_0+\psi)}\right] U_2 = 0,$$

where there is designated

$$U_1 = S_\varphi + S_{\theta\varphi}, \quad U_2 = S_\varphi - S_{\theta\varphi}.$$

The obtained equations allow separation of variables and therefore are integrated simply:

$$\ln U_1 = \ln C_1 - \int\left[2\operatorname{ctg}(\varphi_0+\psi) + \frac{1}{\sin(\varphi_0+\psi)}\right] d\psi,$$

$$\ln U_2 = \ln C_1 - \int\left[2\operatorname{ctg}(\varphi_0+\psi) - \frac{1}{\sin(\varphi_0+\psi)}\right] d\psi.$$

After integration of these equations we obtain

$$U_1 = \frac{C_1}{\sin^2(\varphi_0+\psi)\operatorname{tg}\frac{\varphi_0+\psi}{2}},$$

$$U_2 = \frac{C_2 \operatorname{tg}\frac{\varphi_0+\varphi}{2}}{\sin^2(\varphi_0+\psi)}.$$

Now it is possible to write the expressions for internal forces

$$N_\varphi = \frac{\cos\theta}{2\sin^2(\varphi_0+\psi)}\left(\frac{C_1}{\operatorname{tg}\frac{\varphi_0+\psi}{2}} + C_2 \operatorname{tg}\frac{\varphi_0+\psi}{2}\right),$$

$$N_{\theta\varphi} = \frac{\sin\theta}{2\sin^2(\varphi_0+\psi)}\left(\frac{C_1}{\operatorname{tg}\frac{\varphi_0+\psi}{2}} - C_2 \operatorname{tg}\frac{\varphi_0+\psi}{2}\right),$$

$$N_\theta = -N_\varphi. \qquad (5.52)$$

the

For determination of constants C_1 and C_2 let us use conditions:

1) moment of forces N_ϕ when $\psi = \alpha$ (Fig. 79) relative to the axis perpendicular to the drawing and passing through point A, is equal to assigned moment:

$$M = \int_0^{2\pi} [N_\varphi ds \sin(\varphi_0+\psi) R \sin(\varphi_0+\psi)\cos\theta]_{\psi=\alpha};$$

2) the sum of projections of forces $N_{\theta\phi}$ and N_ϕ when $\psi = \alpha$ to the line of intersection of the plane of action of the moment with the plane of base of the segment must be equal to zero:

erefore

$$\int_0^{2\pi} (N_\varphi)_{\psi=\alpha} ds \cos(\varphi_0+\alpha)\cos\theta - \int_0^{2\pi} (N_{\theta\varphi})_{\psi=\alpha} ds \sin\theta = 0.$$

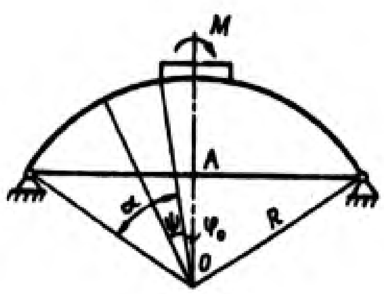

Fig. 79.

From these conditions it follows that

$$C_1 = C_2 = \frac{M}{\pi R^2}.$$

Then forces N_ϕ, N_θ, $N_{\theta\phi}$ are determined according to expressions

$$N_\varphi = \frac{M\cos\theta}{2\pi R^2 \sin^2(\varphi_0+\psi)}\left(\frac{1}{\mathrm{tg}\frac{\varphi_0+\psi}{2}} + \mathrm{tg}\frac{\varphi_0+\psi}{2}\right),$$

$$N_\theta = -N_\varphi,$$

$$N_{\theta\varphi} = \frac{M\sin\theta}{2\pi R^2 \sin^2(\varphi_0+\psi)}\left(\frac{1}{\mathrm{tg}\frac{\varphi_0+\psi}{2}} - \mathrm{tg}\frac{\varphi_0+\psi}{2}\right). \qquad (5.53)$$

Now it is possible to calculate the potential strain energy of the considered segment with its loading by bending moment:

$$Э = \frac{1}{2E\delta}\int_0^{2\pi}\int_0^{\alpha}\left[N_\varphi^2 + N_\theta^2 - 2\mu N_\varphi N_\theta + 2(1+\mu)N_{\theta\varphi}^2\right]\times$$
$$\times R^2\sin(\varphi_0+\psi)\,d\psi\,d\theta.$$

Having substituted here the expressions (5.53) for forces N_ϕ, N_θ, $N_{\theta\phi}$ and integrated within the shown limits, we obtain

$$Э = \frac{(1+\mu)FM^2}{2\pi R^2 E\delta}.$$

Hence according to the Castigliano theorem we find the angle of rotation of the washer:

$$\gamma = \frac{dЭ}{dM} = \frac{(1+\mu)FM}{\pi R^2 E\delta}.$$

Here there is designated

$$F = \frac{\mathrm{ctg}\,\varphi_0}{\sin^3\varphi_0} + \frac{\mathrm{ctg}\,\varphi_0}{2\sin\varphi_0} + \frac{1}{2}\ln\frac{\mathrm{tg}\frac{\varphi_0+\alpha}{2}}{\mathrm{tg}\frac{\varphi_0}{2}} - \frac{\mathrm{ctg}(\varphi_0+\alpha)}{\sin^3(\varphi_0+\alpha)} - \frac{\mathrm{ctg}(\varphi_0+\alpha)}{2\sin(\varphi_0+\alpha)}.$$

If inside the spherical segment there acts internal pressure with intensity q, for determination of forces N_ϕ and N_θ there are obtained expressions:

$$N_\varphi = \frac{qR}{2} + \frac{M\cos\theta}{2\pi R^2\sin^2(\varphi_0+\psi)}\left(\frac{1}{\mathrm{tg}\frac{\varphi_0+\psi}{2}} + \mathrm{tg}\frac{\varphi_0+\psi}{2}\right),$$

$$N_\theta = \frac{qR}{2} - \frac{M\cos\theta}{2\pi R^2\sin^2(\varphi_0+\psi)}\left(\frac{1}{\mathrm{tg}\frac{\varphi_0+\psi}{2}} + \mathrm{tg}\frac{\varphi_0+\psi}{2}\right).$$

By substituting these values for forces and the value for $N_{\theta\phi}$ (5.53) in expression (5.51), we can be certain that internal pressure in the segment does not affect the angle of rotation, appearing from the effect of moment M. Such a result is obtained due to the fact that the momentless state of the shell is considered.

Determination of the angle of rotation of a rigid washer, fastened to spherical segment and loaded by twisting moment. From the condition of equilibrium of the part of the segment lying above the parallel circle, determined by angle $\phi_0 + \psi$ (Fig. 80), we find

$$N_{\theta\varphi} = \frac{M_{\kappa p}}{2\pi R^2 \sin^2(\varphi_0+\psi)}.$$

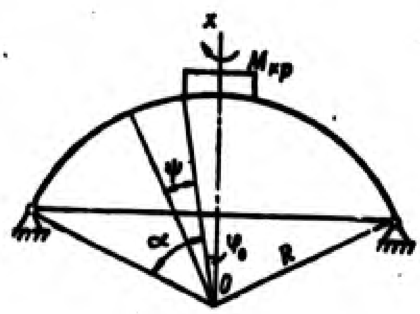

Fig. 80.

In this case the remaining forces of momentless state in the shell are equal to zero.

The expression for potential strain energy will have the form

$$Э = \frac{1+\mu}{E\delta}\int_0^{\alpha}\int_0^{2\pi} N_{\theta\varphi}^2\, dF.$$

After substitution here of values of $N_{\theta\phi}$ and $dF = R^2\sin^2(\varphi_0+\psi)\,d\psi\, d\theta$ and integration within the indicated limits we obtain

$$Э = \frac{(1+\mu) M_{\kappa p}^2}{4\pi R^2 E\delta}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\alpha}{2}}{\operatorname{tg}\frac{\varphi_0}{2}} + \frac{\operatorname{ctg}\varphi_0}{\sin\varphi_0} - \frac{\operatorname{ctg}(\varphi_0+\alpha)}{\sin(\varphi_0+\alpha)}\right].$$

Angle of rotation of the washer around axis Ox will be

$$\vartheta=\frac{d\mathfrak{Э}}{dM_{\kappa p}}=\frac{(1+\mu)M_{\kappa p}}{2\pi R^2E\delta}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\alpha}{2}}{\operatorname{tg}\frac{\varphi_0}{2}}+\frac{\operatorname{ctg}\varphi_0}{\sin\varphi_0}-\frac{\operatorname{ctg}(\varphi_0+\alpha)}{\sin(\varphi_0+\alpha)}\right].$$

Displacement of point of application of tangential concentrated force, applied to rigid washer, which is fastened to a spherical segment at its polar part. This problem can be solved by using the results obtained with loading of the spherical segment by a bending moment.

$$N_\varphi=S_\varphi\cos\theta,\quad N_{\theta\varphi}=S_{\theta\varphi}\sin\theta.$$

Consequently, for forces N_ϕ, N_θ, $N_{\theta\phi}$ it is possible to use formulas (5.52).

For determination of constants of integration in this case we have the following conditions:

1) moment of forces N_ϕ when $\psi = \alpha$ with respect to axes passing through the point of intersection of vertical diameter of the segment with plane of its base will be equal to the moment of force P relative to the same axis:

$$PR[\cos\varphi_0-\cos(\varphi_0+\alpha)]=\int_0^{2\pi}(N_\varphi)_{\psi=\alpha}\,ds\sin(\varphi_0+\alpha)\,r\cos\theta;$$

2) resultant of projections of elastic forces N_ϕ and $N_{\theta\phi}$ when $\psi = \alpha$ to the direction of force P is equivalent to this force:

$$\int_0^{2\pi}(N_\varphi)_{\psi=\alpha}\,ds\cos(\varphi_0+\alpha)\cos\theta-\int_0^{2\pi}(N_{\theta\varphi})_{\psi=\alpha}\,ds\sin\theta=P.$$

By substituting here $ds=R\sin(\psi_0+\alpha)$ and $r=R\sin(\varphi_0+\alpha)$, after simple calculations for constants of integration we obtain the following values:

$$C_1=-\frac{2P}{\pi R}\sin^2\frac{\varphi_0}{2},\qquad C_2=\frac{2P}{\pi R}\cos^2\frac{\varphi_0}{2}.$$

By substituting the found values of constants of integration in formulas (5.52), we obtain the following formulas for internal forces in the spherical segment with its loading by force acting in the plane of the washer:

$$N_{\varphi}=\frac{P\cos\theta}{\pi R\sin^{2}(\varphi_{0}+\psi)}\left(\cos^{2}\frac{\varphi_{0}}{2}\,\mathrm{tg}\,\frac{\varphi_{0}+\psi}{2}-\frac{\sin^{2}\frac{\varphi_{0}}{2}}{\mathrm{tg}\,\frac{\varphi_{0}+\psi}{2}}\right),$$

$$N_{\theta\varphi}=-\frac{P\sin\theta}{\pi R\sin^{2}(\varphi_{0}+\psi)}\left(\frac{\sin^{2}\frac{\varphi_{0}}{2}}{\mathrm{tg}\,\frac{\varphi_{0}+\psi}{2}}+\cos^{2}\frac{\varphi_{0}}{2}\,\mathrm{tg}\,\frac{\varphi_{0}+\psi}{2}\right),$$

$$N_{\theta}=-N_{\varphi}.$$

Now it is possible to calculate the potential energy of deformation:

$$Э=\frac{(1+\mu)R^{2}}{E\delta}\int_{0}^{\alpha}\int_{0}^{2\pi}(N_{\varphi}^{2}+N_{\theta\varphi}^{2})\sin(\varphi_{0}+\psi)\,d\psi\,d\theta.$$

After substitution here of values for N_{ϕ} and $N_{\theta\phi}$ and calculations of integrals we obtain the following expression for energy:

$$Э=\frac{(1+\mu)KP^{2}}{\pi E\delta}.$$

Displacement in the direction of force P will be

$$\delta_{P}=\frac{2(1+\mu)KP}{\pi E\delta}.$$

Here is designated

$$K=\frac{1}{2}(1+\cos^{2}\varphi_{0})\left[\frac{\mathrm{ctg}\,\varphi_{0}}{\sin^{3}\varphi_{0}}+\frac{\mathrm{ctg}\,\varphi_{0}}{2\sin\varphi_{0}}+\frac{1}{2}\ln\frac{\mathrm{tg}\,\frac{\varphi_{0}+\alpha}{2}}{\mathrm{tg}\,\frac{\varphi_{0}}{2}}-\frac{\mathrm{ctg}(\varphi_{0}+\alpha)}{\sin^{3}(\varphi_{0}+\alpha)}-\frac{\mathrm{ctg}(\varphi_{0}+\alpha)}{2\sin(\varphi_{0}+\alpha)}\right]+\left[\frac{1}{\sin^{4}(\varphi_{0}+\alpha)}-\frac{1}{\sin^{4}\varphi_{0}}\right]\cos\varphi_{0}.$$

The remaining designations are clear from Fig. 81.

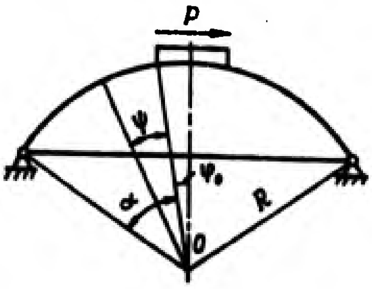

Fig. 81.

Linear and angular displacements of the rigid washer, fastened to spherical segment and loaded by bending moment and tangential force. In this instance the expressions for internal forces can be obtained by superposition of solutions, obtained during loading of the segment by force P and moment M (Fig. 82):

$$\left.\begin{aligned}
N_\varphi = {} & \frac{M\cos\theta}{2\pi R^2\sin^2(\varphi_0+\psi)}\left(\frac{1}{\operatorname{tg}\frac{\varphi_0+\psi}{2}}+\operatorname{tg}\frac{\varphi_0+\psi}{2}\right)- \\
& -\frac{P\cos\theta}{\pi R\sin^2(\varphi_0+\psi)}\left(\frac{\sin^2\frac{\varphi_0}{2}}{\operatorname{tg}\frac{\varphi_0+\psi}{2}}-\cos^2\frac{\varphi_0}{2}\operatorname{tg}\frac{\varphi_0+\psi}{2}\right), \\
N_{\theta\varphi} = {} & \frac{M\sin\theta}{2\pi R^2\sin^2(\varphi_0+\psi)}\left(\frac{1}{\operatorname{tg}\frac{\varphi_0+\psi}{2}}-\operatorname{tg}\frac{\varphi_0+\psi}{2}\right)- \\
& -\frac{P\sin\theta}{\pi R\sin^2(\varphi_0+\psi)}\left(\frac{\sin^2\frac{\varphi_0}{2}}{\operatorname{tg}\frac{\varphi_0+\psi}{2}}+\cos^2\frac{\varphi_0}{2}\operatorname{tg}\frac{\varphi_0+\psi}{2}\right), \\
N_\theta = {} & -N_\varphi.
\end{aligned}\right\} \quad (5.54)$$

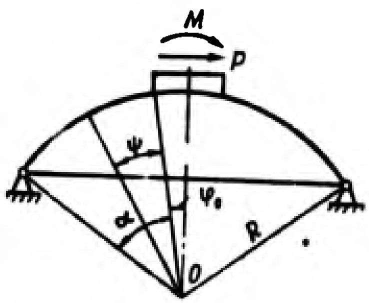

Fig. 82.

For potential strain energy we have expression

$$Э=\frac{(1+\mu)R^2}{E\delta}\int_0^{\alpha}\int_0^{2\pi}(N_{\varphi}^2+N_{\theta\varphi}^2)\sin(\varphi_0+\psi)\,d\psi\,d\theta.$$

After substitution of magnitudes of forces in the expression for Э [see (5.54)] and computation of integrals we obtain

$$Э=\frac{(1+\mu)}{4\pi R^2E\delta}\left[\left(M-2PR\sin^2\frac{\varphi_0}{2}\right)^2(F_1-F_2)+\right.$$
$$\left.+\left(M+2PR\cos^2\frac{\varphi_0}{2}\right)^2(F_1+F_2)\right].$$

For determination of displacement in the corresponding direction the partial derivatives of potential energy must be taken with respect to force P or M.

Displacement in the direction of force P

$$\delta_p=\frac{\partial Э}{\partial P}=\frac{1+\mu}{\pi R^2E\delta}\left\{M(F_1\cos\varphi_0+F_2)+\right.$$
$$\left.+2PR\left[\frac{1}{2}F_1(1+\cos^2\varphi_0)+F_2\cos\varphi_0\right]\right\}.$$

Angle of rotation in the direction of moment M will be

$$\vartheta=\frac{\partial Э}{\partial M}=\frac{1+\mu}{\pi R^2E\delta}[MF_1+PR(F_1\cos\varphi_0+F_2)].$$

In the given formulas there is designated:

$$F_1=\frac{\operatorname{ctg}\varphi_0}{\sin^3\varphi_0}+\frac{\operatorname{ctg}\varphi_0}{2\sin\varphi_0}+\frac{1}{2}\ln\frac{\operatorname{tg}\frac{\varphi_0+\psi}{2}}{\operatorname{tg}\frac{\varphi_0}{2}}-$$
$$-\frac{\operatorname{ctg}(\varphi_0-\alpha)}{\sin^3(\varphi_0+\alpha)}-\frac{\operatorname{ctg}(\varphi_0+\alpha)}{2\sin(\varphi_0+\alpha)},$$
$$F_2=\frac{1}{\sin^4(\varphi_0+\alpha)}-\frac{1}{\sin^4\varphi_0}.$$

C H A P T E R VI

SHELLS, LOADED BY LOCAL AXISYMMETRICAL LINEAR LOAD

Formulas given in the previous chapter, obtained according to momentless theory, accurately determine the stressed and deformed state of thin-walled shells in zones where the load changes smoothly. If the load undergoes discontinuity, i.e., it changes unevenly, then in these sections bending of the shell will occur. Bending of the shell also appears when the cross section is changed unevenly, and also at places of joining of shells of different geometrical shape.

In the shown cases bending will carry a local character, and the area of its distribution will be comparatively small. For these sections the formulas of momentless theory listed in Chapter V seem inadequate. It is necessary to examine bending of a shell and sum up stresses from bending with stresses of momentless state. However, here we should specify that the supporting power of shell in many instances will be determined by the momentless stress condition, and local bending will not play a substantial role. Therefore, in practical calculations the bending stresses are frequently not determined and the entire calculation is performed according to momentless theory.

However, without knowledge of the fundamentals of moment theory in certain cases it is impossible to correctly understand the work of the construction and to solve problems of designing. Therefore, in the following paragraphs we will give basic prerequisites for calculation of shells at local bending.

§ 25. Differential Equations of the Edge Effect with Axisymmetrical Deformation of the Shell

Let us examine a shell of revolution, loaded, as shown in Fig. 83, by force Q_0 and moment M_0.

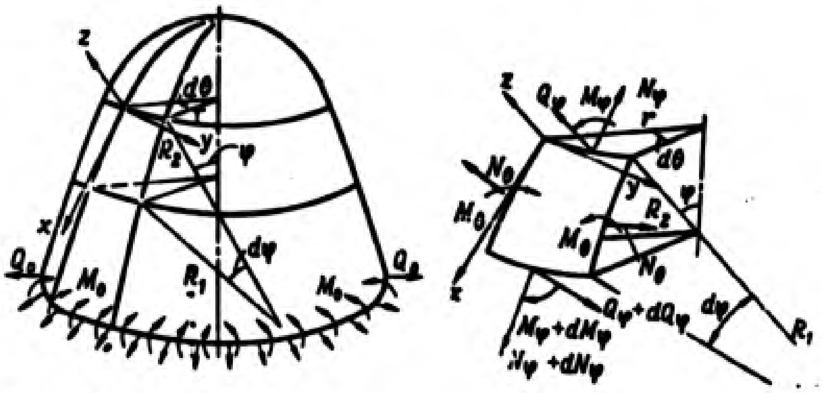

Fig. 83.

From this shell let us cut out an infinitely small element $Rd\varphi rd\theta$, let us apply to its edges as yet unknown to us internal elastic forces and let us formulate equations of its equilibrium.

For equilibrium of a body in space it is necessary to have six equations of statics. In this case from these six equations only three will remain. The remaining three equations will be identically satisfied.

Having formulated the sum of moments of all forces, acting on the element around axis y, we obtain

$$\frac{d}{d\varphi}(R_2 M_\varphi \sin\varphi) - Q_\varphi R_1 R_2 \sin\varphi - M_\theta R_1 \cos\varphi = 0. \qquad (6.1)$$

From condition of equilibrium of forces, acting on the element, in the direction of axis z we will have

$$\frac{d}{d\varphi}(R_2 Q_\varphi \sin\varphi) + N_\theta R_1 \sin\varphi + N_\varphi R_2 \sin\varphi = 0. \qquad (6.2)$$

To get the third equation let us examine the equilibrium of the cutoff part of the shell at angle ϕ (Fig. 84). Let us project linear forces N_ϕ and Q_ϕ to the vertical:

$$\int_0^{2\pi} N_\varphi r\, d\theta \sin\varphi + \int_0^{2\pi} Q_\varphi r\, d\theta \cos\varphi = 0.$$

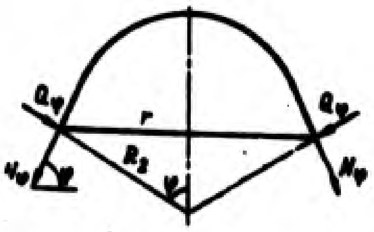

Fig. 84.

Hence for N_ϕ we obtain expression

$$N_\varphi = -Q_\varphi \operatorname{ctg}\varphi. \quad (6.3)$$

Thus, we have three equations, which include five unknown internal forces N_φ, N_θ, Q_φ, M_φ, M_θ.

Having excluded force N_ϕ from equation (6.2) with the aid of expression (6.3) we obtain

$$\frac{d}{d\varphi}(R_2 Q_\varphi \sin\varphi) + N_\theta R_1 \sin\varphi - Q_\varphi R_2 \cos\varphi = 0. \quad (6.4)$$

Let us designate the product of R_2Q_φ in the form of $V = R_2Q_\varphi$. Then equation (6.4) can be written in the form

$$\frac{d}{d\varphi}(V \sin\varphi) + N_\theta R_1 \sin\varphi - V\cos\varphi = 0.$$

After differentiation from this equation we obtain

$$N_\theta = -\frac{dV}{R_1\, d\varphi}.$$

In this case equations (6.1) and (6.3) will have the form

$$N_\varphi = -\frac{V}{R_2}\operatorname{ctg}\varphi,$$

$$\frac{d}{d\varphi}(R_2 M_\varphi \sin\varphi) - VR_1 \sin\varphi - M_\theta R_1 \cos\varphi = 0. \quad (6.5)$$

By having expressions for N_θ and N_ϕ and using Hooke law, we obtain

$$\varepsilon_\varphi=\frac{1}{E\delta}(N_\varphi-\mu N_\theta),\quad \varepsilon_\theta=\frac{1}{E\delta}(N_\theta-\mu N_\varphi).$$

Furthermore, for deformation components we obtained formulas (5.32) and (5.33):

$$\varepsilon_\varphi=\frac{du}{R_1\,d\varphi}+\frac{w}{R_1},$$
$$\varepsilon_\theta=\frac{u}{R_2}\operatorname{ctg}\varphi+\frac{w}{R_2}.$$

Having substituted these expressions for ε_ϕ and ε_θ in formulas of Hooke law, we obtain

$$w+\frac{du}{d\varphi}=\frac{R_1}{E\delta}(N_\varphi-\mu N_\theta),$$
$$w+u\operatorname{ctg}\varphi=\frac{R_2}{E\delta}(N_\theta-\mu N_\varphi). \tag{6.6}$$

By subtracting the second equation from the first, we find

$$\frac{du}{d\varphi}-u\operatorname{ctg}\varphi=\frac{R_1}{E\delta}(N_\varphi-\mu N_\theta)-\frac{R_2}{E\delta}(N_\theta-\mu N_\varphi). \tag{6.7}$$

Let us differentiate the second expression (6.6):

$$\frac{dw}{d\varphi}+\frac{du}{d\varphi}\operatorname{ctg}\varphi-\frac{u}{\sin^2\varphi}=\frac{1}{E\delta}\frac{d}{d\varphi}[R_2(N_\theta-\mu N_\varphi)].$$

By excluding derivative $\frac{du}{d\varphi}$ from this expression with the aid of relationship (6.7), we obtain

$$-\left(u-\frac{dw}{d\varphi}\right)=\frac{1}{E\delta}\frac{d}{d\varphi}[R_2(N_\theta-\mu N_\varphi)]-$$
$$-\frac{1}{E\delta}[R_1(N_\varphi-\mu N_\theta)-R_2(N_\theta-\mu N_\varphi)]\operatorname{ctg}\varphi. \tag{6.8}$$

Let us designate

$$u-\frac{dw}{d\varphi}=R_1U. \tag{6.9}$$

where U - new unknown function, expressing the angle of rotation of the section of shell.

Then equation (6.8) after substitution of N_ϕ and N_θ in it finally obtains the form

$$\frac{R_2}{R_1}\frac{d^2V}{d\varphi^2}+\left[\frac{d}{d\varphi}\left(\frac{R_2}{R_1}\right)+\frac{R_2}{R_1}\operatorname{ctg}\varphi\right]\frac{dV}{d\varphi}+\left(\mu-\frac{R_1}{R_2}\operatorname{ctg}^2\varphi\right)V=E\delta R_1 U. \qquad (6.10)$$

Thus, instead of equations (6.2) and (6.3) we obtain one equation (6.10) with two unknown functions V and U. To get one more equation, which connects these functions, one could use equation (6.5). Let us preliminarily write down the expressions for moments through changes of curvature

$$M_\varphi=-D(\chi_\varphi+\mu\chi_\theta),$$
$$M_\theta=-D(\chi_\theta+\mu\chi_\varphi).$$

Let us express the components of change of curvature through components of displacements u and w.[1]

In our case the contiguity angle between tangents, drawn through points 1 and 2 (Fig. 85), in the position after deformation will be equal to the difference of angles of rotation of the sections passing through these points:

$$\left[\left(\frac{u}{R_1}-\frac{dw}{R_1 d\varphi}\right)+d\left(\frac{u}{R_1}-\frac{dw}{R_1 d\varphi}\right)\right]-\left(\frac{u}{R_1}-\frac{dw}{R_1 d\varphi}\right)=d\left(\frac{u}{R_1}-\frac{dw}{R_1 d\varphi}\right).$$

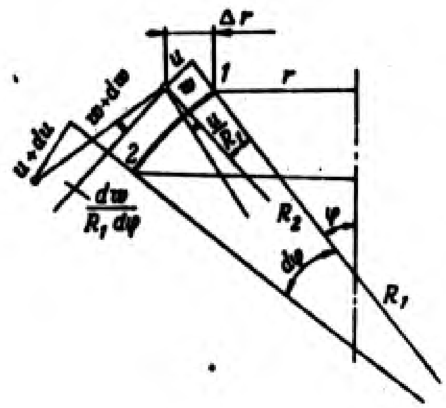

Fig. 85.

[1]By curvature of the curve we mean the relationship of contiguity angle to the length of arc, when the latter approaches zero.

Considering designation (6.9), we find that the contiguity angle will be equal to dU. Therefore, for change of curvature of meridian we obtain formula

$$\chi_\varphi = \frac{dU}{R_1\,d\varphi}. \qquad (6.11)$$

The expression for curvature will be

$$\chi_\theta = \frac{1}{R_2'} - \frac{1}{R_2}.$$

From Fig. 85 we have

$$R_2 = \frac{r}{\sin\varphi}, \quad R_2' = \frac{r+\Delta r}{\sin(\varphi+U)}.$$

Consequently,

$$\chi_\theta = \frac{\sin(\varphi+U)}{r+\Delta r} - \frac{\sin\varphi}{r} \approx \frac{\sin(\varphi+U)}{r} - \frac{\sin\varphi}{r} =$$
$$= \frac{\sin\varphi + U\cos\varphi}{r} - \frac{\sin\varphi}{r} = \frac{U\cos\varphi}{r} = \frac{U}{R_2}\operatorname{ctg}\varphi.$$

Then the expressions for moments will take the form

$$M_\varphi = -D\left(\frac{dU}{R_1\,d\varphi} + \mu\frac{U}{R_2}\operatorname{ctg}\varphi\right),$$
$$M_\theta = -D\left(\frac{U}{R_2}\operatorname{ctg}\varphi + \mu\frac{dU}{R_1\,d\varphi}\right).$$

Having substituted these expressions for moments in (6.5), we obtain

$$\frac{R_2}{R_1}\frac{d^2U}{d\varphi^2} + \underline{\left[\frac{d}{d\varphi}\left(\frac{R_2}{R_1}\right) + \frac{R_2}{R_1}\operatorname{ctg}\varphi\right]\frac{dU}{d\varphi} - \left(\mu + \frac{R_1}{R_2}\operatorname{ctg}^2\varphi\right)U} = -\frac{VR_1}{D}. \qquad (6.12)$$

The two obtained simultaneous equations (6.10) and (6.12) completely solve the problem of calculation of shell of revolution, loaded by edge forces Q_0 and M_0. However, in practical calculations in their complete form these equations are very rarely applied in view of the complexity of their solution. Most often it is necessary to use approximate equations of edge effect. These approximate equations are obtained from equations (6.10) and (6.12), if we discard the underlined terms in their left sides.

The basis of this simplification is formed by St. Venant principle, according to which the action of self-balanced edge radial or moment load carries a rapidly attenuating character. This is confirmed by theoretical investigations of some particular problem, in which it was revealed that the shown process in shells carries, furthermore, an oscillating character. Functions which describe such processes have form $e^{-kx}f(x)$, where $f(x)$ — a limited periodic function. Factor e^{-kx} determines the rapidity of damping; for shells the index of damping k is a large quantity. From properties of such functions it follows that their first derivative is always greater than the function itself, the second derivative is greater than the first derivative and so forth. Therefore, in the shown equations we drop the components containing the first derivative and the function itself in comparison with the second derivative.

Thus simplified edge effect equations will have the form

$$\frac{R_2}{R_1^2}\frac{d^2V}{d\varphi^2}=E\delta U,\quad \frac{R_2}{R_1^2}\frac{d^2U}{d\varphi^2}=-\frac{V}{D}. \tag{6.13}$$

Let us apply these equations to the solution of some particular problems.

§ 26. Semi-Infinite Cylindrical Shell, Loaded by Distributed Lateral Force and Moment on the End

In this instance $R_1=\infty$, $\varphi=90°$, $R_2=R$, $R_1d\varphi=dx$. Then equations (6.13) will take the form

$$\frac{d^2V}{dx^2}=\frac{E\delta}{R}U,\quad \frac{d^2U}{dx^2}=-\frac{V}{DR}.$$

By excluding, for example, function V from these equations, we obtain

$$\frac{d^4U}{dx^4}+4k^4U=0,$$

where

$$k^4=\frac{3(1-\mu^2)}{R^2\delta^2}. \tag{6.14}$$

Furthermore, in this case

$$V=-DR\frac{d^2U}{dx^2},\quad U=-\frac{dw}{dx},\quad Q_\varphi=\frac{V}{R}=-D\frac{d^2U}{dx^2},\quad N_\varphi=0,$$

$$N_\theta=DR\frac{d^3U}{dx^3},\quad M_\varphi=-D\frac{dU}{dx},\quad M_\theta=-\mu M_\varphi,\quad \varepsilon_\varphi=-\frac{\mu DR}{E\delta}\frac{d^3U}{dx^3},$$

$$\varepsilon_\theta=\frac{DR}{E\delta}\frac{d^3U}{dx^3},\quad \varepsilon_\theta=\frac{w}{R},\quad \varepsilon_\varphi=\frac{du}{dx},\quad \chi_\varphi=\frac{dU}{dx},\quad \chi_\theta=0.$$

These formula completely describe the stressed and deformed state of a cylindrical shell with its loading by edge distributed axisymmetrical load.

Solution of equation (6.14) has the form

$$U=e^{kx}(C_1\cos kx+C_2\sin kx)+e^{-kx}(C_3\cos kx+C_4\sin kx). \tag{6.15}$$

Let us apply this expression to calculation of a shell, loaded according to Fig. 86.

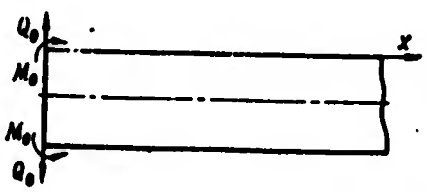

Fig. 86.

In the physical sense of the given problem function U should decrease in proportion to the distance from the place of application of forces. Therefore, in expression (6.15) we should assume

$$C_1=C_2=0.$$

Then

$$U=e^{-kx}(C_3\cos kx+C_4\sin kx).$$

Constants C_3 and C_4 must be found from boundary conditions.

When $x = 0$ there must be

$$(Q_\varphi)_{x=0}=Q_0,\quad (M_\varphi)_{x=0}=M_0.$$

These conditions give us

$$2Dk^2C_1 = Q_0,$$
$$Dk(-C_3 + C_4) = M_0,$$

whence

$$C_3 = \frac{M_0}{Dk} + \frac{Q_0}{2Dk^2}, \quad C_4 = \frac{Q_0}{2Dk^2}.$$

Then

$$
\begin{aligned}
&U = e^{-kx}\left[\frac{Q_0}{2Dk^2}(\cos kx + \sin kx) + \frac{M_0}{Dk}\cos kx\right],\\
&Q_\varphi = e^{-kx}[Q_0(\cos kx - \sin kx) - 2kM_0 \sin kx],\\
&N_\theta = 2Rke^{-kx}[Q_0 \cos kx + kM_0(\cos kx - \sin kx)],\\
&M_\varphi = e^{-kx}\left[M_0(\cos kx + \sin kx) + \frac{Q}{k}\sin kx\right],\\
&M_\theta = \mu M_\varphi,\\
&\varepsilon_\theta = \frac{2Rk}{E\delta}e^{-kx}[kM_0(\cos kx - \sin kx) + Q_0 \cos kx],\\
&w = \frac{2R^2k}{E\delta}e^{-kx}[kM_0(\cos kx - \sin kx) + Q_0 \cos kx],\\
&\frac{dw}{dx} = -e^{-kx}\left[\frac{Q_0}{2Dk^2}(\cos kx + \sin kx) + \frac{M_0}{Dk}\cos kx\right].
\end{aligned}
\tag{6.16}
$$

Deflection and angle of rotation when $x = 0$ will be

$$w = \frac{2R^2k}{E\delta}(kM_0 + Q_0), \quad \frac{dw}{dx} = -\frac{1}{2Dk^2}(2kM_0 + Q_0).$$

Functions, through which components of stressed and deformed state are expressed, are tabulated, and they can be found in mathematics handbooks. Table 5 contains some combinations of these functions, encountered in formulas (6.16).

From this table it is evident that functions, through which components of stressed and deformed state of the shell are expressed in boundary value problems, carry a rapidly damping character. This property, as was stated above, and forms the basis of simplification of original equations (6.10) and (6.12). Rapidly damping functions of edge effect possess the property that their lower derivatives are small in comparison with higher derivatives, and the function itself is smaller than its first derivative.

Table 5.

kx	e^{-kx} $(\cos kx+\sin kx)$	e^{-kx} $(\cos kx-\sin kx)$	e^{-kx} $\sin kx$	e^{-kx} $\cos kx$	kx
0	1,0000	1,0000	0,0000	1,0000	0
0,1	0,9907	0,8100	0,0903	0,9003	0,1
0,2	0,9651	0,6398	0,1627	0,8024	0,2
0,3	0,9267	0,4888	0,2189	0,7077	0,3
0,4	0,8784	0,1564	0,2610	0,6174	0,4
0,5	0,8231	0,2415	0,2908	0,5323	0,5
0,6	0,7628	0,1431	0,3099	0,4530	0,6
0,7	0,6997	0,0599	0,3199	0,3798	0,7
0,8	0,6354	−0,0093	0,3223	0,3131	0,8
0,9	0,5712	−0,0657	0,3185	0,2527	0,9
1,0	0,5083	−0,1108	0,3096	0,1988	1,0
1,1	0,4476	−0,1457	0,2967	0,1510	1,1
1,2	0,3899	−0,1716	0,2807	0,1091	1,2
1,3	0,3355	−0,1897	0,2626	0,0729	1,3
1,4	0,2849	−0,2011	0,2430	0,0419	1,4
1,5	0,2384	−0,2008	0,2226	0,0158	1,5
1,6	0,1959	−0,2077	0,2018	−0,0059	1,6
1,7	0,1576	−0,2047	0,1812	−0,0235	1,7
1,8	0,1234	−0,1985	0,1610	−0,0376	1,8
1,9	0,0932	−0,1899	0,1415	−0,0484	1,9
2,0	0,0667	−0,1794	0,1230	−0,0563	2,0
2,1	0,0439	−0,1675	0,1057	−0,0618	2,1
2,2	0,0244	−0,1548	0,0295	−0,0652	2,2
2,3	0,0080	−0,1416	0,0748	−0,0668	2,3
2,4	−0,0056	−0,1282	0,0613	−0,0669	2,4
2,5	−0,0166	−0,1149	0,0492	−0,0658	2,5
2,6	−0,0254	−0,1019	0,0383	−0,0636	2,6
2,7	−0,0320	−0,0895	0,0287	−0,0608	2,7
2,8	−0,0369	−0,0777	0,0204	−0,0573	2,8
2,9	−0,0403	−0,0666	0,0132	−0,0534	2,9
3,0	−0,0423	−0,0563	0,0071	−0,0493	3,0
3,1	−0,0431	−0,0469	0,0019	−0,0450	3,1
3,2	−0,0431	−0,0383	−0,0024	−0,0407	3,2
3,3	−0,0422	−0,0306	−0,0058	−0,0364	3,3
3,4	−0,0408	−0,0237	−0,0085	−0,0323	3,4
3,5	−0,0389	−0,0177	−0,0106	−0,0283	3,5
3,6	−0,0366	−0,0124	−0,0121	−0,0245	3,6
3,7	−0,0341	−0,0079	−0,0131	−0,0210	3,7
3,8	−0,0314	−0,0040	−0,0137	−0,0177	3,8
3,9	−0,0286	−0,0008	−0,0140	−0,0147	3,9
4,0	−0,0258	+0,0019	−0,0139	−0,0120	4,0

The underlined terms in the shown equations are dropped on this basis.

Let us apply the theory given above to solution of certain particular problems.

Infinitely long cylindrical shell, loaded by distributed radial annular load.

For determination of unknown edge forces Q_0 and M_0 we have conditions (Fig. 87)

$$(Q_\varphi)_{x=0}=\frac{P}{2},\quad \left(\frac{dw}{dx}\right)_{x=0}=0,$$

whence

$$Q_0=\frac{P}{2},\quad \frac{Q_0}{2k}+M_0=0.$$

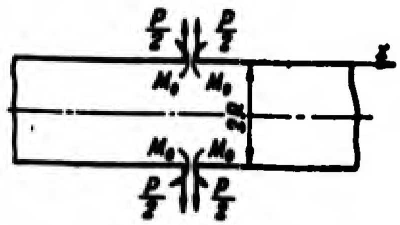

Fig. 87.

Then for this we obtain formulas

$$Q_\varphi=\frac{P}{2}e^{-kx}\cos kx,$$

$$N_0=\frac{PRk}{2}e^{-kx}(\cos kx+\sin kx),$$

$$M_\varphi=-\frac{P}{4k}e^{-kx}(\cos kx-\sin kx),$$

$$M_0=\mu M_\varphi,$$

$$w=\frac{PR^2k}{2Eh}e^{-kx}(\cos kx+\sin kx).$$

Infinitely long cylindrical shell, loaded by distributed annular moment. In this case for determination of edge forces Q_0 and M_0 we have conditions (Fig. 88)

$$(M_t)_{x=0}=\frac{m}{2},\quad (w)_{x=0}=0,$$

whence

$$M_0=\frac{1}{2}m,\quad Q=-\frac{1}{2}km.$$

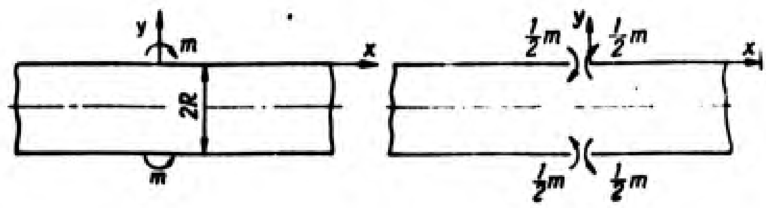

Fig. 88.

Then

$$Q_\varphi = -\frac{1}{2} kme^{-kx} \cos kx,$$
$$N_\theta = Rk^2 me^{-kx} \sin kx,$$
$$M_\varphi = \frac{1}{2} me^{-kx} \cos kx,$$
$$M_\theta = \mu M_\varphi,$$
$$w = -\frac{R^2 k^2 m}{E\delta} e^{-kx} \sin kx.$$

Infinitely long cylindrical shell, one of the sections of which is rotated to angle θ_0. In this case for determination of unknown internal elastic forces Q_0 and M_0 we have conditions (Fig. 89)

$$\left(\frac{dw}{dx}\right)_{x=0} = \theta_0, \quad (w)_{x=0} = 0,$$

whence

$$Q_0 = 2Dk^2\theta_0, \quad M_0 = -2Dk\theta_0.$$

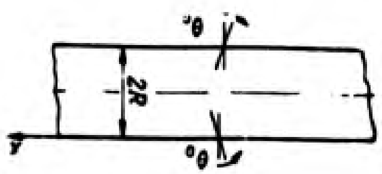

Fig. 89.

Then

$$Q_\varphi = 2Dk^2\theta_0 e^{-kx} (\cos kx + \sin kx),$$
$$N_\theta = 4Dk^3 R\theta_0 e^{-kx} \sin kx,$$
$$M_\varphi = -2Dk\theta_0 e^{-kx} \cos kx,$$
$$M_\theta = \mu M_\varphi,$$
$$w = \frac{4R^2 k^3 D\theta_0}{E\delta} e^{-kx} \sin kx.$$

§ 27. Determination of Force of Interaction Between Frame and Wall of a Tank at Internal Pressure

The theory of edge effect of a cylindrical shell, given in the previous paragraph can be used for determination of force of interaction between the frame and wall of a tank, when the latter, for example, is under the action of internal pressure.

Let us examine two cross sections of frame - rectangular and z-shaped.

Frame with rectangular cross section. Under the action of internal pressure the radius of the tank is increased by (see § 19)

$$\Delta_1 = \frac{(2-\mu)\, q_n R^2}{2E\delta}.$$

As a result of this displacement at the junction between the frame and wall of the tank there appears force of interaction P (Fig. 90). Decrease in the radius of the tank from the action of forces P will be

$$\Delta_2 = (w)_{x=0} = -\frac{PR^2 k}{2E\delta}.$$

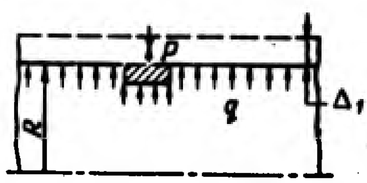

Fig. 90.

Increase in the radius of the frame under the action of force P will be

$$\Delta_3 = \frac{PR^2}{EF},$$

where F - area of cross section of frame.

Let us formulate the condition of equality of radial displacements of the tank and frame:

$$\Delta_1 + \Delta_2 = \Delta_3.$$

From this equation we find

$$P=\frac{(2-\mu)\,q_{н}F}{2\left(1+\frac{kF}{2\delta}\right)\delta}. \qquad (6.17)$$

Frame with z-shaped cross section. Increase in the radius of the tank from internal pressure

$$\Delta_1=\frac{(2-\mu)\,q_{н}R^2}{2E_1\delta_1}.$$

Decrease in the radius of the tank from forces of interaction *P*

$$\Delta_2=-\frac{PR^2k_1}{2E_1\delta_1}.$$

Now let us turn to examination of forces of the frame.

Under the action of forces *P* the radius of the frame will be increased by

$$\Delta_3=\frac{PR^2}{E_2F},$$

where *F* - area of cross section of frame.

Due to the fact that forces *P* are applied to the frame eccentrically, the cross section of the frame will be rotated to angle (Fig. 91)

$$\alpha=\frac{M_tR^2}{E_2J_y},$$

where $M_t=Pa$; J_y — moment of inertia of the frame relative to axis *y-y*.

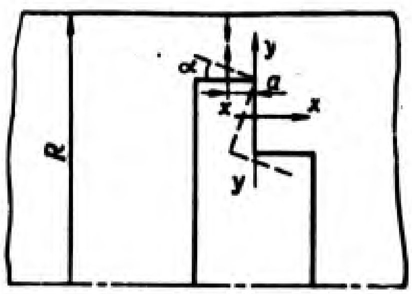

Fig. 91.

From this rotation the point of application of force *P* receives linear displacement, equal to

$$\Delta_4 = \alpha a = \frac{Pa^2R^2}{E_2J_y}.$$

Furthermore, force P will cause bending of the frame web. In this case deflection under force can be determined as for a beam - strip of unit width:

$$\Delta_5 = \frac{Pa^3}{3(1-\mu^2)E_2J},$$

where $J = \frac{\delta_2^3}{12}$ — moment of inertia of beam-strip.

In this case the cross section of the beam-strip under force is rotated to angle

$$\gamma = \frac{6Pa^2}{(1-\mu^2)E_2\delta_2^3}.$$

In actuality there should be no rotation. Considering the beam-strip as a cantilever with nonrotating end, for liquidation of the shown rotation to it under force there must be applied moment

$$M = \frac{Pa}{2}.$$

This moment will cause a decrease of deflection of beam-strip under force by

$$\Delta_6 = -\frac{3Pa^3}{(1-\mu^2)E_2\delta_2^3}.$$

Furthermore, moment M will cause decrease in the angle of rotation of the frame α

$$\Delta\alpha = \frac{MR^2}{E_2J_y} = \frac{PaR^2}{2E_2J_y}$$

and accordingly deflection of the point of application of force P will be decreased to quantity

$$\Delta_7 = -a\Delta\alpha = -\frac{Pa^2R^2}{2E_2J_y}.$$

Now there are all the necessary data for composition of the condition of continuity of the shell-frame system. This condition will have the form

$$\Delta_1+\Delta_2=\Delta_3+\Delta_4+\Delta_5+\Delta_6+\Delta_7.$$

In expanded form this condition will be

$$\frac{(2-\mu)\,q_{н}R^2}{2E_1\delta_1}-\frac{PR^2k_1}{2E_1\delta_1}=\frac{PR^2}{E_2F}+\frac{Pa^2R^2}{E_2J_y}+\frac{Pa^3}{3(1-\mu^2)\,E_2J}-$$
$$-\frac{3Pa^3}{(1-\mu^2)\,E_2\delta_2^3}-\frac{Pa^2R^2}{2E_2J_y}.$$

Hence we obtain

$$P=\frac{\frac{2-\mu}{k_1}q_{н}}{1+\frac{2E_1\delta_1}{E_2Fk_1}\left\{1+\frac{Fa^2}{J_y}\left[0{,}5+\frac{aJ_y}{(1-\mu^2)\,R^2\delta_2^3}\right]\right\}}. \qquad (6.18)$$

By comparing formulas (6.17) and (6.18), it is possible to see that in the latter case the force of interaction P will be less than in the first case, since the z-shaped frame is less rigid than rectangular with the same cross-sectional area. This phenomenon is favorably indicated on the work of welded points, with which the frame is welded to the tank wall.

If in addition to boost pressure we add pressure from hydrostatic liquid column, then we obtain a formula making it possible to determine the force of interaction from total internal pressure in the tank

$$P_0=\frac{\frac{1}{k_1}\left[(2-\mu)\,q_{н}+2\gamma H\right]}{1+\frac{2E_1\delta_1}{E_2Fk_1}\left\{1+\frac{a^2F}{J_y}\left[0{,}5+\frac{aJ_y}{(1-\mu^2)\,R^2-\delta_2^3}\right]\right\}},$$

where γ - specific weight of liquid; H - height of liquid column above the considered frame.

By using this expression, it is possible to compute linear force of interaction between the shell and frame.

If the frame is welded to the shell by spot welding, then force on one point will be

$$Q = P_0 t,$$

where t - spacing of spot welds.

By knowing force Q and the force at which the weld point fails, we can determine the safety factor of the weld

$$\eta = \frac{Q_{раз}}{Q}.$$

For weld points in this case it is possible to take $Q_{раз} = (0,3 \div 0,4) Q_{отр}$, where $Q_{отр}$ — breakaway force for the weld point, obtained on samples loaded according to the diagram of Fig. 92.

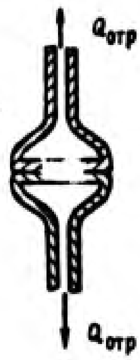

Fig. 92.

§ 28. Calculation of Spherical Shells with an Opening at the Pole

In this instance $R_1 = R_2 = R$. Then the original equations of edge effect (6.13) take the form[1]

$$\frac{d^2V}{d\varphi^2} = E\delta RU, \quad \frac{d^2U}{d\varphi^2} = -\frac{VR}{D}.$$

By excluding function V from these equations, we obtain

$$\frac{d^4U}{d\varphi^4} + 4\beta^4 U = 0, \qquad (6.19)$$

where

---

[1]These equations are applicable to the considred problem in the case when angle ϕ_0 (Fig. 93) is not less than 15°. Otherwise it is necessary to use equations (6.10) and (6.12).

$$\beta=\sqrt[4]{\frac{3(1-\mu^2)R^2}{\delta^2}}.$$

The solution of this equation is also known and has the form

$$U=e^{\beta\varphi}(C_1\cos\beta\varphi+C_2\sin\beta\varphi)+e^{-\beta\varphi}(C_3\cos\beta\varphi+C_4\sin\beta\varphi).$$

Let us apply this solution to a spherical shell, loaded as shown in Fig. 93. Subsequently for convenience it is expedient to introduce a new variable according to formula

$$\varphi=\varphi_0+\psi.$$

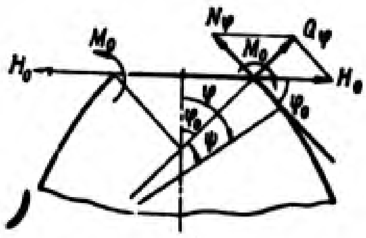

Fig. 93.

Then the solution for U will have the form

$$U=e^{\beta(\varphi_0+\psi)}[C_1\cos\beta(\varphi_0+\psi)+C_2\sin\beta(\varphi_0+\psi)]+$$
$$+e^{-\beta(\varphi_0+\psi)}[C_3\cos\beta(\varphi_0+\psi)+C_4\sin\beta(\varphi_0+\psi)].$$

By dropping the first component, as not satisfying the meaning of the given problem, after some conversions we obtain

$$U=e^{-\beta\psi}(D_1\cos\beta\psi+D_2\sin\beta\psi).$$

In this expression variable ψ is read from the edge of the opening.

For forces and displacements we will have expressions

$$N_\varphi=\frac{D}{R^2}\frac{d^2U}{d\varphi^2}\operatorname{ctg}\varphi,\quad N_\theta=\frac{D}{R^2}\times$$
$$\times\frac{d^3U}{d\varphi^3},\quad Q_\varphi=-\frac{D}{R^2}\frac{d^2U}{d\varphi^2},$$
$$M_\varphi=-\frac{D}{R}\frac{dU}{d\varphi},$$
$$M_\theta=\mu M_\varphi,\quad \varepsilon_\theta=\frac{1}{4\beta^4}\frac{d^3U}{d\varphi^3},$$
$$\frac{u}{R}-\frac{dw}{R\,d\varphi}=U.$$

Through the new variable these expressions will have the form $(d\varphi = d\psi)$

$$N_\varphi = \frac{D}{R^2}\frac{d^2U}{d\psi^2}\operatorname{ctg}(\varphi_0+\psi),$$

$$N_\theta = \frac{D}{R^2}\frac{d^3U}{d\psi^3}, \quad G_\varphi = -\frac{D}{R^2}\frac{d^2U}{d\psi^2}, \quad M_\varphi = -\frac{D}{R}\frac{dU}{d\psi}, \quad M_\theta = \mu M_\varphi,$$

$$\varepsilon_\theta = \frac{1}{4\beta^4}\frac{d^3U}{d\psi^3}, \qquad \frac{u}{R} - \frac{dw}{R\,d\psi} = U.$$

Let us find constants of integration D_1 and D_2 from conditions

$$(N_\varphi)_{\psi=0} = -H_0\cos\varphi_0, \quad (M_\varphi)_{\psi=0} = M_0,$$

whence

$$D_1 = \frac{R}{D\beta}\left(M_0 + \frac{RH_0\sin\varphi_0}{2\beta}\right),$$

$$D_2 = \frac{R^2H_0\sin\varphi_0}{2D\beta^2}.$$

Then

$$N_\varphi = \frac{2\beta e^{-\beta\psi}}{R}\left[\frac{RH_0}{2\beta}(\sin\beta\psi - \cos\beta\psi) - M_0\sin\beta\psi\right]\operatorname{ctg}(\varphi_0+\psi),$$

$$N_\theta = \frac{2\beta e^{-\beta\psi}}{R}\left[\beta M_0(\cos\beta\psi - \sin\beta\psi) + H_0R\sin\varphi_0\cos\beta\psi\right],$$

$$Q_\varphi = \frac{2\beta e^{-\beta\psi}}{R}\left[\frac{RH_0\sin\varphi_0}{2\beta}(\sin\beta\psi - \cos\beta\psi) - M_0\sin\beta\psi\right],$$

$$M_\varphi = e^{-\beta\psi}\left[M_0(\sin\beta\psi + \cos\beta\psi) + \frac{RH_0\sin\varphi_0}{\beta}\sin\beta\psi\right],$$

$$M_\theta = \mu M_\varphi,$$

$$\varepsilon_\theta = \frac{Re^{-\beta\psi}}{2D\beta^2}\left[M_0(\cos\beta\psi - \sin\beta\psi) + \frac{HR\sin\varphi_0}{\beta}\cos\beta\psi\right], \tag{6.20}$$

$$U = \frac{u}{R} - \frac{dw}{R\,d\psi} =$$

$$= \frac{Re^{-\beta\psi}}{D\beta}\left[\frac{H_0R\sin\varphi_0}{2\beta}(\sin\beta\psi + \cos\beta\psi) + M_0\cos\beta\psi\right].$$

The given theory permits examining a number of practically important cases of loading a spherical shell. Let us consider some examples.

1. Through the axis of a cylindrical tank, which has spherical bottoms, proceeds a cylindrical tunnel pipe (Fig. 94). The tank is under the action of internal pressure. Inspect the stressed and deformed state at the place of attachment of the bottom with the tunnel pipe.

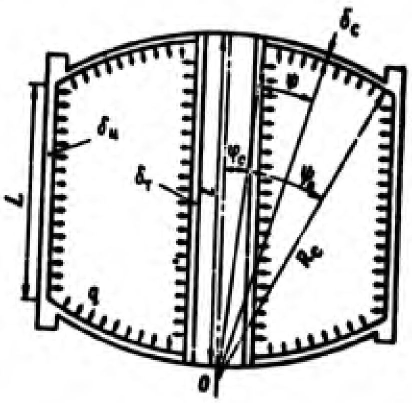

Fig. 94.

In this case there is posed the problem of calculation of shells of various geometric shape during their combined work, loaded by distributed pressure q. Inasmuch as in these shells at the place of their attachment the deformations will be different, at this place additional internal forces should appear. The problem of calculation consists of determining these forces and the deformations caused by them.

During examination of similar problems on utilization of the theory of edge effect at first the appropriate statically determinant momentless problem of shells must be solved, having determined the linear and angular displacements at the juncture position. Then at this junction we apply unknown internal elastic forces H_0 and M_0 and find from them the linear and angular displacements. After this it is possible to formulate conditions of continuity of linear and angular displacements and from these equations to determine H_0 and M_0.

First let us examine momentless problem for the bottom (Fig. 95).

From the condition of equilibrium of forces on the axis of the bottom we obtain

$$N_\varphi^1 = \frac{qR_c[\sin^2(\varphi_0+\psi) - \sin^2\varphi_0]}{2\sin^2(\varphi_0+\psi)}.$$

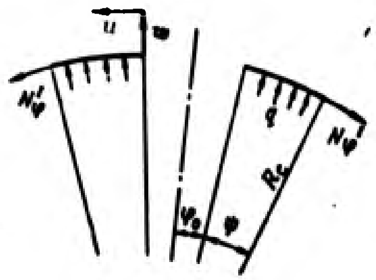

Fig. 95.

The expression for annular force N_θ are determined from Laplace equation

$$N_\theta^{\mathrm{I}}=qR_c-N_\varphi^{\mathrm{I}}=\frac{qR_c\left[\sin^2(\varphi_0+\psi)+\sin^2\varphi_0\right]}{2\sin^2(\varphi_0+\psi)},$$

where R_c - radius of sphere.

By having expressions for forces N_φ^{I} and N_θ^{I}, we can determine the components of deformation:

$$\varepsilon_\varphi=\frac{1}{E\delta_c}(N_\varphi^{\mathrm{I}}-\mu N_\theta^{\mathrm{I}}),\qquad \varepsilon_\theta=\frac{1}{E\delta_c}(N_\theta^{\mathrm{I}}-\mu N_\varphi^{\mathrm{I}}).$$

Increase in the radius of opening

$$\Delta_q^{\mathrm{I}}=(w_\varphi)_{\psi=0}\sin\varphi_0+(u_\varphi)_{\psi=0}\cos\varphi_0.$$

Displacement of the point of the edge of opening of the bottom toward the tank axis from pressure q will be

$$\Delta_q^{\mathrm{II}}=(w_\varphi)_{\psi=0}\cos\varphi_0-(u_\varphi)_{\psi=0}\sin\varphi_0.$$

Let us find the displacements u and w from equations (5.32) and (5.33). When $R_1=R_2=R_c$

$$\frac{du}{d\varphi}-u\operatorname{ctg}\varphi=R_c(\varepsilon_\varphi-\varepsilon_\theta),$$
$$w=R_c\varepsilon_\theta-u\operatorname{ctg}\varphi. \qquad (6.21)$$

The right side of the first equation in th's case will have the form

$$R_c(\varepsilon_\varphi-\varepsilon_\theta)=-\frac{(1+\mu)qR_c^2\sin^2\varphi_0}{E\delta_c\sin^2(\varphi_0+\psi)}.$$

Then when $\varphi=\varphi_0+\psi$ we obtain

$$\frac{du}{d\psi}-u\operatorname{ctg}(\varphi_0+\psi)=-\frac{(1+\mu)qR_c^2\sin^2\varphi_0}{E\delta_c\sin^2(\varphi_0+\psi)}.$$

This equation can be written in the following form:

$$\left[\frac{u}{\sin(\varphi_0+\psi)}\right]' = -\frac{(1+\mu)\,qR_c^2\sin^2\varphi_0}{E\delta_c\sin^3(\varphi_0+\psi)}.$$

Integral of this equation will be

$$u=\left[C-\frac{(1+\mu)\,qR_c^2\sin^2\varphi_0}{E\delta_c}\int\frac{d\psi}{\sin^3(\varphi_0+\psi)}\right]\sin(\varphi_0+\psi),$$

or

$$u=\left\{C-\frac{(1+\mu)\,qR_c^2\sin^2\varphi_0}{2E\delta_c}\left[\ln\operatorname{tg}\frac{\varphi_0+\psi}{2}-\frac{\operatorname{ctg}(\varphi_0+\psi)}{\sin(\varphi_0+\psi)}\right]\right\}\sin(\varphi_0+\psi).$$

For determination of constant C there is condition $u = 0$ when $\psi = \psi_0$ (see Fig. 94). Then finally

$$u_q=\frac{(1+\mu)\,qR_c^2\sin^2\varphi_0}{2E\delta_c}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\psi_0}{2}}{\operatorname{tg}\frac{\varphi_0+\psi_0}{2}}+\frac{\operatorname{ctg}(\varphi_0+\psi)}{\sin(\varphi_0+\psi)}-\right.$$

$$\left.-\frac{\operatorname{ctg}(\varphi_0+\psi_0)}{\sin(\varphi_0+\psi_0)}\right]\sin(\varphi_0+\psi).$$

The expression for deflection w we find from equation (6.21):

$$w=R_c\varepsilon_\theta-u\operatorname{ctg}(\varphi_0+\psi).$$

After substitution of values of ε_θ and u here we obtain

$$w_q=\frac{qR_c^2}{2E\delta_c}\left\{\frac{(1-\mu)\sin^2(\varphi_0+\psi)+(1+\mu)\sin^2\varphi_0}{\sin^2(\varphi_0+\psi)}-\right.$$

$$\left.-(1+\mu)\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\psi_0}{2}}{\operatorname{tg}\frac{\varphi_0+\psi}{2}}+\frac{\operatorname{ctg}(\varphi_0+\psi)}{\sin(\varphi_0+\psi)}-\frac{\operatorname{ctg}(\varphi_0+\psi_0)}{\sin(\varphi_0+\psi_0)}\right]\sin^2\varphi_0\cos(\varphi_0+\psi)\right\}.$$

As a result of loading the tank with internal pressure at the juncture position of the pipe with the bottom there appear forces of interaction, directed along the axis of the pipe. Under the action of these forces the pipe will be elongated, and the bottom collapsed. Stresses and displacements from forces of interaction between the pipe and bottom can be determined in the following manner (Fig. 96). From condition of equilibrium of forces on the axis of tank we obtain

$$N_\varphi^{II} = P\frac{\sin\varphi_0}{\sin^2(\varphi_0+\psi)}. \qquad (6.22)$$

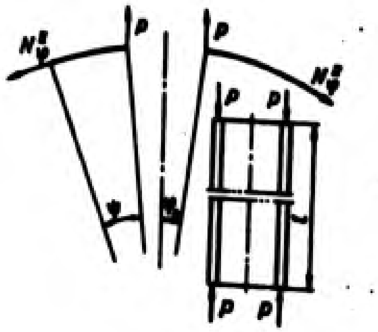

Fig. 96.

For annular forces N_θ^{II} from Laplace equation we find

$$N_\theta^{II} = -N_\varphi^{II} = -P\frac{\sin\varphi_0}{\sin^2(\varphi_0+\psi)}. \qquad (6.23)$$

By expressions (6.22) and (6.23) we can determine the components of deformation:

$$\varepsilon_\varphi = \frac{(1+\mu)P\sin\varphi_0}{E\delta_c\sin^2(\varphi_0+\psi)},$$

$$\varepsilon_\theta = -\frac{(1+\mu)P\sin\varphi_0}{E\delta_c\sin^2(\varphi_0+\psi)}.$$

For determination of displacement u we have equation

$$\frac{du}{d\varphi} - u\,\mathrm{ctg}(\varphi_0+\psi) = \frac{2(1+\mu)PR_c\sin\varphi_0}{E\delta_c\sin^2(\varphi_0+\psi)}.$$

The integral of this equation

$$u = \left\{B + \frac{(1+\mu)PR_c\sin\varphi_0}{E\delta_c}\left[\ln\mathrm{tg}\frac{\varphi_0+\psi}{2} - \frac{\mathrm{ctg}(\varphi_0+\psi)}{\sin(\varphi_0+\psi)}\right]\right\}\sin(\varphi_0+\psi).$$

Let us find constant B from condition $u = 0$ when $\psi = \psi_0$. Then finally we will have

$$u_p = -\frac{(1+\mu) PR_c}{E\delta_c}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\psi_0}{2}}{\operatorname{tg}\frac{\varphi_0+\psi}{2}} + \frac{\operatorname{ctg}(\varphi_0+\psi)}{\sin(\varphi_0+\psi)} - \right.$$

$$\left. -\frac{\operatorname{ctg}(\varphi_0+\psi_0)}{\sin(\varphi_0+\psi_0)}\right]\sin\varphi_0 \sin(\varphi_0+\psi).$$

For deflection w in this case we obtain expression

$$w = \frac{(1+\mu) PR_c \sin\varphi_0}{E\delta_c}\left\{\left[\ln\frac{\operatorname{tg}\frac{\varphi_0+\psi_0}{2}}{\operatorname{tg}\frac{\varphi_0+\psi}{2}} + \frac{\operatorname{ctg}(\varphi_0+\psi)}{\sin(\varphi_0+\psi)} - \frac{\operatorname{ctg}(\varphi_0+\psi_0)}{\sin(\varphi_0+\psi_0)}\right]\times\right.$$

$$\left.\times\cos(\varphi_0+\psi) - \frac{1}{\sin^2(\varphi_0+\psi)}\right\}.$$

Increased in the radius of the opening from forces P

$$\Delta_P^{III} = (w_P)_{\psi=0}\sin\varphi_0 + (u_P)_{\psi=0}\cos\varphi_0.$$

Displacement of edge of opening along the axis of the tank will be

$$\Delta_P^{IV} = (w_P)_{\psi=0}\cos\varphi_0 - (u_P)_{\psi=0}\sin\varphi_0.$$

Now let us examine displacements of the pipe and cylindrical part of the tank.

Under the action of internal pressure q the radius of the pipe is reduced to

$$\Delta_q^{V} = -\frac{qR_T^2}{E\delta_T} = -\frac{qR_c^2\sin^2\varphi_0}{E\delta_T}.$$

Under action of forces P the radius of the pipe will be increased

$$\Delta_P^{VI} = \frac{\mu PR_c\sin\varphi_0}{E\delta_T}.$$

Under the action of these forces the length of the pipe will be shortened to

$$\Delta_P^{VII} = -\frac{Pl_T}{E\delta_T},$$

where $l_т$ — is length of the pipe.

Under action of pressure q the length of the pipe will be increased to

$$\Delta_q^{VIII} = \frac{\mu q R_c l_т \sin \varphi_0}{E \delta_т}.$$

Annular forces at the tank wall is determined from Laplace equation

$$N_\theta^{III} = qR_u = qR_c \sin(\varphi_0 + \psi_0).$$

Now we can determine elongation of the tank in axial direction from internal pressure q and forces P:

$$\Delta_{qP}^{IX} = \varepsilon_\varphi L = \frac{L}{E\delta_u}(N_\varphi^{III} - \mu N_\theta^{III}) =$$

$$= \frac{L}{E\delta_u}\left[qR_c \frac{(1-2\mu)\sin^2(\varphi_0+\psi_0) - \sin^2\varphi_0}{2\sin(\varphi_0+\psi_0)} + \frac{P\sin\varphi_0}{\sin(\varphi_0+\psi_0)}\right].$$

Here L - length of cylindrical part of the tank.

$$N_\varphi^{III} = \frac{qR_c}{2}\frac{\sin^2(\varphi_0+\psi_0) + \sin^2\varphi_0}{\sin(\varphi_0+\psi_0)} + \frac{P\sin\varphi_0}{\sin(\varphi_0+\psi_0)}.$$

Let us determine displacements u and w for the bottom from the action of edge forces H_0 and M_0.

In this case the differential equation for u will have the form

$$\frac{du}{d\varphi} - u \operatorname{ctg}(\varphi_0 + \psi) = \frac{(1+\mu)R_c}{E\delta_c}(N_\varphi - N_\theta).$$

Having substituted here the expressions for forces N_ϕ and N_θ by function U, we obtain

$$\frac{du}{d\psi} - u \operatorname{ctg}(\varphi_0 + \psi) = \frac{(1+\mu)D}{E\delta_c R_c}\left[\frac{d^2U}{d\psi^2}\operatorname{ctg}(\varphi_0+\psi) - \frac{d^3U}{d\psi^3}\right].$$

Such an equation can be simplified, having dropped the lower derivatives in it in comparison with higher and the sought function in comparison with its first derivative. This is admissible here on the same basis as when obtaining simplified equations of boundary value problem.

Thus

$$\frac{du}{d\psi}=-\frac{(1+\mu)D}{E\delta_c R_c}\frac{d^3U}{d\psi^3}.$$

After substitution here of function U we will have

$$\frac{du}{d\psi}=-\frac{(1+\mu)R_c}{E\delta_c}\left[-\frac{2M_0\beta^2}{R}e^{-\beta\psi}(\sin\beta\psi-\cos\beta\psi)+\right.$$
$$\left.+2H_0\beta\sin\varphi_0 e^{-\beta\psi}\cos\beta\psi\right]. \qquad (6.24)$$

As a result of integration of equation (6.24) we obtain

$$u_{H_0,M_0}=-\frac{(1+\mu)R_c}{E\delta_c}\left[\frac{2M_0\beta}{R}e^{-\beta\psi}\sin\beta\psi+H_0\sin\varphi_0 e^{-\beta\psi}(\sin\beta\psi-\cos\beta\psi)\right].$$

Constant of integration in this case will be equal to zero, since there must be $u = 0$ when $\psi \to \infty$.

For displacement w we have equation

$$w_{H_0,M_0}=R_c\varepsilon_0-u\,\mathrm{ctg}(\varphi_0+\psi)=\frac{M_0R_c^2}{D}\left[-\frac{e^{-\beta\psi}}{2\beta^2}(\sin\beta\psi-\cos\beta\psi)+\right.$$
$$\left.+\frac{2(1+\mu)\beta D}{E\delta R_c^2}e^{-\beta\psi}\sin\beta\psi\,\mathrm{ctg}(\varphi_0+\psi)\right]+\frac{H_0R_c^3}{D}\left[\frac{\sin\varphi_0}{2\beta^3}e^{-\beta\psi}\cos\beta\psi+\right.$$
$$\left.+\frac{(1+\mu)D}{E\delta_c R_c^2}e^{-\beta\psi}\mathrm{ctg}(\varphi_0+\psi)(\sin\beta\psi-\cos\beta\psi)\right]\sin\varphi_0.$$

Component of obtained displacements u and w in the direction of the tank axis

$$\Delta^x_{H_0,M_0}=(w_{H_0,M_0})_{\psi=0}\cos\varphi_0-(u_{H_0,M_0})_{\psi=0}\sin\varphi_0.$$

Accordingly perpendicutlar to the axis of tank

$$\Delta^{x1}_{H_0,M_0}=(w_{H_0,M_0})_{\psi=0}\sin\varphi_0+(u_{H_0,M_0})_{\psi=0}\cos\varphi_0.$$

Displacement w of the end of the pipe under the action of edge forces H_0 and M_0 can be determined by formula (6.16), if in it we substitute $Q_0 = -H_0$ and for M_0 we change the sign to opposite:

$$\Delta^{XII}_{H_0,M_0}=(w)_{x=0}=-\frac{2R_c^2 k_T \sin^2\varphi_0}{Et_T}\,(k_T M_0+H_0).$$

Now let us determine angles of rotation of the edge of opening and end of the pipe. The edge of the opening of the bottom will receive angular displacements from the action of internal pressure q, edge forces H_0, M_0 and force of interaction of the pipe with the bottom P.

Angles of rotation from the indicated forces will be determined by formula

$$\theta=U=\frac{u}{R_c}-\frac{dw}{R_c d\psi},$$

by applying which we obtain

$$\theta^{I}_{q}=\left(\frac{u_q}{R_c}\right)_{\psi=0}-\left(\frac{dw_q}{R_c d\psi}\right)_{\psi=0},$$

$$\theta^{II}_{P}=\left(\frac{u_P}{R_c}\right)_{\psi=0}-\left(\frac{dw_P}{R_c d\psi}\right)_{\psi=0},$$

$$\theta^{III}_{H_0,M_0}=\left(\frac{u_{H_0,M_0}}{R_c}\right)_{\psi=0}-\left(\frac{dw_{H_0,M_0}}{R_c d\psi}\right)_{\psi=0}.$$

For the pipe the rotation of the end section will proceed only from edge forces H_0 and M_0:

$$\theta^{IV}_{H_0,M_0}=\left(\frac{dw}{dx}\right)_{x=0}=\frac{1}{D_T k_T}\left(M_0+\frac{1}{2k_T}H_0\right).$$

Now let us formulate conditions of continuity of deformations of the bottom and pipe at their junction point.

With formulation of conditions of continuity attention must be turned to signs of the corresponding displacements.

For positive direction of normal displacement w there is accepted the direction along the external normal to the shell. For angular displacements the following considerations should be followed. With derivation of the expression for change in curvature χ_ϕ (6.11) displacement w was increased with growth of angular coordinate ϕ. In this case the element of meridian $R_1 d\varphi$ was rotated clockwise. This angular displacement is taken as positive.

During the transition to reading the angular coordinate from the side of the equator to the side of the pole the indicated direction of rotation of the element of meridian is not changed, i.e., it will be directed clockwise.

All the above relative to the direction of displacements does not depend at all on the direction of forces applied to the shell.

Conditions of compatibility of linear and angular displacements will have the form

$$\Delta_q^{I} + \Delta_P^{III} + \Delta_{H_0,M_0}^{XI} = \Delta_q^{V} + \Delta_P^{VII} + \Delta_{H_0,M_0}^{XII},$$

$$\Delta_q^{II} + \Delta_P^{IV} + \frac{1}{2}^{*} \Delta_{q,P}^{I} + \Delta_{H_0,M_0}^{X} = \frac{1}{2} \Delta_P^{VII} + \frac{1}{2} \Delta_q^{VIII},$$

$$\theta_q^{I} + \theta_P^{II} + \theta_{H_0,M_0}^{III} = \theta_{H_0,M_0}^{IV}.$$

These equations cannot be solved in general form, and it is best of all to solve them numerically. From this solution we will determine unknown forces P, H_0 and M_0. Then we can determine the stressed and deformed state of the pipe and bottom by the corresponding formulas, given above. Resulting stresses will be equal to the sum of corresponding stresses from internal pressure q and edge forces P, H_0 and M_0.

2. As the second example of calculation of a spherical shell having an opening at the pole, let us examine Fig. 97.

As can be seen from this figure, the spherical shell is loaded by internal pressure q and force P, applied to the shell through a rigid plate. Let us determine the stressed and deformed state of this cover.

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*Numerical coefficient 1/2 in this equation considers the circumstance that total elongations of the pipe and cylindrical part are equally distributed relative to the middle of the tank.

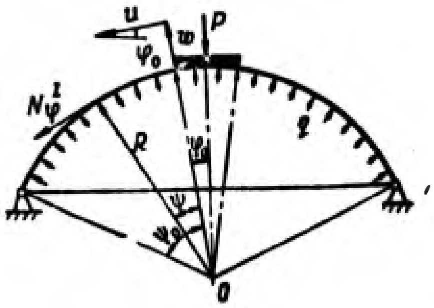

Fig. 97.

From condition of equilibrium of part of the sphere, determined by angle $\phi_0 + \psi$, we find

$$N_\varphi = -\frac{P}{2\pi R \sin^2(\varphi_0 + \psi)}.$$

Let us find the expression for circumferential force from Laplace equation

$$N_\theta = -N_\varphi = \frac{P}{2\pi R \sin^2(\varphi_0 + \psi)}.$$

By having the expression for internal forces, it is possible to switch to determination of displacements. For displacement u we have equation

$$\frac{du}{d\psi} - u \operatorname{ctg}(\varphi_0 + \psi) = -\frac{(1+\mu)P}{\pi E\delta \sin^2(\varphi_0 + \psi)},$$

integral of which is known to us and equal to

$$u = \left\{C - \frac{(1+\mu)P}{2\pi E\delta}\left[\ln \operatorname{tg}\frac{\varphi_0 + \psi}{2} - \frac{\operatorname{ctg}(\varphi_0 + \psi)}{\sin(\varphi_0 + \psi)}\right]\right\}\sin(\varphi_0 + \psi).$$

Constant of integration C is determined from condition $u = 0$ when $\psi = \psi_0$. Then finally

$$u_p = \frac{(1+\mu)P}{2\pi E\delta}\left[\ln\frac{\operatorname{tg}\frac{\varphi_0 + \psi_0}{2}}{\operatorname{tg}\frac{\varphi_0 + \psi}{2}} + \frac{\operatorname{ctg}(\varphi_0 + \psi)}{\sin(\varphi_0 + \psi)} - \frac{\operatorname{ctg}(\varphi_0 + \psi_0)}{\sin(\varphi_0 + \psi_0)}\right]\sin(\varphi_0 + \psi).$$

Let us find the expression for w from equation (6.21):

$$w_p = \frac{(1+\mu)P}{2\pi E\delta}\left\{\frac{1}{\sin^2(\varphi_0 + \psi)} - \left[\ln\frac{\operatorname{tg}\frac{\varphi_0 + \psi_0}{2}}{\operatorname{tg}\frac{\varphi_0 + \psi}{2}} + \frac{\operatorname{ctg}(\varphi_0 + \psi)}{\sin(\varphi_0 + \psi)} - \frac{\operatorname{ctg}(\varphi_0 + \psi_0)}{\sin(\varphi_0 + \psi_0)}\right]\cos(\varphi_0 + \psi)\right\}.$$

Let us determine the angle of rotation from force P. We have

$$\theta=\frac{u}{R}-\frac{dw}{Rd\psi}.$$

After substitution of u and $\frac{dw}{d\psi}$ here we obtain

$$\theta_p=\frac{(1+\mu)P}{2\pi E\delta R}\left[\frac{\operatorname{ctg}(\varphi_0+\psi)}{\sin^2(\varphi_0+\psi)}+\operatorname{ctg}(\varphi_0+\psi)+\operatorname{ctg}^3(\varphi_0+\psi)\right].$$

Projection of displacements u and w when $\psi = 0$ to the direction perpendicular to force P

$$\Delta_p=(w_p)_{\psi=0}\sin\varphi_0+(u_p)_{\psi=0}\cos\varphi_0=\frac{(1+\mu)P}{2\pi E\delta\sin\varphi_0}.$$

Let us determine the components of displacement from internal pressure q when $\psi = 0$. In this case only the following normal displacement will be nonzero

$$w=\frac{(1-\mu)qR^2}{2E\delta}.$$

Displacement u and angle of rotation θ will be equal to zero. This follows from the fact that the right side of the equation for u when loading by constant pressure q becomes zero. Consequently, the solution of this equation will be $u = 0$.

Horizontal projection of displacement w

$$\Delta_q=\frac{(1-\mu)qR^2}{2E\delta}\sin\varphi_0.$$

As a result of linear and angular displacements of the shell along its juncture postition with the plate, which is taken absolutely rigid, a break of continuity is obtained. Because of this, at their junction point there should appear internal forces H_0 and M_0, which compensate this break of continuity (Fig. 98).

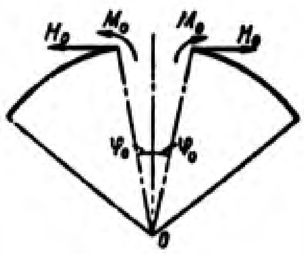

Fig. 98.

Displacement from these forces in direction H_0 will be ($\psi = 0$)

$$\Delta_{H_0,M_0}=(\varepsilon\theta_{H_0,M_0})_{\psi=0}R\sin\varphi_0=\frac{R^2}{2D\beta^2}\left(M_0+\frac{H_0R}{\beta}\sin\varphi_0\right)\sin\varphi_0.$$

Analogously for the angel of rotation when $\psi = 0$

$$\theta_{H_0,M_0}=\frac{R}{D\beta}\left(M_0+\frac{H_0R}{2\beta}\right).$$

For determintation of unknown edge forces H_0 and M_0 we have the following two equations of continuity of deformations:

$$\Delta_q+\Delta_p+\Delta_{H_0,M_0}=0,$$
$$\theta_p+\theta_{H_0,M_0}=0.$$

After solution of this system of equations and determination of H_0, M_0 we can determine all internal forces and deformations in the shell by formulas (6.20).

§ 29. Boundary Value Problem for a Cylindrical Tank. Calculation of a Spherical Shell Without an Opening at the Pole

Let us examine the order of solution of boundary value problem for a cylindrical tank, which has spherical bottoms. Before changing to such a problem, let us derive formulas for components of stresses and deformations of a spherical shell, loaded along the edge by distributed foreces H_0 and M_0 (see Fig. 100). In obtaining these formulas it is expedient to read the angle in equations (6.13) not from the pole, but as shown in Fig. 99:

$$\varphi=\varphi_0-\psi.$$

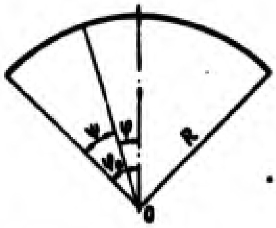

Fig. 99.

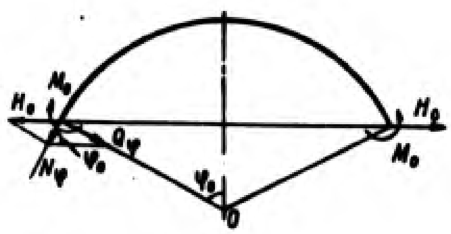

Fig. 100.

Furthermore, we have

$$R_1 = R_2 = R.$$

Then the solution of the original equation (6.19) in accordance with the meaning of the problem will be function

$$U = e^{-\beta\psi}(D_1 \sin\beta\psi + D_2 \cos\beta\psi).$$

For forces and deformations we obtain expressions

$$N_\varphi = \frac{D}{R^2}\,\frac{d^2U}{d\psi^2}\,\mathrm{ctg}(\varphi_0 - \psi),$$

$$N_\theta = -\frac{D}{R^2}\,\frac{d^3U}{d\psi^3}, \qquad Q_\varphi = -\frac{D}{R^2}\,\frac{d^2U}{d\psi^2},$$

$$M_\varphi = \frac{D}{R}\,\frac{dU}{d\psi}, \qquad M_\theta = \mu M_\varphi, \qquad \varepsilon_\theta = -\frac{1}{4\beta^4}\,\frac{d^3U}{d\psi^3}.$$

After substitution here of corresponding derivatives of function U we obtain

$$N_\varphi = \frac{2D\beta^2 e^{-\beta\psi}}{R^2}(D_2 \sin\beta\psi - D_1 \cos\beta\psi)\,\mathrm{ctg}(\varphi_0 - \psi),$$

$$N_\theta = -\frac{2D\beta^3 e^{-\beta\psi}}{R^2}[(D_1 - D_2)\sin\beta\psi + (D_1 + D_2)\cos\beta\psi],$$

$$Q_\varphi = -\frac{2D\beta^2 e^{-\beta\psi}}{R^2}(D_2 \sin\beta\psi - D_1 \cos\beta\psi),$$

$$M_\varphi = \frac{D\beta e^{-\beta\psi}}{R}[-(D_1 + D_2)\sin\beta\psi + (D_1 - D_2)\cos\beta\psi],$$

$$\varepsilon_\theta = -\frac{e^{-\beta\psi}}{2\beta}[(D_1 - D_2)\sin\beta\psi + (D_1 + D_2)\cos\beta\psi].$$

For determination of constants of integration we have conditions (Fig. 100)

$$(N_\varphi)_{\psi=0} = H_0 \cos\varphi_0, \quad (M_\varphi)_{\psi=0} = M_0.$$

Then

$$\left.\begin{aligned}
N_\varphi &= e^{-\beta\psi}\left[H_0 \sin\varphi_0(\cos\beta\psi - \sin\beta\psi) - \frac{2M_0\beta}{R}\sin\beta\psi\right]\mathrm{ctg}(\varphi_0 - \psi),\\
N_\theta &= -2\beta e^{-\beta\psi}\left[\frac{M_0\beta}{R}(\sin\beta\psi - \cos\beta\psi) - H_0 \sin\varphi_0 \cos\beta\psi\right],\\
Q_\varphi &= e^{-\beta\psi}\left[H_0 \sin\varphi_0(\cos\beta\psi - \sin\beta\psi) - \frac{2M_0\beta}{R}\sin\beta\psi\right],
\end{aligned}\right\} \quad (6.25)$$

$$M_\varphi = \frac{R}{\beta} e^{-\beta\psi}\left[\frac{M_0\beta}{R}(\sin\beta\psi + \cos\beta\psi) + H_0 \sin\varphi_0 \sin\beta\psi\right],$$

$$\varepsilon_\theta = -\frac{R^3 e^{-\beta\psi}}{2D\beta^3}\left[\frac{M_0\beta}{R}(\sin\beta\psi - \cos\beta\psi) - H_0 \sin\varphi_0 \cos\beta\psi\right],$$

$$U = -\frac{R^2 e^{-\beta\psi}}{2D\beta^2}\left[H_0 \sin\varphi_0 (\sin\beta\psi + \cos\beta\psi) + \frac{2M_0\beta}{R}\cos\beta\psi\right]. \qquad (6.25 \text{ cont'd})$$

Subsequently these formulas will be used during solution of the problem posed in this paragraph.

Let us make several remarks about so-called thrust forces in the shells. Let us examine a vessel, consisting of two shells (Fig. 101), which is under the action of internal boost pressure. If we mentally cut this vessel along the junction line of the shells (Fig. 102) and balance the load affecting them with the forces S_1 and S_2, directed along the tangent to the middle surface, from condition of equilibrium of these forces on the axis of the vessel we obtain

$$S_1 \cos\varphi_1 = S_2 \cos\varphi_2,$$

i.e., vertical projections of these forces mutually balance each other. Projections of these forces to the plane of junction of shells will be $S_1 \sin\varphi_1$ and $S_2 \sin\varphi_2$.

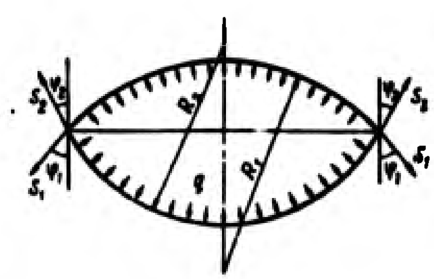

Fig. 101.

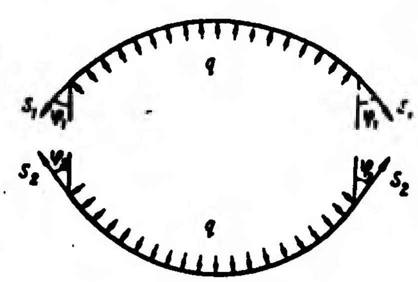

Fig. 102.

From Fig. 102 it is evident that these projections are directed to the same side and therefore they do not balance each other. Each of the thrust forces, applied to one of the shells, can exist only when on the side of the other shell there will take place reactive force of the opposite direction. Since thin shells weakly resist bending from thrust forces, for perception of the latter in constructions of tanks we usually place rings (frames).

Thrust forces in the vessels will be absent only when $\phi_1 = 0$, $\phi_2 = 0$. An example of such a vessel is a cylindrical tank with hemispherical or ellipsoidal bottom.

Now let us turn to solution the posed problem First let us examine the solution of momentless problem for a tank (Fig. 103).

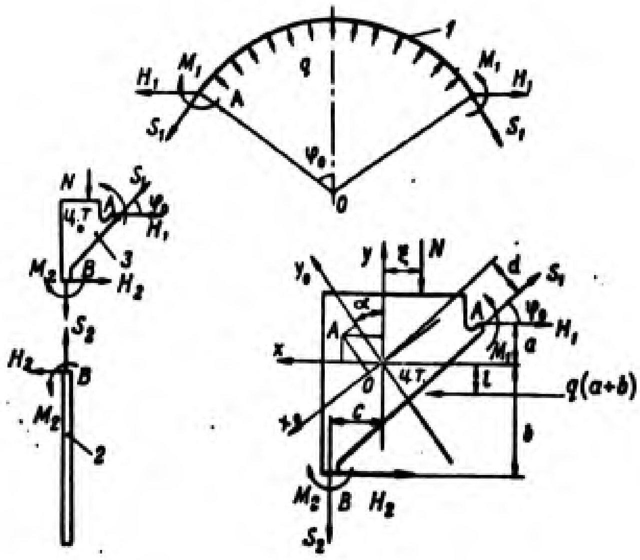

Fig. 103.

For the bottom:

- displacement of point A of the bottom in direction H_1 from internal pressure

$$\Delta_1^{(1)} = \frac{R_1 \sin\varphi_0}{E\delta_1}(N_\theta - \mu N_\varphi) = \frac{(1-\mu) q R_1^2 \sin\varphi_0}{2E\delta_1};$$

- displacement of the same point A in direction H_1 from edge forces H_1 and M_1 will be

$$\Delta_2^{(1)} = (\varepsilon_0)_{\psi=0} R_1 \sin\varphi_0 = \frac{R_1^2 \sin\varphi_0}{2D\beta^3}\left(\frac{M_1\beta}{R_1} + H_1 \sin\varphi_0\right);$$

- angle of rotation of edge section of the bottom from forces H_1 and M_1

$$\theta_1^{(1)} = -\frac{R_1^2}{2D_1\beta^2}\left(H_1 \sin\varphi_0 + \frac{2M_1\beta}{R_1}\right);$$

- angle of rotation of the same section from the action of internal pressure is equal to zero.

For a cylinder:

- displacement of point *B* of the cylinder in direction H_2 from the action of internal pressure and compressive force *N*

$$\Delta_1^{(2)} = \varepsilon_t R_2 = \frac{R_2}{E\delta_2}(N_t - \mu N_\varphi) = \frac{R_2}{E\delta_2}\left[qR_2 - \mu\left(\frac{qR_2}{2} - N\right)\right];$$

- displacement of point *B* from the action of edge forces H_2 and M_2

$$\Delta_2^{(2)} = \frac{2R_2^2 k}{E\delta_2}(kM_2 + H_2);$$

- angle of rotation of edge section from forces acting in this section:

$$\theta_1^{(2)} = -\frac{1}{2D_2 k^2}(2kM_2 + H_2);$$

- angle of rotation of the considered section from internal pressure and compressive force *N* will be equal to zero.

For the frame:

- rotation of the cross section of frame from twisting moment *M* will be expressed by formula

$$\theta_1^{(3)} = \frac{MR_3^2}{EJ_x},$$

where

$$M = -M_1 + M_2 - S_1 d - S_2 c + H_1 a - H_2 b + N\xi + q(a+b)l;$$

N - linear axial compressive force;

- displacement of center of gravity of the cross section of frame under the action of resultant force *P*

$$\Delta_1^{(3)} = -\frac{PR_3^2}{EF},$$

where

$$P = S_1 \cos\varphi_0 + H_1 + H_2 - q(a+b);$$

F - area of cross section of frame;

- displacement of point A of frame in the direction of force H_1 from rotation of the section

$$\Delta_2^{(3)} = -\theta_1^{(3)} a;$$

- analogously for point B

$$\Delta_3^{(3)} = \theta_1^{(3)} b.$$

For determination of unknown edge forces H_1, H_2, M_1, M_2 as a result we obtain the following equations of compatibility of deformations of the system bottom-frame, frame-cylinder:

$$\Delta_1^{(1)} + \Delta_2^{(1)} = \Delta_1^{(3)} + \Delta_2^{(3)}, \quad \theta_1^{(1)} = \theta_1^{(3)},$$
$$\Delta_1^{(2)} + \Delta_2^{(2)} = \Delta_1^{(3)} + \Delta_3^{(3)}, \quad \theta_1^{(2)} = \theta_1^{(3)}.$$

It is necessary to solve this system of equations numerically, since in general form the solution is obtained extremely bulky. After determination of unknown forces it is possible to calculate stresses in the bottom, frame and in the cylindrical part of the tank by corresponding formulas of this paragraph.

When determining the stresses in the frame the position of its principal axes of inertia must be determined. We have (see Fig. 103)

$$x_0 = -y \sin\alpha + x \cos\alpha, \qquad y_0 = y \cos\alpha + x \sin\alpha.$$

Principal moments of inertia with respect to axes x_0, y_0 will be equal to

$$J_{x_0} = \int_F y_0^2 dF = J_x \cos^2\alpha + J_y \sin^2\alpha + J_{xy} \sin 2\alpha,$$
$$J_{y_0} = \int_F x_0^2 dF = J_x \sin^2\alpha + J_y \cos^2\alpha - J_{xy} \sin 2\alpha,$$
$$J_{x_0 y_0} = \frac{1}{2}(J_y - J_x) + J_{xy} \cos 2\alpha.$$

In principal axes the centrifugal moment of inertia must be equal to zero:

$$J_{x_0 y_0} = 0.$$

From this condition we obtain the expression for angle α:

$$\operatorname{tg} 2\alpha = \frac{2J_{xy}}{J_x - J_y}.$$

Then stresses in the frame

$$\sigma = \pm \frac{M_{x_0} y_0}{J_{x_0}} \pm \frac{M_{y_0} x_0}{J_{y_0}} - \frac{PR_3}{F},$$

where

$$M_{x_0} = MR_3 \cos\alpha,$$
$$M_{y_0} = MR_3 \sin\alpha.$$

§ 30. Hemispherical Shell, Loaded Along the Edge by Distributed Lateral Load and Moment

We obtain expressions for internal forces and deformations from formulas (6.25), if in them we substitute $\varphi_0 = \frac{\pi}{2}$ (Fig. 104):

$$N_\varphi = \frac{2D\beta^2 e^{-\beta\psi}}{R^2}(D_2 \sin\beta\psi - D_1 \cos\beta\psi)\operatorname{tg}\psi,$$

$$N_\theta = -\frac{2D\beta^3 e^{-\beta\psi}}{R^2}[(D_1 - D_2)\sin\beta\psi + (D_1 + D_2)\cos\beta\psi],$$

$$Q_\varphi = -\frac{2D\beta^2 e^{-\beta\psi}}{R^2}(D_2 \sin\beta\psi - D_1 \cos\beta\psi),$$

$$M_\varphi = \frac{D\beta e^{-\beta\psi}}{R}[-(D_1 + D_2)\sin\beta\psi + (D_1 - D_2)\cos\beta\psi],$$

$$\varepsilon_\theta = -\frac{e^{-\beta\psi}}{2\beta}[(D_1 - D_2)\sin\beta\psi + (D_1 + D_2)\cos\beta\psi],$$

$$U = e^{-\beta\psi}(D_1 \sin\beta\psi + D_2 \cos\beta\psi).$$

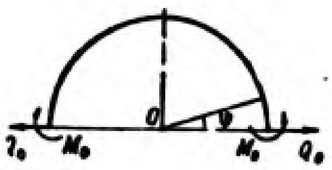

Fig. 104.

For determination of constants D_1 and D_2 we have conditions

$$(Q_\varphi)_{\psi=0} = Q_0, \qquad (M_\varphi)_{\psi=0} = M_0,$$

Then

$$N_\varphi = e^{-\beta\psi}\left[Q_0(\sin\beta\psi - \cos\beta\psi) - \frac{2M_0\beta}{R}\sin\beta\psi\right]\operatorname{tg}\varphi,$$

$$N_\theta = -2\beta e^{-\beta\psi}\left[Q_0\cos\beta\psi + \frac{M_0\beta}{R}(\sin\beta\psi - \cos\beta\psi)\right],$$

$$Q_\varphi = -e^{-\beta\psi}\left[Q_0(\sin\beta\psi - \cos\beta\psi) - \frac{2M_0\beta}{R}\sin\beta\psi\right],$$

$$M_\varphi = -\frac{R}{\beta}e^{-\beta\psi}\left[Q_0\sin\beta\psi - \frac{M_0\beta}{R}(\sin\beta\psi + \cos\beta\psi)\right],$$

$$\varepsilon_\theta = -\frac{2\beta}{E\delta}e^{-\beta\psi}\left[Q_0\cos\beta\psi + \frac{M_0\beta}{R}(\sin\beta\psi - \cos\beta\psi)\right],$$

$$U = \frac{R^2e^{-\beta\psi}}{2D\beta^3}\left[Q_0(\sin\beta\psi + \cos\beta\psi) - \frac{2M_0\beta}{R}\cos\beta\psi\right].$$

Let us apply these expressions for some particular cases of loading of a spherical shell.

Loading of spherical shell by annular load distributed along the equator. In this instance for determination of unknown forces Q_0 and M_0 there are conditions (Fig. 105)

$$(Q_\varphi)_{\psi=0} = \frac{P_0}{2}, \qquad (U)_{\psi=0} = 0.$$

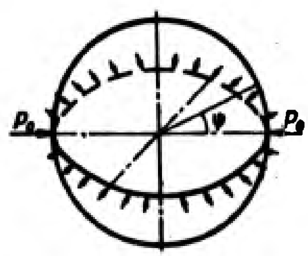

Fig. 105.

Then for internal forces and deformations we obtain the following expressions:

$$N_\varphi = -\frac{P_0}{2}e^{-\beta\psi}\cos\beta\psi\operatorname{tg}\varphi;$$

$$N_\theta = -\frac{P\beta}{2}e^{-\beta\psi}(\sin\beta\psi + \cos\beta\psi),$$

$$Q_\varphi = \frac{P_0}{2}e^{-\beta\psi}\cos\beta\psi,$$

$$M_\varphi = -\frac{P_0R}{4\beta}e^{-\beta\psi}(\sin\beta\psi - \cos\beta\psi),$$

$$\varepsilon_\theta = -\frac{P_0\beta}{2E\delta}e^{-\beta\psi}(\sin\beta\psi + \cos\beta\psi).$$

Loading of spherical shell by linear moment. In this instance for determination of forces Q_0 and M_0 we have conditions (Fig. 106)

$$(M_\varphi)_{\psi=0}=\frac{1}{2}m, \qquad (\varepsilon_\theta)_{\psi=0}=0,$$

Then

$$N_\varphi=-\frac{m\beta}{2R}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi)\operatorname{tg}\psi,$$

$$N_\theta=-\frac{m\beta^2}{R}e^{-\beta\psi}\sin\beta\psi,$$

$$Q_\varphi=\frac{m\beta}{2R}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi),$$

$$M_\varphi=\frac{1}{2}me^{-\beta\psi}\cos\beta\psi,$$

$$\varepsilon_\theta=-\frac{m\beta^2}{E\delta R}e^{-\beta\psi}\sin\beta\psi.$$

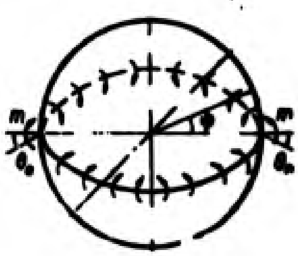

Fig. 106.

Spherical shell, one section of which is turned to angle θ_0. For determination of unknown Q_0 and M_0 we have the following conditions (see Fig. 106):

$$(U)_{\psi=0}=-\theta_0, \quad (\varepsilon_\theta)_{\psi=0}=0.$$

Then

$$N_\varphi=-\frac{2D\beta^2\theta_0}{R^2}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi)\operatorname{tg}\psi,$$

$$N_\theta=-\frac{4D\beta^3\theta_0}{R^2}e^{-\beta\psi}\sin\beta\psi,$$

$$Q_\varphi=\frac{2D\beta^2\theta_0}{R^2}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi),$$

$$M_\varphi=\frac{2D\beta\theta_0}{R}e^{-\beta\psi}\cos\beta\psi,$$

$$\varepsilon_\theta=-\frac{4D\beta^3\theta_0}{E\delta R^2}e^{-\beta\psi}\sin\beta\psi.$$

§ 31. Calculation of Ellipsoidal Doughnut-Shaped Shells for Axisymmetrical Linear Load

In the case of doughnut-shaped shells, loaded by linear axisymmetrical load (Fig. 107), it is also possible to use simplified equations of edge effect to get approximate solutions. We have

$$\frac{d^4U}{d\varphi^4}+4\beta^4U=0,$$

where

$$\beta^4=\frac{3(1-\mu^2)R_1^4}{R_2^2\delta^2}.$$

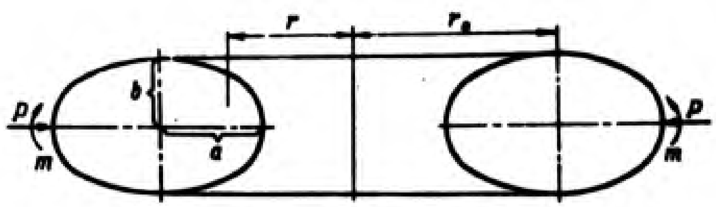

Fig. 107.

In this equation it is convenient to read the angle from the equator $\varphi=\frac{\pi}{2}-\psi$. By omitting all intermediate computations, let us write out basic formulas for internal forces and deformations:

$$N_\varphi=\frac{2D\beta^3e^{-\beta\psi}}{R_1^2}(D_1\sin\beta\psi-D_2\cos\beta\psi)\,\mathrm{tg}\,\psi,$$

$$N_\theta=-\frac{2DR_2\beta^3e^{-\beta\psi}}{R_1^3}[(-D_1+D_2)\sin\beta\psi+(D_1+D_2)\cos\beta\psi],$$

$$Q_\varphi=-\frac{2D\beta^2e^{-\beta\psi}}{R_1^2}(D_1\sin\beta\psi-D_2\cos\beta\psi),$$

$$M_\varphi=\frac{E\beta e^{-\beta\psi}}{R_1}[(-D_1+D_2)\cos\beta\psi-(D_1+D_2)\sin\beta\psi],$$

$$\varepsilon_\theta=-\frac{2DR_2\beta^3e^{-\beta\psi}}{E\delta R_1^3}[(-D_1+D_2)\sin\beta\psi+(D_1+D_2)\sin\beta\psi],$$

$$U=\theta=e^{-\beta\psi}(D_1\cos\beta\psi+D_2\sin\beta\psi).$$

where we will consider that in the zone of application of loads the radii of curvature R_1 and R_2 are constant and equal to their values at the equator. Such an assumption is entirely acceptable for the narrow zone of edge effect. Let us apply the written formulas for solution of some problems.

Loading of ellipsoidal doughnut-shaped shell by linear annular load. In this instance the constants of integration are determined from conditions (see Fig. 107)

$$(Q_\varphi)_{\psi=0}=\frac{P}{2}, \quad (\theta)_{\psi=0}=0.$$

Then for forces and deformations we obtain formulas

$$N_\varphi=-\frac{P}{2}e^{-\beta\psi}\cos\beta\psi\,\mathrm{tg}\,\psi,$$

$$N_\theta=-\frac{PR_2\beta e^{-\beta\psi}}{2R_1}(\sin\beta\psi+\cos\beta\psi),$$

$$Q_\varphi=\frac{Pe^{-\beta\psi}}{2}\cos\beta\psi,$$

$$M_\varphi=\frac{PR_1e^{-\beta\psi}}{4\beta}(\cos\beta\psi-\sin\beta\psi),$$

$$\varepsilon_\theta=-\frac{PR_2\beta e^{-\beta\psi}}{2E\delta R_1}(\sin\beta\psi+\cos\beta\psi),$$

$$\dot{U}=\theta=\frac{PR_1^2e^{-\beta\psi}}{4D\beta^2}\sin\beta\psi.$$

Loading of ellipsoidal doughnut-shaped shell by distributed moment. For determination of constants of integration in this instance we have conditions (see Fig. 107)

$$(M_\varphi)_{\psi=0}=\frac{1}{2}m, \quad (\varepsilon_\theta)_{\psi=0}=0.$$

Then

$$N_\varphi=-\frac{m\beta}{2R_1}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi)\,\mathrm{tg}\,\psi,$$

$$N_\theta=-\frac{mR_2\beta^2}{R_1^2}e^{-\beta\psi}\sin\beta\psi,$$

$$Q_\varphi=\frac{m\beta}{2R_1}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi),$$

$$M_\varphi = \frac{1}{2} m e^{-\beta\psi} \cos \beta\psi,$$

$$\varepsilon_\theta = -\frac{mR_2\beta^2}{E\delta R_1^2} e^{-\beta\psi} \sin \beta\psi,$$

$$U = \theta = \frac{mR_1}{4D\beta} e^{-\beta\psi} (\sin \beta\psi - \cos \beta\psi).$$

Ellipsoidal doughnut-shaped shell, equatorial section of which is turned to angle θ_0. In this instance the constants of integration are found from conditions (Fig. 108)

$$(U)_{\psi=0} = -\theta_0, \qquad (\varepsilon_\theta)_{\psi=0} = 0.$$

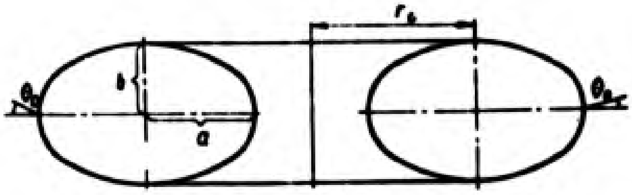

Fig. 108.

Then

$$N_\varphi = -\frac{2D\beta^2\theta_0}{R_1^2} e^{-\beta\psi} (\sin \beta\psi + \cos \beta\psi) \operatorname{tg} \varphi,$$

$$N_\theta = -\frac{4DR_2\theta_0}{R_1^3} e^{-\beta\psi} \sin \beta\psi,$$

$$Q_\varphi = \frac{2D\beta^2\theta_0}{R_1^2} e^{-\beta\psi} (\sin \beta\psi + \cos \beta\psi),$$

$$M_\varphi = \frac{2D\beta\theta_0}{R_1} e^{-\beta\psi} \cos \beta\psi,$$

$$\varepsilon_\theta = -\frac{4DR_2\beta\theta_0}{E\delta R_1^3} e^{-\beta\psi} \sin \beta\psi,$$

$$U = \theta = \theta_0 e^{-\beta\psi} (\sin \beta\psi - \cos \beta\psi).$$

In conclusion let us note that after solution of the corresponding boundary value problem an estimation of the strength of the given construction of the shell should be obtained. Usually this process is reduced to summing up the corresponding stresses and comparison of this total stress with breaking stress, i.e.,

$$\sigma_1=\left(\frac{N_\varphi}{\delta}\right)_q+\left(\frac{N_\varphi}{\delta}\right)_p+\left(\frac{N_\varphi}{\delta}\right)_H+\left(\frac{N_\varphi}{\delta}\right)_M\pm\frac{6M_\varphi}{\delta^2},$$

$$\sigma_2=\left(\frac{N_\theta}{\delta}\right)_q+\left(\frac{N_\theta}{\delta}\right)_p+\left(\frac{N_\theta}{\delta}\right)_H+\left(\frac{N_\theta}{\delta}\right)_M\pm\frac{6M_\theta}{\delta^2}.$$

The experience from operation of constructions, made from plastic materials, shows that components of stress from moments M_ϕ and M_θ should be disregarded in this estimation and the safety factor derived only with respect to membrane stresses. Thus, the safety factor will be equal to the ratio of $\sigma_{разр}$ to the greatest membrane stress.

Above we examined boundary value problems for a cylinder, sphere and torus. In calculation practice shells and other geometric shapes can be encountered. In these cases with accuracy sufficient for practice it is possible to use formulas obtained for a sphere, if the considered shell is divided into several small sections and use these formulas for each of them.

Within each such section the radius of curvature can be considered constant, equal to the average radius of the section. In this case the stresses on the boundary of each section will be initial conditions for the subsequent, i.e., calculation of such a shell will be accomplished by successive passages from one section to another until the stresses of edge effect become small. For practical purposes it is sufficient to break down the shell into not more than 3-4 sections.

CHAPTER VII

GENERAL CASE OF CALCULATION OF SHELLS

In this chapter are considered questions of calculation of shells without any simplifying assumptions relative to the stressed states and the character of the effective load. The differential equations obtained here are applicable to the solution of a wide circle of problems, which are encountered in practical calculations.

For derivation of equations of equilibrium the apparatus of calculus of variations is used.

§ 32. Brief Information from Calculus of Variations

Calculus of variations is a mathematical discipline, occupied with finding the maximum or minimum values of functionals, which correspond to prescribed boundary conditions of the given physical problems.

In this sense the problems of calculus of variations are formally analogous to problems of finding maximums and minimums of functions in differential calculus.

If the problem about extremumof a given function is solved, then everything is reduced to determination of those values of independent variables, at which this function obtains maximum or minimum. In the case of the function of one variable for this it is necessary to solve equation

$$y'(x)=0,$$

roots of which determine the extremum.

In this case function $y(x)$ is known to us and it is required to determine only its extremal values. Calculus of variations deals with so-called functionals.

By functional we mean definite integral from a complex function, the form of which is known to us. The problem of calculus of variations consists of determining the differential equation which this function must satisfy. The obtained equation will possess the property that the function, which satisfies it, turns the assigned functional to maximum or minimum, i.e., in this case extremal values of functional are determined not by coordinates of separate points, as for the assigned function in differential calculus, but the entire differential equation.

As the simplest example let us find the differential equation for deflection of a beam, loaded by uniform load. Total potential energy for a straight beam has the form

$$Э=\int_0^l\left[\frac{EJ}{2}\left(\frac{d^2y}{dx^2}\right)^2-qy\right]dx, \qquad (7.1)$$

where the first component expresses the work of bending deformation, and the second – work of external forces.

In this case we deal with a functional in the form of integral of total potential energy. It is required to determine the differential equation which is satisfied by function $y(x)$, turning the given integral into extremum.

Let us apply to this expression the total energy of origin of virtual displacements (Lagrange principle).

By applying the origin of virtual displacements, we suppose that deflections of the beam received infinitely small increases. Then the change of energy of deformation of the beam should be equal to the work of external forces at the same increases of deflection.

This condition can be written in the following manner:

$$\delta Э = \delta \int_0^l \left[\frac{EJ}{2} \left(\frac{d^2 y}{dx^2} \right)^2 - qy \right] dx = 0,$$

where index δ designates infinitely small increase of curve of deflection from its equilibrium state and is called variation. The idea of variation is identical to the idea of differential in analysis. Therefore, the following are valid

$$\delta \left(\frac{dy}{dx} \right) = \frac{d}{dx} (\delta y),$$

$$\delta \left(\frac{d^2 y}{dx^2} \right) = \frac{d}{dx} \left[\delta \left(\frac{dy}{dx} \right) \right] = \frac{d^2}{dx^2} (\delta y).$$

Thus we have

$$\delta \int_0^l \left[\frac{EJ}{2} \left(\frac{d^2 y}{dx^2} \right)^2 - qy \right] dx = \int_0^l \left[EJ \frac{d^2 y}{dx^2} \delta \left(\frac{d^2 y}{dx^2} \right) - q \delta y \right] dx = 0.$$

Let us integrate with respect to parts expression

$$\frac{1}{2} \int_0^l \delta \left(\frac{d^2 y}{dx^2} \right)^2 dx = \int_0^l \frac{d^2 y}{dx^2} \delta \left(\frac{d^2 y}{dx^2} \right) dx =$$

$$= \int_0^l \frac{d^2 y}{dx^2} \frac{d}{dx} \left(\delta \frac{dy}{dx} \right) dx = \left[\frac{d^2 y}{dx^2} \delta \left(\frac{dy}{dx} \right) \right]_0^l -$$

$$- \int_0^l \frac{d^3 y}{dx^3} \frac{d}{dx} (\delta y) dx = \left[\frac{d^2 y}{dx^2} \delta \left(\frac{dy}{dx} \right) \right]_0^l - \left[\frac{d^3 y}{dx^3} \delta y \right]_0^l + \int_0^l \frac{d^4 y}{dx^4} \delta y \, dx.$$

Then for variation $\delta Э$

$$\delta Э = \left[EJ \frac{d^2 y}{dx^2} \delta \left(\frac{dy}{dx} \right) \right]_0^l - \left[EJ \frac{d^3 y}{dx^3} \delta y \right]_0^l + \int_0^l \left(EJ \frac{d^4 y}{dx^4} - q \right) \delta y \, dx = 0,$$

where the first two components refer to end sections of the beam and determine boundary conditions, and the third component determines the type of differential equation solution of which turns the integral of total potential energy of the beam into extremum.

Inasmuch as all three components in the obtained expression do not depend on each other in view of the arbitrariness of variations δy and $\delta y'$, each of them must be equal to zero. Thus, we obtain

$$\left[EJ\frac{d^2y}{dx^2}\delta\left(\frac{dy}{dx}\right)\right]_0^l=0,$$
$$\left[EJ\frac{d^3y}{dx^3}\delta y\right]_0^l=0, \tag{7.2}$$
$$\int_0^l\left(EJ\frac{d^4y}{dx^4}-q\right)\delta y\,dx=0.$$

With arbitrariness of variation δy from the last expression we obtain the known equation of bending of beams with straight axis

$$EJ\frac{d^4y}{dx^4}-q=0,$$

and the first two expressions determine the boundary conditions at its ends.

Thus, methods of calculus of variations as applied to problems of structural mechanics, by passing the usual method of formation of differential equations from the condition of equilibrium of an infinitely small element, permit obtaining them by purely formal means.

In a more general form the given computations could have been presented in the following manner.

We have functional

$$J=\int_0^l F(x,\,y,\,y'')\,dx, \tag{7.3}$$

variation of which will be equal to zero in the case of extremum:

$$\delta J=\int_0^l\left(\frac{\partial F}{\partial y}\delta y+\frac{\partial F}{\partial y''}\delta y''\right)dx=0.$$

Here F – integrand of expression (7.3); $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial y''}$ – derivatives of function F in terms of y and y''.

Let us integrate by parts component:

$$\int_0^l \frac{\partial F}{\partial y''}\delta y''\,dx = \int_0^l \frac{\partial F}{\partial y''}\frac{d}{dx}\left(\delta\frac{dy}{dx}\right)dx =$$
$$= \left[\frac{\partial F}{\partial y''}\delta\left(\frac{dy}{dx}\right)\right]_0^l - \left[\frac{d}{dx}\frac{\partial F}{\partial y''}\delta y\right]_0^l + \int_0^l \frac{d^2}{dx^2}\frac{\partial F}{\partial y''}\delta y\,dx.$$

Then

$$\delta J = \left[\frac{\partial F}{\partial y''}\delta\left(\frac{dy}{dx}\right)\right]_0^l - \left[\frac{d}{dx}\frac{\partial F}{\partial y''}\delta y\right]_0^l + \int_0^l \left(\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} + \frac{\partial F}{\partial y}\right)\delta y\,dx = 0.$$

Hence, let us obtain expressions equivalent to (7.2)

$$\left[\frac{\partial F}{\partial y''}\delta\left(\frac{dy}{dx}\right)\right]_0^l = 0, \qquad \left[\frac{d}{dx}\frac{\partial F}{\partial y''}\delta y\right]_0^l = 0,$$
$$\frac{d^2}{dx^2}\frac{\partial F}{\partial y''} + \frac{\partial F}{\partial y} = 0,$$

of which the first two refer to end sections of the beam and determine boundary conditions, and the latter is a Euler equation of the given variation problem.

The given reasonings can be extended even to cases when the functional depends on several functions of many variables. All problems of the theory of elasticity, theory of plates, and theory of shells, for example, lead to such functionals.

§ 33. Expression of Total Potential Energy for Shells

By total potential energy of an elastic system we mean the work that forces of the system accomplish during its transition from deformed state to nondeformed. In this case internal elastic forces accomplish work on the elastic displacements, which were caused by these forces.

In the case of thin shells the elementary value for total energy

$$
\begin{aligned}
d\mathcal{E} = {} & \frac{1}{2} N_x dy \varepsilon_x dx + \frac{1}{2} N_y dx \varepsilon_y dy + \frac{1}{2} N_{xy} dy \gamma dx + \\
& + \frac{1}{2} N_{yx} dx \alpha dy + \frac{1}{2} M_x dy \chi_x dx + \frac{1}{2} M_y dx \chi_y dy + \\
& + \frac{1}{2} M_{xy} dy \chi_{xy} dx + \frac{1}{2} M_{yx} dx \chi_{yx} dy - q_x dx\, dy u - \\
& - q_y dx\, dy v - q_z dx\, dy w.
\end{aligned}
$$

The first four components represent the work of membrane forces, evenly distriubted along the wall thickness of the shells. The second group of four components gives work of bending and twisting moments. The last three components express the work of external load on corresponding displacements.

The work of all internal forces is taken with coefficient 1/2, since by definition of potential energy the internal forces increase from zero to their infinite value gradually according to linear law. Therefore, the work of each internal force will be expressed by the area of a triangle. Figure 109 shows internal and external forces, applied to the element of shell with area $dxdy$, and displacements corresponding to them. Since for an infinitely small element all the curved lines can be replaced by straight segments, curvature of the shell is not shown in Fig. 109.

Further it is convenient to express all internal force factors through corresponding deformations according to Hooke law. For a biaxial stressed state Hooke law has the form

$$
\begin{aligned}
N_x &= \frac{E\delta}{1-\mu^2} (\varepsilon_x + \mu \varepsilon_y), \\
N_y &= \frac{E\delta}{1-\mu^2} (\varepsilon_y + \mu \varepsilon_x), \\
N_{xy} &= G\varepsilon_{xy} = \frac{E}{2(1+\mu)} \varepsilon_{xy}.
\end{aligned}
$$

Analogously

$$M_x = \frac{E\delta^3}{12(1-\mu)^2}(\chi_x + \mu\chi_y),$$

$$M_y = \frac{E\delta^3}{12(1-\mu^2)}(\chi_y + \mu\chi_x),$$

$$M_{xy} = (1-\mu)\frac{E\delta^3}{12(1-\mu^2)}\chi_{xy} = (1-\mu)\frac{E\delta^3}{12(1-\mu^2)}\chi_{yx}.$$

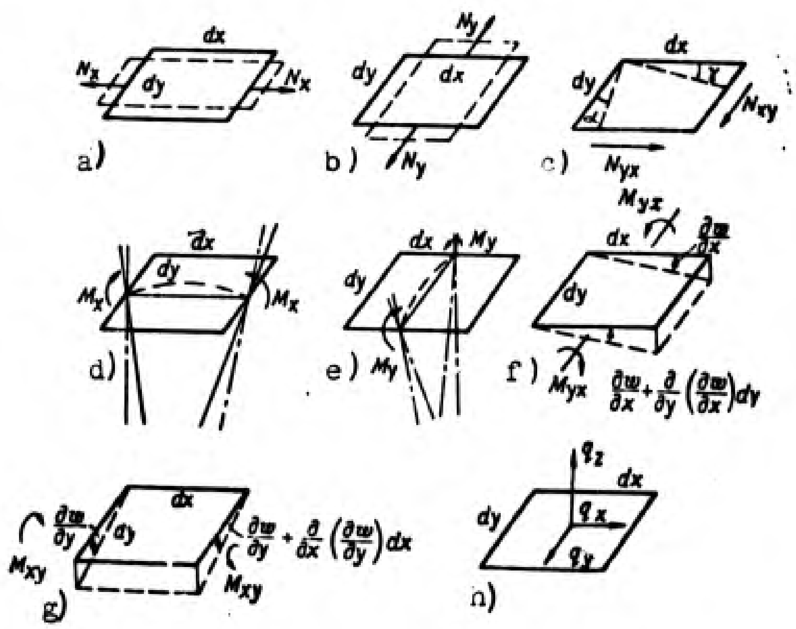

Fig. 109.

The correctness of the last equality ensues from equality of angles of rotation according to Fig. 109f, g.

Let us express deformation components and changes of curvature through displacements u, v, and w, directed along axes x, y, z of mobile trihedron (Figs. 110, 111).

For this purpose from the shell let us isolate element $dxdy$, sides of which are directed along line of principal curvatures, and represent the positions of the sides of this element in deformed state. Expressions for deformation components can be obtained if we project closed three-dimensional polygon 1-2-3-4-5-6-7-8-1 to axis x, and polygon 4-9-10-11-12-1-2-3-4- to axis y.

Then, we obtain

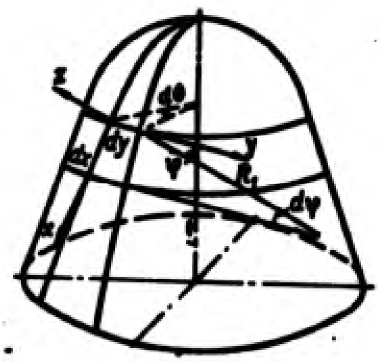

Fig. 110.

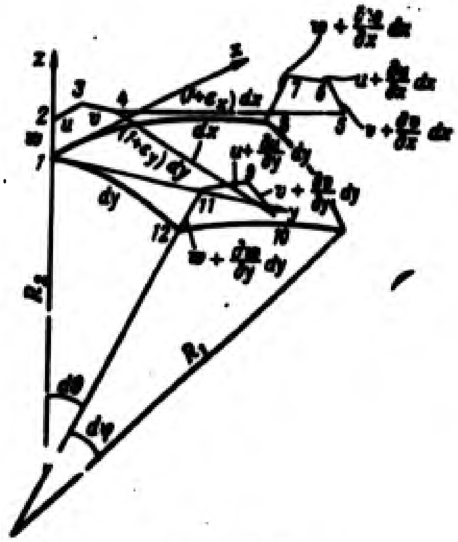

Fig. 111.

$$u+(1+\varepsilon_x)dx-u-\frac{\partial u}{\partial x}dx-dx-\left(w+\frac{\partial w}{\partial x}dx\right)\frac{dx}{R_1}=0,$$
$$v+(1+\varepsilon_y)dy-v-\frac{\partial v}{\partial y}dy-dy-\left(w+\frac{\partial w}{\partial y}dy\right)\frac{dy}{R_2}=0.$$

With formulation of these equations we assumed

$$\cos d\varphi\approx 1,\quad \cos[dx,\ (1+\varepsilon_x)dx]\approx 1,$$
$$\cos d\psi\approx 1,\quad \cos[dy,\ (1+\varepsilon_y)dy]\approx 1.$$

For change of the angle between directions dx and dy in deformed state we have expression

$$\varepsilon_{xy}=\frac{\frac{\partial v}{\partial x}dx}{(1+\varepsilon_x)dx}+\frac{\frac{\partial u}{\partial y}dy}{(1+\varepsilon_y)dy}.$$

Here we should assume $1+\varepsilon_x\approx 1$, $1+\varepsilon_y\approx 1$, since for metals deformation $\varepsilon_x,\ \varepsilon_y\approx 0{,}003\div 0{,}005$.

Thus, with accuracy to smalls of the first order we obtain

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{w}{R_1},\qquad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{w}{R_2},\qquad \varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}.$$

Let us obtain formulas for change of curvature of the shell.

In the plane of meridian

$$\chi_x=-\frac{\left[\frac{\partial w}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)dx\right]-\frac{\partial w}{\partial x}}{(1+\varepsilon_x)dx}+\left(\frac{1}{R_1+w}-\frac{1}{R_1}\right),$$

where the first component gives the relationship of contiguity angle between the tangents to element dx in its deformed state to the length of this element;* the second component gives change of curvature of the element as a result of normal displacement w. Having expanded fraction $\frac{1}{R_1+w}$, into series in terms of Newton binomial formula, with accuracy to small quantities of the first order we obtain

*The minus sign before this fraction indicates that rotation of element dx during transition from nondeformed state to deformed is accomplished in the direction opposite the positive direction of reading angle $d\phi$ (see Fig. 111).

$$\frac{1}{R_1+w}=\frac{1}{R_1}\left(1-\frac{w}{R_1}\right).$$

Then for change of curvature χ_x we obtain expression

$$\chi_x=-\frac{\partial^2 w}{\partial x^2}-\frac{w}{R_1^2}.$$

Analogously in the plane perpendicular to the meridian:

$$\chi_y=-\frac{\partial^2 w}{\partial y^2}-\frac{w}{R_2^2}.$$

As a result of twisting there are changed the angles of slope of tangents, drawn to opposite sides of element $dxdy$. For change of curvature of twisting we obtain expressions

$$\chi_{xy}=-\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)=-\frac{\partial^2 w}{\partial x\,\partial y},$$

$$\gamma_{yx}=-\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right)=-\frac{\partial^2 w}{\partial x\,\partial y}.$$

Expressions for forces and moments is expressed through components of displacement:

$$N_x=\frac{E\delta}{1-\mu^2}\left[\frac{\partial u}{\partial x}+\frac{w}{R_1}+\mu\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)\right],$$

$$N_y=\frac{E\delta}{1-\mu^2}\left[\frac{\partial v}{\partial y}+\frac{w}{R_2}+\mu\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)\right],$$

$$N_{xy}=\frac{E\delta}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

$$M_x=D\left[\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\mu\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}\right)\right],$$

$$M_y=D\left[\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}\right)\right],$$

$$M_{xy}=(1-\mu)D\frac{\partial^2 w}{\partial x\,\partial y}.$$

Here $D=\frac{E\delta^3}{12(1-\mu^2)}$ - cylindrical rigidity of the shell.

By having these expressions for internal forces, the potential energy of shell can be written as a function of components of displacements. Let us preliminarily rewrite the expression for Э in the following form:

$$Э=\frac{1}{2}\iint_s (N_x\varepsilon_x+N_y\varepsilon_y+N_{xy}\varepsilon_{xy})\,dx\,dy+$$
$$+\frac{1}{2}\iint_s (M_x\chi_x+M_y\chi_y+2M_{xy}\chi_{xy})\,dx\,dy-$$
$$-\iint_s (q_xu+q_yv+q_zw)\,dx\,dy.$$

Here the first two double integrals are extended to the entire surface of the shell; the last integral is extended only to the sections of the shell, which are affected by components of external load.

Let us express potential energy through deformation components and change of curvature:

$$Э=\frac{1}{2}B\iint_s \left(\varepsilon_x^2+\varepsilon_y^2+2\mu\varepsilon_x\varepsilon_y+\frac{1-\mu}{2}\varepsilon_{xy}\right)dx\,dy+$$
$$+\frac{1}{2}D\iint_s \left[\chi_x^2+\chi_y^2+2\mu\chi_x\chi_y+2(1-\mu)\chi_{xy}^2\right]dx\,dy-$$
$$-\iint_s (q_xu+q_yv+q_zw)\,dx\,dy, \qquad (7.4)$$

where $B=\frac{Et}{1-\mu^2}$ – tensile rigidity.

If here we substitute the expressions for ε_x, ..., χ_x, ..., we obtain

$$Э=\frac{1}{2}B\iint_s\left[\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)^2+\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)^2+\right.$$
$$\left.+2\mu\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)+\frac{1-\mu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]dx\,dy+$$
$$+\frac{1}{2}D\iint_s\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}\right)^2+\right.$$
$$\left.+2\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}\right)\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}\right)+2(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy-$$
$$-\iint_s (q_xu+q_yv+q_zw)\,dx\,dy.$$

§ 34. Variation Equations of Equilibrium of Shells and Boundary Conditions

Expression obtained in the previous paragraph for total potential energy of a shell is the functional of functions u, v and w and their derivatives. By using this functional and applying methods of calculus of variations to it, the differential equations and boundary conditions necessary for solution of the problems can be obtained. For convenience of computations let us rewrite the expression for energy in the following form:

$$Э=\iint_s F(u, u_x, x_y, v, v_x, v_y, w, w_{xx}, w_{yy}, w_{xy})\, dx\, dy,$$

where

$$u_x=\frac{\partial u}{\partial x}, \ldots, \quad w_{xy}=\frac{\partial^2 w}{\partial x\, \partial y}.$$

Inasmuch as the expression of total energy is written for a shell, being in equilibrium deformed state, the sum of works of all forces on problems will be equal to zero:

$$\delta Э=\iint_s \left(\frac{\partial F}{\partial u}\delta u+\frac{\partial F}{\partial u_x}\delta u_x+\frac{\partial F}{\partial u_y}\delta u_y+\frac{\partial F}{\partial v}\delta v+\frac{\partial F}{\partial v_x}\delta v_x+\right.$$
$$\left.+\frac{\partial F}{\partial v_y}\delta v_y+\frac{\partial F}{\partial w}\delta w+\frac{\partial F}{\partial w_{xx}}\delta w_{xx}+\frac{\partial F}{\partial w_{yy}}\delta w_{yy}+\frac{\partial F}{\partial w_{xy}}\delta w_{xy}\right) dx\, dy=0.$$

Let us integrate each component of the last expression by parts and sum up the results. Let us show the fulfillment of such operations for some components entering this expression. Let us assume $a-b$, $c-d$ are boundaries of the integration range. Then

$$\iint_s \frac{\partial F}{\partial u_x}\delta u_x\, dx\, dy=\int_a^b\int_c^d \frac{\partial F}{\partial u_x}\frac{\partial}{\partial x}(\delta u)\, dx\, dy=$$
$$=\int_c^d\left[\frac{\partial F}{\partial u_x}\delta u\right]_a^b dy-\int_a^b\int_c^d \frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}\delta u\, dx\, dy;$$

$$\iint_s \frac{\partial F}{\partial w_{xx}}\delta w_{xx}\, dx\, dy=\int_a^b\int_c^d \frac{\partial F}{\partial w_{xx}}\frac{\partial}{\partial x}\left[\delta\left(\frac{\partial w}{\partial x}\right)\right] dx\, dy=$$

$$= \int_c^d \left[\frac{\partial F}{\partial w_{xx}} \delta\left(\frac{\partial w}{\partial x}\right)\right]_a^b dy - \int_a^b\int_c^d \frac{\partial}{\partial x}\,\frac{\partial F}{\partial w_{xx}}\,\frac{\partial}{\partial x}(\delta w)\,dx\,dy =$$

$$= \int_c^d \left[\frac{\partial F}{\partial w_{xx}} \delta\left(\frac{\partial w}{\partial x}\right)\right]_a^b dy - \int_c^d \left[\frac{\partial}{\partial x}\,\frac{\partial F}{\partial w_{xx}}\,\delta w\right]_a^b dy +$$

$$+ \int_a^b\int_c^d \frac{\partial^2}{\partial x^2}\,\frac{\partial F}{\partial w_{xx}}\,\delta w\,dx\,dy.$$

After analogous conversion of each component of variation of total energy we obtain

$$\delta Э = \int_c^d \left[\frac{\partial F}{\partial u_x}\delta u\right]_a^b dy + \int_c^d \left[\frac{\partial F}{\partial v_x}\delta v\right]_a^b dy + \int_c^d \left(\frac{\partial F}{\partial w_{xx}}\delta\left(\frac{\partial w}{\partial x}\right)\right]_a^b dy +$$

$$+ \int_a^b \left[\frac{\partial F}{\partial u_y}\delta u\right]_c^d dx + \int_a^b \left[\frac{\partial F}{\partial v_y}\delta v\right]_c^d dx + \int_a^b \left[\frac{\partial F}{\partial w_{yy}}\delta\left(\frac{\partial w}{\partial y}\right)\right]_c^d dx -$$

$$- \int_c^d \left[\left(\frac{\partial}{\partial x}\,\frac{\partial F}{\partial w_{xx}} + \frac{\partial}{\partial y}\,\frac{\partial F}{\partial w_{xy}}\right)\delta w\right]_a^b dy -$$

$$- \int_a^b \left[\left(\frac{\partial}{\partial y}\,\frac{\partial F}{\partial w_{yy}} + \frac{\partial}{\partial x}\,\frac{\partial F}{\partial w_{xy}}\right)\delta w\right]_c^d dx + \left[\left(\frac{\partial F}{\partial w_{xy}}\right)_a^b \delta w\right]_c^d -$$

$$- \iint_s \left(\frac{\partial}{\partial x}\,\frac{\partial F}{\partial u_x} + \frac{\partial}{\partial y}\,\frac{\partial F}{\partial u_y} - \frac{\partial F}{\partial u}\right)\delta u\,dx\,dy -$$

$$- \iint_s \left(\frac{\partial}{\partial y}\,\frac{\partial F}{\partial v_y} + \frac{\partial}{\partial x}\,\frac{\partial F}{\partial v_x} - \frac{\partial F}{\partial v}\right)\delta v\,dx\,dy +$$

$$+ \iint_s \left(\frac{\partial^2}{\partial x^2}\,\frac{\partial F}{\partial w_{xx}} + \frac{\partial^2}{\partial y^2}\,\frac{\partial F}{\partial w_{yy}} + \frac{\partial^2}{\partial x\,\partial y}\,\frac{\partial F}{\partial w_{xy}} + \frac{\partial F}{\partial w}\right)\delta w\,dx\,dy = 0.$$

The first nine components of this equation refer to boundaries of integration range and characterize boundary conditions of the given problem. The remaining three components, standing under signs of double integrals, must be spread throughout the entire integration range.

This variation of total energy according to the origin of virtual displacements will be equal to zero with any combination of components entering it. However, it is possible to impose the following requirements: 1) all contour integrals, which enter the expression of variation of total energy at any arbitrary point of the boundary range, must be equal to zero; this requirement must subsequently be fulfilled with solution of particular problems, proceeding from boundary conditions; 2) in view of the arbitrariness of variation δu, δv, δw the sum of the last three components can be equal to zero when each function, standing under the sign of double integral in parentheses, at an arbitrary point of the range will be equal to zero. In view of the aforesaid we obtain the following three groups of equations:

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}+\frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}-\frac{\partial F}{\partial u}=0,$$

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial v_x}+\frac{\partial}{\partial y}\frac{\partial F}{\partial v_y}-\frac{\partial F}{\partial v}=0,$$

$$\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{xx}}+\frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial w_{yy}}+\frac{\partial^2}{\partial x\,\partial y}\frac{\partial F}{\partial w_{xy}}+\frac{\partial F}{\partial w}=0,$$

$$\left.\begin{aligned}
&\frac{\partial F}{\partial u_x}\,\delta u=0,\\
&\frac{\partial F}{\partial v_x}\,\delta v=0,\\
&\frac{\partial F}{\partial w_{xx}}\,\delta\left(\frac{\partial w}{\partial x}\right)=0,\\
&\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial w_{xx}}+\frac{\partial}{\partial y}\frac{\partial F}{\partial w_{xy}}\right)\delta w=0.
\end{aligned}\right\}\ \text{when}\ x=a,\ x=b$$

$$\left.\begin{aligned}
&\frac{\partial F}{\partial u_y}\,\delta u=0,\\
&\frac{\partial F}{\partial v_y}\,\delta v=0,\\
&\frac{\partial F}{\partial w_{yy}}\,\delta\left(\frac{\partial w}{\partial y}\right)=0,\\
&\left(\frac{\partial}{\partial y}\frac{\partial F}{\partial w_{yy}}+\frac{\partial}{\partial x}\frac{\partial F}{\partial w_{xy}}\right)\delta w=0.
\end{aligned}\right\}\ \text{when}\ y=c,\ y=d$$

The first group gives equilibrium equations of the shell. The two other groups of equations determine boundary conditions, which must be fulfilled during solution of concrete problem.

Each equation of the last two groups is a product of force factor and displacement corresponding to it. Therefore, on the contour the force factor of displacement corresponding to it should be equal to zero.

Values of partial derivatives of function F have the form

$$\frac{\partial F}{\partial u} = -q_x,$$
$$\frac{\partial F}{\partial u_x} = \frac{E\delta}{1-\mu^2}\left[\frac{\partial u}{\partial x} + \frac{w}{R_1} + \mu\left(\frac{\partial v}{\partial y} + \frac{w}{R_2}\right)\right] = N_x,$$
$$\frac{\partial F}{\partial u_y} = \frac{E\delta}{2(1+\mu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = N_{xy},$$
$$\frac{\partial F}{\partial v} = -q_y,$$
$$\frac{\partial F}{\partial v_x} = \frac{E\delta}{2(1+\mu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = N_{xy},$$

$$\frac{\partial F}{\partial v_y} = \frac{E\delta}{1-\mu^2}\left[\frac{\partial v}{\partial y} + \frac{w}{R_2} + \mu\left(\frac{\partial u}{\partial x} + \frac{w}{R_2}\right)\right] = N_y,$$
$$\frac{\partial F}{\partial w_{xx}} = \frac{E\delta^3}{12(1-\mu^2)}\left[\frac{\partial^2 w}{\partial x^2} + \frac{w}{R_1^2} + \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R_2^2}\right)\right] = M_x,$$
$$\frac{\partial F}{\partial w_{yy}} = \frac{E\delta^3}{12(1-\mu^2)}\left[\frac{\partial^2 w}{\partial y^2} + \frac{w}{R_2^2} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{w}{R_1^2}\right)\right] = M_y,$$
$$\frac{\partial F}{\partial w_{xy}} = 2(1-\mu)\frac{E\delta^3}{12(1-\mu^2)}\,\frac{\partial^2 w}{\partial x\,\partial y} = 2M_{xy},$$
$$\frac{\partial F}{\partial w} = \frac{E\delta}{(1-\mu^2)R_1}\left[\frac{\partial u}{\partial x} + \frac{w}{R_1} + \mu\left(\frac{\partial v}{\partial y} + \frac{w}{R_2}\right)\right] + \frac{E\delta}{(1-\mu^2)R_2}\left[\frac{\partial v}{\partial y} + \frac{w}{R_2} + \right.$$
$$\left. + \mu\left(\frac{\partial u}{\partial x} + \frac{w}{R_1}\right)\right] + \frac{E\delta^3}{12(1-\mu^2)R_1^2}\left[\frac{\partial^2 w}{\partial x^2} + \frac{w}{R_1^2} + \mu\left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R_2^2}\right)\right] +$$
$$+ \frac{E\delta^3}{12(1-\mu^2)R_2^2}\left[\frac{\partial^2 w}{\partial y^2} + \frac{w}{R_2^2} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{w}{R_1^2}\right)\right] -$$
$$-q_z = \frac{N_x}{R_1} + \frac{N_y}{R_2} + \frac{M_x}{R_1^2} + \frac{M_y}{R_2^2} - q_z.$$

Let us represent boundary conditions and equations of equilibrium in the following form:

$$\left.\begin{aligned} N_x\delta u &= 0,\\ N_{xy}\delta v &= 0,\\ M_x\delta\left(\frac{\partial w}{\partial x}\right) &= 0,\\ \left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right)\delta w &= 0,\end{aligned}\right\}\ \text{when}\ x=a,\ x=b$$

$$\left.\begin{aligned} N_y\delta v &= 0,\\ N_{xy}\delta u &= 0,\\ M_y\delta\left(\frac{\partial w}{\partial y}\right) &= 0,\\ \left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}\right)\delta w &= 0,\end{aligned}\right\}\ \text{when}\ y=c,\ y=d$$

$$
\left.
\begin{aligned}
&\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial}{\partial x}\left(\frac{w}{R_1}\right)+\\
&\qquad+\mu\frac{\partial}{\partial x}\left(\frac{w}{R_2}\right)=-\frac{(1-\mu^2)\,q_x}{E\delta},\\
&\frac{\partial^2 v}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial}{\partial y}\left(\frac{w}{R_2}\right)+\\
&\qquad+\mu\frac{\partial}{\partial y}\left(\frac{w}{R_1}\right)=-\frac{(1-\mu^2)\,q_y}{E\delta},\\
&D\left\{\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{\partial^2}{\partial x^2}\left(\frac{w}{R_1^2}+\mu\frac{w}{R_2^2}\right)+\frac{\partial^2}{\partial y^2}\left(\frac{w}{R_2^2}+\mu\frac{w}{R_1^2}\right)+\right.\\
&+\frac{1}{R_1^2}\left[\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\mu\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}\right)\right]+\frac{1}{R_2^2}\left[\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\right.\\
&\left.\left.+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}\right)\right]\right\}+B\left\{\frac{1}{R_1}\left[\frac{\partial u}{\partial x}+\frac{w}{R_1}+\mu\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)\right]+\right.\\
&\left.+\frac{1}{R_2}\left[\frac{\partial v}{\partial y}+\frac{w}{R_2}+\mu\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)\right]\right\}=q_z.
\end{aligned}
\right\}
\tag{7.5}
$$

From these equations we can easily obtain equations of both the two-dimensional problem of the theory of elasticity ($w = 0$), and equations of bending plates when $R_1 = \infty$, $R_2 = \infty$, $u = 0$, $v - 0$, and also bending of beams.

If components of surface load q_x and q_y are absent and radii of curvature of the shell do not depend on x and y, the given three equations allow considerable simplifications. Let us introduce the function of stresses for this case according to formulas

$$
\left.
\begin{aligned}
N_x&=\frac{E\delta}{1-\mu^2}\left[\frac{\partial u}{\partial x}+\frac{w}{R_1}+\mu\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)\right]=\frac{\partial^2\varphi}{\partial y^2},\\
N_y&=\frac{E\delta}{1-\mu^2}\left[\frac{\partial v}{\partial y}+\frac{w}{R_2}+\mu\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)\right]=\frac{\partial^2\varphi}{\partial x^2},\\
N_{xy}&=\frac{E\delta}{2(1+\mu)}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=-\frac{\partial^2\varphi}{\partial x\,\partial y}.
\end{aligned}
\right\}
\tag{7.6}
$$

If we substitute these expressions in (7.5), the first two of them will be satisfied, and the third takes the form

$$
\begin{aligned}
&\frac{1}{R_1}\frac{\partial^2\varphi}{\partial y^2}+\frac{1}{R_2}\frac{\partial^2\varphi}{\partial x^2}+D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\right.\\
&\left.+2\left(\frac{1}{R_1^2}+\frac{\mu}{R_2^2}\right)\frac{\partial^2 w}{\partial x^2}+2\left(\frac{1}{R_2^2}+\frac{\mu}{R_1^2}\right)\frac{\partial^2 w}{\partial y^2}+\left(\frac{1}{R_1^4}+\frac{1}{R_1^2R_2^2}+\frac{1}{R_2^4}\right)w\right]=q_z.
\end{aligned}
\tag{7.7}
$$

The solution of many concrete problems shows that the specific weight of underlined components, standing in brackets, in comparison with the remaining components is comparatively small. Therefore, for the sake of simplicity (7.7) it is possible to drop these components and replace the equation by approximate:

$$\frac{1}{R_1}\frac{\partial^2\varphi}{\partial y^2}+\frac{1}{R_2}\frac{\partial^2\varphi}{\partial x^2}+D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q_z. \qquad (7.8)$$

The second equation, connecting functions ϕ and w, can be obtained from expressions (of 7.6) by exclusion of displacements u and v from them:

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=E\delta\left(\frac{1}{R_1}\frac{\partial^2 w}{\partial y^2}+\frac{1}{R_2}\frac{\partial^2 w}{\partial x^2}\right). \qquad (7.9)$$

Let us write equations (7.8) and (7.9) in symbolic form:

$$\begin{aligned}\nabla_k^2\varphi+D\nabla^2\nabla^2 w&=q_z,\\ \nabla^2\nabla^2\varphi&=E\delta\nabla_k^2 w,\end{aligned} \qquad (7.10)$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},$$

$$\nabla_k^2=\frac{1}{R_1}\frac{\partial^2}{\partial y^2}+\frac{1}{R_2}\frac{\partial^2}{\partial x^2}.$$

To the first of the obtained equations let us apply operation $\nabla^2\nabla^2$, and to the second – ∇_k^2, and then let us subtract the second equation from the first. Then we obtain one resolvent of the eighth order

$$E\delta\nabla_k^2\nabla_k^2 w+D\nabla^2\nabla^2\nabla^2\nabla^2 w=\nabla^2\nabla^2 q_z.$$

With introduction of the function of stresses the blundary conditions for membrane forces and tangential displacements will take the form

[1]This equation was obtained for the first time by V. Z. Vlasov.

$$\left.\begin{aligned}\frac{\partial^2\varphi}{\partial y^2}\,\delta u=0,\\ \frac{\partial^2\varphi}{\partial x\,\partial y}\,\delta v=0,\end{aligned}\right\}\ \text{when } \begin{aligned}x=a\\ x=b\end{aligned}$$

$$\left.\begin{aligned}\frac{\partial^2\varphi}{\partial x^2}\,\delta v=0,\\ \frac{\partial^2\varphi}{\partial x\,\partial y}\,\delta u=0.\end{aligned}\right\}\ \text{when } \begin{aligned}y=c\\ y=d\end{aligned}$$

Equations (7.7) and (7.9) together with the corresponding boundary conditions permit solving an extensive circle of practically important problems on the strength and stability of shells. However, one should note that these equations allow accurate solution within limits of hypotheses of applied theory of shells only in certain particular cases. In most cases they can be solved either by known approximate methods or with utilization of computers.

Let us examine some particular problems, in which utilization of equations (7.10) is shown.

§ 35. Calculation of a Cylindrical Shell from the Action of Concentrated Forces and Moments

Let us examine the calculation of a cylindrical shell, closed on both ends with bottoms, to which through a rigid boss, inserted into the wall of the shell, concentrated moment M is applied (Fig. 112). It is also asssumed that the ends of shell have hinged support. Let us determine the stressed and deformed state of the shell for the shown case.

When $R_1 = \infty$, $R_2 = R$ differential equations (7.8) and (7.9) will have the form

$$\begin{gathered}\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=\frac{E\delta}{R}\,\frac{\partial^2 w}{\partial x^2},\\ \frac{1}{R}\,\frac{\partial^2\varphi}{\partial x^2}+D\left(\frac{\partial^4 w}{\partial x^4}+2\,\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q_z.\end{gathered}\qquad(7.11)$$

We will seek the solution of these equations in the form of

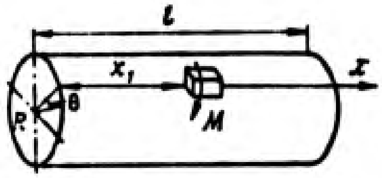

Fig. 112.

double trigonometric series, having placed the origin of coordinates (Fig. 112) at the left end of the shell:

$$w_1 = \sum_n \sum_m A_{1mn} \sin\frac{m\pi x}{l} \sin n\theta,$$

$$\varphi_1 = \sum_n \sum_m B_{1mn} \sin\frac{m\pi x}{l} \sin n\theta.$$

These functions satisfy the following boundary conditions:

$$\left.\begin{aligned} w&=0,\\ M_x&=0, \end{aligned}\right\} \text{when } x=0,\ x=l$$

$$\left.\begin{aligned} v&=0,\\ N_x&=\frac{\partial^2\varphi}{R\partial\theta^2}=0, \end{aligned}\right\} \text{when } x=0,\ x=l$$

Furthermore, when $\theta = 0$ deflection and the function of stresses, as follows from the character of loading, become zero and are odd functions of angle θ.

For determination of coefficients A_{1mn} and B_{1mn} it is necessary to substitute expressions w_1 and ϕ_1 in equations (7.11). Before this is done, let us also expand the external effective load into double trigonometric series in terms of the sought functions. In our case the external load is represented in the form of concentrated moment, which is statically equivalent to force couple with arm $d=[(2\pi-\beta_2)+\beta_1]R$ (Fig. 113). Therefore, it is possible to write

$$P_1 = \frac{0{,}5M_{\text{об}}}{R\beta_1}, \quad P_2 = \frac{0{,}5M_{\text{об}}}{(2\pi-\beta_2)R}.$$

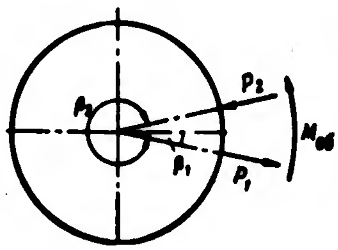

Fig. 113.

Thus, in the considered case the external load is represented in the form of two concentrated forces $|P_1| = |P_2|$. In order to bring this load to the dimension of distributed pressure q_z it is necessary to represent it in the form

$$q_z = \frac{P_1}{\Delta F},$$

where $\Delta F = \Delta s_1 \Delta x_1$ – small area on which force P_1 is applied; $\Delta s_1, \Delta x_1$ – dimensions of this area in circumferential and axial directions.

Then

$$q_z = \frac{P_1}{\Delta s_1 \Delta x_1} = \sum_n \sum_m C_{mn} \sin \frac{m\pi x}{l} \sin n\theta.$$

Let us multiply the right and left sides of this expression by $\sin m\pi x/l \sin n\theta d\theta dx$ and integrate the right side with respect to x from 0 to l and with respect to θ from 0 to 2π, and the left side with respect to x from value x_1 to $x_1 + \Delta x_1$ and with respect to θ from value β_1 to $\beta_1 + \Delta\beta_1$. By solving the equation obtained after this integration relative to C_{mn} we find

$$C_{mn} = \frac{2\frac{P_1}{\Delta s_1 \Delta x_1}}{\pi R l} \frac{l}{\pi m}\left[\cos \frac{m\pi (x_1 + \Delta x_1)}{l} - \cos \frac{m\pi x_1}{l}\right] \times$$
$$\times \frac{R}{n}\left[\cos n(\beta_1 + \Delta\beta_1) - \cos n\beta_1\right].$$

Converting to the limit when $\Delta s_1 \to 0$, $\Delta x_1 \to 0$ (where $\Delta\beta_1 = \frac{\Delta s_1}{R}$, $\beta_1 = \frac{s_1}{R}$), we obtain

$$C_{mn}=\frac{2P_1}{\pi^2 mn}\lim_{\Delta x_1\to 0}\frac{\cos\frac{m\pi(x_1+\Delta x_1)}{l}-\cos\frac{m\pi x_1}{l}}{\Delta x_1}\times$$

$$\times\lim_{\Delta s_1\to 0}\frac{\cos\frac{n(s_1+\Delta s_1)}{R}-\cos\frac{ns}{R}}{\Delta s_1},$$

or

$$C_{mn}=\frac{2P_1}{\pi Rl}\sin\frac{m\pi x_1}{l}\sin n\beta_1$$

and

$$q_z=\frac{2P_1}{\pi Rl}\sum_n\sum_m \sin\frac{m\pi x_1}{l}\sin n\beta_1\sin\frac{m\pi x}{l}\sin n\theta.$$

Here x_1 and β_1 – coordinates of the point of application of force P_1.

Having substituted the corresponding derivatives of functions ϕ_1 and w_1 and the value of external load q_z in equations (7.11), we obtain

$$B_{1mn}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2=-E\delta RA_{1mn}\left(\frac{m\pi R}{l}\right)^2,$$

$$-\frac{B_{1mn}}{R^3}\left(\frac{m\pi R}{l}\right)^2+\frac{DA_{1mn}}{R^4}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2=\frac{2P_1}{\pi Rl}\sin\frac{m\pi x_1}{l}\sin n\beta_1.$$

By solving these equations relative to parameters A_{1mn} and $B1_{mn}$ we find

$$A_{1mn}=\frac{2P_1R\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\sin n\beta_1}{\pi E\delta l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}},$$

$$B_{1mn}=-\frac{2P_1R^2\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}\sin n\beta_1}{\pi l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}}.$$

Having substituted the values of force P_1 in these expressions by formula $P_1=\frac{0.5M_{o6}}{R\beta_1}$, we obtain [Trans. Note: об = shell].

$$A_{1mn}=\frac{M_{o6}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\,\frac{\sin n\beta_1}{\beta_1}}{\pi E\delta l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}},$$

$$B_{1mn}=-\frac{M_{o6}R\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}\,\frac{\sin n\beta_1}{\beta_1}}{\pi l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}}.$$

For force P_2 we obtain analogous expressions, but with opposite signs and replacements of β_1 by β_2, and $n\beta_1$ by $n(2\pi-\beta_2)$, i.e.,

$$A_{2mn}=-\frac{M_{o6}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\,\frac{\sin n\beta_2}{2\pi-\beta_2}}{\pi E\delta l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^2\right\}},$$

$$B_{2mn}=\frac{M_{o6}R\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}\,\frac{\sin n\beta_2}{2\pi-\beta_2}}{\pi l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}}.$$

By converting in the obtained expressions for A_{1mn}, B_{1mn}, A_{2mn} B_{2mn} to the limit $\beta_1 \to 0$, $\beta_2 \to 2\pi$, we have

$$A_{1mn}=\frac{M_{o6}n\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}}{\pi E\delta l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}},$$

$$A_{2mn}=\frac{M_{o6}n\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}}{\pi E\delta l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}},$$

$$B_{1mn}=-\frac{M_{o6}Rn\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}}{\pi l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}},$$

$$B_{2mn}=-\frac{M_{o6}Rn\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}}{\pi l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}}.$$

Total solution of the problem is found by super impositions of obtained solutions

$$w = w_1 + w_2 =$$

$$= \frac{2M_{об}}{\pi E\delta l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^2 \sin\frac{m\pi x_1}{l} \sin\frac{m\pi x}{l} \sin n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^4 + \left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi = \varphi_1 + \varphi_2 =$$

$$= -\frac{2M_{об}R}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{n\left(\frac{m\pi R}{l}\right)^2 \sin\frac{m\pi x_1}{l} \sin\frac{m\pi x}{l} \sin n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^4 + \left(\frac{m\pi R}{l}\right)^4}.$$

By having expressions for deflection and functions of stresses, all the internal force factors, which appear in the shell from moment $M_{об}$ can be obtained:

$$N_x = \frac{\partial^2\varphi}{R^2\partial\theta^2}, \quad N_y = \frac{\partial^2\varphi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2\varphi}{R\partial\theta\,\partial x},$$

$$M_x = D(\chi_x + \mu\chi_y), \quad M_y = D(\chi_y + \mu\chi_x).$$

In the case of action of two diametrically opposite bending moments (Fig. 114) the expressions for w and ϕ take the form

$$w = \frac{4M_{об}}{\pi E\delta l} \sum_{m=1}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^2 \sin\frac{m\pi x_1}{l} \sin\frac{m\pi x}{l} \sin n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^4 + \left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi = -\frac{4M_{об}R^2}{\pi l} \sum_{m=1}^{\infty} \sum_{n=1,3,5,\ldots}^{\infty} \frac{n\left(\frac{m\pi R}{l}\right)^2 \sin\frac{m\pi x_1}{l} \sin\frac{m\pi x}{l} \sin n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^4 + \left(\frac{m\pi R}{l}\right)^4}.$$

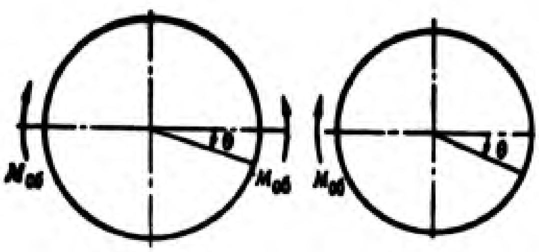

Fig. 114.

These expressions are also obtained by superposition of solutions found separately for two cases of loading (Fig. 115).

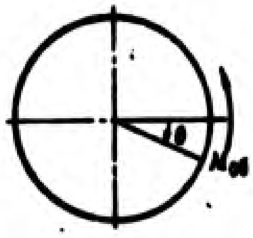

Fig. 115.

Sometimes at attachment points of longerons to the body of the aircraft the latter can have support of various type in the form of rod frames or frames.

As illustration of the calculation of constructions of similar type let us suppose that the shell examined above is reinforced by a square or annular frame (frame) or straight rod, passed through the shell along the diameter of its cross section.

Let us determine, in the first place, what part of the bending moment the shell takes and what part the support, and, secondly, let us determine the force of interaction between the shown supports and the shell, if the latter is under the action of internal pressure.

Distribution of moment M between the shell and support is found from equations

$$(\varphi)_{под} = \left(\frac{w}{R\partial\vartheta}\right)_{\substack{\vartheta=0\\ \vartheta=\pi}}, \qquad M_{под} + M_{об} = M, \tag{7.12}$$

where $(\phi)_{под}$ – angle of rotation of reinforcement point; $\left(\frac{w}{R\partial\vartheta}\right)$ – angle of rotation of the tangent to the shell at the place of application of moment M; $M_{под}$ – moment perceived by support; $M_{об}$ – moment perceived by shell.

Rotation of tangent

$$\left|\frac{\partial w}{R\partial\vartheta}\right|_{\substack{\vartheta=0\\ x=x_1}} = \left|\frac{\partial w}{R\partial\vartheta}\right|_{\substack{\vartheta=\pi\\ x=x_1}} =$$

$$=\frac{4M_{об}}{\pi E\delta Rl}\sum_{m=1}^{\infty}\sum_{n=1,3,5\ldots}^{\infty}\frac{n^2\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin^2\frac{m\pi x_1}{l}}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4}.$$

Now the angle of rotation $(\psi)_{под}$ must be determined. First let us cut the frame (Fig. 116) mentally along the vertical diameter; at the place of cut let us apply force N and moment m_0, and restrain the lower section. Let us find unknown internal forces N and m_0 by using Castigliano theorem. Thus, we have the following equations:

$$\frac{\partial Э}{\partial N}=0,\quad \frac{\partial Э}{\partial m_0}=0,\quad (\psi)^{ш}_{под}=\frac{\partial Э}{\partial M_{ш}}, \tag{7.13}$$

where Э – internal potential energy of semiring. For its computation we have expression

$$Э=\frac{R}{2EJ}\int_0^{\frac{\pi}{2}}M_1^2d\beta+\frac{R}{2EJ}\int_{\frac{\pi}{2}}^{\pi}M_2^2d\beta,$$

where the moment for sections 1 and 2 of the frame

$$M_1=NR(1-\cos\beta)-m_0;$$
$$M_2=NR(1-\cos\beta)-m_0-M_{ш};$$

J – intrinsic moment of inertia of the frame relative to axis, parallel to generatrix of the shell.

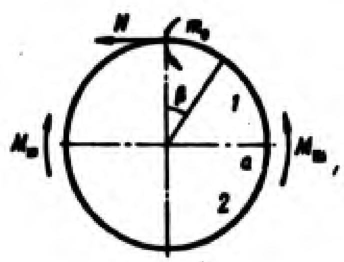

Fig. 116.

After taking the quadrature in the expression for Э and use of equation (7.13) we obtain

$$N=\frac{2M_{ш}}{\pi R},\quad m_0=\frac{1}{2}\left(\frac{4}{\pi}-1\right)M_{ш},$$
$$(\psi)^{ш}_{под}=0{,}149\,\frac{M_{ш}R}{EJ}.$$

Now as the supporting element let us examine a square frame (Fig. 117). For this instance the equations for determination of angle of rotation $(\psi)^{P}_{под}$ have the form

$$\frac{\partial Э}{\partial N}=0, \quad \frac{\partial Э}{\partial P}=0, \quad \frac{\partial Э}{\partial m_0}=0, \quad (\psi)^{P}_{под}=\frac{\partial Э}{\partial M_p}, \tag{7.14}$$

where

$$Э=\frac{1}{2EJ}\int_0^a M_1^2 dx+\frac{1}{2EJ}\int_0^a M_2^2 dx,$$

$$M_1=-m_0+Nx,$$

$$M_2=-m_0-M_p+Na-Px;$$

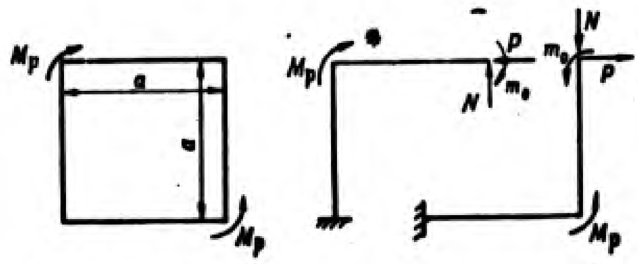

Fig. 117.

J – moment of inertia of cross section of rod relative to its own axis.

Having completed the shown integration and using conditions (7.14) we obtain

$$N=P=\frac{3}{4\sqrt{2}}\frac{M_p}{R}; \quad m_0=\frac{M_p}{4}; \quad (\psi)^{P}_{под}=\frac{\sqrt{2}RM_p}{8EJ}.$$

Let us examine the effect of rod support (Fig. 118). In this instance the angle of rotation will be determined by formula

$$(\psi)^{c}_{под}=\frac{RM_c}{EJ}.$$

By comparing the obtained expressions for angles of rotation at various supports, it is possible to see that the most "rigid" support is the frame.

Fig. 118.

By using conditions (7.12), the distribution of applied moment M between the shell and support can be obtained. For example, in the case of support by a frame

$$M'_{об}=\frac{0,149M}{0,149+\frac{4J}{\pi^3 R^4 l}\sum_{m=1}^{\infty}\sum_{n=1,3,5\ldots}^{\infty}\frac{n^2\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin^2\frac{m\pi x_1}{l}}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4}},$$

$$M''_{под}=M-M'_{об}.$$

Analogously we can obtain expressions for $M_{об}$ and $M_{под}$ for the remaining cases of support.

Let us examine the problem of determination of forces of interaction between the support and shell, when the latter is under the action of internal pressure. We will consider that in this case the shell is loaded by four concentrated forces according to Fig. 119. Subsequently for convenience let us designate the effective forces by indices 1, 2, 3, 4.

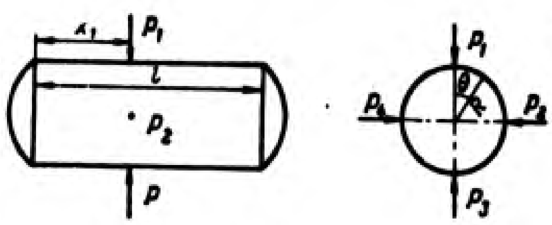

Fig. 119.

For solution of this problem let us again use equations (7.11). In this case the expressions for function of deflection and function of stresses must be odd relative to angle θ, i.e., $w(+\theta)=w(-\theta)$, $\varphi(+\theta)=\varphi(-\theta)$. Proceeding from this we take

$$w_1 = \sum\sum A_{1mn} \sin\frac{m\pi x}{l} \cos n\theta,$$
$$\varphi_1 = \sum\sum B_{1mn} \sin\frac{m\pi x}{l} \cos n\theta.$$

Let us begin solution of problem with examination of the case of action of one force P_1. Let us represent external load q_z in the form

$$q_z = \frac{P_1}{\Delta s_1 \Delta x_1}$$

and expand it into double series in terms of the sought functions:

$$q_z = \frac{P_1}{\Delta s_1 \Delta x_1} \sum\sum C_{mn} \sin\frac{m\pi x}{l} \cos n\theta.$$

Let us multiply the right and left sides of this expression by $\sin m\pi x/l \cos n\theta dx d\theta$ and integrate – the right side within limits with respect to x from 0 to l, with respect to θ – from 0 to 2π, and the left side with respect to x within limits from x_1, to $x_1 + \Delta x_1$ and with respect to θ within limits from β_1 to $\beta_1 + \Delta\beta_1$. Then when $\Delta x_1 \to 0$, $\Delta\beta_1 \to 0$ we obtain

$$C_{mn} = \frac{2P_1}{\pi R l} \sin\frac{m\pi x_1}{l} \cos n\beta_1.$$

With allowance for the value of coefficient C_{mn} for q_z we have expression

$$q_z = \frac{2P_1}{\pi R l} \sum\sum \sin\frac{m\pi x_1}{l} \cos n\beta_1 \sin\frac{m\pi x}{l} \cos n\theta.$$

By substituting the expressions for functions ϕ_1 and w_1, and also q_z in equations (7.11), we obtain two equations for determining parameters A_{1mn} and B_{1mn}

$$B_{1mn}\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^2 = -E\delta R A_{1mn}\left(\frac{m\pi R}{l}\right)^2,$$
$$-\frac{B_{1mn}}{R^3}\left(\frac{m\pi R}{l}\right)^2 + \frac{DA_{mn}}{R^4}\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^2 = \frac{2P_1}{\pi R l} \sin\frac{m\pi x_1}{l} \cos n\beta_1.$$

From these equations we find

$$A_{1mn}=\frac{2P_1R\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\cos n\beta_1}{\pi E\delta l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}},$$

$$B_{1mn}=-\frac{2P_1R^2\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}\cos n\beta_1}{\pi l\left\{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right\}}.$$

Having substituted these values of A_{1mn} and B_{1mn} in expressions for ϕ_1 and w_1, we obtain

$$w_1=\frac{2P_1R}{\pi E\delta l}\sum_n\sum_m\frac{\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\cos n\beta_1\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi_1=-\frac{2P_1R^2}{\pi l}\sum_n\sum_m\frac{\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}\cos n\beta_1\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4}.$$

By analogy with the obtained solution for force P_1 it is possible to write expressions for deflection and function of stresses for the remaining forces by replacement of angle β_1 by β_2, β_3, β_4:

$$w_2=\frac{2P_2R}{\pi E\delta l}\sum_n\sum_m\frac{\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\cos n\beta_2\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi_2=-\frac{2P_2R^2}{\pi l}\sum_n\sum_m\frac{\left(\frac{m\pi R}{l}\right)^2\sin\frac{m\pi x_1}{l}\cos n\beta_2\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4},$$

$$w_3=\frac{2P_3R}{\pi E\delta l}\sum_n\sum_m\frac{\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2\sin\frac{m\pi x_1}{l}\cos n\beta_3\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi_3 = -\frac{2P_3R^2}{\pi l}\sum_n\sum_m \frac{\left(\frac{m\pi R}{l}\right)^2 \sin\frac{m\pi x_1}{l}\cos n\beta_3 \sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4\right)},$$

$$w_4 = \frac{2P_4R}{\pi E\delta l}\sum_n\sum_m \frac{\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2 \sin\frac{m\pi x_1}{l}\cos n\beta_4 \sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi_4 = -\frac{2P_4R^2}{\pi l}\sum_n\sum_m \frac{\left(\frac{m\pi R}{l}\right)^2 \sin\frac{m\pi x_1}{l}\cos n\beta_4 \sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4}.$$

Final expressions for deflection of the function of stresses are found by superposition of obtained solutions. Assuming that the shell works within limits of elasticity, and taking $P_1 = P_2 = P_3 = P_4 = P$, $\beta_1 = 0$, $\beta_2 = \frac{\pi}{2}$, $\beta_3 = \pi$, $\beta_4 = \frac{3\pi}{2}$, we find

$$w = w_1 + w_2 + w_3 + w_4 =$$

$$= \frac{8PR}{\pi E\delta l}\sum_{m=1}^{\infty}\sum_{n=0,4,8,12}^{\infty} \frac{\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2 \sin\frac{m\pi x_1}{l}\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4},$$

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 =$$

$$= -\frac{8PR^2}{\pi l}\sum_{m=1}^{\infty}\sum_{n=0,4,8,12}^{\infty} \frac{\left(\frac{m\pi R}{l}\right)^2 \sin\frac{m\pi x_1}{l}\sin\frac{m\pi x}{l}\cos n\theta}{\frac{D}{E\delta R^2}\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^4+\left(\frac{m\pi R}{l}\right)^4}.$$

By using the obtained expressions for functions ϕ and w, we can determine all internal forces in the cylindrical shell, which are under the action of one, two, three and four concentrated radial forces, by using the method of superposition.

Now let us derive expressions for displacements of frame and rod, loaded by concentrated forces, in the direction of radius of the shell. When considering the frame (Fig. 120) as an annular frame it was established that radial displacement under forc

$$w_ш = 0{,}006\frac{PR^3}{EJ}.$$

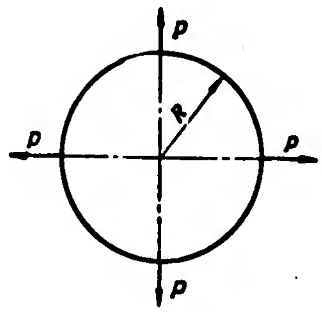

Fig. 120.

In the case of a square frame (Fig. 121) we obtained

$$w_p = 0{,}707\frac{PR}{EF}.$$

In the case of tension of rod

$$w_c = \frac{PR}{EF}.$$

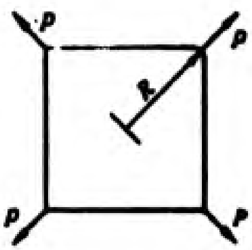

Fig. 121.

Increase in the radius of cylinder under the action of internal pressure

$$w_{\text{ц}} = \frac{(1-0{,}5\mu)\,qR^2}{E\delta}.$$

Condition of compatibility of deformations of the system cylindrical shell-support will have the form

$$w_{\text{ц}} - (w_{\text{об}})_{\substack{x=x_1\\ \theta=0}} = w_{\text{кол}}.$$

Thus, for instance, in the case of reinforcement of shell by a frame for force of interaction P we obtain expression

$$P=\frac{(1-0.5\mu)\,qR^{2}}{\frac{0{,}006R^{3}\delta}{J}+\frac{8R}{\pi l}\sum_{m=1}^{\infty}\sum_{n=0,4,8,12}^{\infty}\frac{\left[\left(\frac{m\pi R}{l}\right)^{2}+n^{2}\right]^{2}\sin^{2}\frac{m\pi x_{1}}{l}}{\frac{D}{E\delta R^{2}}\left[\left(\frac{m\pi R}{l}\right)^{2}+n^{2}\right]^{4}+\left(\frac{m\pi R}{l}\right)^{4}}}.$$

The expression for force of interaction at the other types of supports, examined above, can be obtained analogously.

§ 36. Loading of a Cylindrical Shell by Local Circumferential and Axial Bending Moments

Let us examine loading of a cylindrical shell by local circumferential or axial bending moments. Problems of such type must be encountered, for example, during calculation of suspended cylinders or tanks, resting on a number of brackets, or steel pipeline framework (flanges, branch pipes and so forth) fastened by some means.

As can be seen from provided diagrams of loading (Figs. 122 and 123) the problem is reduced to calcu ıtion of a shell for distributed pressure q_z, equal to the specific pressure on the surface of the shell on the part of applied moment M.

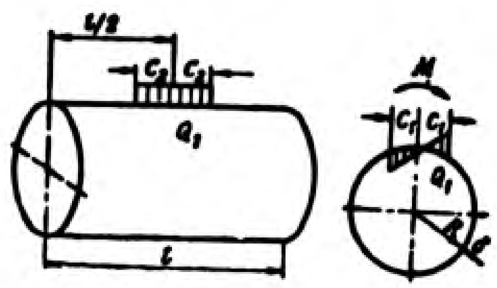

Fig. 122.

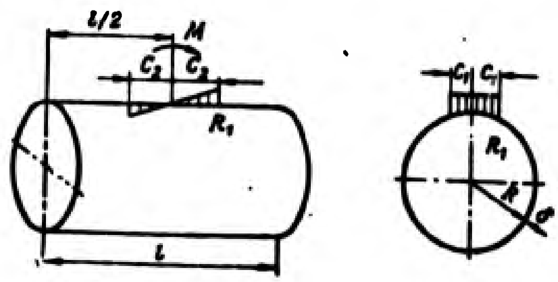

Fig. 123.

For calculation it is possible to accept that pressure q_z is changed according to linear law, as shown in Figs. 122, 123. Having expanded this load into double Fourier series in terms of the sought functions ϕ and w and having substituted all these expressions in (7.11), we obtain the desired solution of the problem.

In this case we are limited to listing the finished results of such calculation.[1]

With loading of the shell by circumferential moment M of the magnitudes of internal elastic moments M_x and M, and also magnitudes of internal membrane forces N_x and N_θ are listed in Figs. 124 and 125. These magnitudes are calculated for point Q_1 (see Fig. 122) with square support areas $C_1 = C_2 = C$. The charts are valid when $l/R \geq 1$.

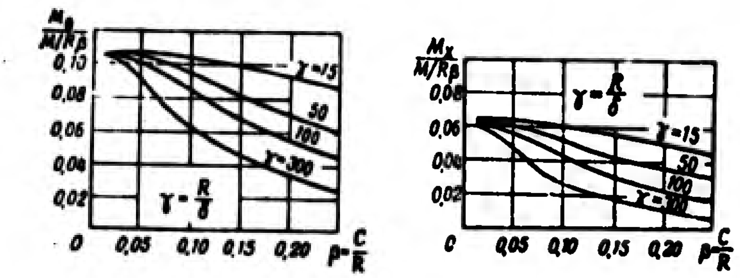

Fig. 124.

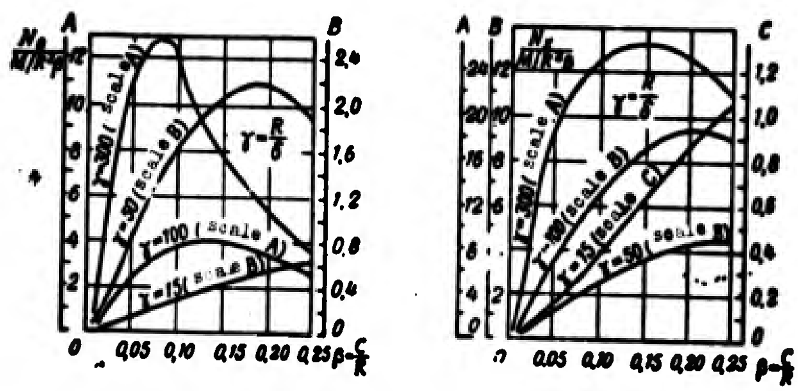

Fig. 125.

From the loading diagram of Fig. 122 it is evident that at point Q_1 the stresses from bending moments will be compressive in the external fiber; the membrane forces also cause compressive stresses.

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[1]Collection. "Questions of strength of cylindrical shells", translated from English, Oborongiz, 1960.

Figure 126 shows charts of angles of rotation θ. This angle is equal to the ratio of deflection at point Q_1 (see Fig. 122) to dimension C_1. Curves *A*, *B*, *C*, *D* are constructed for $l/R = 4$. At other quantities of l/R angle θ can be determined by using a correcting curve, placed in the right lower corner of Fig. 126. For this purpose first we determine the angle of rotation according to curves *A*, *B*, *C*, *D* when $l/R = 4$. Then this angle is refined, using correcting curve in percentage in comparison with $l/R = 4$.

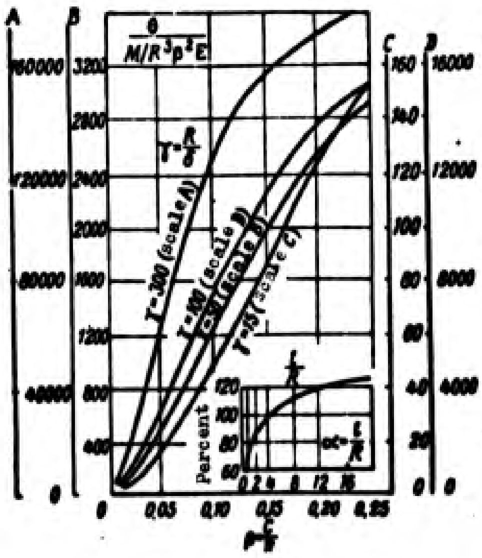

Fig. 126.

If the support area of application of circumferential moment will be rectangular, then angle of rotation θ, and also bending moments M_θ and M_x at point Q_1 can be determined in the following manner.

First we find quantity $\sqrt[3]{\beta_1^2\beta_2} = \sqrt[3]{\left(\frac{C_1}{R}\right)^2\left(\frac{C_2}{R}\right)}$ for the given rectangular area. Then this quantity is multiplied by coefficient k_c and quantity β is obtained equal to

$$\beta = k_c\sqrt[3]{\beta_1^2\beta_2},$$

where coefficient k_c is taken from Table 6 for the given ratio C_1/C_2 and R/δ. By the obtained value of β in Figs. 124 or 126 the bending moments or angle of rotation are found.

Table 6.

C_1/C_2	R/δ	k_c for θ	k_c for M_θ	k_c for M_x	C_c for N_θ	C_c for N_x
1/4	15	1,09	1,31	1,84	0,31	0,49
1/4	50	1,04	1,24	1,62	0,21	0,46
1/4	100	0,97	1,16	1,45	0,15	0,44
1/4	300	0,92	1,02	1,17	0,09	0,46
1/2	15	1,00	1,09	1,36	0,64	0,75
1/2	50	0,98	1,08	1,31	0,57	0,75
1/2	100	0,94	1,04	1,26	0,51	0,76
1/2	300	0,95	0,99	1,13	0,39	0,77
2	15	1,00	1,20	0,97	1,70	1,30
2	100	1,19	1,10	0,95	1,43	1,12
2	300	—	1,00	0,90	1,30	1,00
4	15	1,00	1,47	1,08	1,75	1,31
4	100	1,49	1,38	1,06	1,49	0,84
4	300	—	1,27	0,98	1,35	0,74

During determination of membrane forces N_θ and N_x first we calculate

$$\beta = \sqrt[3]{\beta_1^2 \beta_2} = \sqrt[3]{\left(\frac{C_1}{R}\right)^2 \left(\frac{C_2}{R}\right)}.$$

According to the obtained value of β in Fig. 125 for specified value $\gamma = R/\delta$ we find quantities N_θ and N_x, which we then multiply by coefficient C_c, shown in Table 6.

With loading of the shell by axial moment (see Fig. 123) the internal forces and moments, and also angle of rotation are determined according to curves of Figs. 127, 128, 129. These curves can be used, if $l/R \geq 1$.

During computation of angles of rotation (ratio of deflection at point R_1 to dimension C_2) and bending moments M_θ and M_x for rectangular regions at first we determine quantity

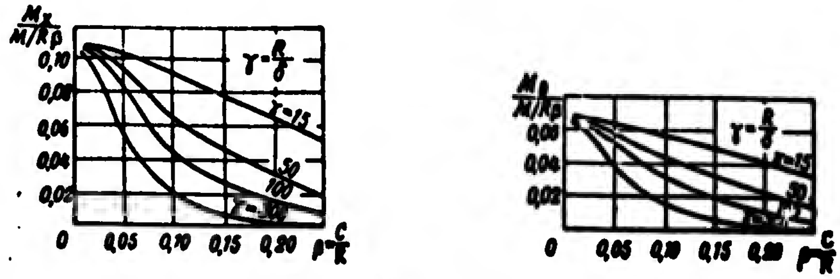

Fig. 127.

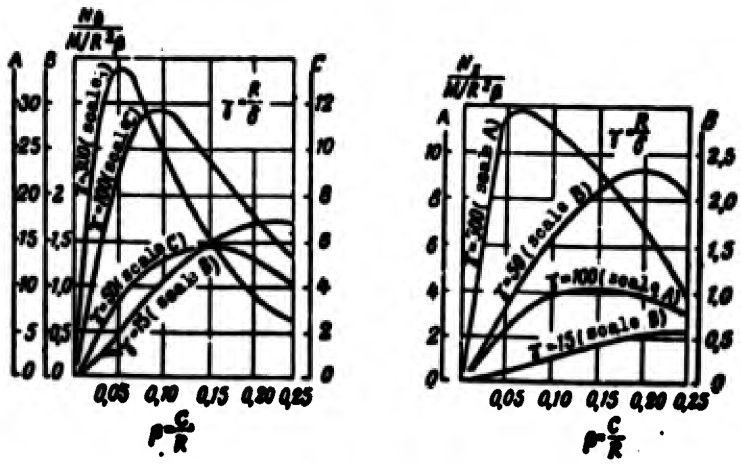

Fig. 128.

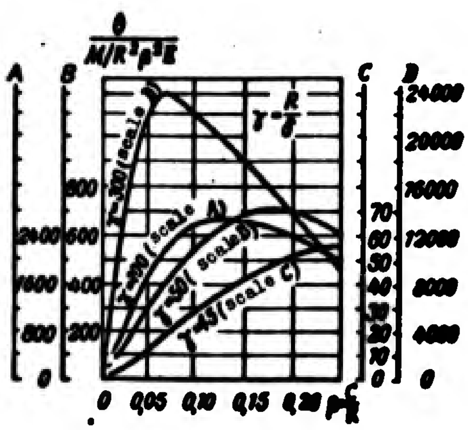

Fig. 129.

$$\sqrt[3]{\beta_1\beta_2^2} = \sqrt[3]{\left(\frac{C_1}{R}\right)\left(\frac{C_2}{R}\right)^2},$$

which we then multiply by coefficient k_l and obtain

$$\beta = k_l\sqrt[3]{\left(\frac{C_1}{R}\right)\left(\frac{C_2}{R}\right)^2}.$$

By the obtained quantity of β according to Figs. 127 and 129 we determine the angle of rotation and bending moments M_θ and M_x. Coefficient k_l in this case is taken from Table 7. When determining membrane forces N_θ, N_x first we determine

$$\beta = \sqrt[3]{\beta_1\beta_2^2} = \sqrt[3]{\left(\frac{C_1}{R}\right)\left(\frac{C_2}{R}\right)^2},$$

and by the obtained quantity β according to Fig. 128 for specified value $\gamma = R/\delta$ we find forces N_θ and N_x. Then these forces are multiplied by coefficient C_l, taken from Table 7.

Table 7.

C_1/C_2	R/δ	k_l for θ	k_l for M_θ	k_l for M_x	C_l for N_θ	C_l for N_x
1/4	15	1,14	1,80	1,24	0,75	0,43
1/4	50	1,13	1,65	1,16	0,77	0,33
1/4	100	1,18	1,59	1,11	0,80	0,24
1/4	300	1,31	1,56	1,11	0,90	0,07
1/2	15	1,00	1,08	1,04	0,90	0,76
1/2	100	1,00	1,06	1,02	0,97	0,68
1/2	300	1,00	1,05	1,02	1,10	0,60
2	15	—	0,94	1,12	0,87	1,3
2	100	1,09	0,89	1,07	0,81	1,15
2	300	—	0,79	0,90	0,80	1,50
4	15	1,39	0,90	1,24	0,68	1,20
4	100	1,18	0,54	1,12	0,51	1,03
4	300	—	0,64	0,83	0,50	1,33

§ 37. Calculation of a Cylindrical Shell for Axisymmetric Load

Let us examine a cylindrical shell, loaded by axisymmetrical lateral load.

Differential equations for the solution of this problem can be obtained from equations (7.11), having assumed in them all the derivatives with respect to y are equal to zero. Then, converting from partial derivatives to usual, we will have

$$\frac{d^4\varphi}{dx^4}=\frac{E\delta}{R}\frac{d^2w}{dx^2},$$
$$\frac{1}{R}\frac{d^2\varphi}{dx^2}+D\frac{d^4w}{dx^4}=q_z. \qquad (7.15)$$

Let us write the first of these equations in the form

$$\frac{d^2}{dx^2}\left(\frac{d^2\varphi}{dx^2}-E\delta\frac{w}{R}\right)=0.$$

After double integration we obtain

$$\frac{d^2\varphi}{dx^2}-E\delta\frac{w}{R}=C_1x+C. \qquad (7.16)$$

Let us show that $C_1x + C$ in this case is equal to zero. The expression for circumferential deformation with axisymmetric loading of a cylindrical shell has the form

$$\varepsilon_y=\frac{w}{R}=\frac{1}{E}(\sigma_y-\mu\sigma_x).$$

Since the considered shell is loaded only by lateral pressure, axial stresses $\sigma_x = 0$. Then

$$\varepsilon_y=\frac{w}{R}=\frac{\sigma_y}{E}.$$

Therefore,

$$\sigma_y-E\frac{w}{R}=0.$$

After multiplication of this equation by shell thickness δ and introduction of the function of stress we obtain

$$\frac{d^2\varphi}{dx^2} - E\delta \frac{w}{R} = 0.$$

By comparing this equation with equation (7.16), we see that $C_1 x + C = 0$. By excluding the function of stresses from the second equation (7.15), we have

$$\frac{d^4 w}{dx^4} + 4k^4 w = \frac{q_z}{D}, \qquad (7.17)$$

where

$$4k^4 = \frac{E\delta}{DR^2}.$$

When $q_z = 0$ we obtain homogeneous equation of boundary effect for a cylindrical shell (6.14), already known to us, where it is written relative to function U, equal to angle of rotation dw/dx.

The solution of equation (7.17) will be made up of the solution of homogeneous equation and some particular solution, i.e.,

$$w = \bar{w} + e^{kx}(C_1 \cos kx + C_2 \sin kx) + e^{-kx}(C_3 \cos kx + C_4 \sin kx).$$

Constants of integrations in each particular case must be determined from boundary conditions. Expressions for internal forces in this case will have the form

$$\sigma_x = 0, \qquad \sigma_y = E\frac{w}{R}, \qquad M_x = D\frac{d^2 w}{dx^2}, \qquad M_y = \mu M_x.$$

Let us apply the obtained solution for the case of loading of a shell by a load consisting of boosting and hydrostatic column of liquid of specific weight γ:

$$q_z = q_n + \gamma(H - x),$$

where H – height of liquid column; x – flowing coordinate, read from the lower end of the shell.

For the given load the particular solution of equation (7.17) should be sought in the form

$$\bar{w}=A_0+A_1x.$$

After substitution of expressions q_x and $\bar{w}$ in equation (7.17) we obtain

$$4k^4(A_0+A_1x)=\frac{1}{D}[q_n+\gamma(H-x)].$$

By equating the coefficients with identical powers x in this equation, we find

$$A_0=\frac{1}{4Dk^4}(q_n+\gamma H), \qquad A_1=-\frac{\gamma}{4Dk^4}.$$

Then

$$\bar{w}=\frac{q_n+\gamma(H-x)}{4Dk^4}.$$

Consequently, general solution of equation (7.17) in this case will have the form

$$w=\frac{[q_n+\gamma(H-x)]R^2}{E\delta}+e^{kx}(C_1\cos kx+C_2\sin kx)+$$
$$+e^{-kx}(C_3\cos kx+C_4\sin kx).$$

Let us assume that the shell is rather long and boundary conditions on its end do not affect each other. In this instance there must be $C_1 = 0$, $C_2 = 0$. Then

$$w=\frac{[q_n+\gamma(H-x)]R^2}{E\delta}+e^{-kx}(C_3\cos kx+C_4\sin kx).$$

Let us suppose that the lower end of the shell is rigidly restrained in an absolutely rigid frame. Then the boundary conditions on this end

$$(w)_{x=0}=0, \quad \left(\frac{dw}{dx}\right)_{x=0}=0.$$

These conditions lead to equations

$$C_3 + \frac{(q_n + \gamma H) R^2}{E\delta} = 0,$$

$$k (C_4 - C_3) - \frac{\gamma R^2}{E\delta} = 0.$$

Hence we find

$$C_3 = -\frac{(q_n + \gamma H) R^2}{E\delta},$$

$$C_4 = \frac{(1 - Hk) \gamma R^2}{KE\delta} - \frac{q_n R^2}{E\delta}.$$

Then the expression for w takes the form

$$w = \frac{[q_n + \gamma (H - x)] R^2}{E\delta} + e^{-kx} \left\{ \left[\frac{(1 - Hk) \gamma R^2}{kE\delta} - \frac{q_n R^2}{E\delta} \right] \sin kx - \right.$$
$$\left. - \frac{(q_n + \gamma H) R^2}{E\delta} \cos kx \right\}.$$

For circumferential force and bending moment N_y and bending moment M_x we obtain formulas

$$N_y = E\delta \frac{w}{R} = [q_n + \gamma (H - x)] R + e^{-kx} \left\{ \left[\frac{(1 - Hk) \gamma R}{k} - q_n R \right] \sin kx - \right.$$
$$\left. - (q_n + \gamma H) R \cos kx \right\},$$

$$M_x = D \frac{d^2 w}{dx^2} = -\frac{R\delta e^{-kx}}{2\sqrt{3(1 - \mu^2)}} \left\{ \left[\frac{1 - Hk) \gamma}{k} - q_n \right] \cos kx + \right.$$
$$\left. + (q_n + \gamma H) \sin kx \right\}.$$

By using these formulas, we can determine stresses at an arbitrary point of shell. For example, in framing $x = 0$

$$N_y = 0,$$

$$(M_x)_{x=0} = -\frac{R\delta}{2\sqrt{3(1 - \mu^2)}} \left[\frac{(1 - Hk) \gamma}{k} - q_n \right].$$

Stresses from this moment will be found from expression

$$\sigma_x = \pm \frac{6 (M_x)_{x=0}}{\delta^2} = \pm \frac{3R\delta}{\sqrt{3(1 - \mu^2)}} \left[\frac{(1 - Hk) \gamma}{k} - q_n \right].$$

§ 38. Application of Origin of Virtual Displacements For Problems of Calculation of Shells

In many cases the energy method can be useful for the approximate solution of problems of strength and rigidity of shells. General fundamentals of this method were presented in Section I in examining rods and plates. During calculation of shells by this method only the selection of approximating functions is complicated. But with some experience this difficulty can be easily overcome. In this case the expression of total energy is a function of three components of displacement u, v and w, and with solution of particular problem it is necessary to select expressions for these functions, which would satisfy the prescribed boundary conditions and be in accordance with the physical sense of the problem. The more fully these requirements are satisfied, the more accurate the result of solution of the posed problem will be. Having provided the selected functions of displacements with indeterminate coefficients, which do not depend on current coordinates, it is possible to calculate the total potential energy of the shell, which will be expressed in the function of these coefficients. For determination of the latter we should formulate partial derivatives of each of them and equate these derivatives to zero. In this case we obtain a number of equations, corresponding to the number of unknown coefficients. After determination of coefficients we can determine all internal force factors by formulas of Hooke law. Let us illustrate the application of this method on a concrete example.

Let us assume there is a very long weightless cylindrical shell, one end of which is rigidly restrained, and the second is free. In some arbitrary section of this shell let us place supporting ring a (Fig. 130) with rectangular cross section. In this case the center of gravity of the ring section is arranged on the middle surface of the shell.

Let us apply concentrated force P to this ring.

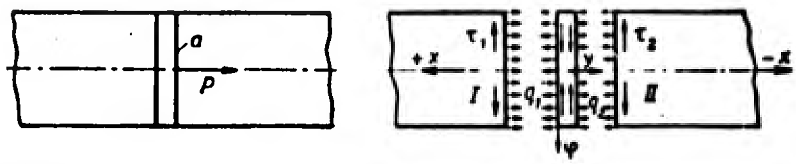

Fig. 130.

For solution of the problem let us mentally cut the shell, as is shown in Fig. 130. Here there is shown the selected system of coordinates both for shells *I*, *II*, and for ring *a*. At places of cuts there are applied unknown normal forces q_1, q_2 and tangential forces τ_1 and τ_2. The shell is considered momentless.

Let us begin with examination of shell *I*. The potential energy of the shell can be written in the following manner:

$$Э=\frac{E\delta R}{2(1-\mu^2)}\int_0^l\int_0^{2\pi}\left(\varepsilon_x^2+\varepsilon_\varphi^2+2\mu\varepsilon_x\varepsilon_\varphi+\frac{1-\mu}{2}\varepsilon_{x\varphi}^2\right)dx\,d\varphi-T,$$

where

$$\varepsilon_x=\frac{\partial u}{\partial x},\qquad \varepsilon_\varphi=\frac{\partial v}{R\partial\varphi},\qquad \varepsilon_{x\varphi}=\frac{\partial u}{R\partial\varphi}+\frac{\partial v}{\partial x},$$

u, v – components of displacement of points of the middle surface in the direction of axes x and ϕ. Let us disregard for now component w/R in ε_ϕ; E, μ – modulus of elasticity and Poisson bracket; R, δ – radius and thickness of shell; T – work of forces q_1 and τ_1; l – length of shell. It is considered that $l \gg R$.

Further in accordance with the accepted method of solution it is necessary to assign suitable expressions of u and v with indeterminate coefficients and to express potential energy as a function of these coefficients. As such suitable functions let us take

$$u=A_0x+\sum_n(A_{1n}+A_{2n}x)e^{-\frac{nx}{R}}\cos n\varphi,$$

$$v=\sum_n(B_{1n}+B_{2n}x)e^{-\frac{nx}{R}}\sin n\varphi.$$

The first term in the expression for u does not depend on angle ϕ and is proportional to distance x from the origin of coordinates; the second origin of this expression rapidly attenuates with withdrawal from the origin of coordinates and is introduced for account of the local effect from concentrated force.

In the expression for v there is taken only the rapidly damping term, since this displacement should be diminished with withdrawal from the place of application of force.

Furthermore, displacement u must be an even function relative to coordinate ϕ, and displacement v – odd.

Before writing out the expression of total energy, let us calculate the work of forces q_1 and τ_1. For convenience of determination of these forces let us subsequently represent them in the form of Fourier series:

$$q_1 = q_0' + \sum_n q_n' \cos n\varphi, \qquad \tau_1 = \sum_n \tau_n' \sin n\varphi. \tag{7.18}$$

Forces q_1 should be an even function relative to ϕ, therefore expansion is taken with respect to cosines; forces τ_1 should be an odd function, therefore expansion is taken in terms of sines. In the first expansion we take an absolute term, which is easily determined from conditions of statics

$$\int_0^{2\pi} q_1 R d\varphi = P,$$

whence

$$q_0' = \frac{P}{2\pi R}. \tag{7.19}$$

Absolute term in the second expansion is equal to zero.

Now let us calculate the work of forces q_1 and τ_1:

$$T=\int_0^l\int_0^{2\pi} q_1\frac{\partial u}{\partial x}\,dx\,R\,d\varphi-\int_{-\pi}^{\pi}\tau_1(v)_{x=0}P\,d\varphi.$$

The minus sign before the second component is placed because force τ_1 and displacement v are directed to opposite sides.

After calculation of the shown integrals we obtain

$$T=2\pi RA_0q_0'\int_0^l dx-\pi Rq'A_{1n}-\pi R\tau'B_{1n}.$$

Now it is possible to write out the expression for total potential energy of shell I:

$$Э=\frac{\pi R\delta E}{2(1-\mu^2)}\left(2A_0^2\int_0^l dx+\frac{3-\mu}{4}\frac{n}{R}A_{1n}^2+\frac{3-\mu}{8}\frac{R}{n}A_{2n}^2+\right.$$
$$+\frac{3-\mu}{4}\frac{n}{R}B_{1n}^2+\frac{3-\mu}{8}\frac{R}{n}B_{2n}^2-\frac{1+\mu}{4}A_{1n}A_{2n}+\frac{1+\mu}{4}B_{1n}B_{2n}+$$
$$\left.+\frac{1-3\mu}{2}\frac{n}{R}A_{1n}B_{1n}+\frac{1+\mu}{4}A_{2n}B_{1n}-\frac{1+\mu}{4}A_{1n}B_{2n}\right)-$$
$$-2\pi RA_0q_0'\int_0^l dx+\pi Rq'A_{1n}+\pi R\tau'B_{1n}.$$

With calculation of integrals in view of the very great length of shell we assumed

$$\int_0^l e^{-\frac{nx}{R}}dx\approx\frac{R}{n}.$$

Thus, potential energy of shell I is expressed as a function of coefficients $A_0, A_{1n}, A_{2n}, B_{1n}, B_{2n}$. For their determination let us formulate the partial derivatives of total energy $\frac{\partial Э}{\partial A_0}$, $\frac{\partial Э}{\partial A_{1n}}$, ..., $\frac{\partial Э}{\partial B_{2n}}$ and equate them to zero. In this case the number of equations will correspond to the number of unknown parameters, which are determined from the solution of this system of equations.

Thus, we have

$$\left.\begin{array}{c}\left(\frac{A_0E\delta}{(1-\mu^2)}-q_0'\right)\int\limits_0^l dx=0,\\ \frac{3-\mu}{2}\frac{n}{R}A_{1_n}-\frac{1+\mu}{4}A_{2_n}+\frac{1-3\mu}{2}\frac{n}{R}B_{1_n}-\frac{1+\mu}{4}B_{2_n}+\\ +\frac{2(1-\mu^2)}{E\delta}q_n'=0,\\ (1+\mu)A_{1_n}-(3-\mu)\frac{R}{n}A_{2_n}-(1+\mu)B_{1_n}=0,\\ \frac{3-\mu}{2}\frac{n}{R}B_{1_n}+\frac{1+\mu}{4}B_{2_n}+\frac{1-3\mu}{2}\frac{n}{R}A_{1_n}+\frac{1+\mu}{4}A_{2_n}+\\ +\frac{2(1-\mu^2)}{E\delta}\tau_n'=0,\\ (1+\mu)B_{1_n}+(3-\mu)\frac{R}{n}B_{2_n}-(1+\mu)A_{1_n}=0.\end{array}\right\} \quad (7.20)$$

From the first equation (7.20) follows

$$A_0=\frac{(1-\mu^2)q_0'}{E\delta}.$$

After substitution here of values of q_0' from (7.19) we obtain

$$A_0=\frac{(1-\mu^2)P}{2\pi R\delta E}.$$

From solution of the remaining equations of system (7.20) follows:

$$A_{1_n}=-\frac{2Rq_n'}{E\delta n}+(1-\mu)\frac{R\tau_n'}{E\delta n},$$

$$B_{1_n}=-\frac{2R\tau_n'}{E\delta n}+(1-\mu)\frac{Rq_n'}{E\delta n},$$

$$A_{2_n}=B_{2_n}=-\frac{(1+\mu)(q_n'-\tau_n')}{E\delta}.$$

By substituting the found values of constants $A_0, A_{1n}, A_{2n}, B_{1n}, B_{2n}$ in expressions for u and v, we find

$$\left.\begin{array}{c}u_1=\frac{(1-\mu^2)Px}{2\pi R\delta_1E_1}+\frac{R}{E_1\delta_1}\sum\limits_n\frac{q_n'}{n}\left\{-\left[2+\frac{(1+\mu)nx}{R}\right]+\right.\\ \left.+\frac{\tau_n'}{q_n'}\left[1-\mu+\frac{(1+\mu)nx}{R}\right]\right\}e^{-\frac{nx}{R}}\cos n\varphi,\end{array}\right\} \quad (7.21)$$

$$\left.\begin{aligned} v_1 = -\frac{R}{E_1\delta_1}\sum_n \frac{q'_n}{n}\left\{-\left[1-\mu-\frac{(1+\mu)nx}{R}\right]+\right. \\ \left.+\frac{\tau'_n}{q'_n}\left[2-\frac{(1+\mu)nx}{R}\right]\right\} e^{-\frac{nx}{R}}\sin n\varphi. \end{aligned}\right\} \quad \text{(7.21 cont'd)}$$

Index 1 designated quantities pertaining to shell *I*, and index 2 – to shell *II*.

By having expressions for displacements u_1 and v_1, it is possible to write the expressions for stresses, using known relationships ensuing from Hooke law:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\frac{\partial u}{\partial x}+\mu\frac{\partial v}{R\partial\varphi}\right),$$
$$\sigma_\varphi = \frac{E}{1-\mu^2}\left(\frac{\partial v}{R\partial\varphi}+\mu\frac{\partial u}{\partial x}\right),$$
$$\sigma_{x\varphi} = \frac{E}{2(1+\mu)}\left(\frac{\partial u}{R\partial\varphi}+\frac{\partial v}{\partial x}\right).$$

By substituting here the appropriate derivatives of u_1 and v_1 we obtain

$$\left.\begin{aligned} \sigma_{x_1} &= \frac{P}{2\pi R\delta_1}+\frac{1}{\delta_1}\sum_n q'_n\left[1+\left(1-\frac{\tau_{1n}}{q'_n}\right)\frac{nx}{R}\right]e^{-\frac{nx}{R}}\cos n\varphi, \\ \sigma_{\varphi_1} &= \frac{\mu P}{2\pi R\delta_1}+\frac{1}{\delta_1}\sum_n q'_n\left[\left(1-\frac{2\tau'_n}{q'_n}\right)-\right. \\ &\quad \left.-\left(1-\frac{\tau'_n}{q'_n}\right)\frac{nx}{R}\right]e^{-\frac{nx}{R}}\cos n\varphi, \\ \sigma_{x\varphi_1} &= \frac{1}{\delta_1}\sum_n q'_n\left[\frac{nx}{R}+\frac{\tau'_n}{q'_n}\left(1-\frac{nx}{R}\right)\right]e^{-\frac{nx}{R}}\sin n\varphi. \end{aligned}\right\} \quad (7.22)$$

Having obtained the expressions for displacements and stresses of shell *I*, by analogy, not repeating intermediate computations, it is possible to write the appropriate expressions even for shell *II*, bearing in mind in this case that the absolute term in the expansion for normal force $q2n$ is equal to zero ($q''_0 = 0$) and that coordinate

x is read from zero to negative values. Furthermore, tangential force τ_n'' is directed to the side opposite τ_n'.

Having made these remarks, for shell *II* it is possible to write

$$\left.\begin{aligned}
u_2 &= -\frac{R}{E_2 b_2}\sum_n \frac{q_n''}{n}\left\{-\left[2-\frac{(1+\mu)nx}{R}\right]-\right.\\
&\left.-\frac{\tau_n''}{q_n''}\left[1-\mu-\frac{(1+\mu)nx}{R}\right]\right\}e^{\frac{nx}{R}}\operatorname{con} n\varphi,\\
v_2 &= \frac{R}{E_2 b_2}\sum_n \frac{q_n''}{n}\left\{-\left[1-\mu+\frac{(1+\mu)nx}{R}\right]-\right.\\
&\left.-\frac{\tau_n''}{q_n''}\left[2+\frac{(1+\mu)nx}{R}\right]\right\}e^{\frac{nx}{R}}\sin n\varphi,\\
\sigma_{x_2} &= -\frac{1}{b_2}\sum_n q_n''\left[1-\left(1+\frac{\tau_n''}{q_n''}\right)\frac{nx}{R}\right]e^{\frac{nx}{R}}\cos n\varphi,\\
\sigma_{\varphi_2} &= -\frac{1}{b_2}\sum_n q_n''\left[\left(1+\frac{2\tau_n''}{q_n''}\right)+\left(1+\frac{\tau_n''}{q_n''}\right)\frac{nx}{R}\right]e^{\frac{nx}{R}}\cos n\varphi,\\
\sigma_{x\varphi_2} &= -\frac{b}{b_2}\sum_n q_n''\left[-\frac{nx}{R}-\frac{\tau_n''}{q_n''}\left(1+\frac{nx}{R}\right)\right]e^{\frac{nx}{R}}\sin n\varphi.
\end{aligned}\right\} \quad (7.23)$$

In these expressions the values of x should be taken with minus.

Now let us turn to examination of the stressed state of supporting ring. In this case we will proceed from the usual theory of bending of beams. This can be considered entirely permissible in this problem if only because the zone of effect of concentrated force in circumferential direction has a sharply expressed local character and concentration of forces q_n', q_n'', τ_n', τ_n'' occurs on a comparatively small part of the ring. Taking this remark into account, let us write the expression for total energy of the ring.

$$Э=\frac{EJ}{2R^3}\int_0^{2\pi}\left(\frac{d^2 v_\kappa}{d\varphi^2}\right)^2 d\varphi+R\int_0^{2\pi}(q_1+q_2)\,v_\kappa\,d\varphi-$$

$$-\int_{-\pi}^{\pi}(\tau_1+\tau_2)\,R\,d\varphi\cdot h\,\frac{dv_\kappa}{R\,d\varphi}-P(v_\kappa)_{\varphi=0},$$

where J, R – central inertia moment and radius of the ring. Inertia moment is calculated with respect to the axis, lying in the plane of the ring.

The first term here represents elastic energy of bending deformation of the ring. The remaining components represent the work of external forces: normal forces q_1, q_2 tangential forces τ_1, τ_2 and external force P.

Normal forces q_1, q_2 permit work of rings $v_{к}$ at deflections from its plane; tangential forces τ_1, τ_2 – elongation of extreme fibers of the ring, equal to $h\frac{dv_{к}}{Rd\varphi}$, where h – half the height of the ring; force P works on displacement of ring $v_{к}$ at point $\phi = 0$. The plus sign in the second component is taken because forces q_1, q_2 is directed to the side opposite positive displacement $v_{к}$. The last two components are taken with minus sign, inasmuch as the direction of forces τ_1 and τ_2 and P coincide with corresponding displacement.

Now it is necessary to select a suitable expression for $v_{к}$ with indeterminate coefficients and then determine them just as when examining a shell. This displacement can be represented by Fourier series

$$v_{к}=\sum_{n} C_n \cos n\varphi.$$

The absolute term here is omitted, since it reflects displacement of the ring as a solid body.

Having substituted the expression for $v_{к}$ and also expressions for q_1 and τ_1 and expressions for q_2 and τ_2 analogous to them, which can be represented in the form

$$q_2=\sum_{n} \dot{q}_n \cos n\varphi, \qquad \tau_2=\sum_{n} \dot{\tau}_n \sin n\varphi \tag{7.24}$$

(and which were already used when determining displacements u_2, v_2), in the expression for energy of the ring, after taking the quadratures we obtain

$$Э=\frac{\pi JEC_n^2n^4}{2R^3}+\pi R(q'_n+q''_n)C_n-\pi h(\tau'_n+\tau''_n)nC_n-PC_n.$$

Let us find parameters C_n from condition $\frac{dЭ}{dC_n}=0$:

$$C_n=\frac{[P-\pi R(q'_n+q''_n]+\pi h(\tau'_n+\tau''_n)n]R^3}{\pi JEn^4}.$$

Having substituted C_n in expression $v_к$ we obtain

$$v_к=\frac{R^3}{\pi JE}\sum_n\frac{[P-\pi R(q_n+q''_n)+\pi h(\tau'_n+\tau''_n)n]\cos n\varphi}{n^4}. \qquad (7.25)$$

Thus, having obtained all the necessary data for unknown forces q'_n, q''_n, τ'_n and τ''_n, for their determination it is possible to formulate the following four conditions of conjugation of shells *I* and *II* with the ring:

$$-(u_1)_{x=0}=v_к; \qquad -(\varepsilon_{\varphi_1})_{x=0}=\varepsilon_к;$$
$$(u_2)_{x=0}=v_к, \qquad (\varepsilon_{\varphi_2})_{x=0}=\varepsilon_к,$$

where $\varepsilon_к$ – relative elongation of the most elongated fibers of the ring, which adjoin shells *I* and *II*.

The first two conditions express the equality of linear displacements of shells *I* and *II* with displacements of the ring at their junction point, the second two – equality of relative deformations of shells and ring.

Expressions for $(\varepsilon_{\varphi_1})_{x=0}$, $(\varepsilon_{\varphi_2})_{x=0}$ and $\varepsilon_к$ have the form

$$(\varepsilon_{\varphi_1})_{x=0}=\left(\frac{dv_1}{Rd\varphi}\right)_{x=0}=-\frac{1}{E_1\delta_1}\sum_n[-(1-\mu)q'_n+2\tau'_n]\cos n\varphi,$$

$$(\varepsilon_{\varphi_2})_{x=0}=\left(\frac{dv_2}{Rd\varphi}\right)_{x=0}=\frac{1}{E_2\delta_2}\sum_n[-(1-\mu)q''_n-2\tau''_n]\cos n\varphi,$$

$$\varepsilon_к=\frac{\sigma}{E}=\frac{M}{WE}=\frac{R}{\pi EW}\sum_n\frac{1}{n^2}[P-\pi R(q'_n+q''_n)+$$
$$+\pi h(\tau'_n+\tau''_n)n]\cos n\varphi,$$

where W – moment of resistance of the ring to bending.

The last expression is obtained from the following relationship, known in theories of bending of beams,

$$M = -EJ\frac{d^2 v_x}{R^2 d\varphi^2}.$$

The minus sign is taken here because the load distributed along the ring is directed to the side opposite the accepted positive direction of coordinate y for the ring (see Fig. 130).

Conditions of conjugation of the shell and ring in expanded for can be written in the following manner:

$$-\frac{1}{E_1\delta_1}\left[-2q'_n+(1-\mu)\tau'_n\right]=\frac{R^2A}{\pi E_0 J n^3},$$

$$-\frac{1}{E_2\delta_2}\left[-2q''_n-(1-\mu)\tau''_n\right]=\frac{R^2A}{\pi E_0 J n^3},$$

$$\frac{1}{E_1\delta_1}\left[-(1-\mu)q'_n+2\tau'_n\right]=\frac{RA}{\pi E_0 W n^2},$$

$$\frac{1}{E_2\delta_2}\left[-(1-\mu)q''_n-2\tau''_n\right]=\frac{RA}{\pi E_0 W n^2},$$

where

$$A=P-\pi R(q'_n+q''_n)+\pi h(\tau'_n+\tau''_n)n.$$

In these equations index 0 pertains to the ring, the remaining indices were explained earlier.

By solving the obtained equations relative to q'_n, q''_n, τ'_n and τ''_n we obtain

$$q'_n=\frac{P}{2\pi R\frac{E_0}{E_1}\left[\frac{1}{2}\frac{E_1}{E_0}\left(1+\frac{E_2\delta_2}{E_1\delta_1}\right)+\frac{Jn^3}{R^3\delta_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{E_1 hn}{2E_0 Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]},$$

$$q''_n=\frac{E_2\delta_2}{E_1\delta_1}q'_n,\qquad \tau'_n=\frac{q''_n}{k},\qquad \tau''_n=-\frac{q''_n}{k},$$

where

$$k=\frac{1-\mu+\frac{2RW}{Jn}}{2+\frac{(1-\mu)RW}{Jn}}=\frac{1-\mu+\frac{2R}{nh}}{2+\frac{(1-\mu)R}{nh}}.$$

By substituting the found values of q_n', q_n'', τ_n' and τ_n'' in expressions (7.18)-(7.25), we will have

$$\left.\begin{aligned}
&q_1=\frac{P}{2\pi R}+\frac{P}{\pi R}\sum_{n=1}^{\infty}\frac{\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)};\\
&\tau_1=\frac{P}{\pi R}\sum_{n=1}^{\infty}\frac{\sin n\varphi}{k\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]};\\
&q_2=\frac{PE_2\delta_2}{\pi RE_1\delta_1}\sum_{n=1}^{\infty}\frac{\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)};\\
&\tau_2=-\frac{PE_2\delta_2}{\pi RE_1\delta_1}\sum_{n=1}^{\infty}\frac{\sin n\varphi}{k\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]};\\
&u_1=\frac{(1-\mu^2)Px}{2\pi R\delta_1E_1}+\\
&+\frac{P}{\pi E_1\delta_1}\sum_{n=1}^{\infty}\frac{\left\{-\left[2+\frac{(1+\mu)nx}{R}\right]+\frac{1}{k}\left[1-\mu+\frac{(1+\mu)nx}{R}\right]\right\}e^{-\frac{nx}{R}}\cos n\varphi}{n\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]};\\
&v_1=-\frac{P}{\pi E_1\delta_1}\times\\
&\times\sum_{n=1}^{\infty}\frac{\left\{-\left[1-\mu-\frac{(1+\mu)nx}{R}\right]+\frac{1}{k}\left[2-\frac{(1+\mu)nx}{R}\right]\right\}e^{-\frac{nx}{R}}\sin n\varphi}{n\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]};\\
&\sigma_{x_1}=\frac{P}{2\pi R\delta_1}+\\
&+\frac{P}{\pi R\delta_1}\sum_{n=1}^{\infty}\frac{\left[1+\left(1-\frac{1}{k}\right)\frac{nx}{R}\right]e^{-\frac{nx}{R}}\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)};\\
&\sigma_{\varphi_1}=\frac{\mu P}{2\pi R\delta_1}+\\
&+\frac{P}{\pi R\delta_1}\sum_{n=1}^{\infty}\frac{\left[1-\frac{2}{k}-\left(1-\frac{1}{k}\right)\frac{nx}{R}\right]e^{-\frac{nx}{R}}\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)};\\
&\sigma_{x\varphi_1}=\frac{P}{\pi R\delta_1}\sum_{n=1}^{\infty}\frac{\left[\frac{nx}{R}+\frac{1}{k}\left(1-\frac{nx}{R}\right)\right]e^{-\frac{nx}{R}}\sin n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)};
\end{aligned}\right\}\qquad(7.26)$$

$$
\left.
\begin{aligned}
u_2 &= -\frac{P}{\pi E_1\delta_1}\sum_{n=1}^{\infty}\frac{\left\{-\left[2-\frac{(1+\mu)nx}{R}\right]+\frac{1}{k}\left[1-\mu-\frac{(1+\mu)nx}{R}\right]\right\}e^{\frac{nx}{R}}\cos n\varphi}{n\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]},\\
v_2 &= \frac{P}{\pi E_1\delta_1}\sum_{n=1}^{\infty}\frac{\left\{-\left[1-\mu+\frac{(1+\mu)nx}{R}\right]+\frac{1}{k}\left[2+\frac{(1+\mu)nx}{R}\right]\right\}e^{\frac{nx}{R}}\sin n\varphi}{n\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{Rk}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]},\\
\sigma_{x_2} &= -\frac{PE_2}{\pi R\delta_1E_1}\sum_{n=1}^{\infty}\frac{\left[1-\left(1-\frac{1}{k}\right)\frac{nx}{R}\right]e^{\frac{nx}{R}}\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)},\\
\sigma_{\varphi_2} &= -\frac{PE_2}{\pi R\delta_1E_1}\sum_{n=1}^{\infty}\frac{\left[1-\frac{2}{k}+\left(1-\frac{1}{k}\right)\frac{nx}{R}\right]e^{\frac{nx}{R}}\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JF_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)},\\
\tau_{x\varphi_2} &= -\frac{PE_2}{\pi R\delta_1E_1}\sum_{n=1}^{\infty}\frac{\left[1-\frac{nx}{R}+\frac{1}{k}\left(1+\frac{nx}{R}\right)\right]e^{\frac{nx}{R}}\sin n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)},\\
v_k &= \frac{2P}{\pi E_1\delta_1}\sum_{n=1}^{\infty}\frac{\left(1-\frac{1-\mu}{2k}\right)\cos n\varphi}{n\left[1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)\right]},\\
M &= \frac{2JE_0P}{\pi R^2\delta_1E_1}\sum_{n=1}^{\infty}\frac{n\left(1-\frac{1-\mu}{2k}\right)\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)}.
\end{aligned}
\right\}
\quad (7.27)
$$

The presence of the first term in the expression for $\sigma_{\varphi 1}$ is connected with the assumption about the absence of deflection of shell ($w = 0$) during its loading by axial concentrated force. It is assumed that the shell is as if under the action not only of force P, but also constant internal pressure with intensity $q = \frac{\mu P}{2\pi R^2}$, from which the constant component of stress $\sigma_{\varphi 1}$ was obtained. In actuality internal pressure is absent. Therefore, it is necessary to the obtained expression for $\sigma_{\varphi 1}$ to add annular stresses from external uniform pressure with intensity $q = -\frac{\mu P}{2\pi R^2}$ with allowance for the fact that one end of the considered shell is rigidly restrained in the ring.

In this instance for calculation of the shell we will proceed from differential equation (7.17)

$$\frac{d^4w}{dx^4} + 4k^4w = \frac{q_z}{D},$$

where

$$4k^4 = \frac{E_1\delta_1}{D_1R} = \frac{12(1-\mu^2)}{\delta_1^2R^2}.$$

Solution of the given equation in the considered case will be

$$w = \frac{q_z}{4k^4D_1} + e^{-kx}(C_3\cos kx + C_4\sin kx).$$

Let us place the origin of coordinates in the section coinciding with framing of the shell. By determining constants C_3 and C_4 from conditions $w=0$, $\frac{dw}{dx}=0$ when $x = 0$, we obtain

$$w = \frac{q_z}{4k^4D_1}[1 - e^{-kx}(\cos kx + \sin kx)].$$

Annular stresses in this case will be

$$\sigma_\varphi^\bullet = E_1\frac{w}{R} = \frac{E_1q_z}{4Rk^4D_1}[1 - e^{-kx}(\cos kx + \sin kx)].$$

By substituting here $q_z = -\frac{\mu P}{2\pi R^2}$, we obtain

$$\sigma_\varphi^\bullet = -\frac{\mu P}{2\pi R\delta_1}[1 - e^{-kx}(\cos kx + \sin kx)]. \qquad (7.28)$$

By assuming these stresses up with stress $\sigma_{\phi 1}$ we will have

$$\sigma_{\varphi_1} = \frac{\mu P}{2\pi R\delta_1}e^{-kx}(\cos kx + \sin kx) + $$
$$+\frac{P}{\pi R\delta_1}\sum_{n=1}^{\infty}\frac{\left[1-\frac{2}{k}-\left(1-\frac{1}{k}\right)\frac{nx}{R}\right]e^{-\frac{nx}{R}}\cos n\varphi}{1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{R^3\delta_1E_1}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right)}. \qquad (7.29)$$

Formulas (7.26)-(7.29) represent all components of stressed and deformed state of the shell, and also internal forces in the juncture position of shell and ring and the bending moment, acting in the ring.

As numerical calculations show, the obtained series converge rather rapidly, with the exception of the series entering the expression for bending moment of the ring.

Let us obtain another expression for bending moment of the ring, more convenient for practical calculation. For this let us use known differential relationship, establishing the connection between derivative of moment and shearing force.

In our case this will be

$$Q=\frac{dM}{Rd\varphi},$$

where

$$Q=\int qRd\varphi+D_1.$$

Here by q we must mean the load distributed along the ring, i.e.,

$$q=q_1+q_2+q_3,$$

where

$$q_1+q_2=q_0'+\sum_{n=1}^{\infty}(q_n'+q_n'')\cos n\varphi,$$

and q_3 – projection of tangential forces τ_1 and τ_2 to direction of axis y. This component

$$q_3=\tau_1\frac{dv_\kappa}{Rd\varphi}-\tau_2\frac{dv_\kappa}{Rd\varphi}=-\frac{C_n(\tau_n'-\tau_n'')\,n\sin^2 n\varphi}{R}.$$

Here we whould substitute absolute values of τ_n' and τ_n'', since their signs have already been taken into account during formulation of the expression for q_3, and in expression C_n they should be taken according to formula (7.26).

Then we will have

$$q=q_0'+\sum_{n=1}^{\infty}(q_n'+q_n'')\cos n\varphi-\sum_{n=1}^{\infty}\frac{C_n(\tau_n'-\tau_n'')\,n\sin^2 n\varphi}{R};$$

$$Q=R\int\Big[q_0'+\sum_{n=1}^{\infty}(q_n'+q_n'')\cos n\varphi-$$

$$-\frac{1}{R}\sum_{n=1}^{\infty}C_n n\,(\tau_n'-\tau_n'')\sin^2 n\varphi\Big]\,d\varphi+D_1.$$

After integration of this expeession we obtain

$$Q=R\Big[q_0'\varphi+\sum_{n=1}^{\infty}\frac{1}{n}(q_n'+q_n'')\sin n\varphi-$$

$$-\frac{1}{R}\sum_{n=1}^{\infty}C_n n\,(\tau_n'-\tau_n'')\left(\frac{\varphi}{2}-\frac{1}{4n}\sin 2n\varphi\right)\Big]+D_1.$$

The constant of integrations D_1 is determined from the condition that when $\phi = 0$ there must be $Q=-\frac{P}{2}$. Then

$$Q=R\Big[q_0'\varphi+\sum_{n=1}^{\infty}\frac{1}{n}(q_n'+q_n'')\sin n\varphi-$$

$$-\frac{1}{2R}\sum_{n=1}^{\infty}C_n n\,(\tau_n'-\tau_n'')\left(\varphi-\frac{1}{2n}\sin 2n\varphi\right)\Big]-\frac{P}{2}.$$

Now it is possible to calculate the bending moment:

$$M=R\int Q\,d\varphi+D_2.$$

Having substituted here the value of Q and integrating, we obtain expression

$$M=R\Big\{R\Big[\frac{1}{2}q_0'\varphi^2-\sum_{n=1}^{\infty}\frac{1}{n^2}(q_n'+q_n'')\cos n\varphi-$$

$$-\frac{1}{4R}\sum_{n=1}^{\infty}C_n n\,(\tau_n'-\tau_n'')\left(\varphi^2-\frac{1}{2n^2}\cos 2n\varphi\right)\Big]-\frac{1}{2}P\varphi\Big\}+D_2.$$

For determination of the constant of integration D_2 let us use the condition that when $\phi = 0$ there should be $M = M_0$, where M_0 – moment in the root section of the ring. From this condition we obtain

$$D_2 = M_0 + R^2 \sum_{n=1}^{\infty} \frac{1}{n^2}(q_n' + q_n^{\cdot}) + \frac{R}{8} \sum_{n=1}^{\infty} \frac{C_n(\tau_n' - \tau_n^{\cdot})}{n}.$$

Consequently,

$$M = \frac{1}{2} q_0' R^2 \varphi^2 - \frac{1}{2} PR\varphi + 2R^2 \sum_{n=1}^{\infty} \frac{1}{n^2}(q_n' + q_n^{\cdot}) \sin^2 \frac{n\varphi}{2} + \\ + \frac{1}{4} R \sum_{n=1}^{\infty} \frac{C_n}{n}(\tau_n' - \tau_n^{\cdot})(\sin^2 n\varphi - n^2\varphi^2) + M_0.$$

Moment M_0 as yet remains unknown. In order to determine it, we will proceed from the condition that the root section of the ring is not turned as a result of symmetry of loading. For determination of this moment let us formulate the expression of energy of deformation of a semiring

$$Э = \frac{R}{2E_0 J} \int_0^{\pi} M^2 d\varphi,$$

or

$$Э = \frac{R}{2E_0 J} \int_0^{\pi} \left[\frac{1}{2} q_0' R^2 \varphi^2 - \frac{PR\varphi}{2} + 2R^2 \sum_{n=1}^{\infty} \frac{1}{n^2}(q_n' + q_n^{\cdot}) \sin^2 \frac{n\varphi}{2} + \\ + \frac{1}{4} R \sum_{n=1}^{\infty} \frac{C_n}{n}(\tau_n' - \tau_n^{\cdot})(\sin^2 n\varphi - n^2\varphi^2) + M_0 \right]^2 d\varphi.$$

Let us differentiate the expression under the sign of integral with respect to M_0 and equate the derivative to zero:

$$\int_0^{\pi} \left[\frac{1}{2} q_0' R^2 \varphi^2 - \frac{1}{2} PR\varphi + 2R^2 \sum_{n=1}^{\infty} \frac{1}{n^2}(q_n' + q_n^{\cdot}) \sin^2 \frac{n\varphi}{2} + \\ + \frac{1}{4} R \sum_{n=1}^{\infty} \frac{C_n}{n}(\tau_n' - \tau_n^{\cdot})(\sin^2 n\varphi - n^2\varphi^2) + M_0 \right] d\varphi = 0.$$

From this expression after integration we obtain

$$M_0 = \frac{\pi^2 R^2 q_0'}{6} + \frac{\pi R P}{4} - R^2 \sum_{n=1}^{\infty} \frac{1}{n^2}(q_n' + q_n^{\cdot}) -$$

$$- \frac{1}{4} R \sum_{n=1}^{\infty} \frac{C_n}{n}(\tau_n' - \tau_n^{\cdot})\left(\frac{1}{2} - \frac{\pi^2 n^2}{3}\right).$$

Then, considering the expressions given above for q_n', $q_n^{\cdot}$ q_0', τ_n', $\tau_n^{\cdot}$, finally

$$M = \frac{\pi R P}{2}\left(\frac{1}{3} - \frac{\varphi}{\pi} + \frac{\varphi^2}{2\pi^2}\right) -$$

$$- \frac{PR\left(1 + \frac{E_2\delta_2}{E_1\delta_1}\right)}{\pi} \sum_{n=1}^{\infty} \frac{\cos n\varphi}{n^2\left[1 + \frac{E_2\delta_2}{E_1\delta_1} + \frac{2JE_0 n^3}{R^3\delta_1 E_1}\left(1 - \frac{1-\mu}{2k}\right) - \frac{hn}{Rk}\left(1 - \frac{E_2\delta_2}{E_1\delta_1}\right)\right]} -$$

$$- \frac{P^2\left(1 - \frac{E_2\delta_2}{E_1\delta_1}\right)}{4\pi E_1\delta_1} \sum_{n=1}^{\infty} \frac{\left(1 - \frac{1-\mu}{2k}\right)\left[\cos 2n\varphi + 2n^2\left(\varphi^2 - \frac{1}{3}\pi^2\right)\right]}{n^2 k\left[1 + \frac{E_2\delta_2}{E_1\delta_1} + \frac{2JE_0 n^3}{R^3\delta_1 E_1}\left(1 - \frac{1-\mu}{2k}\right) - \frac{hn}{Rk}\left(1 - \frac{E_2\delta_2}{E_1\delta_1}\right)\right]}.$$

As calculations show, in the fast expression for moment M the series converge faster than in formula (7.27).

The structure of all the given formula was obtained rather complex and therefore it is not possible to make any general conclusions about the stressed and deformed state of the considered system. Therefore, calculations of certain constructions were performed. Figures 131-135 for illustration show curves for stresses σ_{x1} and $\sigma_{\phi 1}$ obtained during calculation of the shell, reinforced by a ring, under various assumptions relative to material of the ring and shell, and also with and without allowance for tangential forces, acting at the juncture point of the ring with shell. If tangential forces are not considered, then in all formulas the terms containing quantity k should be omitted.

Curves of stresses are constructed for the case of action of four forces simultaneously, located at 90° angle to each other.

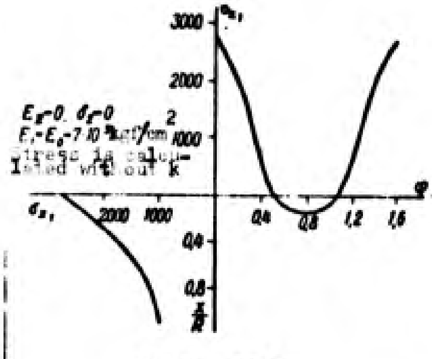

Fig. 131.

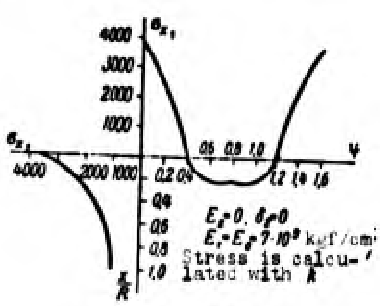

Fig. 132.

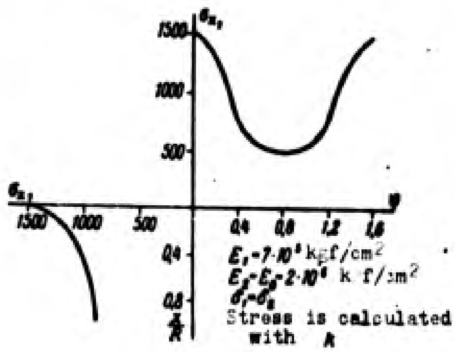

Fig. 133.

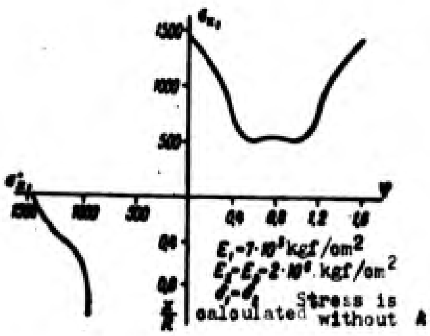

Fig. 134.

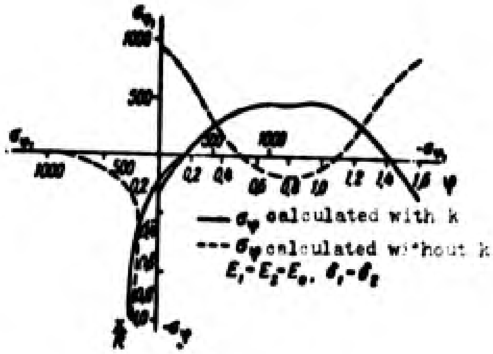

Fig. 135.

If shell *II* is absent ($E_2 = 0$, $\delta_2 = 0$) and tangential forces at the place of shell and ring are not considered, then formulas for shell *I* and for bending moment of the ring (7.26) and (7.27) convert into formulas for a beam on an elastic base. Moreover in these formulas we should substitute $R = l/\pi$, $\phi = \pi y/l$, $\delta_1 = 1$, where l – half of the width of the plate.

From the given curves of stresses σ_{x1}, $\sigma_{\phi 1}$ some practical conclusions can be made.

1. The presence of shell *II* sharply lowers the concentration of membrane stresses. Therefore, with the presence of this shell in the construction it must be checked for rigidity against the action of concentrated force *P*.

2. The account of tangential forces at the juncture point of ring with the shell is substantially indicated by the amount and character of stress distribution in the shell.

3. At a distance of the radius from the point of application of force *P* there occurs almost total equalizing of stresses along the cross section of the shell. Hence follows the conclusion that the obtained formulas are applicable to shells if their length satisfies condition $l \geq R$.

CHAPTER VIII

CALCULATION OF REINFORCED CYLINDRICAL SHELLS UNDER AXIAL AND LATERAL LOADS

In this chapter the sequence of calculating a cylindrical shell, reinforced by stringers and by frames and loaded with bending moments, by axial and lateral forces, are examined. We will consider that the sheathing of the examined is sufficiently shell thin, able to lose regidity long before the failure of the entire construction on the whole (of the elements of the superstructure). After the loss of rigidity the sheathing hardly functions at all and does not respond to normal stresses from bending moments and axial forces. Only narrow strips of the sheathing, which are adjacent to the stringers, will respond to normal stresses. However, the sheathing will become operational under lateral loads, by exerting a shearing force on the reinforced structure.

From the aforementioned it follows that the thin sheathing is in effectively used as a supporting element of the construction.

In spite of this, shells with a thin sheathing find application in technology in view of the fact that comparative calculations indicate that reinforced shells by weight ratio are more suitable than nonreinforced ones with a thick sheathing.

§ 39. Effective Width of the Sheathing of a Reinforced Cylindrical Shell, Responding Under the Action of Axial Compression and Inside Pressure

In § 12 a formula was obtained for the effective width of a plate based on the assumption that it was free of a lateral load.

In this paragraph an analogous formula is derived for a cylindrical panel with the allowance for the action of a lateral distributed load on it. We will consider the effect of this load for a given width through meridional and circumferential stresses, which appear in a cylindrical shell due to booster pressure and the hydrostatic column of liquid. For the solution of the problem posed here let us use the equations (7.10)

$$\nabla^2\nabla^2\varphi = \frac{E\delta}{R}\frac{\partial^2 w}{\partial x^2},$$

$$\frac{1}{R}\frac{\partial^2\varphi}{\partial x^2} + D\nabla^2\nabla^2 w + N_x^0\frac{\partial^2 w}{\partial x^2} + N_y^0\frac{\partial^2 w}{\partial y^2} = 0,$$

where

$$N_x^0 = -\frac{q_{\text{н}} R}{2} + N_{\text{кр}},$$

$$N_y^0 = -qR, \quad q = q_{\text{н}} + \gamma h,$$

$q_{\text{н}}$ – booster pressure; h – height of the liquid column above the investigated section; $N_{\text{кр}}$ – critical value of axial compressive force.

We will propose that the sheathing is rigidly attached to the stringers in its plane, but can freely rotate together with the stringer around its axis. The following expressions for the deflection and function of stresses will satisfy these conditions, along the contour of the panel:

$$w = A\sin\frac{m\pi x}{l}\sin\frac{\pi\theta}{\theta_0},$$

$$\varphi = B\sin\frac{m\pi x}{l}\sin\frac{\pi\theta}{\theta_0},$$

where θ_0 – central angle, which limits the sought width of the sheathing, adjoining two adjacent stringers.

Let us substitute the accepted expressions for w and ϕ, and likewise N_x^0 and N_y^0 in the original equations. From the condition of equality the zero of the determinant of these equations is derived

$$N_{кр} = -\frac{E\delta\lambda^2}{R^2\left[\lambda^2+\left(\frac{\pi}{\theta_0}\right)^2\right]^2} + D\frac{\left[\lambda^2+\left(\frac{\pi}{\theta_0}\right)^2\right]^2}{R^2\lambda^2} + qR\frac{\left(\frac{\pi}{\theta_0}\right)^2}{\lambda^2} + \frac{q_н R}{2},$$

where $\lambda=\left(\frac{m\pi R}{l}\right)^2$ -- parameter of wave formation.

The right side of the derived expression consists of a component, characteristic of the energy of tension and bending of the shell and of the work of the lateral load (pressure). During the examination of the narrow strip of panel adjoining the stringer, its potential energy will basically consist of the energy of bending. Therefore, for the approximate solution of the problem one can discard the first term in the right part of this expression. Then

$$N_{кр} = \frac{D}{R^2}\cdot\frac{\left[x+\left(\frac{\pi}{\theta_0}\right)^2\right]^2}{x} + qR\frac{\left(\frac{\pi}{\theta_0}\right)^2}{x} + \frac{q_н R}{2},$$

where

$$x=\lambda^2.$$

Minimum of force $N_{кр}$ along parameter x will be

$$\sigma_{кр} = \frac{N_{кр}}{\delta} = \frac{2D}{R^2\delta}\left(\frac{\pi}{\theta_0}\right)^2 + \frac{2D}{R^2\delta}\,\frac{\pi}{\theta_0}\sqrt{\left(\frac{\pi}{\theta_0}\right)^2+\frac{qR^3}{D}} + \frac{q_н R}{2\delta}.$$

For determination of angle θ_0 we will have the condition

$$\varepsilon_{об}^{кр} = \varepsilon_{стр}^{кр},$$

expressing the equality of relative deformations of the stringer and the sheathing adjacent to it. In the developed form this condition is

$$\frac{2D}{R^2\delta}\,\frac{\pi}{\theta_0}\left[\frac{\pi}{\theta_0}+\sqrt{\left(\frac{\pi}{\theta_0}\right)^2+\frac{qR^3}{D}}\right] + \frac{q_н R}{2\delta} = \sigma_{стр}.$$

By solving the derived equation relative to θ_0, we will find when $\mu = 0.3$, the following value for the effective width of the sheathing:

$$R\theta_0 = 2C = 1{,}9\delta\sqrt{\frac{E}{\sigma_{стр}}}\sqrt{\frac{1+\frac{qR}{\delta\sigma_{стр}}-\frac{q_н R}{2\delta\sigma_{стр}}}{1-\frac{q_н R}{\delta\sigma_{стр}}}}.$$

By inserting $q = 0$, $q_н = 0$ in this formula, we will obtain the known formula of the effective width of a flat plate.

§ 40. Determination of Stresses in a Reinforced Cylindrical Shell Under a Load by its Bending Moment, and by Axial and Lateral Forces

The calculation of a cylindrical reinforced shell due to the action of bending moments and axial forces (Fig. 136) is based on the unknown formula

$$\sigma = \pm\frac{M_x y_i}{J_{x\,пр}} \pm \frac{M_y x_i}{J_{y\,пр}} - \frac{N}{F_{пр}},$$

where M_x, M_y – bending moments in the section relative to the main axis Ox, Oy; N – resultant axial force in the section; $J_{xпр}$, $J_{yпр}$, $F_{пр}$ – given inertia moments and the area of the lateral section; x_i, y_i – coordinates of the centers of gravity of the elements of the section relative to the main axis Ox, Oy.

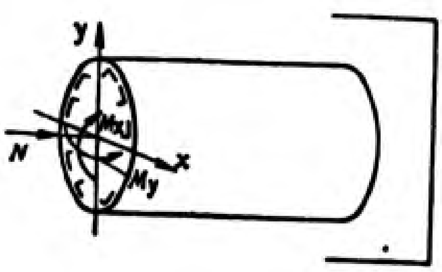

Fig. 136.

The given area and moments of inertia for the stretched zone of the section are determined according to the expressions

$$F_{пр} = \sum_{i=1}^{n} F_{пр\,i},$$

$$F_{пр\,i} = F_{стр\,i} + 0{,}8 b_i \delta_i, \qquad (8.1)$$

$$J_{x\,пр} = \sum_{i=1}^{n} F_{пр\,i} y_i^2,$$

$$J_{y\,пр} = \sum_{i=1}^{n} F_{пр\,i} x_i^2,$$

where $H_{стрi}$ – area of the section of the ith stringer; b_i – distance between the stringers; δ_i – thickness of the sheathing; i – ordinal number of the stringer.

Coefficient[1] 0.8 in the second of the given formulas is considered under the circumstance that the sheathing, having acquired initial irregularities during the manufacture of the shell, does not completely respond to action loads.

In the compressed zone of the section for the given area and for the moments of inertia, we have the following expressions:

$$F_{пр} = \sum_{i=1}^{n} F_{пр\,i}, \quad F_{пр\,i} = F_{стр\,i} + (2C)_{пр\,i} \delta_{об\,i}, \qquad (8.2)$$

$$J_{x\,пр} = \sum_{i=1}^{n} F_{пр\,i} y_i^2, \quad J_{y\,пр} = \sum_{n} F_{пр\,i} x_i^2,$$

where $(2C)_{прi}$ – effective width of the sheathing, which is determined according to the latter formula of the previous paragraph. If the sheathing in the investigated zone loses rigidity due to shear, then the effective width is usually reduced by half.

In order to complete the above given calculation, the position of main central axis of inertia of the x, y sections must be known. During the calculation as a first approximation one first determines the compressive stresses in the investigated shell due to action of loading, equal to

[1]The value of this coefficient was established experimentally.

$$T^{р}_{экв}=\left(\frac{4M_x}{D}+\frac{4M_y}{D}+N\right)f, \qquad (8.3)$$

where f – accepted for the calculation of the safety factor; D – diameter of the shell.

Then, the stress of compression is

$$\sigma^{р}_{экв}=\frac{T^{р}_{экв}}{F},$$

where F – complete area of the section of the shell.

Quadrupled actions of the bending moments, allotted to the diameter of the shell, in the right part of the expression (8.3), for the effect in the compressed zone of the shell are equivalent to the uniform compressive forces.

Following the calculation of the stress $\sigma^{р}_{экв}$ it is compared with the critical stress of compression for the cylindrical panel, which is determined by the formula

$$\sigma_{кр}=0{,}15E\frac{\delta}{R}+3{,}6E\left(\frac{\delta}{b}\right)^2, \qquad (8.4)$$

where δ, R – thickness and radius of the shell; b – distance between the stringers.

This formula, apparently, is semi-empirical. It has found wide application in calculations of aviation constructions.

If the investigated cover is subjected to the action of internal pressure, then for the determination of the critical stress of compression one can obtain the formula

$$\sigma_{кр}=\sigma^{0}_{кр}+\frac{qR}{\delta}\left(1-\frac{\frac{qR}{\delta}}{\sigma_s-\sigma^{0}_{кр}}\right)+\frac{q_{н}R}{2\delta}.$$

where $\sigma^0_{кр}$ is determined according to the expression (8.4). The remaining part of this formula is constructed on the basis of the theorem of convex surfaces of rigidity (see Chapter XVII).

If tangential stresses also act on the investigated part of the shell, then in this instance it is possible to construct a hyperplane in three-dimensional space.

From the comparison of $\sigma^p_{экв}$ and $\sigma_{кр}$ the possibility of a loss in rigidity of the sheathing in the compressed zone of the shell is established, whereupon one can determine the coordinates of the center of gravity of the entire section according to formula

$$x_{ц.т}=\frac{\sum_{i=1}^{n} F_{пр\,i}x_i}{\sum_{i=1}^{n} F_{пр\,i}}; \quad y_{ц.т}=\frac{\sum_{i=1}^{n} F_{пр\,i}y_i}{\sum_{i=1}^{n} F_{пр\,i}},$$

where $x_{ц.т}$, $y_{ц.т}$ – distance from the geometrical center of the section of the shell in the direction axes x, y up to the center of gravity of the section after the loss in rigidity of the sheathing; $F_{пр}$ – area of section of the shell, calculated by the formulas (8.1)-(8.2).

After the determination $x_{ц.т}$, $y_{ц.т}$ it is possible to approach the determination of stresses in a reinforced shell during the action of bending moments and axial forces on it.

The determination of tangential and additional normal stresses in the elements of a reinforced cylindrical shell under its loading by a lateral force. If the sheathing of a reinforced shell loses rigidity due to the action of a lateral force, then additional loading of the superstructure is carried out with normal stresses.

For the determination of tangential stresses in the sheathing of a reinforced shell one can make use of the differential equations of the momentless theory of the shells:

$$\frac{\partial N_z}{\partial z}+\frac{\partial N_{zs}}{\partial s}=0,$$
$$\frac{\partial N_{zs}}{\partial z}+\frac{\partial N_\theta}{\partial s}=0, \quad (8.5)$$
$$N_\theta=q\rho,$$

which in this case is expressed by coordinates z, s (Fig. 137).

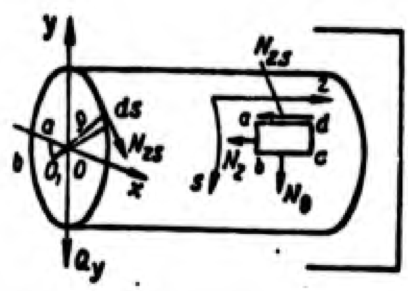

Fig. 137.

By inserting q = const here, we will obtain;

$$N_{zs}=f(s),$$

where $f(s)$ – a certain unknown function of arc s.

From the first equation (8.5) it follows that

$$f(s)=-\int\frac{\partial N_z}{\partial z}ds+N_{zs}^0$$

and

$$N_{zs}=-\int\frac{\partial N_z}{\partial z}ds+N_{zs}^0, \quad (8.6)$$

where N_{zs}^0 – constant of the integration.

Under the sign of the integral in the equation (8.6) is derived from the normal force, which appeared in the shell due to action of the lateral force. For the determination of N_z let us use the formula, known from the course of the strength of materials:

$$N_z = \frac{M_x h y_l}{J_{x\,np}} = \frac{Q_y z h y_l}{J_{x\,np}}.$$

Then,

$$N_{zs} = -\frac{Q_y}{J_{x\,np}} \int \delta ds y_l + N^0_{zs}. \qquad (8.7)$$

This expression is an equation of the equilibrium of the element *abcd* in the direction of axis z (Fig. 137).

The indefinite integral in (8.7) is area moment ratio $ab\delta$ with respect to axis x. Thus, it is possible to write

$$N_{zs} = -\frac{Q_y S_x}{J_{x\,np}} + N^0_{zs}.$$

For the determination of constant N^0_{zs} let us form the equation of moments of all the forces in the section relative to a certain point O_1:

$$Q_y a - \oint N_{zs} \varrho ds = 0$$

or

$$Q_y a + \frac{Q_y}{J_{x\,np}} \oint S_x \varrho\, ds - N^0_{zs} \oint \varrho ds = 0,$$

whence

$$N^0_{zs} = \frac{Q_y a + \frac{Q_y}{J_{x\,np}} \oint S_x \varrho ds}{2F},$$

where $2F = \oint \varrho ds$ — the doubled area, limited by the average line of the transverse section of the shell.

Since during the composition of the equation of moments point $_1$ is selected randomly, then, substituting for convenience this point on the line of the effective force Q_y, we will obtain $Q_y a = 0$. Then

$$N^0_{zs} = \frac{Q_y}{2EJ_{x\,np}} \oint S_x \varrho ds.$$

Ultimately, for current value of the tangential force we will have the formula

$$N_{zs}=\frac{Q_y}{2FJ_{x\,\text{пр}}}\oint S_x\varrho ds-\frac{Q_yS_x}{J_{x\,\text{пр}}}.$$

An analogical relationship can be written even for the case of the action of the lateral force Q_x

$$N'_{zs}=\frac{Q_x}{2FJ_{y\,\text{пр}}}\oint S_y\varrho ds-\frac{Q_xS_y}{J_{y\,\text{пр}}}.$$

The tangential stresses in the shell

$$\tau^{p}=\frac{1}{\delta}(N_{zs}+N'_{zs})f,$$

where f – accepted safety factor.

During the calculation of the total stress τ the directions of action of forces Q_x and Q_y must be considered.

After the calculation of the tangential stress τ^p it is necessary to compare it with the critical shear stress for the given shell. This stress can be determined according to the semi-empirical formula

$$\tau_{\text{кр}}=0{,}1E\frac{\delta}{R}+5E\left(\frac{\delta}{b}\right)^2.$$

If the normal stresses of compression from the longitudinal force and the bending moment as well as from the internal pressure act in the investigated part of the shell, then they should be taken into account during the determination of $\tau_{\text{кр}}$, having made use of the theorem of convex surfaces of rigidity for this purpose (Chapter XVII), on the basis of which a hyperplane is built in three-dimensional space. Figure 138 is given as an example.

$\sigma_{\text{кр}}$ – axial critical stress of compression; σ_2 – annular tensile stress from the internal pressure.

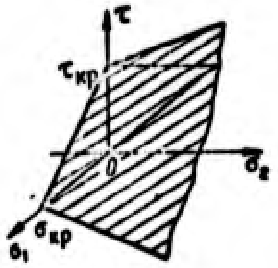

Fig. 138.

On those sections of the shell, these losses in rigidity of the strengthening occur due to shear (which is evident from a comparison of τ^p and $\tau_{кр}$), the superstructure is additionally loaded b cause of the difference in tangential stresses ($\tau^p - \tau_{кр}$). The sheathing "strethces" along the stringers and frames and additionally loads them. In this case the shell in the shear zone will operate as a beam with a thin wall (Fig. 139 .

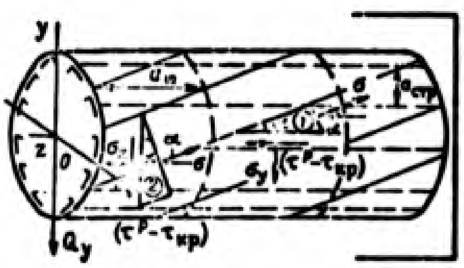

Fig. 139.

From the condition of equilibrium of element 1 in the direction of axis z one can obtain

$$(\tau^p - \tau_{кр}) s\delta = \sigma s\delta \sin\alpha \cos\alpha,$$

whence for the tensile stress in the sheathing

$$\sigma = \frac{\tau^p - \tau_{кр}}{\sin\alpha \cos\alpha}.$$

The stress, calculated according to this formula, is used in the check of the sheathing. In contrast to this, stresses of compresion will appear in the stringers. The sheathing, compressing will appear in the stringers. The force, compressing the stringer, will be

$$Z = a_{\text{стр}} \cos\alpha \cdot \delta\sigma \cos\alpha = a_{\text{стр}} \delta (\tau^{p} - \tau_{\text{кр}}) \,\mathrm{ctg}\,\alpha.$$

Then, the additional stress of compression in the stringer can be determined according to the expression

$$\sigma^{\text{д}}_{\text{стр}} = \frac{Z}{F_{\text{стр}}} = \frac{a_{\text{стр}} \delta (\tau^{p} - \tau_{\text{кр}}) \,\mathrm{ctg}\,\alpha}{F_{\text{стр}}}.$$

Furthermore, from the stresses σ the stringers are also loaded by the lateral bend. The linear load due to action of these stresses can be determined from the condition of equilibrium of element 1 in the direction tangent to the lateral section of the shell:

$$\sigma_{y} s\delta = s \sin\alpha\, \delta\sigma \sin\alpha,$$

whence

$$\sigma_{y} = (\tau^{p} - \tau_{\text{кр}}) \,\mathrm{tg}\,\alpha.$$

Analogically, from the condition of equilibrium of element 2 one can obtain

$$\sigma_{z} = (\tau^{p} - \tau_{\text{кр}}) \,\mathrm{ctg}\,\alpha.$$

The stress in the frame

$$\sigma_{\text{ш}} = \frac{a_{\text{ш}} \delta \sigma_{y}}{F_{\text{ш}}} = \frac{a_{\text{ш}} \delta (\tau^{p} - \tau_{\text{кр}}) \,\mathrm{tg}\,\alpha}{F_{\text{ш}}}.$$

Loading and stringers cause stresses σ_y. The radial component from these stresses will be

$$q = \sigma_{y} \delta \frac{a_{\text{стр}}}{R} = (\tau^{p} - \tau_{\text{кр}}) \delta \frac{a_{\text{стр}}}{R} \,\mathrm{tg}\,\alpha.$$

Figure 140 represents a diagram of the loading of stringers by a distributed load q. For a more precise definition of the given problem one should examine the combined work of stringers and frames due to the action of load q. However, this refinement is associated with serious difficulties in calculation. In the practice of engineering calculations, simplifying this problem, the plastic supports of the stringers are replaced with hinged ones and the diagram of the calculation of a stringer for a lateral load q is reduced to an elementary problem of calculating a beam on two supports (Fig. 141). Sometimes, the theorem of three moments is applied to the calculation of a stringer.

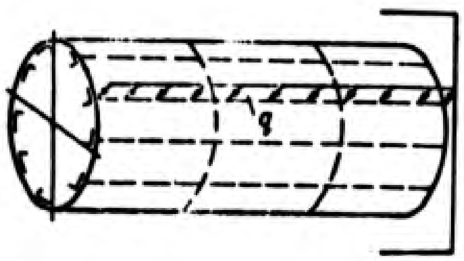

Fig. 140.

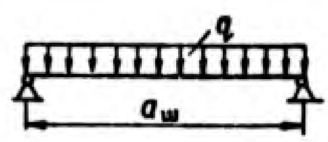

Fig. 141.

The total stresses in the stringers are

$$\sigma^{\Sigma}_{стр} = \sigma_{M_x} + \sigma_{M_y} + \sigma_N + \sigma_z^{Q_y} + \sigma_q^{Q_y},$$

where σ_{Mx}, σ_{My}, σ_N – stresses from the bending moments and axial forces; $\sigma_z^{Q_y}$, $\sigma_q^{Q_y}$ – additional stresses from the lateral force Q_y.

If a lateral force Q_x will act on the shell, then it is likewise necessary to determine from it the additional stresses in stringers.

The greatest obtained compressive stresses can be compared with the critical stresses for stringers, which are usually calculated by the formula for a flat plate:

$$\sigma_{кр} = \frac{0.9kE}{\left(\frac{b}{\delta}\right)^2}.$$

If the critical stresses are obtained higher than the limit of proportionality according to this formula, then their calculation can be made in the expression

$$\sigma_{кр} = \sigma_в - (\sigma_в - \sigma_p)\frac{\frac{b}{\delta}}{\left(\frac{b}{\delta}\right)_{пр}},$$

where

$$\left(\frac{b}{\delta}\right)_{пр} = \sqrt{\frac{0.9kE}{\sigma_p}};$$

b – width regiments to stringer; k – coefficients, depending on the boundary conditions of the plate.

The obtained values of the critical stresses according to the given formula during the determination of the safety factor are somewhat reduced, considering that this is an unfavourable effect on the work of the stringer of the sheathing, which has lost rigidity. This decrease in the critical stresses on the basis of the experiment can be taken within the following limits:

by 10% with a thickness of the sheathing δ = 0.5-1.0 mm by 15% with a thickness of the sheathing δ = 1.0-1.5 mm by 15-20% with a thickness of the sheating δ = 1.5-2.0 mm.

Stresses in the frames, which appear in connection with the loss of the rigidity of the sheathing, are usually not calculated and do not affect the value of the safety factor.

Safety factor for stringers is

$$\eta = \frac{\sigma_{кр}}{\sigma^{\delta}_{стр}} \geq 1.$$

This safety factor should be obtained with an allowance for the temperatures of the sheathing and superstructure. The calculation of the temperature is conducted by means of the corresponding change in the mechanical characteristics of the materials based on the data of the experiments.

All the results of the calculations are entered in a table (Table 8) for the convenience of calculations and checking.

Table 8.

№ stringer	a_i	y_i	$F_{np\,i}$	$F_{np\,i}y_i$	y_i'	$F_{np\,i}y_i'^2$	$\Delta S_i = F_{np\,i}y_i'$	$S_i = \sum \Delta S_i$	$ds = b$	$\varrho\,ds$	$S_i\varrho\,ds$	$\overbrace{\frac{Q_y^p S_i}{J_{x\,np}}}^{I}$
1	2	3	4	5	6	7	8	9	10	11	12	13

Continuation

$\overbrace{\frac{Q_y^p}{2EJ_{x\,np}}\sum S_i\varrho\,ds}^{II}$	$N_{zs}^p = II - I$	$\tau^p = \frac{N_{zs}^p}{\delta}$	$\tau_{kp\,i}$	σ_{M_x}	σ_N	σ_{Q_z}	σ_{Q_y}	$\sigma_{kp\,i}^{cTp}$	η_i
14	15	16	17	18	19	20	21	22	23

$$y_{u.T} = \frac{\sum_{i=1}^{n} F_{np\,i}\,y_i}{\sum_{i=1}^{n} F_{np\,i}}.$$

To complete the calculation from the above given dependences it is necessary to know angle of inclination of wave α after the loss of rigidity of the sheathing due to shear. For the determination of this angle let us employ the energy method.

The potential energy of deformation of a separately taken panel with the superstructure attached to it can be written in the form

$$\Im = \frac{\delta}{2E}\int_0^{a_{cTp}}\int_0^{a_{m}}[\sigma_y^2 + \sigma_z^2 - 2\mu\sigma_y\sigma_z + 2(1+\mu)\tau^2]\,dz\,ds +$$

$$+\int_0^{a_{m}}\frac{N_{cTp}^2\,dz}{2EF_{cTp}} + \int_0^{a_{cTp}}\frac{N_{m}^2\,ds}{2EF_{m}} + \int_0^{a_{m}}\frac{M^2\,dz}{2EJ_{cTp}},$$

where the energy of deformation of the sheathing is determined by the first double integral, and the energy of deformation of the sections of the stringer and frame – by the second and third. The last integral determines the energy of bending of the stringer by the distributed load q in the section between the frames.

In the given expression for the energy of deformation one should substitute the values of stresses and force:

$$\sigma_y=(\tau^p-\tau_{кр})\operatorname{tg}\alpha, \quad \sigma_z=(\tau^p-\tau_{кр})\operatorname{ctg}\alpha, \quad \tau=\tau^p-\tau_{кр},$$

$$N_{стр}=(\sigma_M+\sigma_N)F_{стр}+a_{стр}\delta(\tau^p-\tau_{кр})\operatorname{ctg}\alpha,$$

$$N_ш=a_ш\delta(\tau^p-\tau_{кр})\operatorname{tg}\alpha,$$

$$M=\frac{a_{стр}\delta}{2R}(\tau^p-\tau_{кр})(a_ш z-z^2)\operatorname{tg}\alpha.$$

After simple calculations we obtain

$$Э=\frac{\delta a_{стр}a_ш}{2E}(\tau^p-\tau_{кр})^2(\operatorname{tg}\alpha+\operatorname{ctg}\alpha)^2+$$

$$+\frac{a_ш}{2EF_{стр}}\left[(\sigma_M+\sigma_N)F_{стр}+\frac{(\tau^p-\tau_{кр})a_{стр}\delta}{\operatorname{tg}\alpha}\right]^2+$$

$$+\frac{a_{стр}a_ш^2\delta^2}{2EF_ш}(\tau^p-\tau_{кр})^2\operatorname{tg}^2\alpha+\frac{\delta^2 a_{стр}^2 a_ш^5}{240EJ_{стр}R^2}(\tau^p-\tau_{кр})^2\operatorname{tg}^2\alpha.$$

tg α is the variable in this expression. Let us select it so that the potential energy would become minimum. Using the condition

$$\frac{dЭ}{d(\operatorname{tg}\alpha)}=0,$$

we obtain the following equation for the determination of angle α:

$$\operatorname{tg}^4\alpha-\frac{(\sigma_M+\sigma_N)\operatorname{tg}\alpha}{(\tau^p-\tau_{кр})\left(1+\frac{a_ш\delta}{F_ш}+\frac{a_{стр}\delta a_ш^4}{120J_{стр}R^2}\right)}-$$

$$-\frac{1+\frac{a_{стр}\delta}{F_{стр}}}{1+\frac{a_ш\delta}{F_ш}+\frac{a_{стр}\delta a_ш^4}{120J_{стр}R^2}}=0,$$

where $J_{стр}$ – moment of inertia of the stringer relative to its axis, parallel tangent to the circumference of the shell.

C H A P T E R IX

CALCULATION OF CERTAIN JOINTS AND PARTS OF SHELLS OF ROTATION

§ 41. Calculation of the Elements, Which Reinforce Openings Apertures in Spherical Shells

Nonreinforced openings. Let us examine spherical shell, loaded with an internal excess pressure q. Let us suppose that this cover has been weakened by a circular opening, determined by angle ϕ_0 (by Fig. 142).

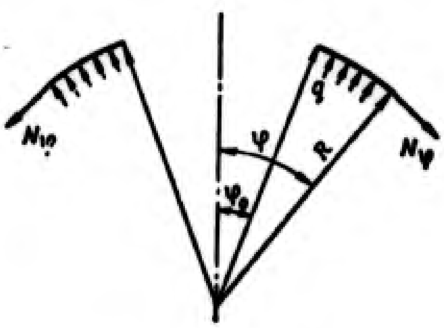

Fig. 142.

The balance of equations for a momentless spherical shell under an axisymmetrical load has the form

$$\frac{dN_\varphi}{d\varphi}+2N_\varphi \operatorname{ctg}\varphi=qR\operatorname{ctg}\varphi,$$
$$N_\theta+N_\varphi=qR. \qquad (9.1)$$

By integrating the first equation of this group, we obtain

$$N_\varphi=\frac{C}{\sin^2\varphi}-\frac{qR}{4}(\operatorname{ctg}^2\varphi-1). \qquad (9.2)$$

Let us find the constant integration C from the condition that

$$(N_\varphi)_{\varphi=\varphi_0}=0.$$

Then

$$N_\varphi=\frac{qR}{2}\left(1-\frac{\sin^2\varphi_0}{\sin^2\varphi}\right).$$

From the solution of the second equation (9.1) we obtain

$$N_\theta=\frac{qR}{2}\left(1+\frac{\sin^2\varphi_0}{\sin^2\varphi}\right),$$

from which it is clear that this **force attains its greatest value** on the contour of the opening:

$$N_{\theta\,max}=qR.$$

One should note that the expression for $N_{\theta\ max}$ does not depend on the diameter of the opening.

Openings, reinforced with a flange (Fig. 143). If a spherical shell has a reinforced opening, then at the joining of this support (flange) with the shell a force appears, whose value is designated by N_0 (Fig. 143b). It is considered that the force at the joint is directed along a tangent to the shell.

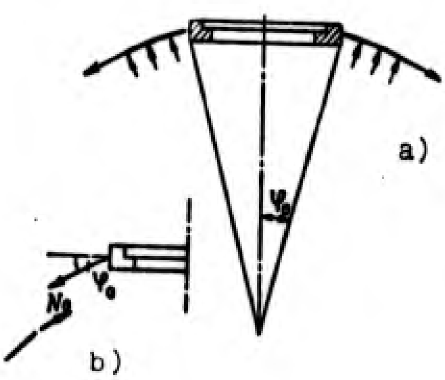

Fig. 143.

Distribution of forces in the shell in this case can be determined from the expression (9.2), if in it one assumes $q = 0$. Then

$$N_\varphi = \frac{C}{\sin^2\varphi}.$$

When $\phi = \phi_0$ there should be $N_\phi = N_0$. Therefore

$$C = N_0 \sin^2\varphi_0.$$

Then

$$N_\varphi = \frac{N_0 \sin^2\varphi_0}{\sin^2\varphi}.$$

The expression for N_θ has the form

$$N_\theta = -N_\varphi = -\frac{N_0 \sin^2\varphi_0}{\sin^2\varphi}.$$

For the determination of an unknown force N_0 let us set up the condition of continuity of displacements in a shell-flange system.

The radius of the opening due to internal pressure when $\phi = \phi_0$ will increase by

$$\Delta r_1 = \frac{R\sin\varphi_0}{Et}(N_\theta - \mu N_\varphi) = \frac{qR^2\sin^2\varphi_0}{Et}.$$

The decrease in the radius of the opening due to boundary regional condition of N_0, will be

$$\Delta r_2 = \frac{R\sin\varphi_0}{Et}(N_\theta - \mu N_\varphi) = -\frac{(1+\mu)N_0 R\sin\varphi_0}{Et}.$$

The increase in the radius of the flange due to force N_0 can be found from the expression

$$\Delta r_3 = \frac{N_0 R^2 \sin^2\varphi_0 \cos\varphi_0}{EF},$$

where F – area of the section of the flange.

Since

$$\Delta r_1 + \Delta r_2 = \Delta r_3, \qquad (9.3)$$

when, considering that the shell and the flange are made from identical material, we obtain the expression, which determines the force N_0:

$$N_0 = \frac{qR^2}{\left[\frac{R^2 \sin 2\varphi_0}{2F} + \frac{(1+\mu)R}{\delta}\right]}.$$

Now, the stresses in the shell and flange can be calculated.

The stresses in the shell will amount to the stresses from the internal pressure and the stresses from the boundary condition N_0:

$$\sigma_\varphi = \frac{qR}{2\delta}\left[1 - \frac{\sin^2\varphi_0}{\sin^2\varphi} + \frac{2\frac{R}{\delta}\frac{\sin^2\varphi_0}{\sin^2\varphi}}{\frac{R^2 \sin 2\varphi_0}{2F} + \frac{(1+\mu)R}{\delta}}\right],$$

$$\sigma_\theta = \frac{qR}{2\delta}\left[1 + \frac{\sin^2\varphi_0}{\sin^2\varphi} - \frac{2\frac{R}{\delta}\frac{\sin^2\varphi_0}{\sin^2\varphi}}{\frac{R^2 \sin 2\varphi_0}{2F} + \frac{(1+\mu)R}{\delta}}\right].$$

The stress in the flange

$$\sigma = \frac{qR^3 \sin 2\varphi_0}{2\delta F\left[\frac{R^2 \sin 2\varphi_0}{2F} + \frac{(1+\mu)R}{\delta}\right]}.$$

An opening, reinforced with an annular plate (Fig. 144). In this case in order to determine the stresses in a reinforced annular plate one can make use of Lamé's formula

$$\sigma_r = \frac{N_0 b^2}{h(b^2 - a^2)}\left(1 - \frac{a^2}{r^2}\right), \quad \sigma_\theta = \frac{N_0 b^2}{h(b^2 - a^2)}\left(1 + \frac{a^2}{r^2}\right),$$

where h – thickness of the plate.

The expressions for stresses in the cover from the internal pressure q and force N_0 will be

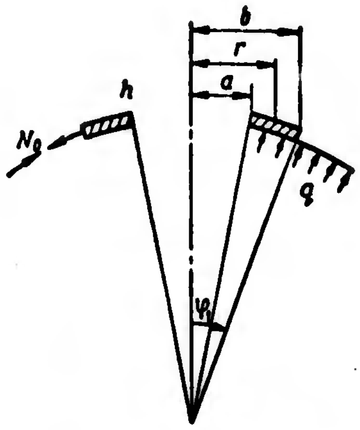

Fig. 144.

$$N_\varphi = \frac{qR}{2}\left(1 - \frac{\sin^2\varphi_1}{\sin^2\varphi}\right),\quad N_\theta = \frac{qR}{2}\left(1 + \frac{\sin^2\varphi_1}{\sin^2\varphi}\right),$$
$$N_\varphi = \frac{N_0 \sin^2\varphi_1}{\sin^2\varphi},\quad N_\theta = -\frac{N_0 \sin^2\varphi_1}{\sin^2\varphi}.$$

For the composition of the equation of compatibility of displacements we will have the following expressions, obtained on the basis of Hooke's law:

$$\Delta r_1 = \frac{qR^2 \sin\varphi}{E\delta},$$
$$\Delta r_2 = -\frac{(1+\mu) N_0 R \sin\varphi_1}{E\delta},$$
$$\Delta r_3 = \frac{N_0 b^2}{Eh(b^2 - a^2)}\left[(1-\mu) + (1+\mu)\frac{a^2}{b^2}\right].$$

Then, from the condition (9.3) we obtain

$$N_0 = \frac{\dfrac{qR^2 \sin\varphi_1}{\delta}}{\dfrac{b^3}{h(b^2 - a^2)}\left[(1-\mu) + (1+\mu)\dfrac{a^2}{b^2}\right] + \dfrac{(1+\mu) R \sin\varphi_1}{\delta}}.$$

After the determination of force N_0 one can calculate the stresses in the reinforced plate based on Lamé's formula, and stresses in the shell – according to the expressions

$$\sigma_\varphi = \frac{qR}{2\delta}\left(1 - \frac{\sin^2\varphi_1}{\sin^2\varphi}\right) + \frac{N_0 \sin^2\varphi_1}{\delta \sin^2\varphi},$$
$$\sigma_\theta = \frac{qR}{2\delta}\left(1 + \frac{\sin^2\varphi_1}{\sin^2\varphi}\right) - \frac{N_0 \sin^2\varphi_1}{\delta \sin^2\varphi}.$$

An opening, reinforced with an annular plate and a flange (Fig. 145). In this case in order to determine stresses in a reinforced annular plate the following formulas of Lamé are used:

$$\sigma_r = \frac{a^2 b^2 (N_2 - N_1)}{h (b^2 - a^2) r^2} + \frac{N_1 b^2 - N_2 a^2}{h (b^2 - a^2)},$$

$$\sigma_\theta = -\frac{a^2 b^2 (N_2 - N_1)}{h (b^2 - a^2) r^2} + \frac{N_1 b^2 - N_2 a^2}{h (b^2 - a^2)}.$$

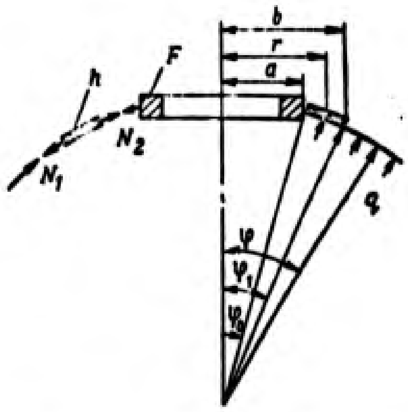

Fig. 145.

By applying Hooke's law, let us find the expressions for the change of the internal and external radius of the plate:

$$\Delta a_{\text{пл}} = \frac{a}{Eh}\left\{N_1 \frac{2b^2}{b^2 - a^2} - N_2\left[(1+\mu)\frac{b^2}{b^2 - a^2} + (1-\mu)\frac{a^2}{b^2 - a^2}\right]\right\},$$

$$\Delta b_{\text{пл}} = \frac{b}{Eh}\left\{N_1\left[(1+\mu)\frac{a^2}{b^2 - a^2} + (1-\mu)\frac{b^2}{b^2 - a^2}\right] - N_2\frac{2a^2}{b^2 - a^2}\right\}.$$

The change in radius of the opening of the shell will be

$$\Delta b_{\text{об}} = -\frac{(1+\mu) R N_1 \sin\varphi_1}{E\delta} + \frac{q R^2 \sin\varphi_1}{E\delta}.$$

The change in the radius of the flange can be determined according to the formula

$$\Delta a_{\text{ф}} = \frac{N_2 R^2 \sin^2\varphi_0}{EF}.$$

Since

$$\Delta a_{\text{ф}} = \Delta a_{\text{пл}}, \quad \Delta b_{\text{об}} = \Delta b_{\text{пл}},$$

then, considering that the material of the shell, of the plates and of the flange are the same, let us determine forces N_1 and N_2:

$$N_1 = \frac{A_3(A_1 + A_2)\dfrac{qR^2 \sin\varphi_1}{\delta}}{A_3(A_1 + A_2)(A_4 + A_5) - \dfrac{a}{bA_3}},$$

$$N_2 = \frac{\dfrac{qR^2 \sin\varphi_1}{\delta}}{A_3(A_1 + A_2)(A_4 + A_5) - \dfrac{a}{bA_3}},$$

where

$$A_1 = \frac{a}{h}\left[\frac{(1+\mu)b^2}{b^2 - a^2} + \frac{(1-\mu)a^2}{b^2 - a^2}\right],$$

$$A_2 = \frac{R^2 \sin^2\varphi_0}{F},$$

$$A_3 = \frac{h(b^2 - a^2)}{2ab^2},$$

$$A_4 = \frac{b}{h}\left[\frac{(1+\mu)a^2}{b^2 - a^2} + \frac{(1-\mu)b^2}{b^2 - a^2}\right],$$

$$A_5 = \frac{(1+\mu)R \sin\varphi_1}{\delta},$$

After the determination of forces N_1 and N_2 one can calculate the stresses in the shell, plate and flange.

The stresses in the flange

$$\sigma_\phi = \frac{N_2 R \sin\varphi_0}{F}.$$

The stresses in the shell

$$\sigma_\varphi = \frac{qR}{2\delta}\left(1 - \frac{\sin^2\varphi_1}{\sin^2\varphi}\right) + \frac{N_1 \sin^2\varphi_1}{\delta \sin^2\varphi},$$

$$\sigma_\theta = \frac{qR}{2\delta}\left(1 + \frac{\sin^2\varphi_1}{\sin^2\varphi}\right) - \frac{N_1 \sin^2\varphi_1}{\delta \sin^2\varphi}.$$

§ 42. Calculation of the Bottom Sides of the Frames for Concentrated Radial Forces

Let us examine the frame of a cylindrical vessel, loaded with two diametrical opposed compressive forces (Fig. 146). Inasmuch as the cylindrical shell adjacent to the frame weakly resists the radial loads, then in the calculation only the bottom, which we will consider to be rather gently sloping should be considered.

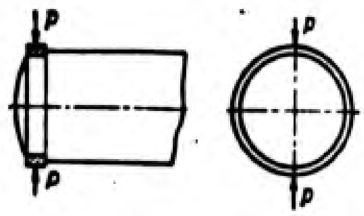

Fig. 146.

Let us conduct the calculation of the frame based on the limiting state, based on the diagram of an ideally plastic material.

An investigated calculated diagram, confirmed by an experiment, is presented in Fig. 147. The frame has a rectangular section.

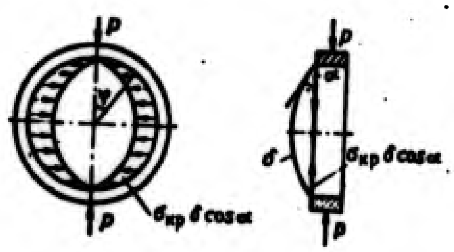

Fig. 147.

Failure of the frame by loading it with forces P will proceed in the following manner. With the shortening of the vertical diameter, the horizontal diameter of the frame will lengthen simultaneously. At this point the bottom will be loaded with compressive stresses at a critical value of which the bottom will lose rigidity. As a result of this a distributed load will act on the frame from the bottom side

$$q=(\sigma_{кр}\delta\cos\alpha)\sin\varphi,$$

where δ — thickness of the bottom; $\sigma_{кр}$ — critical stress of compression for a spherical shell.

This load has been directed perpendicular to that diameter whereby a shortening is obtained.

Prior to the moment of failure of the frame one can distinguish two stages of its work. The first stage will be characterized by the appearance of the first plastic hinge under forces P. After this, the frame still does not rotate as a mechanism and will be in a state to accept an additional load up to the moment of the appearance of the second plastic hinge at points, which lie on the horizontal diameter. After this, the frame rotates as a mechanism, and its bearing capacity will be depleted.

The limiting moment under force P will be

$$M'_{np}=2\left[\left(\frac{h}{2}-a_1\right)b\right]\left(\frac{\frac{h}{2}-a_1}{2}+a_1\right)\sigma_s=$$

$$=\sigma_s b\left(\frac{h^2}{4}-a_1^2\right)=2Q\left(\frac{\frac{h}{2}-a_1}{2}+a_1\right)=2Q\left(\frac{h}{4}+\frac{a_1}{2}\right),$$

where

$$Q=\left(\frac{h}{2}-a_1\right)b\sigma_s.$$

Forces Q are applied at the center of gravity of the non-shaded areas (Fig. 148).

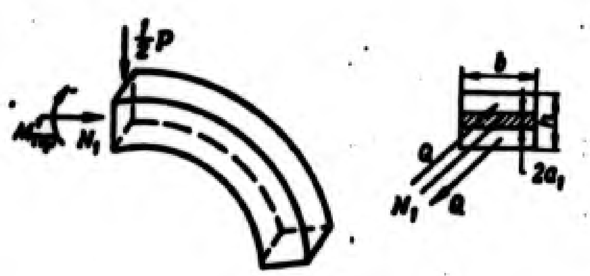

Fig. 148.

Besides M'_{np}, force N_1, distributed over the shaded area, will be applied in the investigated section the limiting value of this force

$$N_1=2a_1b\sigma_s. \quad (9.4)$$

On the other hand, one can determine this force from the condition of equilibrium of a quarter of the ring. For this purpose let us set up the sum of projections of all forces in the direction of N_1:

$$N_1=\int_0^{\frac{\pi}{2}} q\,ds\sin\varphi=\int_0^{\frac{\pi}{2}}(\sigma_{\kappa p}\delta\cos\alpha)\sin\varphi(R\,d\varphi\sin\varphi)=$$

$$=\sigma_{\kappa p}R\delta\cos\alpha\int_0^{\frac{\pi}{2}}\sin^2\varphi\,d\varphi=\frac{\pi}{4}\sigma_{\kappa p}R\delta\cos\alpha. \qquad (9.5)$$

By equating the expressions (9.4) and (9.5), we obtain

$$a_1=\frac{\pi R\delta\sigma_{\kappa p}\cos\alpha}{8b\sigma_s}.$$

Then

$$M'_{np}=\sigma_s b\left[\left(\frac{h}{2}\right)^2-\left(\frac{\pi R\delta\sigma_{\kappa p}\cos\alpha}{8b\sigma_s}\right)^2\right],$$

$$N_1=\frac{\pi}{4}R\delta\sigma_{\kappa p}\cos\alpha.$$

At the end of this stage of loading in the section at an angle $\phi = \pi/2$ (on a horizontal diameter) the plastic hinge is not completely developed and a part of the section continues to remain in the elastic stage. The stresses of fluidity begin to appear first in the fibers, furthest from the neutral axis. Therefore, a further increase in the force P is possible only because of the distribution of the plastic zone for the section located on the horizontal diameter. In this case the moment in the section, located on the vertical diameter, will reach the limiting value and with a further increase in the force P it does not increase.

Let us examine the equilibrium of a quarter of the ring in the limiting state in the presence of plastic hinges in sections along the horizontal and vertical diameters (Fig. 149).

Sum of the moments relative to the I-I axis will be

$$M'_{np}+M''_{np}+N_1R-\frac{PR}{2}-\left[\int_0^{\varphi} q\,d\xi\sin\xi(R\cos\xi-R\cos\varphi)\right]_{\varphi=90^\circ}=0.$$

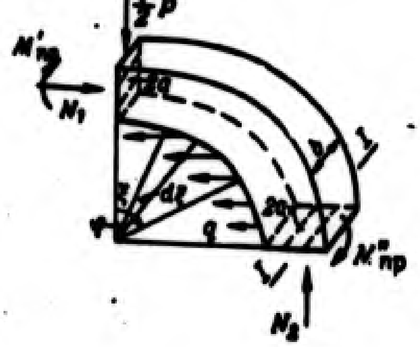

Fig. 149.

Hence, we will obtain

$$M'_{np}+M''_{np}+N_1R-\frac{PR}{2}-\frac{1}{3}R^2\delta\sigma_{np}\cos\alpha=0. \qquad (9.6)$$

Limiting value for M''_{np} will be

$$M''_{np}=2\left[\left(\frac{h}{2}-a^2\right)b\right]\left(\frac{\frac{h}{2}-a_2}{2}+a_2\right)\sigma_s=\sigma_s b\left(\frac{h^2}{4}-a_2^2\right).$$

Expression for a_2 can be determined in the following manner. From the condition of equilibrium of a quarter of the ring in the direction of the vertical diameter, we will obtain

$$N_2=\frac{P}{2}.$$

On the other hand

$$N_2=2\sigma_s a_2 b.$$

From these expressions we will find

$$a_2=\frac{P}{4\sigma_s b}.$$

Then

$$M''_{np}=\sigma_s b\left(\frac{h^2}{4}-\frac{P^2}{16\sigma_s^2 b^2}\right).$$

Now, the equation (9.6) can be expressed in the following form:

$$\sigma_s b\left[\left(\frac{h}{2}\right)^2-\left(\frac{\pi R\delta\sigma_{кр}\cos\alpha}{8b\sigma_s}\right)^2\right]+\sigma_s b\left[\left(\frac{h}{2}\right)^2-\left(\frac{P}{4b\sigma_s}\right)^2\right]+$$
$$+\frac{\pi}{4}R^2\delta\sigma_s\cos\alpha-\frac{PR}{2}-\frac{1}{3}R^2\delta\sigma_{кр}\cos\alpha=0$$

where

$$P^2+8Rb\sigma_s P-8b^2h^2\sigma_s^2+\frac{\pi^2}{4}R^2\delta^2\sigma_{кр}^2\cos^2\alpha-$$
$$-16R^2b\delta\sigma_{кр}\sigma_s\left(\frac{\pi}{4}-\frac{1}{3}\right)\cos\alpha=0.$$

By solving this equation relative to P, we will obtain

$$P=4Rb\sigma_s\left[\sqrt{1+\frac{1}{2}\left(\frac{h}{2}\right)^2-\left(\frac{\pi}{8}\right)^2\left(\frac{\sigma_{кр}}{\sigma_s}\right)^2\left(\frac{\delta}{b}\right)^2\cos^2\alpha+\left(\frac{\pi}{4}-\frac{1}{3}\right)\left(\frac{\sigma_{кр}}{\sigma_s}\right)\left(\frac{\delta}{b}\right)\cos\alpha}-1\right].$$

Making use of formula $\sqrt{1+x}\simeq 1+\frac{1}{2}x$, where $x \ll 1$, finally we will have

$$P_{кр}=2Rb\sigma_s\left[\frac{1}{2}\left(\frac{h}{R}\right)^2-\left(\frac{\pi}{8}\right)^2\left(\frac{\sigma_{кр}}{\sigma_s}\right)^2\left(\frac{\delta}{b}\right)^2\cos^2\alpha+\right.$$
$$\left.+\left(\frac{\pi}{4}-\frac{1}{3}\right)\left(\frac{\sigma_{кр}}{\sigma_s}\right)\left(\frac{\delta}{b}\right)\cos\alpha\right].$$

This formula determines the limiting value of the compressive force, acting on the frame, reinforced with a gently sloping spherical bottom.

Following from the experiments, in this formula one can assume that

$$\sigma_{кр}=0{,}2E\frac{\delta}{R},$$

where R and δ – radius and thickness of the bottom.

§ 43. Twisting of a Ring, Weakened by Openings

Let us examine a circular enclosed ring, weakened by openings and loaded with a distributed twisting moment.

Let us consider the sizes of the cross section of such a ring to be small in comparison with its radius. Among the number of such parts there are flanges of all sizes, superstructures of vessels, which have openings for the joining of coupling units or assemblies. In certain cases for the purpose of economy of weight, the flanges are made with pattern cuts.

Let us examine the effect of openings on the strength of a ring loaded by a twisting moment.

The full potential energy of such a ring is

$$Э=\frac{1}{2}\iint\limits_F\int\limits_0^{2\pi R}\sigma\varepsilon\, dF\, ds-\int\limits_0^{2\pi R}M_t\theta\, ds.$$

Here[1]

$$\sigma=\frac{E\theta y}{R};\quad \varepsilon=\frac{\theta y}{R};$$

θ – angle of rotation of a section of the ring; M_t – linear twisting moment.

Then

$$Э=\frac{E}{2}\iint\limits_F y^2 dF\int\limits_0^{2\pi R}\left(\frac{\theta}{R}\right)^2 ds-\int\limits_0^{2\pi R}M_t\theta\, ds=$$
$$=\frac{E}{2}\int\limits_0^{2\pi R}J\left(\frac{\theta}{R}\right)^2 ds-\int\limits_0^{2\pi R}M_t\theta\, ds.$$

[1] S. P. Timoshenko, Strength of materials, part II, 1934.

Let us assume that the moment of inertia of a section of the ring J is a quantitative variable[1]:

$$J = f(\theta) = J_0 \pm \Delta J_0.$$

Then, the expression for $Э$ takes the form

$$Э = \frac{EJ_0}{2R^2}\int_0^{2\pi R} \theta^2 ds \pm \frac{E}{2R^2}\int_0^{2\pi R} \Delta J_0 \theta^2 ds - \int_0^{2\pi R} M_t \theta ds.$$

After the integration we obtain

$$Э = \frac{\pi E J_0 \theta^2}{R} \pm \frac{E\theta^2}{2R^2}\sum \Delta J_0 (\Delta s) - 2\pi R M_t \theta.$$

Here, under the sign of the summation is the work, consisting of the magnitude, characteristic of the change in the mom- t of inertia of a section of the frame on that length of the : -al line, which changes the moment of inertia. The summation is made based on all the weakened (reinforced) sites of the frame.

To the last expression let us apply the origin of vertical displacements $\delta Э = 0$.

By solving the obtained equation relative to θ from the last condition, we will find that

$$\theta = \frac{M_t R^2}{EJ_0\left[1 \pm \frac{\sum \Delta J (\Delta s)}{2\pi R J_0}\right]}.$$

By making use of this expression, it is possible to evaluate the effect of the weakenings (reinforcements) of separate sections of the ring on its bearing capacity.

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[1]A. N. Dinnik, Longitudinal bending, State Joint Scientific and Technical Publishing House [GONTI] (ГОНТИ), 1939.

§ 44. Calculations of Circular Rings

Let us examine the calculation of a circular ring, attached to a thin cylindrical shell and loaded with forces P, T, M (Fig. 150). The joining shown in this figure is typical for a number of sheathings.

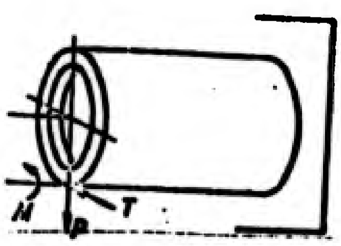

Fig. 150.

When loading the ring with concentrated forces P, T, M along the line of its joining with the shell, there appears a flow of tangential forces, the law of distribution of which will depend upon the character of the applied load. In this case let us consider the shell momentless.

When loading the ring with a concentrated moment this flow will be constant along the perimeter of the ring, and during the action of force P its greatest value will be at a diameter, perpendicular to the line of action of force P. When loading the ring with force T, the balancing flow of tangential forces can be obtained by means of the application of flows shown above.

For the solution of the problem we will consider form of load separately. Loading of the ring with concentrated force P is presented in Fig. 151. In this instance the law of change of tangential forces is determined by the expression

$$\tau = \tau_0 \sin \varphi,$$

where

$$\tau_0 = \frac{P}{\pi R}.$$

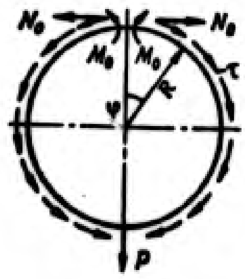

Fig. 151.

In this instance for the bending moment in an arbitrary section of the ring, with a determined angle ϕ, one can obtain the expression

$$M_p = M_0 - N_0 R(1-\cos\varphi) + \tau_0 R^2\left(\frac{1}{2}\varphi\sin\varphi + \cos\varphi - 1\right),$$

where M_0, N_0 – unknown internal forces in the transverse section.

For the determination of the unknown internal power factors, let us employ the principle of least work.

The potential energy of deformation of a semi-ring

$$Э = \frac{R}{2RJ}\int_0^{\pi} M_p^2\, d\varphi.$$

For the determination of forces M_0 and N_0 we will have the conditions

$$\frac{\partial Э}{\partial M_0} = 0, \quad \frac{\partial Э}{\partial N_0} = 0.$$

From these equations we will find

$$M_0 = -\frac{1}{4}\tau_0 R^2, \quad N_0 = -\frac{3}{4}\tau_0 R.$$

Then

$$M_p = \frac{PR}{2\pi}\left(\frac{1}{2}\cos\varphi + \varphi\sin\varphi - 1\right).$$

For the normal and shear forces in section ϕ we will obtain the following expressions:

$$N_p = \frac{P}{2\pi}\left(\frac{3}{2}\cos\varphi - \varphi\sin\varphi\right),$$

$$Q_p = \frac{P}{2\pi}\left(\varphi\cos\varphi + \frac{1}{2}\sin\varphi\right).$$

The function, which determined the change in the sags of the ring, will be found from the solution of the differential equation of the form

$$\frac{d^2w}{d\varphi^2} + w = \frac{-M_pR^2}{EJ} = \frac{-PR^3}{2\pi EJ}\left(\frac{1}{2}\cos\varphi + \varphi\sin\varphi - 1\right).$$

Here, the positive direction of the deflection is taken at the center of the ring. By applying the method of the variation of constant integrations to this equation, we will obtain

$$w = A\sin\varphi + B\cos\varphi -$$
$$-\frac{PR^3}{8\pi EJ}(\varphi^2\cos\varphi - 2\varphi\sin\varphi - \cos\varphi) - \frac{PR^3}{2\pi EJ},$$

where A and B – constants, determined from the boundary conditions.

In this case the boundary conditions will have the form

$$\frac{dw}{Rd\varphi} = 0 \text{ when } \varphi = 0, \quad \varphi = \pi;$$
$$w = -\delta_p \text{ when } \varphi = \pi,$$

where δ_P – sag of the axis of the ring at the point of application of force P.

For the determination of this deflection let us employ the Castigliano theorem. By setting up the derivative based on force $1/2P$ from the above given expression of potential energy, we obtain[1]

$$\frac{d\text{Э}}{d\left(\frac{P}{2}\right)} = \delta_p$$

or

$$\delta_p = \frac{0.013PR^3}{EJ}.$$

[1]The expression for the energy should be integrated within the limits of 0–π.

The values of constants A and B will be equal to

$$A=0, \quad B=-0,152\frac{PR^3}{EJ}.$$

Thus, for the deflection w we will obtain the expression

$$w_p=\frac{PR^3}{EJ}(0,16+0,04\varphi^2\cos\varphi-0,192\cos\varphi-0,08\varphi\sin\varphi).$$

Now, let us examine the loading of the ring with a concentaated moment (Fig. 152). In this instance the tangential stresses are determined by the formula

$$\tau=\frac{M}{2\pi R^2}.$$

Fig. 152.

The current values of the moment, of the normal and shear forces can be determined according to the expressions

$$M_u=\frac{M}{\pi}\left(\sin\varphi-\frac{1}{2}\varphi\right),\quad N_u=\frac{M}{\pi R}\sin\varphi,$$
$$Q_u=\frac{M}{\pi R}\left(\frac{1}{2}-\cos\varphi\right).$$

Just in the preceding case of loading, for the determination of deflections in the ring a differential equation is used

$$\frac{d^2w}{d\varphi^2}+w=-\frac{MR^2}{\pi EJ}\left(\sin\varphi-\frac{1}{2}\varphi\right).$$

The complete integral of this equation has the form

$$w_u=C_1\sin\varphi+C_2\cos\varphi+\frac{MR^2}{2\pi EJ}\left(\varphi+\varphi\cos\varphi-\frac{1}{2}\sin\varphi\right).$$

For the determination of constants C_1 and C_2, we will assume the following boundary conditions:

$$w=0 \quad \text{when} \quad \varphi=\pi;$$
$$\frac{dw}{R\,d\varphi}=\gamma_M \quad \text{when} \quad \varphi=\pi,$$

where γ_M – angle of rotation of the section of the ring at the site of application of the concentrated moment M.

For the determination of angle γ_M let us apply Castigliano's theorem

$$\gamma_M=\frac{d\vartheta}{dM},$$

where

$$\vartheta=\frac{R}{2EJ}\int_{-\pi}^{\pi} M_u^2\,d\varphi$$

and

$$\gamma_M=\left(\frac{\pi^2}{6}-1\right)\frac{MR}{\pi EJ}.$$

Then for C_1 and C_2 we will obtain the values

$$C_1=-\frac{0{,}305\,MR^2}{\pi EJ}, \quad C_2=0.$$

In this case the function of the deflection has the form

$$w_M=\frac{MR^2}{EJ}(0{,}5\varphi+0{,}5\varphi\cos\varphi-0{,}645\sin\varphi).$$

Let us examine, finally, the loading of the ring tangent to force T (Fig. 153).

Fig. 153.

In order to get the expression of tangential forces in this case one can proceed in the following manner. Let us place two equal and oppositely opposed forces of T in the center ring, parallel to the assigned force. Then, the ring will be loaded with force T, passing through the horizontal diameter, and by a moment. For the equilibrium of such a ring it is necessary to apply tangential forces, examined in the first two cases, one upon the other, whereupon

$$\tau = \frac{T}{\pi R}\left(\cos\varphi - \frac{1}{2}\right).$$

The flow of these forces is shown in Fig. 153.

For the bending moment, the normal and shear forces in this case we have the following expressions:

$$M_T = \frac{TR}{2\pi}\left(\frac{3}{2}\sin\varphi - \varphi\cos\varphi - \varphi\right),$$

$$N_T = -\frac{T}{2\pi}\left(\varphi\cos\varphi + \frac{1}{2}\sin\varphi\right),$$

$$Q_T = \frac{T}{2\pi}\left(1 - \varphi\sin\varphi - \frac{1}{2}\cos\varphi\right).$$

Let us obtain the expression for the deflection from the equation

$$\frac{d^2w}{d\varphi^2} + w = -\frac{MR^2}{EJ} = -\frac{TR^3}{2\pi EJ}\left(\frac{3}{2}\sin\varphi - \varphi\cos\varphi - \varphi\right),$$

the integral of which is equal to

$$w_T = A\sin\varphi + B\cos\varphi - \frac{TR^3}{2\pi EJ}\left(\frac{1}{2}\sin\varphi - \varphi\cos\varphi - \frac{1}{4}\varphi^2\sin\varphi - \varphi\right).$$

For the determination of the constants of integration A and B let us insert the following boundary conditions:

$$w = -\delta_T \quad \text{when } \varphi = \pi;$$

$$\frac{dw}{R\,d\varphi} = \gamma_T \quad \text{when } \varphi = \pi,$$

where δ_T – the radial deflection from force T at the point of its application; γ_T – angle of rotation of the section at the site of the application of force T.

For the determination of δ_T and γ_T anew let us apply Castigliano's theorem:

$$\delta_T = \left(\frac{d\text{Э}_{T+P}}{dP}\right)_{P=0}, \quad \gamma_T = \left(\frac{d\text{Э}_{T+M}}{dM}\right)_{M=0},$$

where

$$\text{Э}_{T+P} = \frac{R}{2EJ}\int_{-\pi}^{\pi}(M_T + M_P)^2\,d\varphi,$$

$$\text{Э}_{T+M} = \frac{R}{2EJ}\int_{-\pi}^{\pi}(M_T + M_M)^2\,d\varphi.$$

One arrives the values M_T, M_P, M_M by the above given expressions.

From these conditions we will find that

$$A = -0{,}279\,\frac{TR^3}{EJ}, \quad B = 0.$$

Then

$$w_T = \frac{TR^3}{EJ}(0{,}16\varphi + 0{,}16\varphi\cos\varphi + 0{,}04\varphi^2\sin\varphi - 0{,}36\sin\varphi).$$

C H A P T E R X

CALCULATION OF CONTAINERS MADE FROM FIBERGLASS

In this chapter the problems of calculating the strength of containers made from fiberglass and which operate on internal pressure is examined. Such balloons are manufactured by means of winding the fiberglass, impregnated with a special resin, to a mandrel or to an extracted model.[1]

After completion of the winding, the obtained vessel is subjected to thermal treatment at a defined temperature. In this case the resin hardens and the wound fiberglass is glued together, throughout the thickness of the wall.

The supporting element of the obtained construction in a glass filament. The resin plays the role of the binder and in view of its low mechanical characteristics, the binder is not taken in the calculation.

§ 45. Determination of the Optimum Winging Angles of the Glass Filaments and of Required Thickness of the Wall of Balloons

First let us examine the cylindrical part of the vessel, made from longitudinally aligned and spirally wound glass filaments (Fig. 154). Let's put this container under the action of internal

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[1]For the bibliography on this problem, see [31].

pressure. In this case the fibers of threads will operate only under tension. The tangential forces in a crosswise direction to the fibers will be absorbed by the resin. It is possible to show that at the sites determined by a certain value of the angle of inclination of glass filaments α to the axis of the containers, the shearing forces are absent. From Fig. 155a one can obtain the following relationships between the forces in rectangular and oblique systems of coordinates:

$$N_x \sin\beta - S_1 \cos\beta - N_\xi \cos\alpha - N_{xy} \cos\beta = 0,$$
$$N_y \cos\beta + S_1 \sin\beta + N_\xi \sin\alpha - N_{yx} \sin\beta = 0,$$
$$N_x \sin\alpha + S_2 \cos\alpha + N_\eta \cos\beta - N_{xy} \cos\alpha = 0,$$
$$N_y \cos\alpha - S_2 \sin\alpha - N_\eta \sin\beta - N_{yx} \sin\alpha = 0.$$

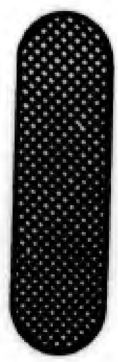


Fig. 154.

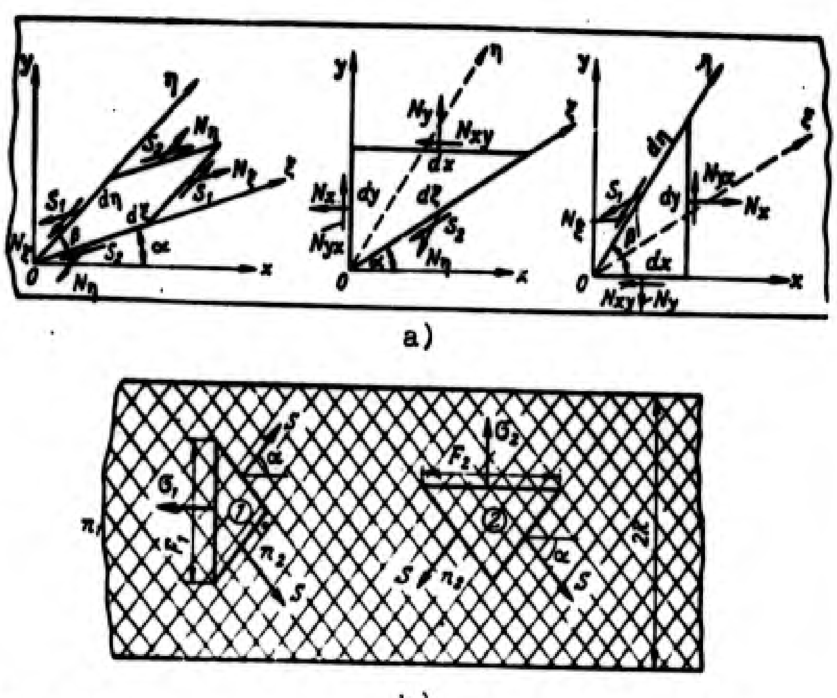


Fig. 155.

In this problem the bounds of dx and dy as a symmetry of loading will be free of tangential stresses ($N_{xy} = N_{yx} = 0$).

Furthermore, from the condition of equality to zero of the moments of forces, applied to the separate element, relative to the axis, perpendicular to the sketch and passing through the origin of coordinates, one can obtain $S_1 = S_2 = S$. Then, when $\beta = \pi - \alpha$ from the given equations we will find

$$N_\xi = N_\eta = \frac{N_x \sin^2\alpha + N_y \cos^2\alpha}{\sin 2\alpha},$$

$$S = \frac{N_y \cos^2\alpha - N_x \sin^2\alpha}{\sin 2\alpha}.$$

For the case of a cylindrical shell we will have

$$N_x = \frac{qR}{2}, \quad N_y = qR.$$

Then

$$N_\xi = N_\eta = \frac{qR(\sin^2\alpha + 2\cos^2\alpha)}{2\sin 2\alpha},$$

$$S = \frac{qR(2\cos^2\alpha - \sin^2\alpha)}{2\sin 2\alpha}.$$

In view of the fact that the binder (resin) possesses very low destructive stresses under shear, it is expedient to select such an angle of winding of glass filaments α so that the shear in the resin will be minimum. In an extreme case, when $S = 0$, let us take $\alpha = 54°40'$. In this case for the tensile forces in the threads we will find values, equal to

$$N_\xi = N_\eta = 1{,}42\,\frac{qR}{2} = 0{,}71\,qR,$$

which will be considerably lower, than at other angle of inclination of the glass filaments (i.e., the found angle α is the most suitable). Therefore, subsequently we will consider that the compressive construction of a cylindrical container is made with the angle of inclination of the threads, close to the optimum. It is easy to prove that this angle is $\alpha = 45°$ for a spherical container.

Now, let us turn to Fig. 155b. Let us set up the condition of equilibrium of element 2 in a circumferential direction. In this case we obtain

$$F_2\delta\sigma_2^p = Sn_2\sin\alpha + Sn_2\sin\alpha = 2Sn_2\sin\alpha, \qquad (10.1)$$

where S – strength of the elements of the glass filament in kgf; n_2 – number of elemental threads at a length F_2, lying in one direction; $\delta\sigma_2^p$ – linear calculated force, acting in a circumferential direction; F_2 – pitch of the winding; α – angle of inclination of the threads to the axis of the cylinder.

From Fig. 156 we will have

$$F_1 = F_2 \operatorname{tg}\alpha = 2\pi R.$$

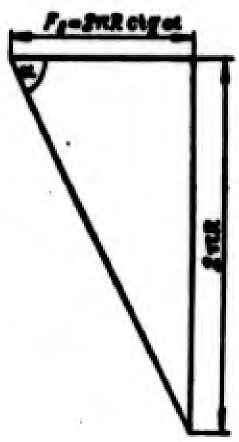


Fig. 156.

Furthermore,

$$\delta\sigma_2^p = q^p R.$$

Then, from the equation (10.1) we will find

$$n_2 = \frac{\pi R^2 q^p \operatorname{ctg}\alpha}{S\sin\alpha}. \qquad (10.2)$$

Now, let us examine the equilibrium of element 1 in an axial direction (see Fig. 155b):

$$F_1\delta\sigma_1^p = Sn_2\cos\alpha + Sn_2\cos\alpha + Sn_1 = 2Sn_2\cos\alpha + Sn_1. \qquad (10.3)$$

Here n_1 – number of elemental threads, longitudinal aligned at a length F_1; $\delta\sigma_1^p$ – linear calculated force in an axial direction from the internal pressure; F_1 – size of the separate element 1 in a circumferential direction.

The relationship between sizes F_1 and F_2 can be obtained from Fig. 157:

$$F_1 = F_2 \operatorname{tg}\alpha = 2\pi R.$$

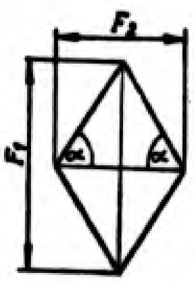


Fig. 157.

Furthermore, we will have

$$\delta\sigma_1^p = \frac{q^p R}{2}.$$

Then, from the equation (10.3), using expression (10.2), we obtain

$$n_1 = \frac{\pi R^2 q^p}{S}(1 - 2\operatorname{ctg}^2\alpha). \qquad (10.4)$$

Let us set up the ratio of the number of threads, aligned per unit length in a circumferential direction n_1/F_1, to the number of threads, aligned per unit length at an angle $\frac{2n_2}{F_2}$, that is

$$\frac{n_1 F_2}{2F_1 n_2} = (1 - 2\operatorname{ctg}^2\alpha)\sin\alpha,$$

whence

$$\frac{n_1}{2n_2} = (1 - 2\operatorname{ctg}^2\alpha)\sin\alpha. \qquad (10.5)$$

From equation (10.5) one can obtain the angle of inclination of the winding depending on the ratio of the number of threads n_1 to n_2. When $n_1 = 0$, we obtain a construction of the cylindrical part of the container, made only from the spirally aligned threads. In this case the angle of winding will be

$$1-2\,\mathrm{ctg}^2\alpha=0,$$

whence $\alpha = 54°40'$.

As shown above, at such an angle of winding of the threads to the axis of the cylinder, the inclined sites of elements 1 and 2 (see Fig. 155b) will be free of tangential stresses. This is a very important factor for prolonging the service life of containers, reinforced with fiberglass.

According to expressions (10.2) and (10.4) one can determine the required thickness of wall of the cylindrical part of a container.

The overall thickness of layers of the longitudinal threads (Fig. 158)

$$\frac{\pi d^2}{4} n_1 = \delta_{np} F_1,$$

where d – diameter of the elemental fiber,

whence

$$\delta_{np} = \frac{d^2 n_1}{8R}.$$

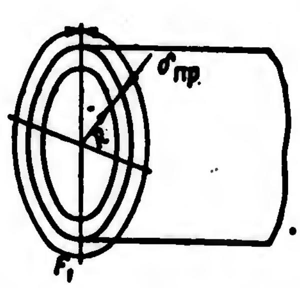


Fig. 158.

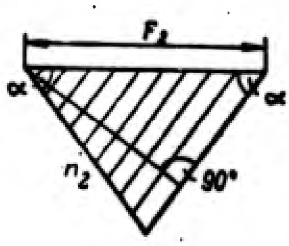
The overall thickness of layers of spirally aligned threads can be obtained from Fig. 159:

$$\frac{\pi d^2}{4} n_2 = F_2 \sin\alpha \frac{\delta_{спр}}{2},$$

whence

$$\delta_{спр} = \frac{d^2 n_2}{4R\cos\alpha}.$$

Fig. 159.



The complete thickness of the wall of the cylindrical part of the container

$$\delta = \delta_{пр} + \delta_{спр} = \frac{d^2}{8R}\left(n_1 + \frac{2n_2}{\cos\alpha}\right).$$

Following the substitution here of expressions for n_1 and n_2 (10.2) and (10.4) we obtain

$$\delta = \frac{3\pi R d^2 q^p}{8S}. \qquad (10.6)$$

From this formula it is evident that the wall thickness of the container under the combined packing of the longitudinal and spiral threads depends on the angle of inclination of the letter. However, from this it still does not follow tHat it is possible to make the cylindrical part of the vessel only from some longitudinally aligned threads. The angle of winding of the threads can be select randomly, but with the adherence to the relationship (10.5).

If in the formula (10.6) one substitutes

$$S=\frac{\pi d^2}{4}\sigma_{в}^{ст}.$$

where $\sigma_{в}^{ст}$ – ultimate strength of the glass filament under tension, then we obtain

$$\delta=1{,}5\frac{q^{p}R}{\sigma_{в}^{ст}}.$$

After the substitution of values n_1 and n_2 in the expressions for $\delta_{пр}$ and $\delta_{спр}$ and after the replacement of S with $\sigma_{в}^{ст}$ we will have

$$\delta_{пр}=\frac{q^{p}R}{2\sigma_{в}^{ст}}(1-2\operatorname{ctg}^2\alpha),$$

$$\delta_{спр}=\frac{q^{p}R}{\sigma_{в}^{ст}\sin^2\alpha}.$$

If in these formulas one assumes $\alpha = 90°$, then

$$\delta_{пр}=\frac{q^{p}R}{2\sigma_{в}^{ст}},\quad \delta_{спр}=\frac{q^{p}R}{\sigma_{в}^{ст}}.$$

At $\alpha = 54°40'$ we will find that

$$\delta_{пр}=0,\quad \delta_{спр}=1{,}5\frac{q^{p}R}{\sigma_{в}^{ст}}.$$

From these examples it is evident that the sum total thickness of the wall remains the same.

Let us calculate the weight per unit length of cylindrical part of the container:

$$G=2\pi R\delta l\gamma_{ст},$$

whence

$$\left(\frac{G}{l}\right)_{ст}=\frac{3\pi R^2\gamma_{ст}q^{p}}{\sigma_{в}^{ст}}. \qquad (10.7)$$

If the cylindrical part were made of metal, then for the determination of the weight per unit length we obtain the formula

$$\left(\frac{G}{l}\right)_{M}=\frac{2\pi R^{2}\gamma_{M}q^{p}}{\sigma_{B}^{M}\varphi_{CB}}, \tag{10.8}$$

where ϕ_{CB} – coefficient of a welded seam.[1]

From a comparison of expressions (10.7) and (10.8), we will find that

$$\eta=\frac{G_{M}}{G_{CT}}=\frac{2}{3}\frac{\gamma_{M}}{\gamma_{CT}}\frac{\sigma_{B}^{CT}}{\sigma_{B}^{M}\varphi_{CB}}.$$

By this ratio it is possible to make use of the weight estimation of containers, made from metal and fiberglass.

Now, let us examine the case of a spherical vessel, made from glass filaments (Fig. 160). From the condition of equilibrium of elements 1 and 2, we obtain

$$F_{1}\delta\sigma_{1}=2n_{1}S\sin\alpha,\quad F_{2}\delta\sigma_{2}=2n_{2}S\cos\alpha. \tag{10.9}$$

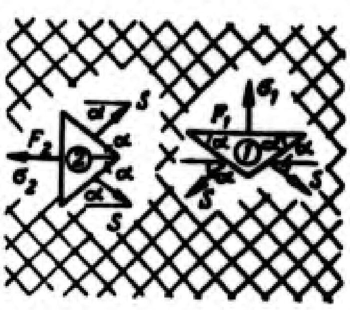


Fig. 160.

For a spherical shell we will have

$$\sigma_{1}=\sigma_{2}=\frac{q^{p}R}{2\delta}.$$

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[1]This coefficient is equal to the ratio of the ultimate strength under tension of the welded sample to the ultimate strength of the sample made from the original material.

Furthermore, let us assume that $F_1 = F_2$. Consequently, n_1 should be equal to n_2. Then

$$\sin\alpha = \cos\alpha.$$

From this relationship it is evident that for a spherical container, the angle of winding is $\alpha = 45°$, i.e., threads should intersect at an angle of 90°.

The required thickness of the wall is

$$\delta_{ст} = \frac{q^p R}{2\sigma_в^{ст}}.$$

This formula can be derived on the basis of any of the equations (10.9) when $\alpha = 45°$.

$$n = \frac{q^p R F}{2\sqrt{2}S} = \frac{2q^p R F}{\pi\sqrt{2}d^2\sigma_в^{ст}}.$$

Since the cross-sectional area of the bounds of the separate element 1 or 2 is

$$F \sin\alpha\delta_{ст} = \frac{\pi d^2}{4} n,$$

when following the exception of the last two expressions of size F we obtain the formula for δ_{CT}.

For estimation of the weight of the spherical container we will have the formula

$$G = 4\pi R^2 \delta\gamma.$$

The ratio of weights of spherical containers made from metal and from fiberglass, will be

$$\eta = \frac{\gamma_м}{\gamma_{ст}} \frac{\sigma_в^{ст}}{\sigma_в^м \varphi_{св}}.$$

In the above given calculation the work of the internal hermetically sealed layer was not considered, without which it is hardly expedient to employ containers which operate under internal pressure.

As a hermetically sealing layer, one can use either special resins or thin metallic shells. One can consider that the above given calculation of a hermetically sealing layer of resin is meaningless, because the bearing capacity of this material will be insignificant compared to that of glass filaments. The effect of the metallic layer can be taken into account in the following manner. Let us first examine this problem using the cylindrical part of the container as an example. One can write the expression for the relative deformation of this layer in a circumferential direction at the moment of failure of the tank.

$$\varepsilon_M = \frac{1}{\bar{E}_M}(\sigma_{2M} - \mu\sigma_{1M}) = \frac{(1 - 0,5\mu)\, q_M R}{\bar{E}_M \delta_M}.$$

In this instant the equality $\varepsilon_M = \varepsilon_{CT}$ – should be satisfied, where ε_{CT} – relative deformation of the fiberglass upon failure.

Furthermore, at the moment of failure of the container the metallic hermetically sealing layer will be in the state of fluidity. Consequently, $\mu = 0.5$. Then, the metallic shell will undergo the following internal pressure:

$$q_M = \frac{4\bar{E}\delta_M \varepsilon_{CT}}{3R}.$$

The supporting part of the construction,oof the container, consisting of fiberglass, will undergo a pressure, equal to $(q - q_M)$, which should be worked into the calculation of the container. The value of the tangential modulus E (at the point, where $\varepsilon_M = \varepsilon_{CT}$) must be taken from the chart in Fig. 162. The thickness of the wall of the hermetically sealing shell should be selected from condition of rigidity under loading by the external pressure from the side of the taut glass filaments.

In the case of a spherical vessel, the hermetically sealing shell will be loaded by internal pressure, which can be determined by the expression

$$q_{м}=\frac{4\bar{E}_{м}\delta_{м}\varepsilon_{ст}}{R}.$$

A further calculation of the supporting part of the container is made just as for the case of a cylindrical shell.

§ 46. Calculation of a Combined Cylindrical Container, Reinforced with Glass Filaments Only in a Circumferential Direction

As it is known, the thickness of the wall of a cylindrical tank can be determined according to the magnitude of the circumferential stresses $\sigma_2=\frac{qR}{\delta}$. At the same time in an axial direction of the tank stresses will act, by one-half as much:

$$\sigma_1=\frac{qR}{2\delta}. \qquad (10.10)$$

Thus, the material of the tank in an axial direction becomes underloaded and the construction in an axial and circumferential direction seems to be of unequal strength.

In order to make the construction of the container of uniform strength, it is necessary to determine the thickness of the wall based on the strength of stresses σ_1, and to determine the deficient thickness of the wall in the circumferential direction to compensate for the winding of the glass filaments, which have a higher specific strength in comparison to metal. Such a construction is more advantageous by weight ratio.

The calculation of a container of this design should be done in this sequence.

From the formula (10.10) when $\sigma_1 = \sigma_{в}\phi_{св}$ we obtain the expression for the thickness of the wall

$$\delta_{м} = \frac{q^{р} R}{2\sigma_{в}^{м} \varphi_{св}},$$

where ϕ_{CB} – coefficient of the welded seam.

For the determination of the required number of glass filaments n_2 in the circumferential direction let us turn to Fig. 161. From the condition of equilibrium of all the forces in the direction S we obtain

$$2\sigma_2 \delta_{м} l + 2Sn_2 = 2Rlq^{р},$$

whence the required number of fibers will be

$$n_2 = \frac{Rlq^{р} - \sigma_2 \delta_{м} l}{S}.$$

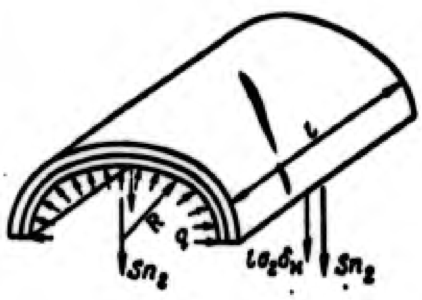


Fig. 161.

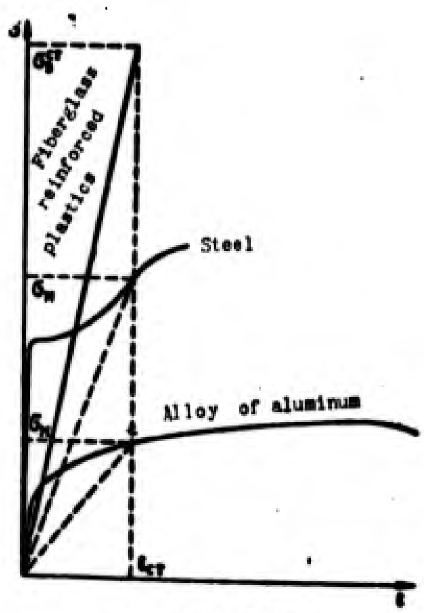


Fig. 162.

According to the graph in Fig. 162 it is evident that the stresses σ_2 at the time of failure will be equal to σ_M. Therefore

$$n_2 = \frac{Rlq^{р} - \sigma_{м} \delta_{м} l}{S},$$

where S — destructive force for an elemental glass filament.

The last formula can be conveniently rewritten in the form

$$\frac{n_2}{l}=\frac{Rq^P-\sigma_{\text{м}}\delta_{\text{м}}}{S}, \qquad (10.11)$$

where n_2/l — number of glass filaments, found per unit length of the tank.

After excluding the quantities n_2 and S from (10.11) which can be obtained from the relationships

$$S=\frac{\pi d^2}{4}\sigma_{\text{в}}^{\text{ст}},$$

$$\frac{\pi d^2}{4}n_2=\delta_{\text{ст}},$$

we obtain the expression for the required thickness of winding of the glass filaments

$$\delta_{\text{ст}}=\frac{q^P R}{\sigma_{\text{в}}^{\text{ст}}}\left(1-\frac{\sigma_{\text{м}}}{2\sigma_{\text{в}}^{\text{м}}\varphi_{\text{св}}}\right). \qquad (10.12)$$

The weight of the cylindrical part of the combined tank will be

$$G_{\text{комб}}=2\pi Rl(\delta_{\text{м}}\gamma_{\text{м}}+\delta_{\text{ст}}\gamma_{\text{ст}}).$$

After the substitution here of values $\delta_{\text{м}}$ and $\delta_{\text{ст}}$ according to the formulas (10.11) and (10.12), we obtain

$$\left(\frac{G}{l}\right)_{\text{комб}}=\frac{\pi R^2 q^P\gamma_{\text{м}}}{\sigma_{\text{в}}^{\text{м}}\varphi_{\text{св}}}\left(1+2\frac{\sigma_{\text{в}}^{\text{м}}\varphi_{\text{св}}\gamma_{\text{ст}}}{\sigma_{\text{в}}^{\text{ст}}\gamma_{\text{м}}}-\frac{\gamma_{\text{ст}}\sigma_{\text{м}}}{\gamma_{\text{м}}\sigma_{\text{в}}^{\text{ст}}}\right). \qquad (10.13)$$

During the manufacture of the cylindrical part of the tank made from metal for the determination of the weight per unit length, the formula (10.8) is derived.

By setting up the ratio of weights from the formulas (10.8) and (10.13), we obtain

$$\eta = \frac{G_м}{G_{комб}} = \frac{2}{1 + 2\frac{\sigma_в^м \varphi_{св} \gamma_{ст}}{\sigma_в^{ст} \gamma_м} - \frac{\gamma_{ст} \sigma_м}{\gamma_м \sigma_в^{ст}}}.$$

By making use of this ratio, one can determine how many times heavier the metallic container is compared to the combined one.

§ 47. Determination of the Tensile Force of Glass Filaments

In order for the glass filaments to operate evenly along the thickness of the wall of the container under a load of the latter by internal pressure, it is necessary to insure their required tensile force in the manufacturing process of the vessel. For the determination of this tension let us examine the construction of the cylindrical part of a vessel, having *n* layers of glass filaments (Fig. 163) in the thickness of the wall; moreover, extreme internal layer is considered to be hermetically sealing and made of metal (M). Let us first examine a two-ply construction (Fig. 164).

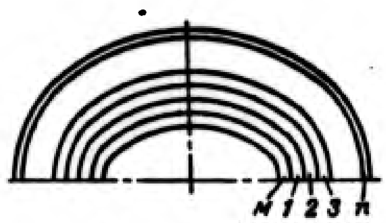


Fig. 163.

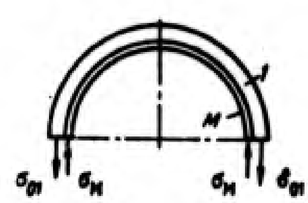


Fig. 164.

From the condition of equilibrium of forces, acting in the external and internal layers, in the direction of a vertical diameter of the cylinder, we obtain

$$\sigma_м \delta_м - \sigma_{01} \delta_{01} = 0, \qquad (10.14)$$

where $\delta_м$, δ_{01} – thickness of the metallic layer and the layer of winding; σ_{01} – tensile stress in the layer of winding; $\sigma_м$ – compression in the hermetically sealed layer.

From equation (10.14) we will find that

$$\sigma_{01} = \sigma_{\text{м}}^{\text{доп}} \frac{\delta_{\text{м}}}{\delta_{01}}, \qquad (10.15)$$

where $\sigma_{\text{м}}^{\text{доп}}$ – allowable compression stress in the metallic layer.

After the winding of the second layer of fiberglass we will have (Fig. 165)

$$(\sigma_{\text{м}} + X_1)\delta_{\text{м}} - (\sigma_{01} + X_1')\delta_{01} - \sigma_{02}\delta_{02} = 0$$

or

$$\underline{\sigma_{\text{м}}\delta_{\text{м}} - \sigma_{01}\delta_{01}} + X_1\delta_{\text{м}} - X_1'\delta_{01} - \sigma_{02}\delta_{02} = 0. \qquad (10.16)$$

The first two underlined terms in the equation (10.16) on the basis of the equation (10.14), is equal to zero.

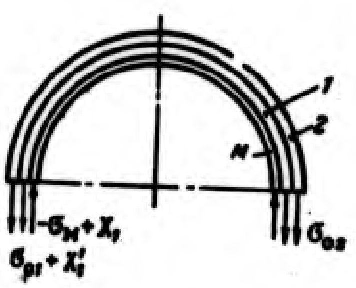


Fig. 165.

Then, from (10.16) for an unknown quantity X_1, we obtain the expression

$$X_1 = X_1' \frac{\delta_{01}}{\delta_{\text{м}}} + \sigma_{02} \frac{\delta_{02}}{\delta_{\text{м}}},$$

where X_1 – additional compression stress in the layer of the metal from the winding of the second layer of fiberglass; X_1' – additional stress in the first layer of the winding following the winding of the second layer.

Then, the sum total of stresses in the metallic layer will be

$$\sigma_{м}^{\Sigma}=\sigma_{м}+X_{1}=\sigma_{01}\frac{\delta_{01}}{\delta_{м}}+X_{1}'\frac{\delta_{01}}{\delta_{м}}+\sigma_{02}\frac{\delta_{02}}{\delta_{м}}=\sigma_{м}^{доп}.$$

Hence, we obtain

$$X_{1}'=\sigma_{м}^{доп}\frac{\delta_{м}}{\delta_{01}}-\sigma_{01}-\sigma_{02}\frac{\delta_{02}}{\delta_{01}}. \qquad (10.17)$$

The sum total of stresses in layer 1 of the winding

$$\sigma_{01}^{\Sigma}=\sigma_{01}+X_{1}'=\sigma_{м}^{доп}\frac{\delta_{м}}{\delta_{01}}-\sigma_{02}\frac{\delta_{02}}{\delta_{01}}.$$

Let us assume that the container has only two layers of windings of the fiberglass. Let us impose the requirement that in both layers the stresses should be identical:

$$\sigma_{02}=\sigma_{01}^{\Sigma}.$$

From this condition we obtain the magnitude of tension in the second layer of the winding

$$\sigma_{02}=\frac{\sigma_{м}^{доп}\dfrac{\delta_{м}}{\delta_{01}}}{1+\dfrac{\delta_{02}}{\delta_{01}}}. \qquad (10.18)$$

By comparing the formulas (10.15) and (10.18), it is possible to see that the tension in the second layer has diminished in comparison with the tension in the first layer.

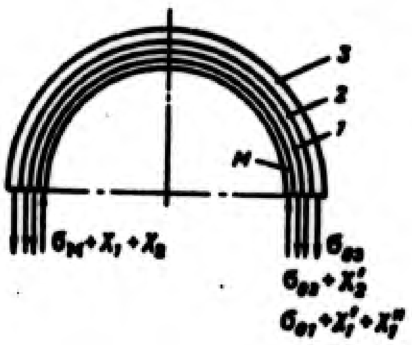
Now let us assume that the construction has three layers of winding (1, 2, 3 in Fig. 166):

$$(\sigma_{м}+X_{1}+X_{2})\delta_{м}-(\sigma_{01}+X_{1}'+X_{1}'')\delta_{01}-(\sigma_{02}+X_{2}')\delta_{02}-\sigma_{03}\delta_{03}=0$$

or

$$\underline{\sigma_M\delta_M - \sigma_{01}\delta_{01} + X_1\delta_M - X_1'\delta_{01} - \sigma_{02}\delta_{02} +}$$
$$+ X_2\delta_M - X_1''\delta_{01} - X_2'\delta_{02} - \sigma_{03}\delta_{03} = 0. \qquad (10.19)$$

Fig. 166.



Sum of the underlined terms on the basis equation (10.16) is equal to zero. From the remaining part of the equation (10.19) we obtain

$$X_2 = X_1''\frac{\delta_{01}}{\delta_M} + X_2'\frac{\delta_{02}}{\delta_M} + \sigma_{03}\frac{\delta_{03}}{\delta_M}.$$

Sum of stresses in layers will be equal to:
in the layer of the metal

$$\sigma_M^з = \sigma_M + X_1 + X_2 = (\sigma_{01} + X_1' + X_1'')\frac{\delta_{01}}{\delta_M} + (\sigma_{02} + X_2')\frac{\delta_{02}}{\delta_M} + \sigma_{03}\frac{\delta_{03}}{\delta_M} = \sigma_M^{доп},$$

in the first layer of winding

$$\sigma_{01}^з = \sigma_{01} + X_1' + X_1'' = \sigma_M^{доп}\frac{\delta_M}{\delta_{01}} - \sigma_{02}\frac{\delta_{02}}{\delta_{01}} + X_1'',$$

in the second layer of winding

$$\sigma_{02}^з = \sigma_{02} + X_2'.$$

Let us assume a condition of uniform strength in the layers:

$$\sigma_{01}^з = \sigma_{02}^з, \quad \sigma_{02}^з = \sigma_{03}, \quad \sigma_M^з = \sigma_M^{доп}.$$

In the developed form these conditions will have the form

$$\sigma_{\text{м}}^{\text{доп}} \frac{\delta_{\text{м}}}{\delta_{01}} - \sigma_{02} \frac{\delta_{02}}{\delta_{01}} + X_1'' = \sigma_{02} + X_2',$$

$$\sigma_{02} + X_2' = \sigma_{03},$$

$$(\sigma_{01} + X_1' + X_1'') \frac{\delta_{01}}{\delta_{\text{м}}} + (\sigma_{02} + X_2') \frac{\delta_{02}}{\delta_{\text{м}}} + \delta_{03} \frac{\delta_{03}}{\delta_{\text{м}}} = \sigma_{\text{м}}^{\text{доп}}.$$

By solving this system of equations, we obtain

$$X_2' = \sigma_{03} - \sigma_{02},$$

$$X_1'' = \sigma_{03} + \sigma_{02} \frac{\delta_{02}}{\delta_{01}} - \sigma_{\text{м}}^{\text{доп}} \frac{\delta_{\text{м}}}{\delta_{01}},$$

$$X_1' = 2\sigma_{\text{м}}^{\text{доп}} \frac{\delta_{\text{м}}}{\delta_{01}} - \sigma_{01} - \sigma_{02} \frac{\delta_{03}}{\delta_{01}} - \sigma_{03}\left(1 + \frac{\delta_{02}}{\delta_{01}} + \frac{\delta_{03}}{\delta_{01}}\right). \tag{10.20}$$

From the comparison of expressions (10.17) and (10.20) we will find

$$\sigma_{03} = \frac{\sigma_{\text{м}}^{\text{доп}} \frac{\delta_{\text{м}}}{\rho_{01}}}{1 + \frac{\delta_{0,}}{\delta_{01}} + \frac{\delta_{03}}{\delta_{01}}}. \tag{10.21}$$

From the comparison of structure of formulas (10.15), (10.18) and (10.21), it is possible to see that for any *n*th layer of winding, the stress is

$$\sigma_{0n} = \frac{\sigma_{\text{м}}^{\text{доп}} \frac{\delta_{\text{м}}}{\delta_{01}}}{1 + \frac{\delta_{02}}{\delta_{01}} + \frac{\delta_{03}}{\delta_{01}} + \ldots + \frac{\delta_{0n}}{\delta_{01}}}. \tag{10.22}$$

Thus, if the stresses in the layers following the winding were determined by the formula (10.22), then in each of the layers the tension would be identical. In this case all the layers under the loading of a container by internal pressure will simultaneously engage in work which is especially important for the maximum utilization of the material.

From formulas (10.18)-(10.22) it is evident that the tension in each subsequent layer should be less, than in the previous one. If the thickness of all the layers is identical, then it is possible to write

$$\sigma_{01}=\frac{A}{1},\quad \sigma_{02}=\frac{A}{2},\quad \sigma_{03}=\frac{A}{3},\ \ldots,\ \sigma_{0n}=\frac{A}{n},$$

where

$$A=\sigma_{\text{м}}^{\text{доп}}\frac{\delta_{\text{м}}}{\delta_{01}}. \qquad (10.23)$$

Thus, in order that all the layers should be found under identical tension following the winding, it is necessary to diminish the tension in each subsequent layer by n times in comparison with the first. The aforementioned are graphically presented in Fig. 167.

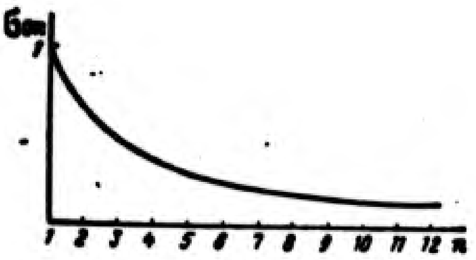


Fig. 167.

The tension of each layer of winding causes compression of the internal hermetically sealed layer. The compression stress in this layer will be

$$\sigma_{\text{м}}=\frac{q_0R}{\delta_{\text{м}}}+\frac{\frac{1}{2}q_0R}{\delta_{\text{м}}}+\frac{\frac{1}{3}q_0R}{\delta_{\text{м}}}+\ldots+\frac{\frac{1}{n}q_0R}{\delta_{\text{м}}}=$$

$$=\frac{q_0R}{\delta_{\text{м}}}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right),$$

where n – ordinal number of the layer of winding.

The permissible stress in the metallic hermetically sealed layer can be calculated in the following manner.

Annular stresses in this layer are

$$\sigma_{кр}=\frac{q_{кр}R}{\delta_{м}},$$

where the critical pressure is determined by Papkovich's formula (11.17)

$$q_{кр}=0{,}92E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}}.$$

Then

$$\sigma_{кр}=0{,}92E\frac{\delta}{l}\sqrt{\frac{\delta}{R}}.$$

From the condition

$$\sigma_{м}=\sigma_{м}^{доп}=\frac{\sigma_{кр}}{f},$$

where f – safety factor, let us find the expression for q_0:

$$\frac{q_0R}{\delta_{м}}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)=0{,}92E\frac{\delta_{м}}{lf}\sqrt{\frac{\delta_{м}}{R}}.$$

whence

$$q_0=\frac{0{,}92E}{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}\frac{\delta_{м}^2}{Rlf}\sqrt{\frac{\delta_{м}}{R}}.$$

By having the expression for q_0, it is possible to calculate the tension in the layers during the winding using formula (10.23), in which

$$A=\sigma_{м}^{доп}\frac{\delta_{м}}{\delta_{01}}=\frac{0{,}92E}{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}\frac{\delta_{м}^2}{\delta_{01}lf}\sqrt{\frac{\delta_{м}}{R}}.$$

Then, the stresses should be: in the first layer

$$\sigma_{01}=0{,}92E\frac{\delta_{м}^2}{\delta_{01}lf}\sqrt{\frac{\delta_{м}}{R}},$$

in the second layer

$$\sigma_{02}=\frac{1}{2}\frac{0{,}92E}{1+\frac{1}{2}}\frac{\delta_{м}^2}{\delta_{01}lf}\sqrt{\frac{\delta_{м}}{R}},$$

in the third layer

$$\sigma_{03}=\frac{1}{3}\,\frac{0{,}92\,E}{1+\frac{1}{2}+\frac{1}{3}}\,\frac{\delta_{м}^{2}}{\delta_{01}lf}\sqrt{\frac{\delta_{м}}{R}}.$$

in the *n*th layer

$$\sigma_{0n}=\frac{1}{n}\,\frac{0{,}92\,E}{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}\,\frac{\delta_{м}^{2}}{\delta_{01}lf}\sqrt{\frac{\delta_{м}}{R}}.$$

Now, let us examine the problem about the tension of glass filaments with an ablique winding.

In this instance on the inside of a hermetically sealed layer a biaxial strained state of compression appears. The magnitude of the compressing stresses in axial and circumferential directions can be determined according to Fig. 168. Let us set up the equations of equilibrium selected in this figure of elements following the winding of the first layer:

$$\sigma_{м}^{пр}\delta_{м}-\sigma_{01}\delta_{01}\cos^{2}\alpha=0,$$
$$\sigma_{м}^{ок}\delta_{м}-\sigma_{01}\delta_{01}\sin^{2}\alpha=0.$$

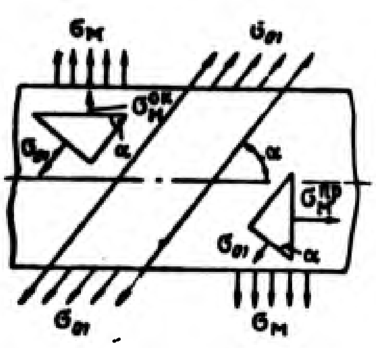


Fig. 168.

From these equations one can determine the tensile stress of tension of the threads in the first layer of the winding

$$\sigma_{01}=\frac{\delta_{м}}{\delta_{01}}\,\frac{\sigma_{м}^{пр}}{\cos^{2}\alpha},\quad \sigma_{01}=\frac{\delta_{м}}{\delta_{01}}\,\frac{\sigma_{м}^{ок}}{\sin^{2}\alpha}.$$

Here $\sigma_{м}^{пр}$ – the intensity of longitudinal compressive stresses in the hermetically sealed layer in the direction of the axis of the cylinder; $\sigma_{м}^{ок}$ – the same in a circumferential direction.

Following the winding of the second layer with a slope on the opposite side relative to the axis of the shell of the equation of equilibrium will take form

$$(\sigma_м^{пр}+X_1)\delta_м-(\sigma_{01}+X_1')\delta_{01}\cos^2\alpha+\sigma_{02}\delta_{02}\cos^2\alpha=0,$$

$$(\sigma_м^{ок}+Y_1)\delta_м-(\sigma_{01}+Y_1')\delta_{01}\sin^2\alpha-\sigma_{02}\delta_{02}\sin^2\alpha=0.$$

Here, X_1, X_1', Y_1, Y_1' – additional stresses in the layer of metal as well as in the first layer of glass filaments following the winding of the second layer.

From these equations during the utilization of expressions for σ_{01} we will find

$$X_1=\left(\frac{\delta_{01}}{\delta_м}X_2'+\frac{\delta_{02}}{\delta_м}\sigma_{02}\right)\cos^2\alpha,$$

$$Y_1=\left(\frac{\delta_{01}}{\delta_м}Y_1'+\frac{\delta_{02}}{\delta_м}\sigma_{02}\right)\sin^2\alpha.$$

The sum total of stresses in the metal following the winding of the second layer

$$\sigma_{м\Sigma}^{пр}=\sigma_м^{пр}+X_1=\left(\frac{\delta_{01}}{\delta_м}\sigma_{01}+\frac{\delta_{01}}{\delta_м}X_1'+\frac{\delta_{02}}{\delta_м}\sigma_{02}\right)\cos^2\alpha,$$

$$\sigma_{м\Sigma}^{ок}=\sigma_м^{ок}+Y_1=\left(\frac{\delta_{01}}{\delta_м}\sigma_{01}+\frac{\delta_{01}}{\delta_м}Y_1'+\frac{\delta_{02}}{\delta_м}\sigma_{02}\right)\sin^2\alpha.$$

From these equalities we obtain

$$X_1'=\frac{\delta_м}{\delta_{01}}\left(\frac{\sigma_{м\Sigma}^{пр}}{\cos^2\alpha}-\frac{\delta_{01}}{\delta_м}\sigma_{01}-\frac{\delta_{02}}{\delta_м}\sigma_{02}\right),$$

$$Y_1'=\frac{\delta_м}{\delta_{01}}\left(\frac{\sigma_{м\Sigma}^{ок}}{\sin^2\alpha}-\frac{\delta_{01}}{\delta_м}\sigma_{01}-\frac{\delta_{02}}{\delta_м}\sigma_{02}\right).$$

Then, the resulting stresses in the first layer of the winding can be examined in the form

$$\sigma_{01\Sigma}=\sigma_{01}+X_1'=\frac{\delta_м}{\delta_{01}}\frac{\sigma_{м\Sigma}^{пр}}{\cos^2\alpha}-\frac{\delta_{02}}{\delta_{01}}\sigma_{02},$$

$$\sigma_{01\Sigma}=\frac{\delta_м}{\delta_{01}}\frac{\sigma_{м\Sigma}^{ок}}{\sin^2\alpha}-\frac{\delta_{02}}{\delta_{01}}\sigma_{02}.$$

By equating the stresses in the first and second layers

$$\sigma_{02}=\sigma_{01\Sigma},$$

we will find that

$$\sigma_{02}=\frac{\dfrac{\delta_{\text{м}}}{\delta_{01}}\,\dfrac{\sigma^{\text{пр}}_{\text{м}\Sigma}}{\cos^2\alpha}}{1+\dfrac{\delta_{02}}{\delta_{01}}};$$

$$\sigma_{02}=\frac{\dfrac{\delta_{\text{м}}}{\delta_{01}}\,\dfrac{\sigma^{\text{ок}}_{\text{м}\Sigma}}{\sin^2\alpha}}{1+\dfrac{\delta_{02}}{\delta_{01}}}.$$

Similarly, one can obtain the expressions for the stresses in the subsequent layers of the winding:

$$\sigma_{03}=\frac{\dfrac{\delta_{\text{м}}}{\delta_{01}}\,\dfrac{\sigma^{\text{пр}}_{\text{м}\Sigma}}{\cos^2\alpha}}{1+\dfrac{\delta_{02}}{\delta_{01}}+\dfrac{\delta_{03}}{\delta_{01}}},$$

$$\sigma_{03}=\frac{\dfrac{\delta_{\text{м}}}{\delta_{01}}\,\dfrac{\sigma^{\text{ок}}_{\text{м}\Sigma}}{\sin^2\alpha}}{1+\dfrac{\delta_{02}}{\delta_{01}}+\dfrac{\delta_{03}}{\delta_{01}}}$$

and so forth.

Thus, we have derived the formulas, associating the stresses in the metallic layer with the stresses in layers, obtained by the winding of threads. For the determination of the permissible tension in the fibers during the winding let us employ the formula (17.7)

$$\frac{\sigma^{\text{пр}}_{\text{м}}}{\sigma^{\text{пр}}_{\text{кр}}}+\frac{\sigma^{\text{ок}}_{\text{м}}}{\sigma^{\text{ок}}_{\text{кр}}}<1,$$

where

$$\sigma^{\text{пр}}_{\text{кр}}=0{,}26E\,\frac{\delta_{\text{м}}}{R};\quad \sigma^{\text{ок}}_{\text{кр}}=\frac{q_{\text{кр}}R}{\delta_{\text{м}}};\quad q_{\text{кр}}=0{,}92\,E\,\frac{\delta^2_{\text{м}}}{Rl}\sqrt{\frac{\delta_{\text{м}}}{R}};$$

R — radius of the cylinder, and l — length.

If we substitute in this formula the values of the longitudinal and circumferential stresses $\sigma_{\text{м}}^{\text{пр}}$ and $\sigma_{\text{м}}^{\text{ок}}$, appearing in the hermetically sealed layer following the winding of each layer, then one can obtain the expressions for the determination of the permissible tension in the layers. For example, following the winding of the first layer we will have

$$\sigma_{\text{м}}^{\text{пр}} = \frac{\delta_{01}}{\delta_{\text{м}}} \sigma_{01} \cos^2 \alpha, \quad \sigma_{\text{м}}^{\text{ок}} = \frac{\delta_{01}}{\delta_{\text{м}}} \sigma_{01} \sin^2 \alpha.$$

Then

$$\frac{\frac{\delta_{01}}{\delta_{\text{м}}} \sigma_{01} \cos^2 \alpha}{\sigma_{\text{кр}}^{\text{пр}}} + \frac{\frac{\delta_{01}}{\delta_{\text{м}}} \sigma_{01} \sin^2 \alpha}{\sigma_{\text{кр}}^{\text{ок}}} < 1.$$

Hence, we will find that

$$\sigma_{01} < \frac{\frac{\delta_{\text{м}}}{\delta_{01}}}{\frac{\cos^2 \alpha}{\sigma_{\text{кр}}^{\text{пр}}} + \frac{\sin^2 \alpha}{\sigma_{\text{кр}}^{\text{ок}}}}.$$

For the second layer we obtain

$$\sigma_{02} < \frac{\frac{\delta_{\text{м}}}{\delta_{01}}}{\left(1 + \frac{\delta_{02}}{\delta_{01}}\right)\left(\frac{\cos^2 \alpha}{\sigma_{\text{кр}}^{\text{пр}}} + \frac{\sin^2 \alpha}{\sigma_{\text{кр}}^{\text{ок}}}\right)}.$$

For the nth layer

$$\sigma_{0n} < \frac{\frac{\delta_{\text{м}}}{\delta_{01}}}{\left(1 + \frac{\delta_{02}}{\delta_{01}} + \frac{\delta_{03}}{\delta_{01}} + \ldots + \frac{\delta_{0n}}{\delta_{01}}\right)\left(\frac{\cos^2 \alpha}{\sigma_{\text{кр}}^{\text{пр}}} + \frac{\sin^2 \alpha}{\sigma_{\text{кр}}^{\text{ок}}}\right)}.$$

If the thickness of the layers were identical, then

$$\sigma_{01} < \frac{\frac{\delta_м}{\delta_{01}}}{\frac{\cos^2\alpha}{\sigma^{пр}_{кр}} + \frac{\sin^2\alpha}{\sigma^{ок}_{кр}}};$$

$$\sigma_{02} < \frac{\frac{\delta_м}{\delta_{01}}}{2\left(\frac{\cos^2\alpha}{\sigma^{пр}_{кр}} + \frac{\sin^2\alpha}{\sigma^{ок}_{кр}}\right)};$$

$$\sigma_{03} < \frac{\frac{\delta_м}{\delta_{01}}}{3\left(\frac{\cos^2\alpha}{\sigma^{пр}_{кр}} + \frac{\sin^2\alpha}{\sigma^{ок}_{кр}}\right)}$$

and so forth.

If we substitute $\frac{\sigma^{пр}_{кр}}{f}$ in place of $\sigma^{пр}_{кр}$, $\frac{\sigma^{ок}_{кр}}{f}$ in place of $\sigma^{ок}_{кр}$ in these formulas, where f – the safety factors, then instead of inequalities, we will obtain equalities

$$\sigma_{01} = \frac{\frac{\delta_м}{\delta_{01}}}{f\left(\frac{\cos^2\alpha}{\sigma^{пр}_{кр}} + \frac{\sin^2\alpha}{\sigma^{ок}_{кр}}\right)},$$

$$\sigma_{02} = \frac{\frac{\delta_м}{\delta_{01}}}{2f\left(\frac{\cos^2\alpha}{\sigma^{пр}_{кр}} + \frac{\sin^2\alpha}{\sigma^{ок}_{кр}}\right)}$$

and so forth.

In the case of a sphere when $\alpha = 45°$ (Fig. 169).

$$\sigma_м\delta_м - \sigma_{01}\delta_{01} = 0,$$

whence

$$\sigma_{01} = \frac{\delta_м}{\delta_{01}}(\sigma_м)_{доп}.$$

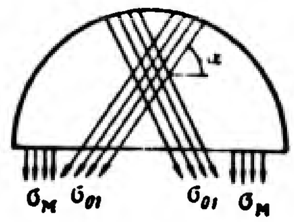


Fig. 169.

Similarly, for the second layer

$$\sigma_{02}=\frac{\frac{\delta_M}{\delta_{01}}(\sigma_M)_{доп}}{1+\frac{\delta_{02}}{\delta_{01}}}.$$

For the n-th layer

$$\sigma_{0n}=\frac{\frac{\delta_M}{\delta_{01}}(\sigma_M)_{доп}}{1+\frac{\delta_{02}}{\delta_{01}}+\frac{\delta_{03}}{\delta_{01}}+\ldots+\frac{\delta_{0n}}{\delta_{01}}}.$$

For a sphere, using $(\sigma_M)_{доп}$ one should assume that

$$(\sigma_M)_{доп}=\frac{\sigma_{кр}}{f}=\frac{0{,}2E\frac{\delta_M}{R}}{f},$$

where f – safety factor.

P A R T III

THE RIGIDITY OF SHELLS

CHAPTER XI

RIGIDITY OF CYLINDRICAL AND CONICAL SHELLS

In the previous chapters problems were examined, which dealt with the calculation of shells for strength. The formulas obtained in these chapters make it possible with sufficient practical accuracy to evaluate the magnitude and character of stress distribution in shells of various geometrical forms from assigned external loadings.

Following the determination of stresses one should compare them with the destructive stresses: in the case of tension - with the yield point or ultimate strength of the material of the shell, in the case of compression - with the critical stresses.

During the action of tensile stresses, the failure of the construction is associated with the distruction of the continuity of the material - with the formation of cracks.

If the normal stresses should be compressive or if in the construction the tangential stresses of defined magnitude act, then the critical stresses may be, basically, the destructive stresses. In this case the failure of the construction is associated with a change in its form, and its surface becomes undulating. In this instance, one refers to the construction as "operating on rigidity."

If the strength characteristics of the material of the construction - ultimate strength and yield point - are obtained from tests of samples for tension, then the critical stresses can be derived only by theoretical means with a subsequent experimental check on them.

In this chapter the various cases of loading the shell are examined and the formulas for the critical stresses are derived.

§ 48. The Formulation of Problems of the Rigidity of Shells

Thin-walled shells following the loss of rigidity are usually covered with pits and bulges. The stress, at which this phenomenon occurs, is called the critical stress.

The method, which will subsequently be used to solve separate partial problems of the rigidity of shells, can be explained by the example of the loss of rigidity of a thin cylindrical shell[1] compressed in an axial direction.

Let us take a specified shell with a certain value of compressive force which loses its rigidity, and its surface is covered with a regular pattern of pits and bulges. Let us explain what the boundary conditions can be like on the contour of pits and bulges. In order to do this let us write the expression of the complete potential energy for some given pits or bulges

$$U=\frac{B}{2}\int_0^a\int_0^b\left(\varepsilon_x^2+\varepsilon_y^2+2\mu\varepsilon_x\varepsilon_y+\frac{1-\mu}{2}\varepsilon_{xy}^2\right)dx\,dy+\frac{D}{2}\int_0^a\int_0^b[\chi_x^2+\chi_y^2+ + 2\mu\chi_x\chi_y+2(1-\mu)\chi_{xy}^2]\,dx\,dy-q\int_0^a\int_0^b w\,dx\,dy. \quad (11.1)$$

The expression (11.1) agrees completely with the expression (7.4), if we assume that in it $q_x = q_y = 0$.

[1]In this chapter the author states his presentation on the mechanism of the loss in rigidity of a thin elastic cover. The method of determining the critical stresses, which the author makes use of, was developed by him for the first time and is not described in known monographs on the rigidity of elastic systems (Note by the author).

In the above given integral of energy, the integration applies only to the surface of pit or bulges. Furthermore, for simplicity of the linings, the external load q in this case is considered constant, because the form of this load does not affect the character of the boundary conditions.

Furthermore, in the given expression of energy, the components, by which the work of the internal reactive force, distributed along the contour of pits and bulges is expressed, is reduced. It is possible to show the work of these forces is equal to zero. Actually let us assume in (11.1) that there should be the components expressing the work of these forces. Then, instead of (11.1) it is possible to write

$$V=U+A, \tag{11.2}$$

where A - work of the reactive forces, and U can be determined by the expression (11.1).

On the other hand, it is known that if we consider the entire shell on the whole, covered with pits and bulges identical in area, then the total energy for it will be equal to the sum of expressions (11.1)

$$Э=U+U+\ldots=2mnU, \tag{11.3}$$

where $2mn$ - overall number of pits and bulges; m - number of pits and bulges in an axial direction; $2n$ - even number of pits and bulges in the circumferential direction.

Since following the loss of rigidity, the shell is found in the state of equilibrium, then any pit or bulge will also be found in the state of equilibrium. On this base we can apply the principle of virtual displacements to the expressions (11.2) and (11.3), according to which, if any mechanical system, including an elastic system, is found in the state of equilibrium, then the sum of the work of all forces, applied to the given system, based on the virtual randomly small displacements, in agreement with the continuities, is equal to zero:

$$\delta V = \delta U + \delta A = 0, \quad \delta \mathcal{E} = 2mn\,\delta U = 0. \tag{11.4}$$

Here, by the symbol δ, one can connote the virtual displacement (variation).

With the application of the first virtual displacements to the expressions of the complete potential energy (11.2) and (11.3), as usually is the case, all the assigned external forces change neither in initial value nor in direction.

From the second equation c. the system (11.4) it follows that $\delta U = 0$, and from the first - $\delta A = 0$, i.e., the work of the reactive forces in expression (11.2) can be reduced to a certain constant A, which subsequently is unessential and can be assumed equal to zero. Then, the expression for the complete energy in the form (11.2) agrees with the expression (11.1). With this one itself it is indicated that in (11.1) the full potential energy for an isolated pit or bulge is represented, and therefore, from this expression using the methods of variational calculus one can derive both the differential balance equations as well as the boundary conditions along the contour of pits and bulges.

We will not present here all the linings special to a cylindrical shell, but will immediately specify that the above expressed reasonings will be valid for any enclosed shell of rotation, which following the loss of rigidity is covered with identical pits and bulges. Therefore, it is possible to immediately make use of the equations in § 34. Only in this case they will pertain to a separately taken pit or bulge:

$$\left.\begin{aligned} &\nabla_k^2\varphi + D\nabla^2\nabla^2 w = q_z, \\ &\Gamma^2\nabla^2\varphi = E\delta\nabla_k^2 w, \end{aligned}\right\} \tag{11.5}$$

$$\left.\begin{aligned} &N_x\delta u = 0, \\ &N_{xy}\delta v = 0, \\ &M_x\delta\left(\frac{\partial w}{\partial x}\right) = 0, \\ &\left(\frac{\partial M_x}{\partial x} + 2\frac{\partial M_{xy}}{\partial y}\right)\delta w = 0 \end{aligned}\right\} \text{I} \qquad \left.\begin{aligned} &N_y\delta v = 0, \\ &N_{xy}\delta u = 0, \\ &M_y\delta\left(\frac{\partial w}{\partial y}\right) = 0, \\ &\left(\frac{\partial M_y}{\partial y} + 2\frac{\partial M_{xy}}{\partial x}\right)\delta w = 0. \end{aligned}\right\} \text{II}$$

In column I the boundary conditions are presented on the side of the pits or bulges, perpendicular to the meridional sections of the shell; in column II the boundary conditions are presented on the side, perpendicular to the parallels.

With the utilization of these boundary conditions in problems of local rigidity of shells the following considerations should be followed. On the contour of pits and bulges, $w = 0$, since at the boundary of pits and bulges, the deflection of the shell is equal to zero. Therefore, the last condition in columns I and II is automatically met.

When selecting the boundary conditions, which are listed in the second line below, one should base them on the physically feasible character of wave formations following the loss of rigidity. Let us explain this by examples. If surface of the shell following loss of rigidity is covered only by pits, inverted inside the shell, then at the boundary of these pits angles of rotation should be absent. The boundary conditions in this case will be as follows:

$$\frac{\partial w}{\partial x}=0, \frac{\partial w}{\partial y}=0.$$

Such a form of wave formation is possible, for example, with an axisymmetrical loss of rigidity of the cylindrical shell under its axial compression.

If the shell can lose rigidity with the simultaneous formation of pits and bulges, then at the boundary of their division, the angles of rotation will differ from zero. In this instance the boundary conditions are expressed in the form

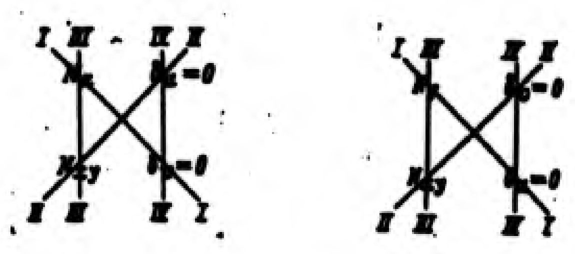
$$M_x=0, M_y=0.$$

Such a form of wave formation can take place under axial compression of a thin cylindrical shell.

From these two examples it is evident that the selection of some given boundary conditions along moments and at angles of rotation should agree with the expected character of wave formation following the loss of rigidity.

Of the remaining two lines in each column, one can obtain the boundary conditions for the membrane forces and tangential movements.

In this case by contrast to the previous one, the expected form of the deformed surface of the shell does not furnish us with any information on the character of the distribution of these forces and displacements along the contour of pits and bulges. Therefore, the selection of boundary conditions based on the membrane forces and the tangential displacements should be subordinate singularly to the possible requirement of obtaining the minimum critical stress. Schematically, the selection of these boundary conditions can be expressed in the following form:



By specifying the various combinations of boundary conditions according to the given schemes under the selected boundary conditions for the moments and angles of rotation, it is possible to detect which one among all these combinations which leads to the least value of the critical force. As a result of such trials it turned out that the boundary conditions III-III gives the lowest value for the critical force.

By solving several problems this assertion will be shown using concrete examples.

In equations (11.5), utilized for the solution of problems of the rigidity of the shell, the lateral load should be replaced by a certain fictitious load, equal to the projection of the internal compressive forces based on a standard for the average surface of the shell,

$$q_z = -N_x^0\chi_x - N_y^0\chi_y - 2N_{xy}^0\chi_{xy},$$

where χ_x, χ_y, χ_{xy} - corresponding changes in curvature; N_x^0, N_y^0, N_{xy}^0 - intensity of the internal forces in the shell, the critical value of which should be determined.

Then, the first equation of the system (11.5) assumes the form

$$\nabla_k^2\varphi + D\nabla^2\nabla^2 w + N_x^0\chi_x + N_y^0\chi_y + 2N_{xy}^0\chi_{xy} = 0.$$

In this case the equation of the compatibility of the deformation remains unchanged.

The given fictitious lateral load can be derived in the following manner.

Let us examine the element of the shell $dx\,dy$ in a position after deformation, and let us set up the projection of the meridional force N_x^0 on axis z. The projection of the line of effect of this force on plane xOz of a moving coordinate system xyz is shown in Fig. 170. Sum of projections on axis z will be equal to

$$-N_x^0\,dy\,\alpha + \left(N_x^0 + \frac{\partial N_x^0}{\partial x}dx\right)dy\left(\alpha + \frac{\partial\alpha}{\partial x}dx\right).$$

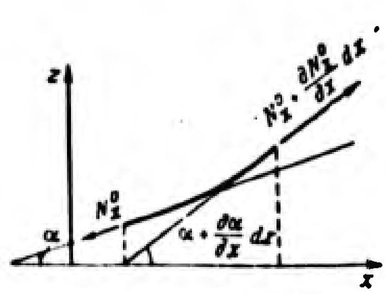


Fig. 170.

By removing the parentheses and discarding the small values, we obtain

$$N_x^0 \frac{\partial \alpha}{\partial x} dx\,dy + \frac{\partial N_x^0}{\partial x} \alpha\, dx\,dy,$$

where α, $\alpha + \frac{\partial \alpha}{\partial x} dx$ - angles of inclination tangent to the element dx.

Similarly also for the force N_y^0 we obtain

$$N_y^0 \frac{\partial \beta}{\partial y} dx\,dy + \frac{\partial N_y^0}{\partial y} \beta\, dx\,dy,$$

where β, $\beta + \frac{\partial \beta}{\partial y} dy$ - angles of inclination tangent to the element dy.

We obtain the projection of the tangential forces N_{xy}^0 to axis z from Fig. 171:

$$-N_{xy}^0 dx\alpha + \left(N_{xy}^0 + \frac{\partial N_{xy}^0}{\partial y} dy\right) dx \left(\alpha + \frac{\partial \alpha}{\partial y} dy\right) = N_{xy}^0 \frac{\partial \alpha}{\partial y} dx\,dy + \frac{\partial N_{xy}^0}{\partial y} dx\,dy\,\alpha.$$

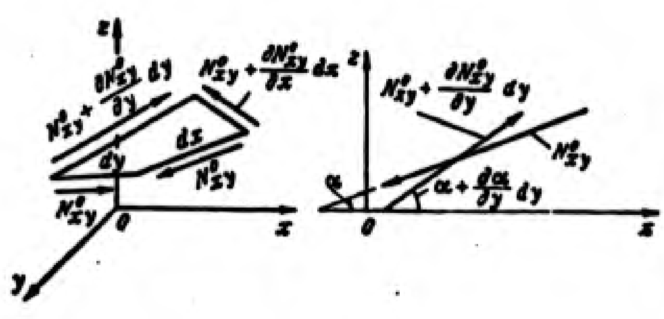


Fig. 171.

Similarly also for the second pair of tangential forces

$$-N_{xy}^0 dy\beta + \left(N_{xy}^0 + \frac{\partial N_{xy}^0}{\partial x} dx\right) dy \left(\beta + \frac{\partial \beta}{\partial x} dx\right) = N_{xy}^0 \frac{\partial \beta}{\partial x} dx\,dy + \frac{\partial N_{xy}^0}{\partial y} dx\,dy\,\beta.$$

After the addition of all projections we will have

$$\left[N_x^0\frac{\partial\alpha}{\partial x}+N_y^0\frac{\partial\beta}{\partial y}+N_{xy}^0\left(\frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}\right)\right]dx\,dy+$$
$$+\left(\frac{\partial N_x^0}{\partial x}+\frac{\partial N_{xy}^0}{\partial y}\right)dx\,dy\,\alpha+\left(\frac{\partial N_y^0}{\partial y}+\frac{\partial N_{xy}^0}{\partial x}\right)dx\,dy\,\beta.$$

From the condition of equilibrium of the element $dx\ dy$ in the plane xOy one can derive these equations:

$$\frac{\partial N_x^0}{\partial x}+\frac{\partial N_{xy}^0}{\partial y}=0,\quad \frac{\partial N_y^0}{\partial y}+\frac{\partial N_{xy}^0}{\partial x}=0.$$

From the differential geometry it is known that the ratio of the angle of contiguity, to the differential of the arc is numerically equal to the curvature at a given point. Therefore

$$\frac{\partial\alpha}{\partial x}=\chi_x,\quad \frac{\partial\beta}{\partial y}=\chi_y.$$

Furthermore, we will have (see § 34)

$$\frac{\partial\alpha}{\partial y}=-\frac{\partial^2 w}{\partial x\,\partial y}=\chi_{xy},\quad \frac{\partial\beta}{\partial x}=-\frac{\partial^2 w}{\partial x\,\partial y}=\chi_{xy}.$$

Then, the sum of projections of the membrane forces on axis z, with reference to unity of the surface, after a change in sign, will be

$$N_x^0\chi_x+N_y^0\chi_y+2N_{xy}^0\chi_{xy}.$$

In the next paragraphs we will turn to the solution of concrete problems

§ 49. Rigidity of a Cylindrical Shell Under Axial Compression

In this instance $R_1=\infty$, $R_2=R$, $N_y^0=0$, $N_{xy}^0=0$, and the differential equations of problem assume the form

$$\nabla^2\nabla^2\varphi - \frac{Eh}{R}\frac{\partial^2 w}{\partial x^2} = 0,$$

$$\frac{1}{R}\frac{\partial^2\varphi}{\partial x^2} + D\nabla^2\nabla^2 w + N_x^0\frac{\partial^2 w}{\partial x^2} = 0. \qquad (11.6)$$

Let us show by an example, this problem of selecting the boundary conditions along the contour of pits and bulges.

Let the shell lose rigidity along an axisymmetrical form with the formation of corrugated folds in a circumferential direction. Then, the functions of stresses and deflections will depend only on the variable x. Because of this, the original equations are considerably simplified and they assume the form

$$\frac{d^4\varphi}{dx^4} - \frac{Eh}{R}\frac{d^2 w}{dx^2} = 0,$$

$$\frac{1}{R}\frac{d^2\varphi}{dx^2} + D\frac{d^4 w}{dx^4} + N_x^0\frac{d^2 w}{dx^2} = 0. \qquad (11.7)$$

The boundary conditions along the contour of the half-waves are reduced due to the absence of the bending moment $M_x = 0$, since the angle of rotation at the points of inflection differ from zero ($\frac{dw}{dx} \neq 0$). It is possible to satisfy this condition, if one takes the expression for the deflection

$$w = A\sin\frac{\pi x}{a},$$

where A - unknown amplitude of the corrugation; a - size of the half-wave of the corrugation.

The boundary conditions for the function of stresses ϕ as a result of the axisymmetrical loss of rigidity can only occur in the form $N_x = 0$. A function, similar to deflection w can satisfy this condition:

$$\varphi = B\sin\frac{\pi x}{a}.$$

After the substitution of the accepted expressions for ϕ and w in the equation (11.7) we obtain

$$\left[B\left(\frac{\pi}{a}\right)^4+A\left(\frac{\pi}{a}\right)^2\frac{E\delta}{R}\right]\sin\frac{\pi x}{a}=0,$$

$$\left[-\frac{B}{R}\left(\frac{\pi}{a}\right)^2+DA\left(\frac{\pi}{a}\right)^4-N_x^0A\left(\frac{\pi}{a}\right)^2\right]\sin\frac{\pi x}{a}=0.$$

After the reduction by $\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}$ we will find that

$$B\left(\frac{\pi}{a}\right)^4+A\frac{E\delta}{R}=0,$$

$$-\frac{B}{R}+A\left[D\left(\frac{\pi}{a}\right)^2-N_x^0\right]=0.$$

Since A and B in these equations are not equal to zero, then from the condition of equality to zero of the determinant of this system, we obtain

$$N_x^0=\frac{E\delta}{R^2}\frac{1}{\left(\frac{\pi}{a}\right)^2}+D\left(\frac{\pi}{a}\right)^2.$$

By considering in this expression the length of a half-wave as a parameter, one can determine the minimum of the critical force N_x^0. Let us designate $\left(\frac{\pi}{a}\right)^2=\xi$. Then,

$$N_x^0=\frac{E\delta}{R^2}\,\frac{1}{\xi}+D\xi. \qquad (11.8)$$

For the determination of the minimum we will have the condition

$$\frac{dN_x^0}{d\xi}=-\frac{E\delta}{R}\,\frac{1}{\xi^2}+D=0,$$

whence

$$\xi=\sqrt{\frac{E\delta}{DR^2}}.$$

Then

$$N^0_{x\,\min}=2\sqrt{\frac{DE\delta}{R^2}}=\frac{E\delta^2}{R\sqrt{3(1-\mu^2)}}\simeq 0{,}6E\,\frac{\delta^2}{R}. \qquad (11.9)$$

Consequently, the critical stress of compression with an axisymmetrical form of loss in rigidity is

$$\sigma_{\kappa p}=0{,}6E\frac{\delta}{R}.$$

Now, let us solve this problem, based on the strength of the assumption that the shell after the loss in rigidity was covered with identical pits and bulges. Such a form of a deformed surface of a shell can be approximately expressed in the following expression for a deflection:

$$w=A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \qquad (11.10)$$

From (11.10) it follows that along the contour of pits and bulges, the deflection and bending moment are equal to zero. Here a and b - the sizes of pits and bulges in an axial and circumferential direction, respectively. Let us take the function of stresses in the form

$$\varphi=B\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

One can be certain that the accepted expression for ϕ satisfies the following boundary conditions.

When $x = 0$, $x = a$

$$N_x=\frac{\partial^2\varphi}{\partial y^2}=-B\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}=0,$$
$$N_{xy}=-\frac{\partial^2\varphi}{\partial x\,\partial y}=-B\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)\cos\frac{\pi y}{a}\cos\frac{\pi y}{b}\neq 0.$$

Since $N_{xy} \neq 0$, then v should be equal to 0. In this way it is easy to prove, if one determines the displacement v according to Hooke's law.

When $y = 0$, $y = b$

$$N_y=\frac{\partial^2\varphi}{\partial x^2}=-B\left(\frac{\pi}{a}\right)^2\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}=0,$$

$$N_{xy}=-\frac{\partial^2\varphi}{\partial x\,\partial y}=-B\left(\frac{\pi}{a}\right)\left(\frac{\pi}{b}\right)\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}\neq 0.$$

Since $N_{xy} \neq 0$, then u should be equal to 0, i.e., the accepted expression for the function of stresses satisfies the boundary conditions I-I (page 321).

Let us substitute the necessary derivatives from the expressions ϕ and w in the equations (11.6). Then, from the condition of equality to zero of the determinant of these equations, we obtain

$$N_x^0=\frac{\frac{Et}{R^2}\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}+D\,\frac{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}{\left(\frac{\pi}{a}\right)^2}.$$

Let us introduce the meaning

$$\xi=\frac{\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2\right]^2}.$$

Then

$$N_x^0=\frac{Et}{R^2}\xi+\frac{D}{\xi}.$$

From the comparison of this expression for N_x^0 with the expression (11.8) it is evident that they completely coincide in structure. Consequently, for the critical stress in this case we obtain the formula (11.9).

Let us examine the boundary conditions for the function of stresses III-III (page 321). In this case the expression for the deflection will be retained in the form (11.10). The boundary conditions III-III will satisfy the following expression[1] for ϕ:

$$\varphi = B_1 \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} + B_2 \sin^2 \frac{\pi x}{a} \sin^2 \frac{2\pi y}{b} + B_3 \sin^2 \frac{2\pi x}{a} \sin^2 \frac{\pi y}{b} + B_4 \sin^2 \frac{2\pi x}{a} \sin^2 \frac{2\pi y}{b}.$$

First, let us examine first approximation when $B_2 = 0$, $B_3 = 0$, $B_4 = 0$. Let us set up the necessary derivatives from the expressions ϕ and w and let us substitute them in the equations (11.6). Since these functions do not satisfy the equations (11.6), then the solution is expediently found, by using the Bubnov-Galerkin method. According to this method, we will multiply the first equation by $\sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{b} dx dy$, and the second one by $\sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dx dy$, and let us integrate within the limits of 0-a, 0-b and let us equate the results to zero. In this case we obtain the following two equations:

$$B_1 = -\frac{16E\delta A\left(\frac{a}{\pi}\right)^2}{9\pi^2 R\left[0,75 + 0,3\left(\frac{a}{b}\right)^2 + 0,75\left(\frac{a}{b}\right)^4\right]},$$

$$AN_x^0 = -\frac{64B_1}{9\pi^2 R} + DA\left(\frac{\pi}{a}\right)^2\left[1 + \left(\frac{a}{b}\right)^2\right].$$

$$N_x = \frac{\partial^2\varphi}{\partial y^2}, \quad N_y = \frac{\partial^2\varphi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2\varphi}{\partial x \partial y}.$$

By excluding the parameters A and B_1 from these equations, we will find

$$\frac{\sigma}{E} = \frac{N_x^0}{E\delta} = \frac{1024\left(\frac{l}{\pi R}\right)^2 m^2}{81\pi^4\left[0,75m^4 + 0,5\left(\frac{l}{\pi R}\right)^2 m^2 n^2 + 0,75\left(\frac{l}{\pi R}\right)^4 n^4\right]} + \alpha\frac{\left[m^2 + \left(\frac{l}{\pi R}\right)^2 n^2\right]^2}{\left(\frac{l}{\pi R}\right)^2 m^2}. \qquad (11.11)$$

---

[1] In this way it is easy to prove, if we set up the expressions.

Connoted here

$$\alpha=\frac{\delta^2}{12(1-\mu^2)R^2};$$

$$m=\frac{l}{a};\quad n=\frac{\pi R}{b};$$

l - length of the shell.

The results of the calculation according to this formula for a shell, which has $\frac{\pi R}{l}$ = 1, $\frac{\delta}{R}$ = 0.005, is listed in Table 9.

Table 9.

$n=2$	$n=3$	$n=4$
$m=15$ $e_x=0{,}00134$	$m=15$ $e_x=0{,}00131$	$m=14$ $e_x=0{,}00136$
$m=16$ $e_x=0{,}00127$	$m=16$ $e_x=0{,}00130$	$m=15$ $e_x=0{,}00132$
$m=17$ $e_x=0{,}00127$	$m=17$ $e_x=0{,}00129$	$m=16$ $e_x=0{,}00131$
$m=18$ $e_x=0{,}00129$	$m=18$ $e_x=0{,}00131$	$m=17$ $e_x=0{,}00132$

From this table it is evident that the least value for a critical stress

$$\left(\frac{\sigma}{E}\right)_{\text{кр}}=0{,}00127.$$

Let us present the formula for the critical stress of compression in the form

$$\sigma_{\text{кр}}=kE\frac{\delta}{R},$$

where k - numerical coefficient, which was found to be equal to 0.6 during our first solutions.

In this case this coefficient is

$$\left(\frac{\sigma}{E}\right)_{кр}=k\frac{\delta}{R}=0{,}00127,$$

whence

$$k=0{,}00127\frac{R}{\delta}=0{,}254. \qquad (11.12)$$

Thus, with the realization of the boundary conditions III-III along the contour of pits and bulges (page 321) one obtains a lower critical stress. Inasmuch as the boundary conditions along the contour of pits and bulges are shaped by the shell itself in the process of losing rigidity, then these boundary conditions will be those whereby the critical load for the specified shell will be the least. The lowering of critical stress under boundary conditions III-III indicates that these boundary conditions, apparently, will be realized even in an experiment. The boundary conditions along the contour of pits and bulges should be natural. Any arbitrary change in these boundary conditions during the solution of the problem implies an increase in the critical stress.

By increasing the number of terms in the expression for the function of stresses in the last solution, one can obtain a more accurate value for the critical stress.[1] With the retention of three components ($B_4 = 0$) in this expression the fourth acquires the same value of critical stress:

$$\sigma_{кр}=0{,}26E\frac{\delta}{R}. \qquad (11.13)$$

During the solution of this problem by a finite difference method under boundary conditions of the III-III form for the function of stresses and with a hinged support along the contour of the pits and bulges based on the function of deflection in the last

[1]The increase in the number of terms in the function of stresses in the first two solutions does not change the magnitude of the critical stress.

formula, a coefficient 0.3 is obtained (both for rectangular, and for diamond-shaped pits and bulges). The critical stress for a case of a long shell is determined by the formula (11.13). By long shells is meant that length at which one half-wave can freely fall. If one half-wave does not fall freely along the length of the shell, then the coefficient in the formula will increase and it will depend on both the length and on the thickness of the shell. In order to come up with a formula for short shells, let us turn to Table 9. From it, it is evident that the minimum of the critical stress is obtained at comparatively small values in numbers of n in comparison with m. Therefore, it becomes possible to discard the components containing n in the original expression (11.11). Then, approximately when $\mu = 0.3$, we obtain

$$\frac{\sigma}{E}=\frac{0{,}0176}{m^2}\left(\frac{l}{R}\right)^2+0{,}9\left(\frac{\delta}{R}\right)^2\left(\frac{R}{l}\right)^2 m^2.$$

If we designate $\xi = m^2\left(\frac{R}{l}\right)^2$ in this formula and determine the minimum of stress, then we will find that

$$\sigma_{кр}=0{,}251E\,\frac{\delta}{R}.$$

The numerical coefficient obtained here is distinguished by its more accurate value (11.12) only in the third sign.

Figure 172 presents curves of the critical stress depending on the length of the shell for a different value of numbers of half-waves in an axial direction.

The numerous experimental results of the investigations on the axial compression of a thin cylindrical shell are very close to that yielded by the formula (11.13). The scattering of experimental data is found within the limits of 0.15-0.3. The more accurate the shell is made, the higher the coefficient k is for it. Therefore, in practical calculations one should consider the corresponding change in k in (11.13).

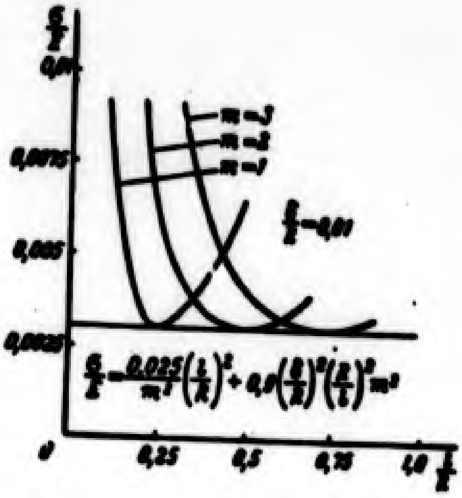


Fig. 172.

Let us look into the solution of this problem, based on the examination of the entire shell as a whole. Let us retain the boundary conditions along the contour of the pits and bulges in the form III-III (page 321). In this instance one can assume that

$$w = A\sin\frac{m\pi x}{l}\sin n\theta,$$

$$\varphi = B_1'\sin^3\frac{m\pi x}{l}\sin n\theta + B_2'\sin^5\frac{m\pi x}{l}\sin n\theta + B_3'\sin^7\frac{m\pi x}{l}\sin n\theta + \ldots$$

These functions at the ends of the shell as well as along the contour of pits and bulges satisfy the following boundary conditions:

$$w = 0,\qquad M_x = 0,\qquad \sigma_x = 0,\qquad \sigma_{x\theta} = 0.$$

Let us examine the first approximation:

$$w = A\sin\frac{m\pi x}{l}\sin n\theta,$$

$$\varphi = B_1'\sin^3\frac{m\pi x}{l}\sin n\theta = B_1'\left(3\sin\frac{m\pi x}{l} - \sin\frac{3m\pi x}{l}\right)\sin n\theta.$$

As a result of the integration of the equations (11.6) according to the Bubnov-Galerkin method we will have

$$\frac{\sigma}{E} = \frac{9\left(\frac{m\pi R}{l}\right)^2}{90\left(\frac{m\pi R}{l}\right)^4 + 36\left(\frac{m\pi R}{l}\right)^2 n^2 + 10n^4} + \alpha\frac{\left[\left(\frac{m\pi R}{l}\right)^2 + n^2\right]^2}{\left(\frac{m\pi R}{l}\right)^2}. \qquad (11.14)$$

where

$$\alpha = \frac{\delta^2}{12(1-\mu^2)R^2}.$$

The calculations, conducted in (11.14), show that the least critical stress is obtained at small values of numbers of n, and the specific weight of components, containing this parameter, is small and they can be discarded without sacrificing accuracy. Then

$$\frac{\sigma}{E} = \frac{0{,}1}{\left(\frac{m\pi R}{l}\right)^2} + \alpha\left(\frac{m\pi R}{l}\right)^2.$$

The minimum of this expression according to parameter m determines $\sigma_{кр}$ in the form

$$\sigma_{кр} = 0{,}192E\,\frac{\delta}{R}.$$

With the second approximation

$$w = A\sin\frac{m\pi x}{l}\sin n\theta,$$

$$\varphi = B_1'\sin^3\frac{m\pi x}{l}\sin n\theta + B_2'\sin^5\frac{m\pi x}{l}\sin n\theta.$$

In this instance

$$\sigma_{кр} = 0{,}22E\,\frac{\delta}{R}.$$

As a result of a subsequent increase in the number of components in the function of stresses (up to $\sin^{13}\frac{m\pi x}{l}\sin n\theta$) with the same expression for the deflection one obtained

$$\sigma_{кр} = 0{,}243E\,\frac{\delta}{R}.$$

A further increase in the components of function ϕ did not change the numerical coefficient in this formula.

If we use the energy method for the solution of the given problem one can be certain of these results.

The full potential energy of the shell will be

$$Э=\frac{1}{2Et}\int_0^l\int_0^{2\pi R}\left[\left(\frac{\partial^2\varphi}{\partial x^2}\right)^2+\left(\frac{\partial^2\varphi}{\partial y^2}\right)^2-2\mu\frac{\partial^2\varphi}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2}+\right.$$

$$\left.+2(1-\mu)\left(\frac{\partial^2\varphi}{\partial x\,\partial y}\right)^2\right]dx\,dy+\frac{D}{2}\int_0^l\int_0^{2\pi R}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2+\left(\frac{\partial^2 w}{\partial y^2}\right)^2+\right.$$

$$\left.+2\mu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]dx\,dy+$$

$$+N_x^0\int_0^l\int_0^{2\pi R}\left[\left(\frac{1}{Et}\left(\frac{\partial^2\varphi}{\partial y^2}-\mu\frac{\partial^2\varphi}{\partial x^2}\right)-\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]dx\,dy.$$

For the connection of the function of stresses ϕ with the function of deflection w, we will have an equation of compatibility

$$\nabla^2\nabla^2\varphi=-\frac{Et}{R}\frac{\partial^2 w}{\partial x^2}. \qquad (11.15)$$

The expressions for ϕ and w can be derived from the previous solution. For the first approximation

$$w=A\sin\frac{m\pi x}{l}\sin n\theta,$$

$$\varphi=B_1'\sin^3\frac{m\pi x}{l}\sin n\theta.$$

After the integration of the equation (11.15) according to the Bubnov-Galerkin method and after obtaining the relationship between the parameters A and B_1' for the full energy we obtain the expression

$$Э=\frac{\pi EtlA^2}{4R^2}\left(\frac{m\pi R}{l}\right)^2\left\{\frac{9\left(\frac{m\pi R}{l}\right)^2}{90\left(\frac{m\pi R}{l}\right)^4+36\left(\frac{m\pi R}{l}\right)^2n^2+10n^4}+\right.$$

$$\left.+a\frac{\left[\left(\frac{m\pi R}{l}\right)^2+n^2\right]^2}{\left(\frac{m\pi R}{l}\right)^2}-\frac{N_x^0}{Et}\right\}.$$

where

$$\alpha = \frac{\delta^2}{12(1-\mu^2)R^2}.$$

From the condition $\delta Э=0$ we obtain the expression for the critical stress, with an accuracy coinciding with the first approximation according to the Bubnov-Galerkin method.

One can be certain that all the subsequent approximations in the energy method will coincide with the corresponding approximations of the previous solution.

Thus, the idea of a separate examination of the pits and bulges - this is merely a convenient means for drawing up the approximate solution, and it is by no means, obligatory. One can also consider the entire shell as a whole, but with the use of natural boundary conditions along the contour of half-waves in the approximate solution.

This idea can be considered as a further more precise definition of the methods of calculating the shells for local rigidity. Critical load depends not only on the selected form of the deformed surface of the shell after the loss of rigidity and physical boundary conditions, but in addition, how the boundary conditions are actually realized along the contour of pits and bulges, i.e., one imposes definite requirements on the selection of the approximating functions.

Remarks on the determination of critical stresses for a cylindrical shell under pure bending. If the cylindrical shell is loaded along the edges with pairs of forces, then the distribution of the axial stresses through the section will change according to the law of the sine or law of cosine (depending upon the origin of coordinates of the angle, see § 23). Because of this one should expect that the critical stress for the compressed zone in contrast with the action of uniform compression should be somewhat higher: within the confines of one pit or bulge stress of compression does not remain constant, and as a consequence of this, the form of the deformed surface will differ from that of pure compression. With bending the boundary conditions on the sides $y = 0$, $y = b$ of pits and bulges, expressed by function w and its derivatives, apparently, will be closer to that of an elastic seal, than that of a hinged support. A reliable theoretical solution of this problem, apparently, is lacking. An experimental check on the bending of a cylindrical shell indicates the fact that coefficient k in this instance in comparison with pure compression is higher by 15-18%.

§ 50. Rigidity of a Cylindrical Shell Under Uniform External Pressure

The method of the division of variables can be used for the solution of this problem. Let us use the expressions for the function of bending w and for the function of stresses ϕ, in the form

$$w = W(x)\cos n\theta, \qquad \varphi = F(x)\cos n\theta,$$

which satisfy the periodicity condition in the circumferential direction. Let us consider the ends of the shell as hinged supports.

Having substituted these expressions in the original equations in § 49, we obtain two combined equations as a function of coordinate x:

$$F^{IV} - \frac{2n^2}{R^2}F'' + \frac{n^4}{R^4}F - \frac{Eh}{R}W'' = 0,$$

$$F'' + DR\left(W^{IV} - \frac{2n^2}{R^2}W'' + \frac{n^4}{R^4}W\right) - \frac{n^2}{R}N_y^0 W + RN_x^0 W'' = 0.$$

Let us express these equations in terms of finite differences:

$$F_k\left[6 + 4\left(\frac{nh}{R}\right)^2 + \left(\frac{nh}{R}\right)^4\right] - \left[4 + 2\left(\frac{nh}{R}\right)^2\right](F_l + F_l) + F_t + \\ + F_s - \frac{Ehh^2}{R}(W_l + W_l - 2W_k) = 0, \qquad (11.16)$$

$$F_l+F_i-2F_k+\frac{DR}{h^2}\left\{W_k\left[6+4\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4\right]-\right.$$
$$\left.-\left[4+2\left(\frac{nh}{R}\right)^2\right](W_l+W_i)+W_l+W_i\right\}-$$
$$-(nh)^2qW_k+RN_x^0(W_l+W_i-2W_k)=0.$$

Designated here is $N_y^0=qR.$

As was stipulated above, the ends of the shell are considered as hinged supports. Using the boundary conditions, for the contour and postcontour points we obtain the following values[1] of the functions W and F:

$$W_k=0,\quad W_l=-W_i,\quad F_k=0,\quad F_l=F_i.$$

Further, let us consider that only one half-wave is generated along the length of the shell, while in the circumferential direction, as this follows from expression w, a $2n$ half-wave is generated.

The first approximation (Fig. 173): $h = l/2$, where h - pitch of the grid.

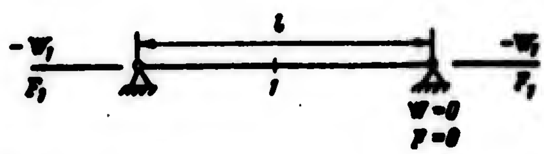


Fig. 173.

Let us set up the equations (11.16) for point 1:

$$F_1\left[6+4\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4\right]+\left[4+2\left(\frac{nh}{R}\right)\right](0+0)+$$
$$+F_1+F_1-\frac{Eh}{R}(0+0-2W_1)=0,$$
$$0+0-2F_1+\frac{DR}{h^2}\left\{W_1\left[6+4\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4\right]-\right.$$
$$\left.-\left[4+2\left(\frac{nh}{R}\right)\right](0+0)-W_1-W_1\right\}-(nh)^2qW_1=0.$$

[1]Additional stresses σ_x and σ_{xy} are absent on the ends of the shell.

After certain conversions these equations can be rewritten in the form

$$F_1(a+2)+\frac{2E\delta h^2}{R}W_1=0,$$

$$F_1=\frac{DR(a-2)W_1}{2h^2}-\frac{1}{2}(nh)^2 qW_1.$$

Designated here is

$$a=6+4\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4.$$

By excluding function F_1 from these equations and assuming that $W_1 \neq 0$, we obtain the expression for $q_{\text{кр}}$:

$$\left(\frac{qR}{E\delta}\right)_{\text{кр}}=\frac{(R\delta)^2\left(4+4n^2\frac{h^2}{R^2}+n^4\frac{h^4}{R^4}\right)}{12(1-\mu^2)n^2h^4}+\frac{4}{n^2\left(8+4n^2\frac{h^2}{R^2}+n^4\frac{h^4}{R^4}\right)}.$$

In order to get the least critical pressure, the obtained expression must be investigated to a minimum through parameter n.

The analytical solution of this problem leads to very cumbersome computations. For the approximate determination of $q_{\text{кр}}$ let us assume that many half-waves are generated along the circumference of the shell after the loss of rigidity and parameter n will be a large number. Then, one can approximately write

$$\left(\frac{qR}{E\delta}\right)_{\text{кр}}=\frac{\delta^2}{12(1-\mu^2)R^2}n^2+\frac{4\left(\frac{R}{h}\right)^4}{n^6}.$$

This expression acquires a minimum during $n^2=\frac{4R}{l}\sqrt{\frac{3R}{l}\sqrt{1-\mu^2}}$. Then, when $\mu = 0.3$ we obtain

$$q^{\text{I}}_{\text{кр}}=0{,}845E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}}.$$

This value of $q^{I}_{кр}$ can be somewhat refined by means of substitution of values of n^2 in the original equation, somewhat greater and somewhat less, than what the formula gives for n^2. In this case one ought to remember that n - integers of natural series.

If one were to take a grid with two points (Fig. 174), then for $q^{II}_{кр}$ we obtain the expression

$$\left(\frac{qR}{E\delta}\right)^{II}_{кр}=\frac{(R\delta)^2\left(1+2n^2\frac{h^2}{R^2}+n^4\frac{h^4}{R^4}\right)}{12(1-\mu^2)\,n^2h^4}+\frac{1}{n^2\left(3+2n^2\frac{h^2}{R^2}+n^4\frac{h^4}{R^4}\right)}.$$

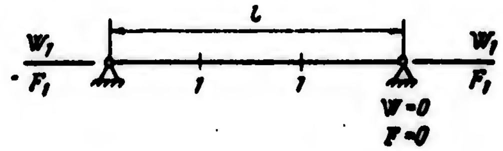


Fig. 174.

An investigation of this expression under the same assumptions as that in the first case, results in the following least value of $q^{II}_{кр}$:

$$q^{II}_{кр}=0{,}884E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}}\quad\text{when}\quad n^2=\frac{3R}{l}\sqrt{\frac{6R}{l}\sqrt{1-\mu^2}}.$$

By applying the extrapolation (§ 2) to the obtained values of $q^{I}_{кр}$ and $q^{II}_{кр}$, we obtain

$$q_{кр.экстр}=0{,}92E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}}\quad\text{when}\quad n^2_{экстр}=7{,}7\,\frac{R}{l}\sqrt{\frac{R}{l}\sqrt{1-\mu^2}}. \qquad (11.17)$$

The obtained formula by accuracy coincides with the known formula of P. F. Papkovich.

In the case of three-dimensional compression of a cylindrical shell by an external pressure under the same boundary conditions as that above, we obtain the following formula.

The first approximation: $h = l/2$:

$$\left(\frac{qR}{E\delta}\right)^{I}_{кр}=\frac{\delta^2}{12(1-\mu^2)h^2}\,\frac{4+4\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4}{1+\left(\frac{nh}{R}\right)^2}+$$
$$+\frac{4h^2}{R^2\left[1+\left(\frac{nh}{R}\right)^2\right]\left[8+4\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4\right]}.$$

The second approximation: $h = l/3$:

$$\left(\frac{qR}{E\delta}\right)^{II}_{кр}=\frac{\delta^2}{6(1-\mu^2)h^2}\,\frac{1+2\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4}{1+2\left(\frac{nh}{R}\right)^2}+$$
$$+\frac{2h^2}{R^2\left[1+2\left(\frac{nh}{R}\right)^2\right]\left[3+2\left(\frac{nh}{R}\right)^2+\left(\frac{nh}{R}\right)^4\right]}.$$

By making the determination of $q^{I}_{кр}$ and $q^{II}_{кр}$ according to these formula we will find a more accurate value of these quantities by means of extrapolation:

$$q_{кр.уточ}=-0.8q^{I}_{кр}+1.8q^{II}_{кр}.$$

Now let us examine the solution with smaller pitch: $h = l/5$ (Fig. 175). In this instance we obtain

$$\left(\frac{qR}{E\delta}\right)^{III}_{кр}=\frac{1}{n^2}\left[\frac{1}{2}\left(\alpha a_6-\frac{a_7}{a_2^2-a_1a_3}\right)\pm\right.$$
$$\left.\pm\sqrt{\frac{1}{4}\left(\alpha a_6-\frac{a_7}{a_2^2-a_1a_3}\right)^2+\alpha^2\left(a_4^2-a_1a_5\right)+\frac{\alpha a_8}{a_2^2-a_1a_3}+\frac{1}{a_2^2-a_1a_3}}\right].$$

Here

$$
\begin{aligned}
a &= \frac{625R^2\delta^2}{12(1-\mu^2)\,l^4},\\
a_1 &= 2+2\left(\frac{nl}{5R}\right)^2+\left(\frac{nl}{5R}\right)^4,\\
a_2 &= 3+2\left(\frac{nl}{5R}\right)^2,\\
a_3 &= 7+4\left(\frac{nl}{5R}\right)^2+\left(\frac{nl}{5R}\right)^4,\\
a_4 &= -1+\left(\frac{nl}{5R}\right)^2,\\
a_5 &= 1+2\left(\frac{nl}{5R}\right)^2+2\left(\frac{nl}{5R}\right)^4,\\
a_6 &= 3a_4+(a_3-1)\,a_5,\\
a_7 &= 6+6\left(\frac{nl}{5R}\right)^2+7\left(\frac{nl}{5R}\right)^4,\\
a_8 &= 7+6\left(\frac{nl}{5R}\right)^2+2\left(\frac{nl}{5R}\right)^4.
\end{aligned}
\tag{11.18}
$$

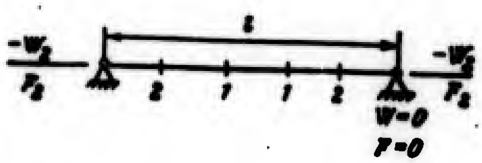


Fig. 175.

A more accurate expression for $q_{\text{кр}}^{\text{III}}$ should be used in those instances when it is necessary to obtain a refined value for the critical pressure. If this is not necessary, then one can make use of Papkovich's formula, which provides entirely reliable results for a shell of average length.

Let us apply the obtained expression for $q_{\text{кр}}^{\text{III}}$ to the determination of the critical pressures of the sheathing in the cells of the reinforced cylindrical shell during the action of a uniform external pressure on it.

We will consider that the subcritical strained state for such a shell can be determined according to the momentless theory of a corresponding smooth shell. Let the panel lean upon two adjacent stringers. In this case the ends of the panel lean on the frames.

In order to convert from an enclosed shell to a panel in the expression $q_{\text{кр}}^{\text{III}}$, it is necessary to substitute n with π/α_0. This follows from the condition $\cos n\alpha = \cos \frac{\pi\alpha}{\alpha_0}$, where α_0 - central angle of the panel (Fig. 176). Then, for the coefficients (11.18) we will have analogous expressions, if we substitute in the latter

$$\frac{nl}{5R} \text{ with } \frac{\pi l}{5a}.$$

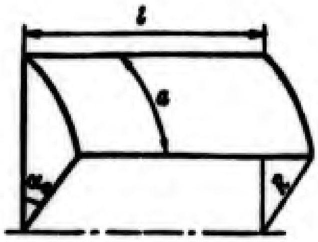


Fig. 176.

Let us examine two cases of a combination of sizes of a cell.

A cell close to a square: $a/l \sim 1$.

In this instance for $q_{\text{кр}}$ we will have

$$\frac{qR}{E\delta} = \left(\frac{a}{\pi R}\right)^2 \left\{\frac{1}{2}\left[\frac{505R^2\delta^2}{(1-\mu^2)a^4} + 0{,}844\right] \pm \right.$$
$$\left. \pm \sqrt{\frac{49\,000R^4\delta^4}{(1-\mu^2)^2a^8} + \frac{136R^2\delta^2}{(1-\mu^2)a^4} + 0{,}0885}\right\}.$$

By substituting here the subradical expression with the approximate quantity

$$\left[\frac{222R^2\delta^2}{(1-\mu^2)a^4} + 0{,}306\right]^2,$$

we obtain

$$\frac{qR}{E\delta} = \left(\frac{a}{\pi R}\right)^2 \left\{\frac{1}{4}\left[\frac{505R^2\delta^2}{(1-\mu^2)a^4} + 0{,}844\right] \pm \left[\frac{222R^2\delta^2}{(1-\mu^2)a^4} + 0{,}306\right]\right\}.$$

By retaining the minus sign in this expression, we obtain

$$\frac{qR}{E\delta}=\frac{3,1\delta^2}{(1-\mu^2)\,a^2}+0,0117\left(\frac{a}{R}\right)^2.$$

By considering the size of panel a in this formula as a parameter, we will find the minimum of critical pressure (μ = 0.3):

$$q_{\text{кр}}=0,4E\left(\frac{\delta}{R}\right)^2 \text{ when } \quad a=4,12\sqrt{R\delta}.$$

A cell strongly elongated in an axial direction:

$$a\ll l,\ \left(\frac{l}{a}\right)^2\gg 1.$$

In this instance the original equation can be approximately written in the form

$$\frac{qR}{E\delta}=\frac{\pi^2\delta^2}{12(1-\mu^2)\,a^2}+\frac{175a^6}{2\pi^6R^2l^4}.$$

By considering the size a as a parameter in this expression, we will find the minimum $q_{\text{кр}}$(μ = 0.3):

$$q_{\text{кр}}=0,89E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \text{ when } \quad a=0,68\sqrt{l\sqrt{R\delta}}.$$

As one would expect, the obtained value $q_{\text{кр}}$ in this instance coincides with the critical pressure for an enclosed shell, which after the loss of rigidity, is divided into a circumferential direction by a number of gently sloping cylindrical panels.

Without a derivation let us present formulas for the external critical pressure of cylindrical panels, obtained under other boundary conditions along the contour.

1. A panel rigidly sealed along its entire contour and with additional stresses lacking (σ_x = 0, σ_y = 0, σ_{xy} = 0).

In this instance for the functions of ϕ and w these expressions are assumed

$$w = W(x)\cos^2\frac{\pi\alpha}{2\alpha_0}, \quad \varphi = F(x)\cos^2\frac{\pi\alpha}{2\alpha_0},$$

where

$$\alpha_0 = \frac{S}{2R};$$

S, R - width and radius of the panel.

As a result of the first ($h = l/2$) and second ($h = l/3$) approximations, one obtains

$$\left(\frac{qR}{E\delta}\right)^{I} = \frac{0{,}75\left(\frac{l}{R}\right)^2}{\left[3+\frac{16}{3}\left(\frac{\pi l}{2S}\right)^2+\frac{16}{3}\left(\frac{\pi l}{2S}\right)^4\right]\left(\frac{\pi l}{2S}\right)^2} + \frac{\left[3+\frac{16}{3}\left(\frac{\pi l}{2S}\right)^2+\frac{16}{3}\left(\frac{\pi l}{2S}\right)^4\right]\left(\frac{\delta}{l}\right)^2}{4(1-\mu^2)\left(\frac{\pi l}{2S}\right)^2}.$$

$$\left(\frac{qR}{E\delta}\right)^{II} = \frac{(l/R)^2}{12\left[3+\frac{8}{3}\left(\frac{\pi l}{3S}\right)^2+\frac{16}{3}\left(\frac{\pi l}{3S}\right)^4\right]\left(\frac{\pi l}{3S}\right)^2} + \frac{9\left[3+\frac{8}{3}\left(\frac{\pi l}{3S}\right)^2+\frac{16}{3}\left(\frac{\pi l}{3S}\right)^4\right]\left(\frac{\delta}{l}\right)^2}{16(1-\mu^2)\left(\frac{\pi l}{3R}\right)^2}.$$

In the case of a square panel ($l \sim S$) it was found that

$$q_{кр}^{I} = 0{,}37E\left(\frac{\delta}{R}\right)^2 \quad \text{when } S^{I} = 5{,}7\sqrt{R\delta},$$

$$q_{кр}^{II} = 0{,}413E\left(\frac{\delta}{R}\right)^2 \quad \text{when } S^{II} = 5{,}83\sqrt{R\delta},$$

$$q_{кр.эксп} = 0{,}45E\left(\frac{\delta}{R}\right)^2 \quad \text{when } S_{эксп} = 5{,}94\sqrt{R\delta}.$$

If the panel is strongly elongated in an axial direction ($l \gg S$), then

$$q^{\mathrm{I}}_{\text{кр}} = 1{,}46E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad S^{\mathrm{I}} = 1{,}8\sqrt{l\sqrt{R\delta}},$$

$$q^{\mathrm{II}}_{\text{кр}} = 1{,}53E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad S^{\mathrm{II}} = 1{,}5\sqrt{l\sqrt{R\delta}},$$

$$q_{\text{кр.экстр}} = 1{,}6E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad S_{\text{экстр}} = 1{,}254\sqrt{l\sqrt{R\delta}}.$$

2. A panel rigidly sealed at the ends and hinged supported along the longitudinal edges. Additional stresses are lacking ($\sigma_x = 0$, $\sigma_y = 0$, $\sigma_{xy} = 0$).

In this instance for functions of ϕ and w these expressions were used

$$w = W(x)\cos\frac{\pi\alpha}{\alpha_0}, \qquad \varphi = F(x)\cos\frac{\pi\alpha}{\alpha_0},$$

where

$$\alpha_0 = \frac{S}{R}.$$

For first and second approximations the following expressions were obtained:

$$\left(\frac{qR}{E\delta}\right)^{\mathrm{I}} = \frac{(l/R)^2}{\left[8+4\left(\frac{\pi l}{2S}\right)^2+\left(\frac{\pi l}{2S}\right)^4\right]\left(\frac{\pi l}{2S}\right)^2} + \frac{\left[8+4\left(\frac{\pi l}{2S}\right)^2+\left(\frac{\pi l}{2R}\right)^4\right]\left(\frac{\delta}{l}\right)^2}{3(1-\mu^2)\left(\frac{\pi l}{2S}\right)^2}$$

$$\left(\frac{qR}{E\delta}\right)^{\mathrm{II}} = \frac{(l/R)^2}{9\left[3+2\left(\frac{\pi l}{3S}\right)^2+\left(\frac{\pi l}{3S}\right)^4\right]\left(\frac{\pi l}{3S}\right)^2} + \frac{3\left[3+2\left(\frac{\pi l}{3S}\right)^2+\left(\frac{\pi l}{3S}\right)^4\right]\left(\frac{\delta}{l}\right)^2}{4(1-\mu^2)\left(\frac{\pi l}{3S}\right)^2}$$

For a panel, close to a square, one obtains

$$q^{\mathrm{I}}_{\text{кр}} = 0{,}495E\left(\frac{\delta}{R}\right)^2 \quad \text{when} \quad S^{\mathrm{I}} = l = 3{,}8\sqrt{R\delta},$$

$$q^{\mathrm{II}}_{\text{кр}} = 0{,}55E\left(\frac{\delta}{R}\right)^2 \quad \text{when} \quad S^{\mathrm{II}} = l = 4{,}15\sqrt{R\delta},$$

$$q_{\text{кр.экстр}} = 0{,}6E\left(\frac{\delta}{R}\right)^2 \quad \text{when} \quad S_{\text{экстр}} = 4{,}5\sqrt{R\delta}.$$

In the case of a strongly elongated panel

$$q^{I}_{кр}=0{,}835E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad S^{I}=1{,}2\sqrt{l\sqrt{R\delta}},$$

$$q^{II}_{кр}=0{,}88E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad S^{II}=1{,}2\sqrt{l\sqrt{R\delta}},$$

$$q_{кр.экстр}=0{,}92E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad S_{экстр}=1{,}2\sqrt{l\sqrt{R\delta}}.$$

§ 51. Rigidity of a Long Cylindrical Shell from the Effect of External Uniform Pressure. Limits of the Applicability of Papkovich's Formula

In this case a balance equation, obtained in § 34, with the retention of all components in it, which depend upon variable y, must be used for the solution of the problem. All the derivatives based on x will disappear since the wave formation for a long shell after the loss of rigidity will not depend upon coordinate x, if we do not take into consideration the small sections at the ends of the shell.

Thus, the solving equations of the problem have the form

$$\frac{d^4\varphi}{dy^4}=0,$$

$$D\left(\frac{d^4w}{dy^4}+\frac{2}{R^2}\frac{\partial^2w}{\partial y^2}+\frac{w}{R^4}\right)+N_y^0\left(\frac{\partial^2w}{\partial y^2}+\frac{w}{R^2}\right)=0. \qquad (11.19)$$

In order to get the critical value of external pressure one must solve the second equation, for which one can assume

$$w=A\cos n\theta=A\cos\frac{ny}{R}.$$

Then, when $A \neq 0$ we obtain

$$N_y^0=\frac{D(n^2-1)}{R^2}.$$

Since $N_y^0 = qR$, then

$$q=\frac{D(n^2-1)}{R^3}.$$

The least value of this pressure will be when $n = 2$:

$$q_{\text{кр}} = \frac{3D}{R^3}.$$

Here let us substitute $D = \frac{E\delta^3}{12(1-\mu^2)}$. Then

$$q_{\text{кр}} = \frac{E}{4(1-\mu^2)}\left(\frac{\delta}{R}\right)^3. \tag{11.20}$$

Thus, for the calculation of the external critical pressure of a cylindrical shell we will have Papkovich's formulas and (11.20), since the latter, as evident from its derivation, is valid only for a very long shell. Therefore, it is necessary to set the limit of applicability of this formula.

From the condition of equality of the critical pressures according to investigated formula when $\mu = 0.3$ we obtain

$$l < 3{,}35R\sqrt{\frac{R}{\delta}}. \tag{11.21}$$

The boundary of applicability of Papkovich's formula sets this expression. If

$$l > 3{,}35R\sqrt{\frac{R}{\delta}},$$

then, one should make use of the formula (11.20).

The condition (11.21) establishes one of the boundaries of the applicability of Papkovich's formula. Let us determine the second boundary of its applicability from the condition that the critical stress in the shell does not exceed the proportional limit of the material, from which it is made, i.e., $\sigma_{\text{кр}} \leq \sigma_{\text{р}}$. Hence, we obtain the second condition

$$l > \frac{0{,}92E\delta}{\sigma_{\text{р}}\sqrt{\frac{R}{\delta}}}.$$

Thus, Papkovich's formula is applicable to all shells, if their length satisfies the condition

$$\frac{0{,}92E\delta}{q_{kp}\sqrt{\frac{R}{\delta}}} < l < 3{,}35R\sqrt{\frac{R}{\delta}}.$$

During the derivation of the formula (11.17) it was proposed that the ends of the shell be hinged supported. If additional connections are added to the ends of the shell which is frequently found in practical constructions, then according to Papkovich's formula for a short shell one obtains an understated value of the critical pressure, since with the more "rigid" sealing of the ends of the shell, the critical pressure must be increased.

Remarks on the calculation of conical shells. When evaluating the rigidity of conical shells, loaded with external uniform pressure, as shown by certain authors, it is possible to make use of Papkovich's formula, if we substitute R with $R/\cos\alpha$ (Fig. 177) in it, and so forth

$$q_{kp} = 0{,}92E\frac{\delta^2\cos\alpha}{Rl}\sqrt{\frac{\delta\cos\alpha}{R}},$$

if $\alpha \leq 25°$.

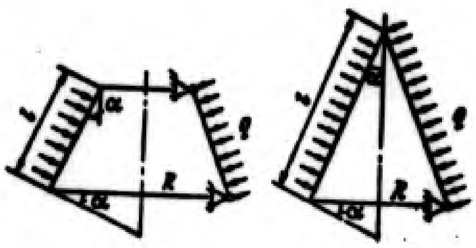


Fig. 177.

§ 52. Rigidity of a Cylindrical Shell Under Uniform External Pressure, if the Half-Waves are Directed Inward After the Loss of Rigidity

This type of loss of rigidity can be expected in constructions, of cylindrical shells designed without a clearance for one another, since there is a grid of small openings on the external shell. During the action of external pressure on such a construction the inside shell can lose rigidity due to the formation of half-waves, directed towards the inside of the shell. Such a type of loss of rigidity can be described by the following system of functions:

$$w = A \sin\frac{\pi x}{l} \sin^2 n\theta,$$
$$\varphi = B \sin\frac{\pi x}{l} \sin^2 n\theta,$$

which furnish even values of deflection and annular stresses that correspond to the character of the wave formation.

The differential equations for the solution of the problem in this case have the form

$$\nabla^2\nabla^2\varphi = \frac{Et}{R}\frac{\partial^2 w}{\partial x^2},$$
$$\frac{1}{R}\frac{\partial^2\varphi}{\partial x^2} + D\nabla^2\nabla^2 w + N_y^0\frac{\partial^2 w}{\partial y^2} = 0.$$

After the substitution of values of functions ϕ and w in these and after their integration by the Bubnov-Galerkin method we obtain

$$B\left(\frac{\pi}{l}\right)^2 = -\frac{3EtA}{8R\left[\frac{3}{8} + n^2\left(\frac{l}{\pi R}\right)^2 - 2n^4\left(\frac{l}{\pi R}\right)^4\right]},$$
$$-\frac{3}{8R}B\left(\frac{\pi}{l}\right)^2 + DA\left(\frac{\pi}{l}\right)^4\left[\frac{3}{8} + n^2\left(\frac{l}{\pi R}\right)^2 + 2n^4\left(\frac{l}{\pi R}\right)^4\right] = \frac{N_y^0 n^2}{2R^2}A.$$

By equating the determinant of this system of equations to zero, we obtain

$$\frac{N_y^0 n^2}{2R^2} = \frac{9E\delta}{64R^2\left[\frac{3}{8}+n^2\left(\frac{l}{\pi R}\right)^2+2n^4\left(\frac{l}{\pi R}\right)^4\right]} +$$
$$+ D\left(\frac{\pi}{l}\right)^4\left[\frac{3}{8}+n^2\left(\frac{l}{\pi R}\right)^2+2n^4\left(\frac{l}{\pi R}\right)^4\right].$$

In order to get the approximate value of $N_{y\,\text{кр}}^0$ we will consider that

$$n^2\left(\frac{l}{\pi R}\right)^2 \gg 1.$$

Then

$$\frac{N_y^0}{E\delta} = \frac{9}{64\left(\frac{l}{\pi R}\right)^4}\,\frac{1}{n^6} + \frac{R^2\delta^2\left(\frac{\pi}{l}\right)^4\left(\frac{l}{\pi R}\right)^4}{3(1-\mu^2)}\,n^2.$$

After the determination of the minimum of force N_y^0 based on the parameter of wave formation n^2 we obtain the formula for the critical external pressure

$$q_{\text{кр}} = 1{,}6E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}} \quad \text{when} \quad n^2 = \frac{3\pi\sqrt[4]{1-\mu^2}}{2\sqrt{2}}\,\frac{R}{l}\sqrt{\frac{R}{\delta}}. \qquad (11.22)$$

Based on structure the given formula agrees with P. F. Papkovich's formula but numerical coefficient, as expected, increases.

§ 53. Rigidity of the Length of a Cylindrical Shell Under an External Uniform Pressure, if the Half-Waves After Loss of Rigidity Are Directed Inward. Limits of the Applicability of the Formula (11.22)

In this case the approximate solution can be obtained, if we assume that the following expression, in agreement with the character of the wave formation, for the deflection is:

$$w = A\sin^2 n\theta.$$

After the substitution of this expression in (11.19) and after the integration of it by the Bubnov-Galerkin method we obtain

$$q=\frac{D(8n^4-4n^2+1,5)}{(2n^2-1,5)R^3}.$$

As can easily be proven, the least critical pressure from this expression is derived when $n = 2$:

$$q_{\text{кр}}=\frac{17,4D}{R^3}.$$

After the substitution of the value of hardness D when $\mu = 0.3$ in this expression

$$q_{\text{кр}}=1,6E\left(\frac{\delta}{R}\right)^3. \qquad (11.23)$$

From the condition of equality of the critical pressures according to formulas (11.22) and (11.23)

$$l<R\sqrt{\frac{R}{\delta}}.$$

If $l>R\sqrt{R/\delta}$, then one should make use of the formula for a long shell.

We will obtain the lower limit of applicability of the formula (11.22) from the condition $\sigma_{\text{кр}} = \sigma_{\text{p}}$, i.e.,

$$l>\frac{E\delta}{\sigma_{\text{p}}\sqrt{\frac{R}{\delta}}}.$$

Thus, the formula (11.22) is applicable to such shells, whose length satisfies the condition

$$\frac{E\delta}{\sigma_{\text{p}}\sqrt{\frac{R}{\delta}}}<l<R\sqrt{\frac{R}{\delta}}.$$

By using the formula (11.22), just as Papkovich's formula is used, one should keep in mind that for a comparatively short shell this formula will give an underrated value of critical pressure, since hinged support assumed in the solution will not always be realized in actuality.

§ 54. Rigidity of a Hinged Support of a Cylindrical Panel Due to Action of an Axial Load, Applied on the Curved Edges and Distributed According to the Law of the Cosine

The problem of the rigidity of a cylindrical panel under the action of compressive loadings, different from a uniform distribution, frequently appears during the calculation of a reinforced shell, because the distribution of the compressive stresses is not always uniform.

Let us examine the problem dealing with the rigidity of a cylindrical panel, loaded in a circumferential direction according to the law of the cosine (Fig. 178). In order to solve the given problem, it is necessary at first to find the distribution of stresses in the plate up to the moment of the loss of rigidity.

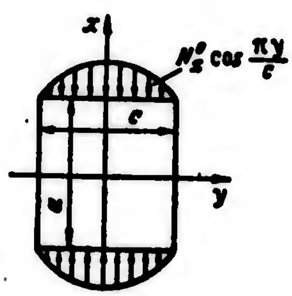


Fig. 178.

For this purpose let us use the balance equations of forces, acting on the average surface of the panel

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,$$

$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0, \qquad N_y=0.$$

Since $N_y = 0$, then $\frac{\partial N_{xy}}{\partial x}=0$, $N_{xy}=f(y)$, where $f(y)$ - function of only variable y. However, since at the ends, $N_{xy} = 0$ everywhere, then it must be that $f(y) = 0$. Consequently, $\frac{\partial N_x}{\partial x}=0$, $N_x=\mathrm{const}$ along the generatrix, i.e., the distribution of forces in the plane of the plate coincides with their distribution at its ends. This derivation, apparently, will be valid only for short plates.

For the solution of an imposed problem let us take the following expressions for the functions of ϕ and w:

$$w=W(x)\cos\frac{\pi y}{c}, \qquad \varphi=F(x)\cos\frac{\pi y}{c}.$$

Then, the original equations of rigidity (11.5) when $R_1 = \infty$, $R_2 = R$ will assume the form

$$\begin{gathered} F^{\mathrm{IV}}-2\left(\frac{\pi}{c}\right)^2 F''+\left(\frac{\pi}{c}\right)^4 F-\frac{Et}{R}W''=0, \\ \left\{F''+DR\left[W^{\mathrm{IV}}-2\left(\frac{\pi}{c}\right)^2 W''+\left(\frac{\pi}{c}\right)^4 W\right]\right\}\cos\frac{\pi y}{c}+ \\ +RN_x^0 W''\cos^2\frac{\pi y}{c}=0. \end{gathered} \tag{11.24}$$

After the multiplication of the second equation by $\cos\frac{\pi y}{c}\,dy$ and after the integration within the limits $-c/2$, $+c/2$, and also after the substitution of the differential operations by the finite differences we obtain

$$\begin{gathered} F_k\left[6+4\left(\frac{\pi h}{c}\right)^2+\left(\frac{\pi h}{c}\right)^4\right]-\left[4+2\left(\frac{\pi h}{c}\right)^2\right](F_i+F_l)+ \\ +F_t+F_s-\frac{Eth^2}{R}(W_i+W_l-2W_k)=0, \\ F_i+F_l-2F_k+\frac{DR}{h^2}\left\{W_k\left[6+4\left(\frac{\pi h}{c}\right)^2+\left(\frac{\pi h}{c}\right)^4\right]-\right. \\ \left.-\left[4+2\left(\frac{\pi h}{c}\right)^2\right](W_i+W_l)+W_t+W_s\right\}+ \\ +\frac{8RN_x^0}{3\pi}(W_i+W_l-2W_k)=0. \end{gathered}$$

Let us consider that the panel is hinged supported and the additional stresses along the contour are equal to zero: $\sigma_x=0$, $\sigma_y=0$, $\sigma_{xy}=0$.

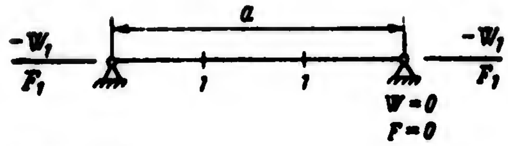


Fig. 179.

The first approximation: $h = \frac{1}{3}a$ (Fig. 179). In this instance, one obtains

$$\frac{8\sigma R}{3\pi E\delta}=\frac{9\left[1+2\left(\frac{\pi}{3n}\right)^2+\left(\frac{\pi}{3n}\right)^4\right]}{12(1-\mu^2)k^2}+\frac{k^2}{9\left[3+2\left(\frac{\pi}{3n}\right)^2+\left(\frac{\pi}{3n}\right)^4\right]},$$

where

$$n=\frac{c}{a},\quad a=k\sqrt{R\delta}.$$

The minimum of this expression based on k^2 will be

$$\frac{\sigma R}{E\delta}=0{,}715\sqrt{\frac{1+2\left(\frac{\pi}{3n}\right)^2+\left(\frac{\pi}{3n}\right)^4}{3+2\left(\frac{\pi}{3n}\right)^2+\left(\frac{\pi}{3n}\right)^4}}.$$

Table 10 presents the values of $\sigma R/E\delta$ depending on n.

Table 10.

n	1	2	3	8	15	∞
$\frac{\sigma R}{E\delta}$	0,592	0,478	0,444	0,418	0,414	0,413

From this table it is evident that the least critical stress is obtained for a strongly elongated panel in a circumferential direction.

Therefore, the problem about the size of the panel in a circumferential direction remains open, and the parameter k will be

$$k = \frac{3}{\sqrt{2\sqrt{3(1-\mu^2)}}}.$$

In examining the second approximation ($h = \frac{1}{5}a$, Fig. 180) it was established that

$$\frac{\sigma R}{E\delta} = 0,37 \quad \text{when} \quad k = 3.$$

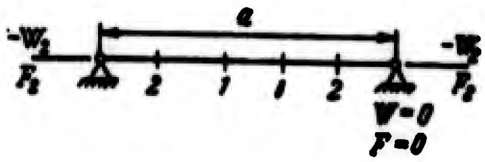


Fig. 180.

By means of extrapolation from the first two approximations one obtains

$$\left(\frac{\sigma R}{E\delta}\right)_{\text{внутр}} = 0,346E\frac{\delta}{R}. \qquad (11.25)$$

Under uniform compression of the cylindrical shell, as shown in § 49, with the solution of the problem by the method of finite differences, the coefficient in (11.25) was obtained equal to 0.3.

Let us compare the limiting values of critical forces for the case of uniform stress distribution at the ends and the stress according to the law of the cosine.

In the first case

$$Q_{кр} = 0,3E\frac{\delta^2}{R}\int_{-c/2}^{c/2} dy = 0,3E\frac{\delta^2 c}{R}.$$

In the second case

$$Q^{*}_{\text{кр}}=0,346E\frac{\delta^2}{R}\int_{-c/2}^{c/2}\cos\frac{\pi y}{c}\,dy=0,22E\frac{\delta^2 c}{R}.$$

From the comparison of these forces it is evident that the loading of the panel according to the law of the cosine is more hazardous than the loading with a uniform load.

By making use of the above given result, one can obtain formulas for the case of loading the panel according to Fig. 181 under the same boundary conditions along the contour, if during a solution of this problem one assumes that

$$w=W(x)\cos\frac{\pi y}{c},$$
$$\varphi=F(x)\cos\frac{\pi y}{c},$$

Then, the first original equation (11.24) remains unchanged, and the second will assume the form

$$F^{*}+DR\left[W^{IV}-2\left(\frac{\pi}{c}\right)^2W''+\left(\frac{\pi}{c}\right)^4W\right]+\frac{2RN_x^0}{3\pi}W''=0.$$

By comparing the coefficients with N_x^0 in these equations, one can see that it is 4 times less. The remaining coefficients in these equations coincide. Therefore, it is possible to make use of the formula (11.25), by increasing the coefficient 4 times in it. Thus

$$\sigma_{\text{кр}}=1,384E\frac{\delta}{R}.$$

The sum total critical force in this instance is

$$Q^{**}_{\text{кр}}=1,384E\frac{\delta^2}{2R}\int_{-c/2}^{c/2}\left(1-\cos\frac{2\pi y}{c}\right)dy=0,692E\frac{\delta^2 c}{R}.$$

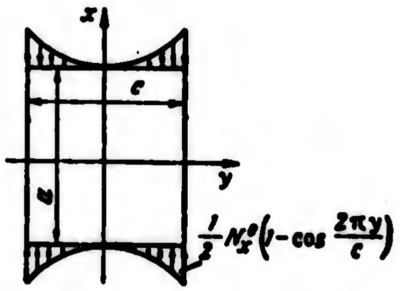


Fig. 181.

As can be seen from the last expression, the limiting force obtained is 2 times higher in comparison with a uniform load and more than 3 times higher in comparison with the loading according to the law of the cosine.

§ 55. Rigidity of a Cylindrical Panel Under the Action of a Concentrated Force

The problem dealing with the calculation of a cylindrical panel due to the effect of a concentrated force on it appears during the designing of the thin-walled constructions, which have hatchways with lids. The lids of the hatchways can be loaded with concentrated forces, transmitted by stringers or by any other elements of rigidity of the construction.

For the determination of the critical value of force P, acting on the panel (Fig. 182), one can use the method of the division of variables. We will consider that the panel is hinged supported and the additional stresses along its contour are equal to zero, i.e., $\sigma_x = 0$, $\sigma_y = 0$, $\sigma_{xy} = 0$.

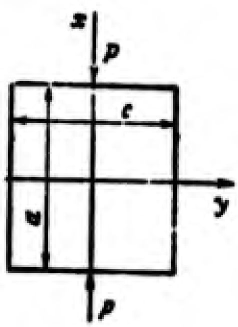


Fig. 182.

For the function of deflection and stresses we will assume that

$$w = W(y)\cos\frac{\pi x}{a},$$
$$\varphi = F(y)\cos\frac{\pi x}{a}.$$

By substituting these expressions in the original equations (11.5) and by replacing the derivatives for y with finite-difference expressions, we obtain

$$F_k\left[6+4\left(\frac{\pi b}{a}\right)^2+\left(\frac{\pi b}{a}\right)^4\right]-\left[4+2\left(\frac{\pi b}{a}\right)\right](F_n+F_m)+$$
$$+F_v+F_u+\left(\frac{\pi b}{a}\right)^2\frac{E\delta b^2}{R}W_k=0,$$
$$-F_k+\frac{D}{\left(\frac{\pi b}{a}\right)^2 b^2}\left\{W_k\left[6+4\left(\frac{\pi b}{a}\right)^2+\left(\frac{\pi b}{a}\right)^4\right]-\right.$$
$$\left.-\left[4+2\left(\frac{\pi b}{a}\right)^2\right](W_n+W_m)+W_v+W_u\right\}-RN_x^0W_k=0,$$

where b - pitch of the grid in the direction of axis y.

The first approximation: $b = \frac{1}{3}c$ (Fig. 183).

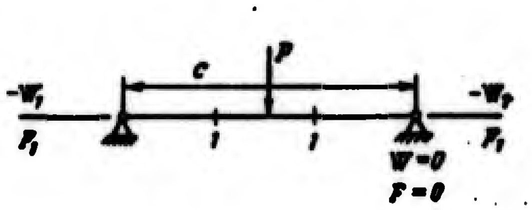


Fig. 183.

Let us consider the concentrated force P as evenly distributed along the length of the pitch $N_x^0 = P/b$. Then, the given equations for any point 1 will be

$$F_1\left[6+4\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4\right]-\left[4+2\left(\frac{\pi n}{3}\right)^2\right](0+F_1)+$$
$$+0+F_1-\frac{E\delta 2n^2k^2}{R}W_1=0,$$

$$-F_1+\left(\frac{3}{\pi h}\right)^2\frac{9D}{\delta n^2k^2}\left\{W_1\left[6+4\left(\frac{\pi n}{3}\right)^2+\left(\frac{n\pi}{3}\right)^4\right]-\right.$$

$$\left.-\left[4+2\left(\frac{\pi n}{3}\right)^2\right](0+W_1)+0-W_1\right\}-\frac{3P}{nk}\sqrt{\frac{R}{\delta}}\,W_1=0,$$

where

$$n=\frac{c}{a},\quad a=k\sqrt{R\delta}.$$

By excluding W_1 and F_1 from these equations, we obtain

$$3P\sqrt{\frac{R}{\delta}}=\frac{\pi^2E\delta n^5k^3}{81\left[3+2\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4\right]}+\frac{81D\left[1+2\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4\right]}{\pi^2\delta n^3k}.$$

Considering here k as a parameter, we will find the minimum P for k:

$$P=0{,}28E\delta^2\sqrt{\frac{\delta}{R}}\,\xi(n),$$

where

$$\xi(n)=\frac{1}{n}\sqrt[4]{\frac{\left[1+2\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4\right]^3}{3+2\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4}}.$$

Figure 184 represents a graph of function $\xi(n)$ depending on n. On the basis of this graph it is possible to write

$$P_{кр}\approx 0{,}3E\delta^2\sqrt{\frac{\delta}{R}}.$$

The problem about the sizes of a panel, at which one obtains the least critical force, in this case, is solved completely.

For the parameter of k we will have the expression

$$k=\frac{9}{\pi n}\sqrt{\frac{1}{6}\sqrt{\left[1+2\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4\right]\left[3+2\left(\frac{\pi n}{3}\right)^2+\left(\frac{\pi n}{3}\right)^4\right]}}.$$

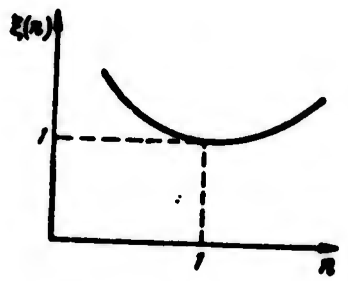


Fig. 184.

After the substitution here of $n = 1$ we obtain $k = 2.63$. Then

$$a=c=2{,}63\sqrt{R\delta}.$$

The second approximation: $b = \frac{1}{5}\,a$ (Fig. 185). In this instance

$$P_{кр}=0{,}3E\delta^{2}\sqrt{\frac{\delta}{R}}.$$

This approximation coincides in accuracy with the first one.

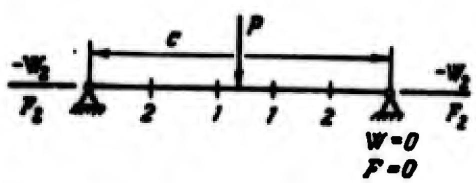


Fig. 185.

The sizes of the panel in this instance are equal: $a=c=3{,}5\sqrt{R\delta}$ and $(a=c)_{эксттр}=4\sqrt{R\delta}$.

§ 56. Rigidity of a Cylindrical Shell Under Torsion with the Account of the Effect of Tensile Stresses in the Axial and Circumferential Directions On It

In the solution of the given problem in a rectangular system of coordinates for the function of deflection it is necessary to take that expression which reflects the spiral form of wave formation following the loss of rigidity. The most suitable function will be

$$w = W(x)\cos\left(\frac{\lambda x}{R} - n\theta\right).$$

Similarly for the function of stresses let us assume that

$$\varphi = F(x)\cos\left(\frac{\lambda x}{R} - n\theta\right).$$

Let us set up the necessary derivatives from these functions and let us substitute them in the original equations (11.5). After the substitution we obtain

$$\begin{aligned}
&\frac{1}{E\delta}\left\{\left[F^{IV} - \frac{2F''}{R^2}(n^2+3\lambda^2) + \frac{F}{R^4}(n^2+\lambda^2)^2\right]\cos\left(\frac{\lambda x}{R} - n\theta\right) + \right.\\
&\left. + \left[\frac{4\lambda}{R^3}(\lambda^2+n^2)F' - \frac{4\lambda}{R}F'''\right]\sin\left(\frac{\lambda x}{R} - n\theta\right)\right\} - \\
&- \frac{1}{R}\left(W'' - \frac{\lambda^2}{R^2}W\right)\cos\left(\frac{\lambda x}{R} - n\theta\right) + \frac{2\lambda}{R}W'\sin\left(\frac{\lambda x}{R} - n\theta\right) = 0,\\
&\frac{1}{R}\left(F'' - \frac{1}{R^2}F\right)\cos\left(\frac{\lambda x}{R} - n\theta\right) - \frac{2}{R^2}F'\sin\left(\frac{\lambda x}{R} - n\theta\right) + \\
&+ D\left\{\left[W^{IV} - \frac{2}{R^2}(n^2+3\lambda^2)W'' + \frac{1}{R^4}(n^2+\lambda^2)^2 W\right]\cos\left(\frac{\lambda x}{R} - n\theta\right) + \right.\\
&\left. + \left[\frac{4\lambda}{R^3}(\lambda^2+n^2)W' - \frac{4\lambda}{R}W'''\right]\sin\left(\frac{\lambda x}{R} - n\theta\right)\right\} + \\
&+ N_x^0\left[\left(W'' - \frac{1}{R^2}W\right)\cos\left(\frac{\lambda x}{R} - n\theta\right) - \frac{2\lambda}{R}W'\sin\left(\frac{\lambda x}{R} - n\theta\right)\right] + \\
&+ \frac{2N_{xy}^0 n}{R}\left[W'\sin\left(\frac{\lambda x}{R} - n\theta\right) + \frac{\lambda}{R}W\cos\left(\frac{\lambda x}{R} - n\theta\right)\right] + \\
&+ \frac{N_y^0}{R^2}\left[-n^2 W\cos\left(\frac{\lambda x}{R} - n\theta\right)\right] = 0.
\end{aligned}$$

In these equations W and F - functions of only variable x. By making useoof the Kantorovich-Vlasov method, let us multiply each of these equations by $\cos\left(\frac{\lambda x}{R} - n\theta\right) d\theta$ and let us integrate within the limits of 0-2π. As a result of these operations we will have

$$F^{IV}-\frac{2}{R^2}(n^2+3\lambda^2)F''+\frac{1}{R^4}(n^2+\lambda^2)^2F-\frac{E\delta}{R}\left(W''-\frac{\lambda^2}{R^2}W\right)=0,$$

$$F''-\frac{\lambda^2}{R^2}F+DR\left[W^{IV}-\frac{2}{R^2}(n^2+3\lambda^2)W''+\frac{1}{R^4}(n^2+\lambda^2)^2W\right]+$$

$$+RN_x^0\left(W''-\frac{\lambda^2}{R^2}W\right)+\frac{2N_{xy}^0\lambda n}{R}W-\frac{N_y^0n^2}{R}W=0.$$

Here by strokes the derivatives with respect to x are designated.

Let us write down the last equations in the finite differences:

$$F_ka_1-a_2(F_i+F_l)+F_t+F_s-\frac{E\delta h^2}{R}(W_i+W_l-a_3W_k)=0,$$

$$F_i+F_l-a_3F_k+\frac{DR}{h^2}[W_ka_1-a_2(W_i+W_l)+W_t+W_s]+$$

$$+RN_x^0(W_i+W_l-a_3W_k)+2RN_{xy}^0\left(\frac{h}{R}\right)^2\lambda nW_k-RN_y^0n^2\left(\frac{h}{R}\right)^2W_k=0.$$

Designated here is:

$$a_1=6+4\left(\frac{h}{R}\right)^2(n^2+3\lambda^2)+\left(\frac{h}{R}\right)^4(n^2+\lambda^2)^2,$$

$$a_2=4+2\left(\frac{h}{R}\right)^2(n^2+3\lambda^2),$$

$$a_3=2+\left(\frac{h}{R}\right)^2\lambda^2.$$

Let us formulate the boundary conditions for the functions of F and W. With the selected expressions for the functions ϕ and w the boundary conditions for the rigid sealing can be written the simplest. In this instance at the ends of the shell it should be

$$w=W(x)\cos\left(\frac{\lambda x}{R}-n\theta\right)=0,$$

$$\frac{\partial w}{\partial x}=W'(x)\cos\left(\frac{\lambda x}{R}-n\theta\right)-\frac{\lambda}{R}W\sin\left(\frac{\lambda x}{R}-n\theta\right)=0$$

or $$W_k=0,\quad W'=\frac{W_l-W_i}{2h}=0,\quad W_i=W_l.$$

For the function of stresses ϕ at the ends of the shell we will assume that the condition that the additional forces N_x and N_{xy} after the loss of rigidity are equal to zero:

$$N_x = \frac{\partial^2 \varphi}{\partial y^2} = -\frac{n^2}{R^2} F \cos\left(\frac{\lambda x}{R} - n\theta\right) = 0,$$

$$N_{xy} = -\frac{\partial^2 \varphi}{\partial x \, \partial y} = -\frac{n}{R} F' \sin\left(\frac{\lambda x}{R} - n\theta\right) - \frac{n\lambda}{R^2} F \cos\left(\frac{\lambda x}{R} - n\theta\right) = 0.$$

Hence

$$F_k = 0, \quad F' = \frac{F_l - F_l}{2h} = 0, \quad F_l = F_l.$$

The first approximation: $h = l/2$ (Fig. 186):

$$F_1 a_1 - a_2(0+0) + F_1 + F_1 - \frac{E\delta h^2}{R}(0 + 0 - a_3 W_1) = 0,$$

$$0 + 0 - a_3 F_1 + \frac{DR}{h^2}[W_1 a_1 - a_2(0+0) + W_1 + W_1] -$$

$$-N_x^0 R(0 + 0 - a_3 W_1) - \frac{M_{\text{кр}}}{\pi R}\left(\frac{h}{R}\right)^2 \lambda n W_1 + N_y^0 R n^2 \left(\frac{h}{R}\right)^2 W_1 = 0.$$

Designated here is: $-N_x^0$ - axial tensile force; $-N_y^0$ - annular tensile force; $M_{\text{кр}} = 2\pi R^2 N_{xy}^0$ - moment, which twists the shell.

By excluding function F_1 from these equations and considering that $W_1 \neq 0$, we obtain the expression for $M_{\text{кр}}$ as a first approximation:

$$\frac{M^1_{\text{кр}}}{\pi E \delta R^2} = \frac{\left[2 + \left(\frac{l}{2R}\right)^2 \lambda^2\right]^2}{\left[8 + 4\left(\frac{l}{2R}\right)^2 (n^2 + 3\lambda^2) + \left(\frac{l}{2R}\right)^4 (n^2 + \lambda^2)^2\right] n\lambda} +$$

$$+ \frac{4R^2\delta^2 \left[8 + 4\left(\frac{l}{2R}\right)^2 (n^2 + 3\lambda^2) + \left(\frac{l}{2R}\right)^4 (n^2 + \lambda^2)^2\right]}{3(1 - \mu^2) n \lambda l^4} +$$

$$+ \frac{4 N_x^0 R^2 \left[2 + \left(\frac{l}{2R}\right)^2 \lambda^2\right]}{E\delta n \lambda l^2} + \frac{N_y^0 n}{E\delta\lambda}. \qquad (11.26)$$

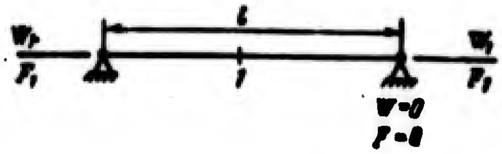


Fig. 186.

Having completed the analogous computations for the second approximation (Fig. 187), $h = \frac{1}{3} l$, we obtain the expression for $N_{кр}$ in the second approximation:

$$\frac{N_{кр}^{II}}{\pi E\delta R^2} = \frac{\left[1+\left(\frac{l}{3R}\right)^2\lambda^2\right]^2}{\left[3+2\left(\frac{l}{3R}\right)^2(n^2+3\lambda^2)+\left(\frac{l}{3R}\right)^4(n^2+\lambda^2)^2\right]n\lambda} + \frac{27R^2\delta^2\left[3+2\left(\frac{l}{3R}\right)^2(n^2+3\lambda^2)+\left(\frac{l}{3R}\right)^4(n^2+\lambda^2)^2\right]}{4(1-\mu^2)n\lambda l^4} + \frac{9N_x^0R^2\left[1+\left(\frac{l}{3R}\right)^2\lambda^2\right]}{E\delta n\lambda l^2} + \frac{N_y^0 R}{E\delta\lambda}. \tag{11.27}$$

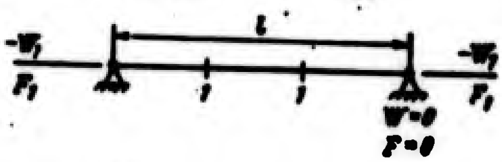


Fig. 187.

For the analysis of expressions (11.26) and (11.27) we will divide the shells into two classes - long and average ones.

In the case of long shells one can consider that $\left(\frac{l}{2R}\right)^2\lambda^2 \gg 1$ and $n^2 >> \lambda^2$. Then, the expression (11.26) can be approximately written in the form

$$\frac{N_{кр}^{I}}{\pi E\delta R^2} = \frac{\lambda^3}{n^5} + A\frac{n^3}{\lambda} + B\frac{\lambda}{n} + C\frac{n}{\lambda}, \tag{11.28}$$

where

$$A = \frac{\delta^2}{12(1-\mu^2)R^2}, \quad B = \frac{N_x^0}{E\delta}, \quad C = \frac{N_y^0}{E\delta}.$$

Let us examine a case, when $N_x^0 = N_y^0 = 0$. Then

$$\frac{M_{\text{кр}}^1}{\pi E\delta R^2} = \frac{\lambda^3}{n^5} + A\frac{n^3}{\lambda}.$$

Let us find the minimum of this expression with respect to λ:

$$\frac{\partial}{\partial\lambda}\left(\frac{M_{\text{кр}}^1}{\pi E\delta R^2}\right) = \frac{3\lambda^2}{n^5} - A\frac{n^3}{\lambda^2} = 0,$$

whence

$$\lambda = \frac{n^2\sqrt{\frac{\delta}{R}}}{\sqrt{6}\sqrt[4]{1-\mu^2}}.$$

Then when $n = 2$ we will have the following least value of $M_{\text{кр}}^1$:

$$M_{\text{кр}}^1 = 1{,}7E\delta^2 R\frac{\sqrt{\frac{\delta}{R}}}{\sqrt[4]{(1-\mu^2)^3}}. \qquad (11.29)$$

This expression for the critical value of the torsional moment is 14% greater than the value $M_{\text{кр}}$, which is cited in published works [8], [26].

Let us examine the case, when $N_x^0 \neq 0$, $N_y^0 \neq 0$. Let us find the minimum of the expression (11.28) with respect to parameters λ and n, whereby we will differentiate this expression with respect to λ and n and equate the derivatives to zero. In this case we obtain the two following equations:

$$3\lambda^4 - An^8 + Bn^4\lambda^2 - Cn^6 = 0,$$
$$5\lambda^4 - 3An^8 + Bn^4\lambda^2 - Cn_4^6 = 0.$$

By solving these equations, we will find that

$$\lambda = \frac{C\sqrt[4]{A}}{2A + B\sqrt{A}}, \quad n = \frac{\sqrt{C}}{\sqrt{2A + B\sqrt{A}}}.$$

By substituting the found values for λ and n in the expression (11.28), we obtain

$$M^{1}_{\text{кр}}=\frac{4{,}75}{\sqrt[4]{1-\mu^2}}E\delta R\sqrt{R\delta}\sqrt{\frac{N_y^0}{E\delta}}\sqrt{1+\frac{\sqrt{3(1-\mu^2)}\,N_x^0 R}{E\delta}}. \qquad (11.30)$$

It is easy to verify that from the expression (11.27) of the second approximation the results of the analysis will agree with the results obtained from the first approximation (11.26). Therefore, for long shells it is inexpedient to go through the investigations of the second approximation.

From the formula (11.30) it is evident that it is inapplicable at any values of annular forces of N_y^0. From it one cannot obtain the formula of the critical forces under torsion of the shell, not loaded by a force of N_y^0. This occured due to the fact that an approximate expression was used for the change in the curvature in a circumferential direction. Therefore, it is necessary to find that least value of N_y^0, the beginning of which one can make use of this formula. For this let us equate the right sides of the formulas (11.29) and (11.30) and from this equality when $N_x^0 = 0$ we will find that the force N_y^0 should satisfy the condition

$$N_y^0>\frac{0{,}128E\delta\left(\frac{\delta}{R}\right)^2}{1-\mu^2}.$$

If $N_y^0 = qR$, where q - intensity of the pressure on the lateral surface of the shell, then

$$q>\frac{0{,}128E}{1-\mu^2}\left(\frac{\delta}{R}\right)^2. \qquad (11.31)$$

That pressure, the beginning of which should make use of the formula (11.30), can be determined by this condition.

From the condition (11.31) it is evident that the least pressure for a thin shell will be very small. This pressure for steel and aluminum is presented in Table 11.

Table 11.

Material	E, kgf/cm^2	q_{min}, kgf/cm^2	
		$\frac{\delta}{R}=0{,}001$	$\frac{\delta}{R}=0{,}01$
Steel	$2\cdot 10^6$	0,00028	0,28
Aluminum	$7\cdot 10^5$	0,0000985	0,0985

If force N_y^0 is less in the determined formula (11.31) or if it is entirely absent, but still with an axial tensile force N_x^0 acting on it, then one can make use of the expression (11.29), multiplied by

$$\sqrt{1+\frac{\sqrt{3(1-\mu^2)}\,N_x^0 R}{E\delta^2}},$$

i.e.,

$$M^1_{кр}=\frac{1{,}7}{\sqrt[4]{(1-\mu^2)^3}}E\delta^2 R\sqrt{\frac{\delta}{R}}\times \times\sqrt{1+\frac{\sqrt{3(1-\mu^2)}\,N_x^0 R}{E\delta^2}}. \qquad (11.32)$$

Let us examine the shells of average length. We will consider $n^2\gg\lambda^2,\ \left(\frac{l}{2R}\right)^4 n^4\gg\left[8+4\left(\frac{l}{2R}\right)^2\lambda^2\right]$. Then, when $N_x^0 = N_y^0 = 0$ the expression for the first approximation can be written in the form

$$\frac{M^1_{кр}}{\pi E\delta R^2}=\frac{4+4\left(\frac{l}{2R}\right)^2\lambda^2+\left(\frac{l}{2R}\right)^4\lambda^4}{\left(\frac{l}{2R}\right)^4 n^4\lambda}+\frac{\delta^2 n^3}{12(1-\mu^2)R^2\lambda}. \qquad (11.33)$$

For the determination of the least value $M^1_{кр}$ let us write down the derivatives from this expression in terms of λ and n and equate them to zero, having obtained in this case two equations:

$$3\left(\frac{l}{2R}\right)^4\lambda^4+4\left(\frac{l}{2R}\right)^2\lambda^2-4-\frac{a}{12}n^8=0,$$
$$\left(\frac{l}{2R}\right)^4\lambda^4+4\left(\frac{l}{2R}\right)^2\lambda^2+4-\frac{a}{20}n^8=0,$$

where

$$a=\frac{\delta^2\left(\frac{l}{2R}\right)^4}{(1-\mu^2)R^2}.$$

From these equations let us find

$$\lambda=\frac{4R}{l}, \quad n=\sqrt[8]{\frac{3600}{5a}}.$$

By substituting the values λ and n in (11.33), we obtain the formula for the least value of torsional moment

$$M^1_{\text{кр}}=\frac{3{,}58}{(1-\mu^2)^{3/8}}\sqrt{\frac{R}{l}}\sqrt[4]{\frac{\delta}{R}}E\delta^2R.$$

The investigation of the expression for the second approximation leads to an analogous formula; therefore, it is not given here.

Now, let us turn to the examination of a case, when $N_x^0 \neq 0$, $N_y^0 \neq 0$, and let us limit ourselves to the examination of the expression for the first approximation, since from the previous one it is clear that this approximation yields entirely reliable results:

$$\frac{M^1_{\text{кр}}}{\pi E\delta R^2}=\frac{4+4\left(\frac{l}{2R}\right)^2\lambda^2+\left(\frac{l}{2R}\right)^4\lambda^4}{\left(\frac{l}{2R}\right)^4 n^5\lambda}+\frac{\delta^2 n^3}{12(1-\mu^2)R^2\lambda}+$$
$$+\frac{4N_x^0R^2\left[2+\left(\frac{l}{2R}\right)^2\lambda^2\right]}{E\delta l^2 n\lambda}+\frac{N_y^0 n}{E\delta\lambda}. \qquad (11.34)$$

Let us differentiate this expression according to λ and n and let us equate the derivatives to zero, obtaining the two following equations for the determination of λ and n, which convert the expression (11.34) to the minimum:

$$3x^2+4x\left(1+\frac{1}{16}\beta n^4\right)-4-\frac{1}{2}\beta n^4-\frac{1}{16}\gamma n^6-\frac{\alpha}{192}n^8=0,$$
$$x^2+4x\left(1+\frac{1}{80}\beta n^4\right)+4+\frac{1}{10}\beta n^4-\frac{1}{80}\gamma n^6-\frac{\alpha}{320}n^8=0.$$

Here, the designations are introduced:

$$\beta=\frac{N_x^0 l^2}{E\delta R^2},\quad \alpha=\frac{\delta^2 l^4}{(1-\mu^2)R^6},\quad \gamma=\frac{N_y^0 l^4}{E\delta R^4},\quad x=\left(\frac{l}{2R}\right)^2\lambda^2.$$

By solving this system, we obtain the following equations for the determination of parameters λ and n:

$$a_5y^5+a_4y^4+a_3y^3-a_2y^2+a_1y+a_0=0,\quad \lambda=\frac{2R}{l}\times$$
$$\times\sqrt{\sqrt{1+\frac{7}{40}\beta n^4+\frac{3\gamma}{160}n^6+\frac{\alpha}{480}\left(1+\frac{27}{40}\frac{\beta^2}{\alpha}\right)n^8}-\left(1+\frac{3}{80}\beta n^4\right)},$$

where

$$y=n^2,\quad a_0=\gamma,\quad a_1=\frac{1}{4}(\alpha-3\beta^2),\quad a_2=\frac{3\gamma\beta}{40},$$
$$a_3=\frac{1}{480}\left(\alpha\beta-3{,}7\beta^3-\frac{3}{8}\gamma^2\right),\quad a_4=\frac{\alpha\gamma}{3840},\quad a_5=\frac{\alpha}{15\,300}\left(\beta^2-\frac{1}{3}\alpha\right).$$

Thus, the problem dealing with the torsion of a shell of an average length with forces N_x^0 and N_y^0 existing in it results in a solution of the equation of the fifth degree relative to n^2. By finding the parameters λ and n with respect to the assigned values α, β, and γ, let us substitute them in the equation (11.34), which also provides the least value of torsional moment. Let us determine the limits of applicability of the obtained formulas.

In the absence of forces N_x^0 and N_y^0 these formulas are obtained:

a) for the long shells

$$M_{\text{кр}}=1{,}7E\delta^2R\frac{\sqrt{\frac{\delta}{R}}}{\sqrt[4]{(1-\mu^2)^3}}.$$

b) for shells of average length

$$M'_{кр}=\frac{3{,}58}{(1-\mu^2)^{5/8}}\sqrt{\frac{R}{l}}\sqrt[4]{\frac{\delta}{R}}E\delta^2R.$$

On the boundary between the average length and long shells one should find that

$$M_{кр}=M'_{кр}.$$

From this condition we will find that if the length of the shell satisfies the condition

$$l<4{,}3R\sqrt{\frac{R}{\delta}},$$

then the shell will belong to the class of average lengths. If $l>4{,}3R\sqrt{\frac{R}{\delta}}$, then to the shell that is long.

If forces N_x^0 and N_y^0 act on the shell, then the limits of applicability of the obtained formulas in the enclosed type will not work. In this instance it is necessary to establish the limits of applicability of the expressions for $M_{кр}$ using actual numbers based on the given right sides of the expressions (11.30) or (11.32) and (11.34) by means of equating, and select that length of l, which divides all of the shells into two classes. In this manner the found length l will represent the longest shell of the class of shells of average length at assigned α, β, γ. For this shell the critical moment, calculated by the formulas (11.31) or (11.32) and (11.34), will be one and the same.

The lower limit of applicability of the formula for shells of average length can be determined from the condition

$$\tau_{кр}<\tau_s,$$

where τ_s - yield point of the material of the shell under shear ($\tau_s \approx 0.6\sigma_s$).

In conclusion let us note that the solution of the given problem with a hinged support of the ends leads to the results, obtained above.

§ 57. Rigidity of the Compressed Zone of a Circular Cylindrical Shell, Reinforced by a Ring During Its Loading with a Concentrated Axial Force

In § 38 the approximate solution for a cylindrical shell under a load by an axial concentrated force was presented. There, the formulas for the membrane stresses in the elongated and compressed zones of the shell were obtained. In this section we will present the approximate solution for the determination of the critical force, at which the loss in rigidity of the compressed zone of the shell will take place. In this case let us propose that the investigated shell is under the action of internal pressure. Then, the formulas for the membrane forces of the compressed zone of such a shell will have the form

$$N_x^0 = \frac{PE_2\delta_2}{\pi R E_1 \delta_1} \sum_{n=1}^{\infty} \frac{\left(1 + a\frac{nx}{R}\right) e^{-\frac{nx}{R}} \cos n\theta}{A} - \sigma_M \delta_2,$$

$$N_y^0 = \frac{PE_2\delta_2}{\pi R E_1 \delta_1} \sum_{n=1}^{\infty} \frac{\left(b - a\frac{nx}{R}\right) e^{-\frac{nx}{R}} \cos n\theta}{A} - \sigma_K \delta_2, \quad \text{a)}$$

$$N_{xy}^0 = -\frac{PE_2\delta_2}{\pi R E_1 \delta_1} \sum_{n=1}^{\infty} \frac{\left[\frac{nx}{R} + \frac{1}{k}\left(1 - \frac{nx}{R}\right)\right] e^{-\frac{nx}{R}} \sin n\theta}{A},$$

where σ_M, σ_K - membrane axial and annular stresses from the internal pressure.

It was experimentally established that the pit after the loss of rigidity forms directly over the force as well as according to a form close to being square (Fig. 188).

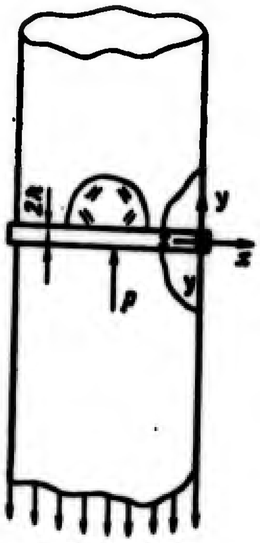


Fig. 188.

The problem on hand can be solved, by making use of the apparatus of finite differences. In this case using the square grid of V. Z. Vlasov's equation one arrives at the form

$$
\begin{gathered}
20F_k-8(F_i+F_l+F_m+F_n)+2(F_p+F_q+F_r+F_o)+F_t+F_s+ \\
+F_v+F_u=\frac{E\delta h^2}{R}(W_i+W_l-2W_k), \\
F_i+F_l-2F_k+\frac{DR}{h^2}[20W_k-8(W_i+W_l+W_n+W_m)+ \\
+2(W_p+W_q+W_r+W_o)+W_t+W_s+W_v+W_u]+ \\
+RN_x^0(W_i+W_l-2W_k)+RN_y^0(W_n+W_m-2W_k)+ \\
+\frac{1}{2}RN_{xy}^0(W_o-W_p+W_r-W_q)=0.
\end{gathered}
\qquad (11.35)
$$

Here h - pitch of the grid; N_x^0, N_y^0, N_{xy}^0 - membrane forces, provided by the formula (a).

We will assume that the boundary conditions along the contour of the pit acquire the following form:

$$
\begin{gathered}
w_k=W_k=0,\ \left(\frac{\partial w}{\partial x}\right)_k=\frac{W_l-W_i}{2h}=0,\ \left(\frac{\partial w}{\partial y}\right)_k=\frac{W_m-W_n}{2h}=0, \\
\sigma_x=\frac{\partial^2\varphi}{\partial y^2}=0,\ \sigma_y=\frac{\partial^2\varphi}{\partial x^2}=0,\ \sigma_{xy}=-\frac{\partial^2\varphi}{\partial x\,\partial y}=0.
\end{gathered}
$$

After the integration of these expressions we will find that

$$\frac{\partial\varphi}{\partial y}=C_1;\quad \varphi=C_1y+C_2;$$
$$\frac{\partial\varphi}{\partial x}=B_1;\quad \varphi=B_1x+B_2;$$
$$\frac{\partial\varphi}{\partial x}=D_1;\quad \varphi=D_1x+D_2.$$

By setting all the constants of the integration here equal to zero (inasmuch as they do not affect the stress condition), we will obtain values of the function of stress for the contour

$$\varphi_k=F_k=0,\quad \frac{\partial\varphi}{\partial x}=\frac{F_e-F_l}{2h}=0,\quad \frac{\partial\varphi}{\partial y}=\frac{F_m-F_n}{2h}=0.$$

Thus, for the contour and postcontour points, we will have the values

$$W_k=0,\qquad F_k=0,$$
$$W_i=W_l,\qquad F_i=F_l,$$
$$W_m=W_n,\qquad F_m=F_n,$$

which correspond to the absence of angles of rotation and of membrane stresses at points of the contour of the pit after the loss of rigidity.

On account of the fact that in the second equation (11.35) the three last components are variables, it is hardly possible to obtain a solution with a rather fine grid, because the calculated formula for the critical force takes on extreme complexity. Therefore, we will limit ourselves to the examination of only one approximation (Fig. 189). The pitch of the accepted grid $h = l/3$, where l - size of the side of the pit.

In order to get the solution of the problem it is necessary to formulate the equations (11.35) for any pair of points 1 and 2, since in each of these points one should take the corresponding values of forces N_x^0, N_y^0, N_{xy}^0. By not reducing the intermediate

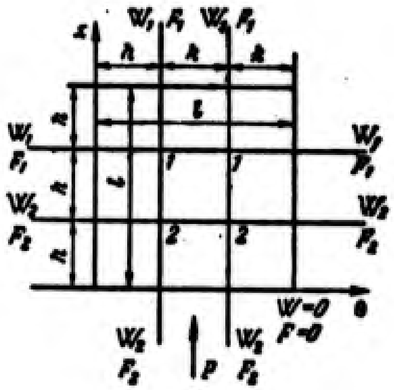


Fig. 189.

computations, associated with the composition of the equations for points 1 and 2, let us write out the final result of this solution:

$$\frac{720P}{\pi E_1 \delta_1 E_2} = k_0^4\left(B_1\frac{F_1+F_4}{F_1F_4-F_2F_3}+B_2\frac{F_2+F_3}{F_1F_4-F_2F_3}\right)\pm$$
$$\pm\sqrt{k_0^8\left(B_1\frac{F_1+F_4}{F_1F_4-F_2F_3}+B_2\frac{F_2+F_3}{F_1F_4-F_2F_3}\right)^2-\frac{1440B_3+1080k_0^4}{F_1F_4-F_2F_3}}. \quad (11.36)$$

Designated here is:

$$B_1=\frac{99}{k_0^2}\left[\frac{1}{6}+\frac{60}{(1-\mu^2)k_0^4}\right]+\frac{360R}{\delta_2E_2k_0^4}(3\sigma_м+2\sigma_к),$$
$$B_2=\frac{9}{k_0^2}\left[\frac{1}{6}-\frac{60}{(1-\mu^2)k_0^4}\right]-\frac{360R\sigma_к}{\delta_2E_2k_0^4},$$
$$B_3=\frac{96\,000}{(1-\mu^2)k_0^4}+\frac{549}{(1-\mu^2)}+\frac{198k_0^2}{\delta_2E_2}\left[\frac{1}{6}+\frac{60}{(1-\mu^2)k_0^4}\right](3\sigma_м+$$
$$+2\sigma_к)+\frac{18Rk_0^2}{\delta_2}\left[\frac{1}{6}-\frac{60}{(1-\mu^2)k_0^4}\right]\frac{\sigma_к}{E_2}+\frac{360R^2}{\delta_2^2}\times$$
$$\times\left[\left(\frac{3\sigma_м+2\sigma_к}{E_2}\right)^2-\left(\frac{\sigma_к}{E_2}\right)^2\right].$$
$$F_1=4f_1+2f_2-f_3-0{,}5f_4,$$
$$F_2=-2f_1+2f_3+f_5+f_6,$$
$$F_3=2f_1+f_2-2f_3-f_4,$$
$$F_4=-f_1+4f_3+2f_5+0{,}5f_6,$$
$$f_1=\sum_{n=1}^{\infty}\frac{\left(1+a\frac{nl}{3R}\right)e^{-\frac{nl}{R}}\cos\frac{n\varphi_0}{3}}{A},$$

$$f_2=\sum_{n=1}^{\infty}\frac{\left(1-\frac{2}{k}-a\frac{nl}{3R}\right)e^{-\frac{nl}{3R}}\cos\frac{n\theta_0}{3}}{A},$$

$$f_3=\sum_{n=1}^{\infty}\frac{\left(1+a\frac{2nl}{R}\right)e^{-\frac{2nl}{3R}}\cos\frac{n\theta_0}{3}}{A},$$

$$f_4=\sum_{n=1}^{\infty}\frac{\left[\frac{2nl}{3R}+\frac{1}{k}\left(1-\frac{2nl}{3R}\right)\right]e^{-\frac{2nl}{R}}\sin\frac{n\theta_0}{3}}{A},$$

$$f_5=\sum_{n=1}^{\infty}\frac{\left(1-\frac{2}{k}-a\frac{2nl}{3R}\right)e^{-\frac{2nl}{3R}}\cos\frac{n\theta_0}{3}}{A},$$

$$f_6=\sum_{n=1}^{\infty}\frac{\left[\frac{nl}{3R}+\frac{1}{k}\left(1-\frac{nl}{3R}\right)\right]e^{-\frac{nl}{3R}}\sin\frac{n\theta_0}{3}}{A},$$

$$\frac{nl}{R}=nk_0\sqrt{\frac{\delta_2}{k}},\quad \theta_0=0{,}5k_0\sqrt{\frac{\delta_2}{k}},\quad l=k_0\sqrt{R\delta_2},\quad a=1-\frac{1}{k},$$

$$k=\frac{1-\mu+\frac{2RW}{nJ}}{2+\frac{(1-\mu)RW}{nJ}},$$

$$A=1+\frac{E_2\delta_2}{E_1\delta_1}+\frac{2JE_0n^3}{E_1\delta_1R^3}\left(1-\frac{1-\mu}{2k}\right)-\frac{hn}{Rk}\left(1-\frac{E_2\delta_2}{E_1\delta_1}\right).$$

$2h$, J, W - height, moment of inertia and moment of rigidity of the ring (see Fig. 188); E_1, δ_1 - Young's modulus and the thickness of elongated zone of the shell; E_2, δ_2 - Young's modulus and the thickness of the compressed zone of the shell; E_0 - Young's modulus of the material of the frame.

If the material of the shell in the compressed and elongated zones, as well as that of the supporting ring is identical, then in the formula (11.36) one should assume that $E_1 = E_2 = E_0 = E$. The same also pertains to the thicknesses of the walls: if they are identical, then $\delta_1 = \delta_2 = \delta$.

The order of determination of the least value of the critical force is as follows. Let us assign a size to the side of the dent l (or k_0) and let us calculate the sums f_1-f_6, F_1-F_4, and then let us determine $P_{кр}$. Such a calculation needs to be repeated for several values of l (or k_0), until one no longer finds the least value of $P_{кр}$.

Figure 190 represents the results of the determination of the critical force for a construction of cylindrical shell with an existing internal pressure and without one. Tests were also conducted on this shell under laboratory conditions, as a result of which it was established that the formula (11.36) gives a value of the critical force, overrated by 1.3 times. This should be expected, because one approximation for the solution of such a problem under a complex strained state is insufficient. However, one can make use of this formula, apparently, in the calculations for a very approximate estimation of the bearing capacity of a shell, having in mind that it gives overrated values.

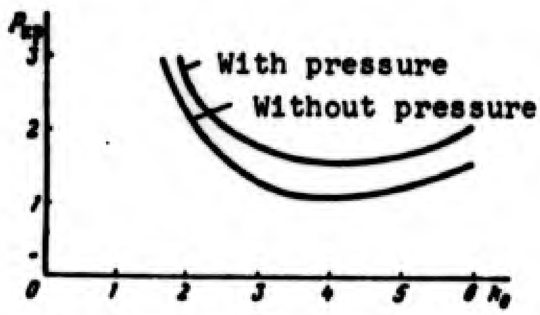


Fig. 190.

CHAPTER XII

RIGIDITY OF SPHERICAL AND ELLIPSOIDAL SHELLS

§ 58. Equations of the Local Loss in Rigidity of Spherical Shells in Various Forms. The Rigidity of Spherical Segments

Proceeding on the basis of the general equations of V. Z. Vlasov where in the case of a sphere under a uniform external pressure, they assume the form $\left(R_1=R_2=R,\ N_x^0=N_y^0=\frac{1}{2}qR\right)$,

$$\frac{1}{E\delta}\nabla^2\nabla^2\varphi=-\frac{1}{R}\nabla^2 w,$$

$$\frac{1}{R}\nabla^2\varphi+D\nabla^2\nabla^2 w+\frac{1}{2}qR\nabla^2 w=0, \qquad (12.1)$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

In the beginning let us examine the axisymmetrical forms of the loss in rigidity of spherical segments. By expressing Laplace's ∇^2 operator in the form

$$\nabla^2=\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr},$$

we will obtain the following entry of the system (12.1):

$$\frac{d^4\varphi}{dr^4}+\frac{2}{r}\frac{d^3\varphi}{dr^3}-\frac{1}{r^2}\frac{d^2\varphi}{dr^2}+\frac{1}{r^3}\frac{d\varphi}{dr}-\frac{E\delta}{R}\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)=0,$$

$$\frac{d^2\varphi}{dr^2}+\frac{1}{r}\frac{d\varphi}{dr}+DR\left(\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}\right)+$$
$$+\frac{1}{2}qR\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)=0. \qquad (12.2)$$

For the bending moments and membrane forces there we will have the expressions

$$M_x = -D(\chi_x + \mu\chi_y), \quad M_y = -D(\chi_y + \mu\chi_x),$$
$$N_x = \frac{\partial^2\varphi}{\partial y^2}, \quad N_y = \frac{\partial^2\varphi}{\partial x^2}.$$

In the case of an axisymmetrical form of loss in rigidity of a spherical shell these expressions take the form

$$\begin{aligned} M_x \sim M_r &= -D\left(\frac{d^2 w}{dr^2} + \mu\frac{1}{r}\frac{dw}{dv}\right), \\ M_y \sim M_\theta &= -D\left(\frac{1}{r}\frac{dw}{dr} + \mu\frac{d^2 w}{dr^2}\right), \\ N_x \sim N_r &= \frac{1}{r}\frac{d\varphi}{dr}, \\ N_y \sim N_\theta &= \frac{d^2\varphi}{dr^2}. \end{aligned} \qquad (12.3)$$

Let us write donw the equations (12.2)-(12.3) as finite differences:

$$\left.\begin{aligned} & F_k\left[6+\left(\frac{h}{r}\right)^2\right] - F_l\left[4+2\left(\frac{h}{r}\right)+\left(\frac{h}{R}\right)^2+\frac{1}{2}\left(\frac{h}{r}\right)^3\right] - \\ & \quad - F_l\left[4-2\left(\frac{h}{r}\right)+\left(\frac{h}{r}\right)^2+\frac{1}{2}\left(\frac{h}{r}\right)^3\right] + \\ & \quad + F_l\left(1+\frac{h}{r}\right) + F_s\left(1-\frac{h}{r}\right) - \\ & \quad - \frac{E\delta h^2}{R}\left[W_l\left(1+\frac{h}{2r}\right) + W'_l\left(1-\frac{h}{2R}\right) - 2W'_k\right] = 0, \\ & F_l\left(1+\frac{h}{2r}\right) + F_l\left(1-\frac{h}{2r}\right) - 2F_k + \frac{DR}{h^2}\left\{W_k\left[6+2\times\right.\right. \\ & \quad \left.\times\left(\frac{h}{r}\right)^2\right] - W_l\left[4+2\left(\frac{h}{r}\right)+\left(\frac{h}{r}\right)^2-\frac{1}{2}\left(\frac{h}{r}\right)^3\right] - \\ & \quad - W'_l\left[4-2\left(\frac{h}{r}\right)+\left(\frac{h}{r}\right)^2+\frac{1}{2}\left(\frac{h}{r}\right)^3\right] + \\ & \quad \left. + W'_l\left(1+\frac{h}{r}\right) + W_s\left(1-\frac{h}{r}\right)\right\} + \frac{1}{2}qR^2\times \\ & \quad \times\left[W'_l\left(1+\frac{h}{2r}\right) + W_l\left(1-\frac{h}{2r}\right) - 2W_k\right] = 0, \\ & M_r = -\frac{D}{h^2}\left[W'_l\left(1+\frac{\mu h}{2r}\right) + W_l\left(1-\frac{\mu h}{2r}\right) - 2W_k\right], \\ & \quad \sigma_r = \frac{1}{2hr}\left(F_l - \dot{F}_l\right). \end{aligned}\right\} \qquad (12.4)$$

Here h - pitch of the grid.

In order to write the original equations (12.2) as finite differences at point $r = 0$, it is first necessary to find the limits of certain differential expressions when $r = 0$. By applying the l'Hopital rule, we will obtain

$$\lim\left(\frac{\frac{d^3\varphi}{dr^3}}{r}\right)_{r\to 0}=\frac{d^4\varphi}{dr^4},$$

$$\lim\left(\frac{\frac{dw}{dr}}{r}\right)_{r\to 0}=\frac{d^2w}{dr^2},$$

$$\lim\left(\frac{\frac{d^2\varphi}{dr^2}}{r^2}\right)_{r\to 0}=\frac{1}{2}\frac{d^4\varphi}{dr^4},$$

$$\lim\left(\frac{\frac{d\varphi}{dr}}{r^3}\right)_{r\to 0}=\frac{1}{6}\frac{d^4\varphi}{dr^4}.$$

Then the original equations (12.2) for point $r = 0$, will assume the form

$$\frac{d^4\varphi}{dr^4}-\frac{3E\delta}{4R}\frac{d^2w}{dr^2}=0,$$

$$\frac{d^2\varphi}{dr^2}+\frac{4DR}{3}\frac{d^4w}{dr^4}+\frac{1}{2}qR^2\frac{d^2w}{dr^2}=0,$$

or as finite differences

$$6F_k-4F_l-4F_i+F_t+F_s-\frac{3E\delta h^2}{4R}(W_l+W_i-2W_k)=0,$$

$$F_l+F_i-2F_k+\frac{4DR}{3h^2}(6W_k-4W_l-4W_i+W_t+W_s)+$$

$$+\frac{1}{2}qR^2(W_l+W_i-2W_k)=0. \qquad (12.5)$$

Let us use the equations (12.4) and (12.5) for the determination of critical external pressure of spherical segments. Let us first examine a case of a hinged support of a segment. The boundary conditions for this case will be

$$M_r=0,\ w=0.$$

We will assume additional radial membrane stresses along the contour of the segment to be $\sigma_r=0$. In the developed form these boundary conditions at a pitch of the grid $h=r_0$ (Fig. 191) will assume the form

$$M_r=-\frac{D}{h^2}[W_1(1+0{,}5\mu)+W_1(1-0{,}5\mu)-0]=0,$$

$$\sigma_r=\frac{1}{2r_0h}(F_1-F_1)=0,$$

whence

$$W_1=-0{,}739W_1,\quad F_1=F_1.$$

Fig. 191.

Along the contour of the segment the function of stresses ϕ will be equal to zero. This stems from the connection of the function of stresses with the bending moment from the load, acting on the contour [4]. In our case contour load $\sigma_r=0$. Consequently, $\varphi_k=F_k=0$.

Balance equations for point 1 will have the form

$$6F_1+F_1+F_1+\frac{3E\delta h^2}{2R}W_1=0,$$

$$-2F_1+\frac{4DR}{3h^2}(6W_1-0{,}739W_1-0{,}739W_1)-qR^2W_1=0.$$

From these equations we will find

$$q=\frac{6D}{Rh^2}+\frac{3E\delta h^2}{8R^3}.$$

Considering in this expression the pitch of grid h as a parameter, we will find $q^1_{кр}$:

$$q^1_{кр}=0{,}9E\left(\frac{\delta}{R}\right)^2 \text{ when } r_0=1{,}1\sqrt{R\delta}.$$

Now, let us examine a grid with two points (Fig. 192)

$$h=\frac{1}{2}r_0,\quad \frac{h}{r_1}=1,\quad \frac{h}{r_0}=\frac{1}{2}.$$

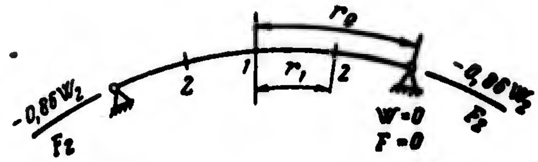


Fig. 192.

In this instance for the postcontour point W_l we will have the equation

$$(M_r)_{r=r_0}=-\frac{D}{h^2}[W_l(1+0,25\mu)+W_1(1-0,25\mu)]=0,$$

whence

$$W_l=-0,86W_1.$$

Point 1:

$$6F_1-4F_2-4F_2-\frac{3E\delta h^2}{2R}(W_2-W_1)=0,$$

$$F_2+F_1-2F_2+\frac{8DR}{3h^2}(3W_1-2W_2)+qR^2(W_2-W_1)=0.$$

Point 2:

$$F_2(6+2)-F_1\left(4-2+1+\frac{1}{2}\right)+F_2(1+1)-$$
$$-\frac{E\delta h^2}{R}\left[W_1\left(1-\frac{1}{2}\right)-2W_2\right]=0,$$

$$F_1\left(1-\frac{1}{2}\right)-2F_2+\frac{DR}{h^2}\left[W_2(6+2)-W_1\left(4-2+1+\frac{1}{2}\right)-\right.$$
$$\left.-0,86W_2(1+1)\right]+\frac{1}{2}qR^2\left[W_1\left(1-\frac{1}{2}\right)-2W_2\right]=0.$$

By excluding F_1 and F_2 from these equations, we will obtain

$$W_1\left(\frac{284}{9}\alpha x^2+1-\frac{34}{9}\beta x\right)+W_2\left(-\frac{391,6}{9}\alpha x^2-1+\frac{40}{9}\beta x\right)=0,$$

$$W_1\left(-\frac{155}{3}\alpha x^2-1+\frac{14,5}{3}\beta x\right)+W_2\left(\frac{248,7}{3}\alpha x^2+4-\frac{34}{3}\beta x\right)=0,$$

where

$$\alpha=\frac{\delta^2}{12(1-\mu^2)R^2},\quad \beta=\frac{qR}{E\delta},\quad x=\left(\frac{R}{h}\right)^2.$$

From the condition of equality to zero of the determinant of these equations we will obtain

$$\left(\frac{qR^2}{E\delta^2}\right)^{II}=\frac{1}{2}\left(\frac{1}{k^2}+0{,}807\,k^2\right)\pm\sqrt{\frac{0{,}104}{k^4}-0{,}084+0{,}23k^4},$$

where

$$k=\frac{h}{\sqrt{R\delta}}.$$

The values $q^{II}_{\text{кр}}$ depending on parameter k in Table 12 are given

Table 12.

k	0,6	0,7	0,8	0,9	1
$\left(\frac{qR^2}{E\delta^2}\right)^{II}$	0,685	0,621	0,617	0,644	0,696

From this table we will find

$$\left(\frac{qR^2}{E\delta^2}\right)^{II}_{\text{кр}}\approx 0{,}62 \text{ when } k=0{,}8 \text{ и } r_0=1{,}6\sqrt{R\delta}.$$

A more accurate value of the critical pressure can be determined by an extrapolation (§ 2):

$$\left(\frac{qR^2}{E\delta^2}\right)_{\text{экстр}}=0{,}527,$$

$$\sigma_{\text{кр}}=0{,}264E\frac{\delta}{R} \text{ when } r_{0\,\text{экстр}}=1{,}8\sqrt{R\delta}.$$

With the solution of this problem for the case of a rigid sealing of the contour (Fig. 193), $\frac{dw}{dr}=0$ and under the same boundary conditions for the function of stresses as a result of the first $(h=r_0)$ and second $\left(h=\frac{1}{2}r_0\right)$ approximations and for the extrapolation we will obtain the following value of critical pressure:

$$\left(\frac{qR^2}{E\delta^2}\right)_{\text{экстр}} = 0{,}73,$$

$$\sigma_{\kappa p} = 0{,}365E\frac{\delta}{R} \text{ when } r_{0\,\text{экстр}} = 3\sqrt{R\delta}.$$

Fig. 193.

Now, let us examine the rigidity of a spherical segment with a rigid undeformable circuit under external pressure.

In this instance, except for the requirment of equality of angle of rotation to zero, it is necessary to impose an additional requirement about the lack of elongation of the circumference of the support contour and so forth.

$$\varepsilon_\theta = \frac{u}{r} = \left(\frac{d^2\varphi}{dr^2} - \frac{\mu}{r_0}\frac{d\varphi}{dr}\right)_{r=r_0} = 0.$$

As finite differences these conditions have the form

$$W_i = W_l,$$

$$\frac{F_i + F_l - 2F_k}{h^2} - \frac{\mu}{r_0}\frac{F_l - F_i}{2h} = 0.$$

In the investigated problem in view of the nondeformability of the support contour after the loss of rigidity additional membrane stresses σ_r will appear. If one regards the axial line of the contour of the segment as a ring, loaded with an even load $N_r = \delta\sigma_r$, where δ — thickness of the segment, then the bending moment in this ring from such load will be equal to zero. Considering the above shown connection of the bending moment with function of stresses, it is possible to show that this function along the contour will be $\varphi_\kappa = F_\kappa = 0$. Then

$$F_i = -\frac{1 + \frac{\mu h}{2r_0}}{1 - \frac{\mu h}{2r_0}} F_l.$$

The first approach: $h = r_0$ (Fig. 194):

$$6F_1 - 0{,}739F_1 - 0{,}739F_1 + \frac{3E\delta h^2}{2R} W_1 = 0,$$
$$-2F_1 + \frac{4DR}{3h^2}(6W_1 + W_1 + W_1) - qR^2 W_1 = 0.$$

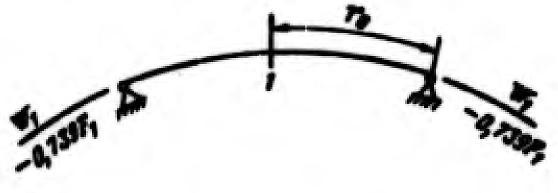


Fig. 194.

From these equations we will obtain

$$q^{I} = \frac{10{,}65\,D}{Rh^2} + \frac{0{,}664E\delta h^2}{R^3}.$$

The minimum of this expression according to parameter h^2

$$q^{I}_{кр} = 1{,}6E\left(\frac{\delta}{R}\right)^2 \text{ when } r_0 = 1{,}1\sqrt{R\delta}.$$

As a result of the second approximation $h = \frac{r_0}{2}$ (Fig. 195) one obtains

$$\left(\frac{qR^2}{E\delta^2}\right)^{II} = \left(\frac{0{,}724}{k^2} + 1{,}25k^2\right) \pm \sqrt{\frac{0{,}044}{k^4} - 0{,}24 + 0{,}633k^4},$$

where

$$k = \frac{r_0}{2\sqrt{R\delta}}.$$

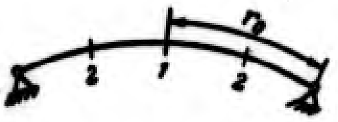


Fig. 195.

The dependence of $q^{II}_{кр}$ on parameter k is shown in Table 13.

Table 13.

k	0,5	1,0	1,2	1,5	2
$\left(\frac{qR^2}{E\delta^2}\right)^{II}$	2,5	1,32	1,26	1,95	2,1

From this table we will find that

$$\left(\frac{qR^2}{E\delta^2}\right)^{II}_{кр} = 1{,}26 \text{ when } r_0 = 2{,}4\sqrt{R\delta}.$$

A more accurate value of the critical pressure can be determined by an extrapolation:

$$\left(\frac{qR^2}{E\delta^2}\right)_{экстр} = 1{,}2 \text{ when } r_{0\,экстр} = 2{,}8\sqrt{R\delta}.$$

$$\sigma_{кр} = 0{,}6\,E\frac{\delta}{R}.$$

§ 59. Rigidity of an Enclosed Spherical Shell Under External Pressure

During the determination of the magnitude of critical external pressure of an enclosed spherical shell let us assume that its surface after the loss of rigidity is covered with identical pits and bulges, having a form close to a square. Such a form of a deformed surface of a sphere with small displacements, apparently, is possible, because at the time of the loss of rigidity the areas of pits and bulges will be small.

Of course, one can assume that the pits and bulges may also have other forms, different from a square.

By analogy with problem of rigidity of cylindrical shell under axial compression let us consider that along the contour of the pits and bulges the following boundary conditions will be realized:

$$w = 0, \quad N_x = 0,$$
$$M_x = 0, \quad N_y = 0,$$
$$M_y = 0, \quad N_{xy} = 0.$$

As is known from the solution of the problem for a cylindrical shell, these boundary conditions satisfy the following expressions for a deflection and for the function of stresses:

$$w = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{a},$$

$$\varphi = B_1 \sin^2\frac{\pi x}{a} \sin^2\frac{\pi y}{a} + B_2 \sin^2\frac{2\pi x}{a} \sin^2\frac{\pi y}{a} + B_3 \sin^2\frac{\pi x}{a} \sin^2\frac{2\pi y}{a} + B_4 \sin^2\frac{2\pi x}{a} \sin^2\frac{2\pi y}{a}.$$

After the substitution of these functions in the equations (12.1) and after their integration by the Bubnov-Galerkin method as a result of the third and fourth approximations according to the function ϕ and with one and the same expression for w we will obtain the following value of critical stress:

$$\sigma_{кр} = 0{,}315E\frac{\delta}{R}.$$

The experimental data according to tests on enclosed spherical shells is lacking in the literature. Therefore, it is difficult to say anything about how reliable the last formula is. If however one were to judge from the formal criteria of the solution itself, completely basing it on the accepted formulation of the boundary conditions along the contour of pits and bulges, then one can hope that the obtained result is acceptable for the practical utilization and is not too over rated in comparison with that known in literature.

§ 60. Rigidity of a Spherical Layer in the Case of External Pressure

Let us examine the question dealing with the rigidity of a spherical band existing under the action of an external uniform pressure q. In this case let us propose that the formed bands are symmetrical relative to axis y, and the sealing of the edges of the bands are hinged (Fig. 196).

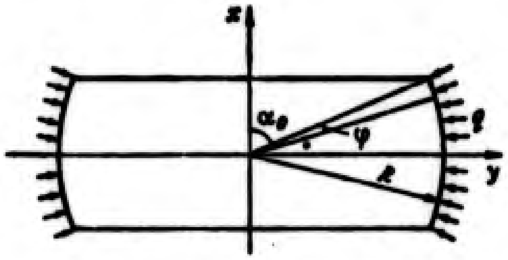


Fig. 196.

Membrane forces in the subcritical state of the shell can be presented by the expressions, obtained from the solution of a momentless problem:

$$N_{\varphi}^{0}=\frac{qR}{2}\,\frac{\sin^2(\alpha_0+\varphi)-\sin^2\alpha_0}{\sin^2(\alpha_0+\varphi)},$$

$$N_{\theta}^{0}=\frac{qR}{2}\,\frac{\sin^2(\alpha_0+\varphi)+\sin^2\alpha_0}{\sin^2(\alpha_0+\varphi)}.$$

Let us conduct all further computations for a comparatively low spherical layer. In this case let us consider that the meridional force N_{φ}^{0} is considerably less than the annular N_{θ}^{0}, and that one can neglect them. Force N_{θ}^{0} can be taken as a safety factor equal to its maximum value when $\varphi=90-\alpha_0$. Then

$$N_{\theta}^{0}\approx qR,\quad N_{\varphi}^{0}\approx 0.$$

The equations of the rigidity in this case will have the form

$$\nabla^2\nabla^2\varphi=\frac{E\delta}{R}\nabla^2 w,$$

$$\frac{1}{R}\nabla^2\varphi+D\nabla^2\nabla^2 w+N_{\theta}^{0}\frac{\partial^2 w}{\partial y^2}=0.$$

By dividing the variables according to the formula

$$w=W(y)\cos\frac{m\pi x}{l},$$

$$\varphi=F(y)\cos\frac{m\pi x}{l}$$

and by converting to finite-difference equations, we will obtain

$$F_n+F_m-a_2F_k+\frac{DR}{b^2\delta}[W_ka_1-2a_2(W_n+W_m)+W_p+W_s]+$$

$$+\frac{N_{\theta}^{0}R}{\delta}(W_n+W_m-2W_k)=0,$$

$$F_ka_1-2a_2(F_n+F_m)+F_s+F_p=\frac{Eb^2}{R}(W_n+W_m-a_2W_k).$$

Designated here is

$$a_1=\left(\frac{m\pi b}{l}\right)^4+4\left(\frac{m\pi b}{l}\right)^2+6;$$

$$a_2=\left(\frac{m\pi b}{l}\right)^2+2;$$

b - size of a half-wave in a circumferential direction; m - number of waves in the direction of axis x.

The height of the layer should be taken as the size l. Let us consider that along the contour of pits and bulges in a circumferential direction one will realize the following boundary conditions:

$$M_y=0,\quad N_y=0,\quad N_{xy}=0.$$

From these conditions for the points along the contour and postcontour, we will obtain the relationships

$$W_k=0,\quad W_l=W_l,\quad F_k=0,\quad F_l=F_l.$$

The first approximation $b=\frac{c}{2}$ (Fig. 197):

$$-a_2F_1+\frac{DR}{b^2\delta}(W_1-W_1-W_1)-\frac{2N_\theta^0 R}{\delta}W_1=0,$$

$$F_1a_1+F_1+F_1=-\frac{Eb^2a_3}{R}W_1.$$

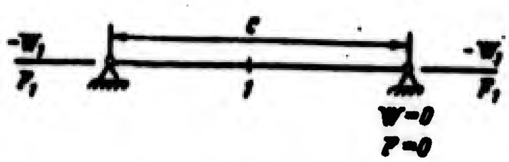


Fig. 197.

By excluding the functions F_1 and W_1 from these equations, we will obtain for N_θ^0 the expression

$$\frac{N_\theta^0}{E\delta}=\frac{qR}{E\delta}=\frac{\left[2+\left(\frac{\pi mb}{l}\right)^2\right]^2\left(\frac{b}{R}\right)^2}{2\left[8+4\left(\frac{\pi mb}{l}\right)^2+\left(\frac{\pi mb}{l}\right)^4\right]}+$$

$$+\frac{\left(\frac{\delta}{R}\right)^2\left[2+\left(\frac{\pi mb}{l}\right)^2\right]^2}{24(1-\mu^2)\left(\frac{b}{R}\right)^2}.$$

From the structure of this expression it is evident that the minimum of critical pressure exists when $m = 1$, i.e., when in longitudinal direction of the band only one half-wave is generated:

$$\frac{qR}{E\delta}=\frac{\left[2+\left(\frac{\pi b}{l}\right)^2\right]^2\left(\frac{l}{\pi R}\right)^2\left(\frac{\pi b}{l}\right)^2}{2\left[8+4\left(\frac{\pi b}{l}\right)^2+\left(\frac{\pi b}{l}\right)^4\right]}+$$

$$+\frac{\left(\frac{\delta}{R}\right)^2\left[2+\left(\frac{\pi b}{l}\right)^2\right]^2}{24\left(1-\mu^2\right)\left(\frac{l}{\pi R}\right)^2\left(\frac{\pi b}{l}\right)^2}.$$

During the analysis of the obtained expression let us consider that the spherical band is short. Therefore, one can expect that the parameter $\left(\frac{\pi b}{l}\right)^2$ will be a large number, a fraction

$$\frac{\left[2+\left(\frac{\pi b}{l}\right)^2\right]^2}{8+4\left(\frac{\pi b}{l}\right)^2+\left(\frac{\pi b}{l}\right)^4}$$

differing slightly unity and can be approximately written as

$$\frac{qR}{E\delta}=\frac{\left(\frac{l}{\pi R}\right)^2}{2}x+\frac{\left(\frac{\delta}{R}\right)^2}{24\left(1-\mu^2\right)\left(\frac{l}{\pi R}\right)^2}\cdot\frac{(2+x)^2}{x}.$$

Designated here is

$$x=\left(\frac{\pi b}{l}\right)^2.$$

For the determination of the least value q it is necessary to take the derivative in terms of x from the last expression and equate it to zero. Having solved the obtained equation in this case relative to x and by substituting it in the expression for q, we will find

$$\left(\frac{qR}{E\delta}\right)_{\text{кр}}=\frac{\delta}{\sqrt{3\left(1-\mu^2\right)}R}\sqrt{1+\frac{\left(\frac{\pi^2R\delta}{l^2}\right)^2}{12\left(1-\mu^2\right)}+\frac{\left(\frac{\pi b}{l}\right)^2}{6\left(1-\mu^2\right)}}.$$

Analogous computations for the smaller grid at a step $b=\frac{1}{3}C$ (Fig. 198) results in a formula, found from the first approximation.

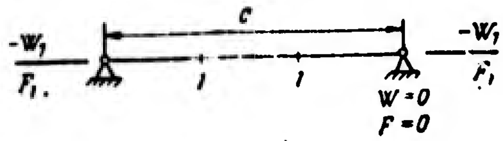


Fig. 198.

§ 61. Rigidity of a Spherical Shell from the Effect of Inside Hydrostatic Pressure

As shown in § 17, during the loading of a spherical shell by hydrostatic pressure the circumferential forces N_y^0 can assume negative values. At a certain value of hydrostatic pressure these forces can reach their critical value and the shell can lose rigidity. For the determination of the critical pressure of the liquid let us consider that the intensity of the internal forces in the compressed zone of the shell can be determined by the formulas in § 18:

$$N_x^0 = -\frac{\gamma R^2}{3}, \quad N_y^0 = \frac{\gamma R^2}{3},$$

where the minus sign indicates tension.

For the solution of the problem of forces N_x^0 and N_y^0 let us take their maximum and equal value at the equator. Such an assumption is entered as a safety factor, and we will obtain a somewhat underrated value of the critical stress.

We will solve V. Z. Vlasov's equations by using the method of Bubnov-Galerkin. By limiting the first approximation, let us use the following expressions for the functions of deflection and stresses:

$$w = A \sin^2 \frac{\pi x}{a} \sin \frac{\pi y}{b},$$

$$\varphi = B \sin \frac{\pi x}{a} \sin^2 \frac{\pi y}{b}.$$

These expressions pertain to one certain pit or bulge (Fig. 199). Because of the symmetry of the load, all the pits and bulges will be completely identical in area. The sizes of their a and b will be determinable from the condition of the minimum of loading. The boundary conditions along the contour of pits and bulges:

$$\left.\begin{aligned} & w=0, \\ & \frac{\partial w}{\partial x}=0, \\ & \sigma_x=\frac{\partial^2\varphi}{\partial y^2}=0, \\ & v=0. \end{aligned}\right\} \begin{aligned} x=0 \\ x=a \end{aligned} \qquad \left.\begin{aligned} & w=0, \\ & M_y=0, \\ & \sigma_y=\frac{\partial^2\varphi}{\partial x^2}=0, \\ & \sigma_{xy}=-\frac{\partial^2\varphi}{\partial x\,\partial y}=0. \end{aligned}\right\} \begin{aligned} y=0 \\ y=b \end{aligned}$$

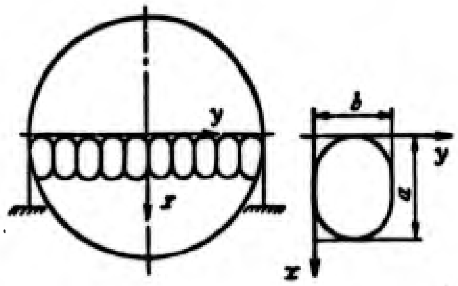


Fig. 199.

The accepted functions of ϕ and w satisfy these boundary conditions. Let us substitute the necessary derivatives from ϕ and w, and likewise the force N_x^0 and N_y^0 in Vlasov's equations and, integrate them within the limits of $0-a$, $0-b$. After the appropriate calculations we will have

$$B=-\frac{16E\delta A\left[1+\left(\frac{a}{b}\right)^2\right]}{9\pi^2R\left[\frac{3}{16}+\frac{1}{2}\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^4\right]\left(\frac{\pi}{a}\right)^2},$$

$$-\frac{16\left[1+\left(\frac{a}{b}\right)^2\right]}{9\pi^2R}B+DA\left(\frac{\pi}{a}\right)^2\left[1+\frac{1}{2}\left(\frac{a}{b}\right)^2+\frac{3}{16}\left(\frac{a}{b}\right)^4\right]-$$
$$-\frac{qR}{12}\left[\frac{3}{4}\left(\frac{a}{b}\right)^2-1\right]A=0.$$

By excluding parameters A and B from these equations, we will obtain expression for $q=(\gamma R)_{\text{кр}}$:

$$\frac{\gamma R^2}{12E\delta}=\frac{256(1+x)^2}{81\pi^4\left(\frac{3}{16}+\frac{1}{2}x+x^2\right)\left(\frac{3}{4}x-1\right)y}+$$
$$+\frac{\alpha\left(1+\frac{1}{2}x+\frac{3}{16}x^2\right)y}{\frac{3}{4}x-1}.$$

Designated here is

$$x=\left(\frac{a}{b}\right)^2, \quad y=\left(\frac{nR}{a}\right)^2, \quad \alpha=\frac{(\delta/R)^2}{12(1-\mu^2)}.$$

The minimum of the given expression in terms of parameter y will be

$$\frac{\gamma R}{E}\left(\frac{R}{\delta}\right)^2=1{,}32\frac{1+x}{\frac{3}{4}x-1}\sqrt{\frac{1+\frac{1}{2}x+\frac{3}{16}x^2}{\frac{3}{16}+\frac{1}{2}x+x^2}}.$$

By attaching various numerical values to parameter x, one can be certain that the minimum for γR will be at large values of x. Within the limit (at $x \to \infty$) we will obtain

$$\left(\frac{\gamma R}{\delta}\right)_{\kappa p}=0{,}251\,E\left(\frac{\delta}{R}\right)^2.$$

The critical value of the hydraulic pressure under the level of the liquid, passing through the equator, is provided by this expression.

§ 62. Rigidity of Ellipsoidal Doughnut-Shaped Shells from the Action of Internal Pressure

Ellipsoidal doughnut-shaped shells, used as independent constructions of containers or as bottoms of tanks, under a specified internal pressure can lose rigidity by wrinkling a circumferential direction. The loss in rigidity in such shells proceeds in zone of compressive annular stresses. As shown in § 14, this zone can be determined by the condition

$$r>\sqrt{-\frac{a^4}{2(b^2-a^2)}}\ \text{when}\ a>b.$$

In the same paragraph the expressions for stresses σ_φ and σ_θ are obtained.

The problem on hand will be solved by the method of finite differences, considering that in this case the form of wave formation after the loss in rigidity, following from the experiment, consists of identical pits and bulges in the circumferential direction of the compressed zone of the cover. Therefore, a solution will be presented in the circumferential direction and of its rigid sealing in the direction of the meridian. V. Z. Vlasov's equations, expressed through finite differences, will have the form

$$
\begin{gathered}
6F_k\left(1+\frac{4}{3}\xi+\xi^2\right)-4(1+\xi)(F_l+F_i+\xi F_n+\xi F_m)+F_t+F_s+ \\
+\xi^2(F_v+F_u)+2\xi(F_p+F_0+F_r+F_q)=\frac{E\delta h^2}{R_2}(W_l+W_i-2W_k)+ \\
+\frac{E\delta\xi h^2}{R_1}(W_n+W_m-2W_k), \\
F_l+F_i-2F_k+\xi\frac{R_2}{R_1}(F_n+F_m-2F_k)+\frac{DR_2}{h^2}\Big[6W_k\left(1+\frac{4}{3}\xi+\xi^2\right)- \\
-4(1+\xi)(W_l+W_i+\xi W_n+\xi W_m)+W_t+W_s+\xi^2(W_v+W_u)+ \\
+2\xi(W_p+W_0+W_r+W_q)\Big]+R_2N_x^0(W_l+W_i-2W_k)+ \\
+\xi R_2N_y^0(W_n+W_m-2W_k)=0.
\end{gathered}
\qquad (12.6)
$$

Designated here is $\xi=\left(\frac{h}{b}\right)^2$.

Expressions for the bending moments and angles of rotation:

$$
\begin{gathered}
M_x=-\frac{D}{h^2}[W_l+W_i+\mu\xi(W_n+W_m)-2(1+\mu\xi)W_k], \\
M_y=-\frac{D}{h^2}[\mu(W_l+W_i)+\xi(W_n+W_m)-2(\xi+\mu)W_k], \\
\frac{\partial w}{\partial x}=\frac{1}{2h}(W_l-W_i), \quad \frac{\partial w}{\partial y}=\frac{1}{2b}(W_n-W_m).
\end{gathered}
$$

These expressions will be subsequently used in the formulation of the boundary conditions along the contour of pits and bulges (see Figs. 201-204).

Let us obtain the boundary conditions for the function of stresses from the solution of the following system of equations relative to the additional stresses, which we will assume to be equal to zero along the contour of pits and bulges:

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2} = 0, \quad \sigma_y = \frac{\partial^2\varphi}{\partial x^2} = 0, \quad \tau_{xy} = -\frac{\partial^2\varphi}{\partial x\,\partial y} = 0.$$

By considering these conditions as differential equations, we will obtain (see § 57)

$$F_k = 0, \quad F_l = 0, \quad F_m = F_n.$$

Let us examine the compressed zone of the eliptical torus (Fig. 200) and set up equations (12.6) for the proposed pits or bulges having sizes l_1 and l_2.

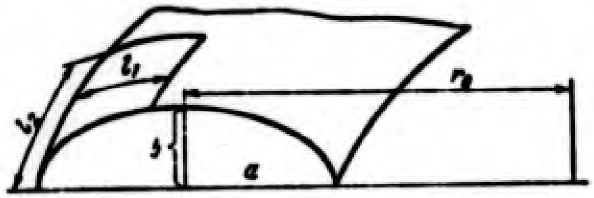


Fig. 200.

The sizes of the dents during the tests of such shells usually are small, and therefore, one can assume that the radii of curvature and of stresses in zone of these dents - constants, equal to their corresponding values on the external contour of the torus. Such an assumption is applied as a safety factor during the determination of the critical value of internal pressure.

1) The first approach (Fig. 201):

$$l_2 = nl_1, \quad h = \frac{1}{2}l_1, \quad b = \frac{1}{2}l_2, \quad \xi = \left(\frac{h}{b}\right)^2 = \frac{1}{n^2},$$

$$6F_1\left(1 + \frac{4}{3}\xi + \xi^2\right) + F_1 + F_1 + \xi^2(F_1 + F_1) = -\frac{E\delta l_1^2}{2R_2}W_1 - \frac{E\delta\xi l_1^2}{2R_1}W_1,$$

$$-2F_1 - 2\xi\frac{R_2}{R_1}F_1 + \frac{4DR_2}{l_1^2}\left[6W_1\left(1 + \frac{4}{3}\xi + \xi^2\right) + W_1 + W_1 + \xi^2(-W_1 - W_1)\right] + 2N_x^0R_2W_1 + 2\xi N_y^0R_2W_1 = 0.$$

By excluding W_1 and F_1 from these equations, we will obtain

$$\frac{qa^2}{2E\delta b} = -\frac{1}{Q_1 + \xi Q_2}\left[\frac{2\delta^2(2 + 2\xi + \xi^2)}{3(1 - \mu^2)l_1^2} + \frac{n^2 l_1^2}{16(1 + \xi + \xi^2)R_2^2}\right].$$

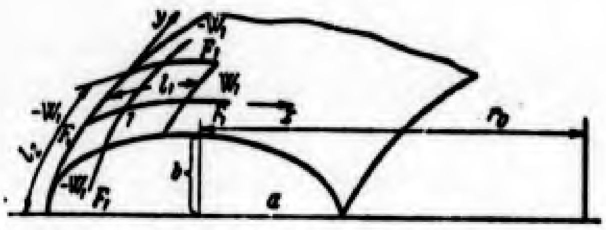


Fig. 201.

Designated here is:

$$Q_1=\left(1+\frac{r_0}{r_0+a}\right)\frac{b}{a},$$

$$Q_2=\left[2\left(\frac{r_0}{a}+1\right)\left(\frac{b^2}{a^2}-1\right)+1\right]\frac{a}{b},$$

$$k=1+\xi\frac{R_2}{R_1},\quad R_1=r_0+a,\quad R_2=\frac{b^2}{a}.$$

By considering the size of the dent l_1 as a parameter in the obtained expression, let us find the relative minimum of pressure q with respect to l_1^2:

$$\frac{qR_2a^2}{E\delta^2 b}=-\frac{0,855k\sqrt{2+2\xi+\xi^2}}{(Q_1+\xi Q_2)\sqrt{1+\xi+\xi^2}}.$$

The calculations, conducted, based on this expression, show that the least value for q is obtained at rather large values of parameter ξ. Therefore, in the right part of the last expression one can disregard unity in comparison with ξ, and Q_1 in comparison with ξQ_2. Then

$$\frac{qa^2}{E\delta^2 b}=-\frac{0,855}{Q_2R_1}$$

or

$$q^{1}_{\text{кр}}=-\frac{0,855E\delta^2}{\left[2\left(\frac{r_0}{a}+1\right)\left(\frac{b^2}{a^2}-1\right)+1\right]a^2}.$$

The second approach (Fig. 202):

$$h=\frac{1}{3}l_1,\quad b=\frac{1}{3}l_2,\quad l_2=nl_1,\quad \xi=\left(\frac{h}{b}\right)^2=\frac{1}{n^2}.$$

Let us set up the equations (12.6) for any of the points 1. In this instance we will obtain the following expression for $q_{\text{кр}}$:

$$\frac{qa^2}{2E\delta} = -\frac{1}{Q_1 + \xi Q_2}\left[\frac{3h^2(3 + 2\xi + \xi^2)}{4(1-\mu^2)l_1^2} + \frac{h^2 l_1^2}{9(3 + 2\xi + 3\xi^2)R_2^2}\right].$$

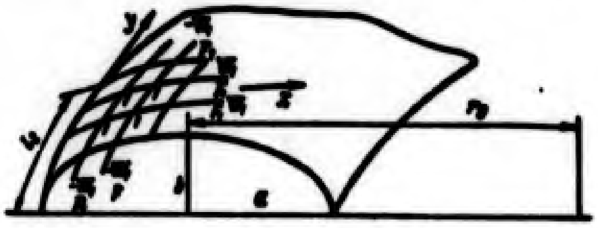


Fig. 202.

An analysis of this expression, conducted by an analogy with the case of the first approximation, leads to the following formula for the second approximation:

$$q_{\text{кр}}^{II} = -\frac{0{,}7E\delta^2}{\left[2\left(\frac{r_0}{a}+1\right)\left(\frac{b^2}{a^2}-1\right)+1\right]a^2}.$$

By not introducing bulky computations for the third approximation (Fig. 203)

$$h = \frac{1}{4}l_1, \quad b = \frac{1}{4}l_2, \; l_2 = nl_1, \; \xi = \left(\frac{h}{b}\right)^2 = \frac{1}{n^2},$$

let us write out the final result, obtained by analogy with the result for the two first approximations:

$$q_{\text{кр}}^{III} = -\frac{0.574E\delta^2}{\left[2\left(\frac{r_0}{a}+1\right)\left(\frac{b^2}{a^2}-1\right)+1\right]a^2}.$$

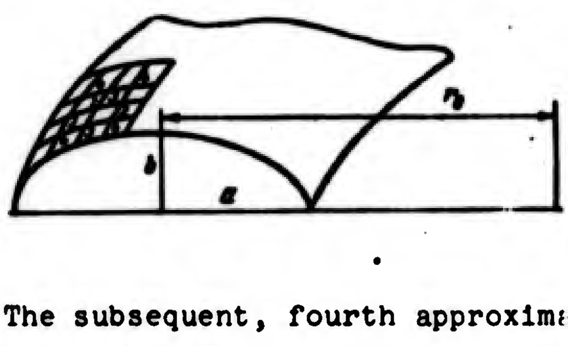


Fig. 203.

The subsequent, fourth approximation can be obtained by extrapolation:

$$q_{\text{кр.экстр}} = 0{,}267q_{\text{кр}}^{\text{I}} - 2{,}314q_{\text{кр}}^{\text{II}} + 3{,}048q_{\text{кр}}^{\text{III}},$$

or finally

$$q_{\text{кр.экстр}} = -\frac{0{,}358E\delta^2}{\left[2\left(\frac{r_0}{a}+1\right)\left(\frac{b^2}{a^2}-1\right)+1\right]a^2}. \qquad (12.7)$$

When $r_0 = 0$ we will obtain the formula for a shell in the form of an ellipsoid of rotation, existing under the action of internal pressure:

$$q_{\text{кр}} = \frac{0{,}358E\delta^2}{b^2\left(\frac{a^2}{b^2}-2\right)}. \qquad (12.8)$$

Hekkeler's formula which occurs rather extensively in the literature has the form (in our designations)

$$q_{\text{кр}} = \frac{1{,}21E\delta^2}{a^2\left(\frac{a^2}{b^2}-2\right)}. \qquad (12.9)$$

In recent years a formula was proposed by Kh. M. Mushtari and V. I. Korolev

$$q_{\text{кр}} = \frac{1{,}21E\delta^2}{\left(\frac{a^2}{b^2}-2\right)b^2}. \qquad (12.10)$$

By comparing the formula (12.10) with (12.8), we will see that they agree according to the structure and can be distinguished only by the coefficient. Hekkeler's formula (12.9), as noted by Kh. M. Mushtari and V. I. Korolev, was erroneous.

Because the same boundary conditions along the contour of pits and bulges were used in obtaining the formula (12.7) that were also used during the investigation of the rigidity of a cylindrical and spherical shells whereby the obtained formulas will agree satisfactorily with the experiment, then the given formula, apparently, will also give satisfactory results. Therefore, the formula (12.8), obtained from (12.7), likewise will give satisfactory results. The formula (12.10), however, was obtained under more rigid boundary conditions along the contour of pits, and therefore, the results of calculations on it will be over rated.

In the case of an ellipsoidal torus (Fig. 204), elongated in the direction of the vertical diameter ($b>a$), the formula for the critical internal pressure assumes the form

$$q_{кр}=\frac{0,358E\delta^2}{\left[2\left(\frac{r_0}{a}-1\right)\left(\frac{b^2}{a^2}-1\right)-1\right]a^2}.$$

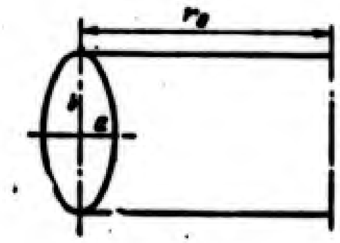


Fig. 204.

CHAPTER XIII

RIGIDITY OF SHELLS OF ROTATION UNDER LOADING OF AN AXISYMMETRICAL LINEAR LOAD AND WITH INTERNAL PRESSURE

§ 63. Rigidity of a Cylindrical Shell Under Loading of an Axisymmetrical Radial Linear Load and with Internal Pressure

In Chapter VI certain problems with respect to the calculation of shells of rotation under loading of a linear axisymmetrical load were examined. For a certain value of forces acting on the shell they can lose rigidity in the compressed zone. During the determination of the critical value of these forces let us consider that the shells are under the action of internal pressure.

1. *A cylindrical cover under a load* according to Fig. 205 for internal forces, we will have the expressions

$$N_x^0 = -\frac{qR}{2},$$

$$N_\varphi^0 = \frac{PRk}{2} e^{-kx} (\sin kx + \cos kx) - qR.$$

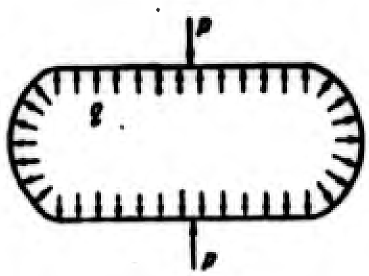


Fig. 205.

The minus sign indicates the fact that the stress is tensile.

For the solution of this and all subsequent problems of this chapter we will make use of the Bubnov-Galerkin method. In order to obtain the approximate solution, let us consider that the deflection of the shell after the loss in rigidity with respect to variable x coincides with the corresponding expression of the subcritical state

$$w = Ae^{-kx}(\sin kx + \cos kx)\cos n\theta. \tag{13.1}$$

Let us use the function of stresses in the form

$$\varphi = e^{-kx}(B_1 \sin kx + B_2 \cos kx)\cos n\theta.$$

From the expression for w it is evident that the deflection of the shell after the loss in rigidity diminishes in proportion to the distance from the place of application of the load and it is a periodic function of angle θ in the circumferential direction, i.e., the shell after the loss in rigidity in the region of application of the load of signs assumes a form, corresponding to Fig. 206.

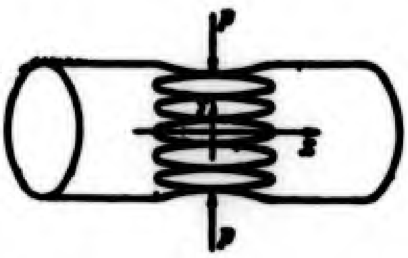


Fig. 206.

From this figure it is evident that the surface of hole because of the axisymmetrical nature of load P will possess the feature of symmetry relative to coordinate $x = 0$. Therefore, it can be said that when $x = 0$ there will be no tangential stresses in the shell after the loss in rigidity. Making use of this circumstance, we can determine one of the parameters of ϕ in the expression from the condition

$$\sigma_{xy} = -\left(\frac{\partial^2 \varphi}{\partial x\, \partial y}\right)_{x=0} = 0.$$

Hence

$$B_1 = B_2 = B.$$

Then, the function of stresses assumes the form

$$\varphi = Be^{-kx}(\sin kx + \cos kx)\cos n\theta.$$

Having substituted these functions in V. Z. Vlasov's equations, after integration them, we will obtain

$$B\left(-12k^4 + \frac{4n^2k^2}{R^2} + \frac{3n^4}{R^4}\right) = -\frac{2k^2E\delta A}{R},$$

$$-\frac{2k^2B}{R} + DA\left(-12k^4 + \frac{4n^2k^2}{R^2} + \frac{3n^4}{R^4}\right) + \frac{4qn^2A}{R} +$$

$$+\frac{qRk^2A}{2} - \frac{6Pkn^2A}{5R} = 0.$$

By excluding parameters A and B from these equations, we will find that

$$\frac{6P}{5E\delta}(Rk)^3 = \frac{4}{x(-12 + 4x + 3x^2)} + \frac{-12 + 4x + 3x^2}{4x} +$$

$$+\frac{3qR^3k^2}{E\delta} + \frac{qR^3k^2}{2x},$$

where

$$x = \left(\frac{n}{Rk}\right)^2.$$

In certain cases the axial thrust is $N_x^0 = 0$. Then, by discarding the last component in the given formula, we will have

$$\frac{6P}{5E\delta}(Rk)^3 = \frac{4}{x(-12 + 4x + 3x^2)} + \frac{-12 + 4x + 3x^2}{4x} + \frac{3qR^3k^2}{E\delta}.$$

A graph of the function is presented in Fig. 207

$$f(x) = \frac{4}{x(-12 + 4x + 3x^2)} + \frac{-12 + 4x + 3x^2}{4x}.$$

From the second branch of this graph it is evident that the least value of function $f(x)$ it is equal to 1.135. Then

$$P_{\text{кр}} = 0{,}444E\delta\sqrt{\left(\frac{\delta}{R}\right)^3} + 1{,}94qR\sqrt{\frac{\delta}{R}} \qquad (13.2)$$

when

$$n = 1{,}73\sqrt{\frac{R}{\delta}}.$$

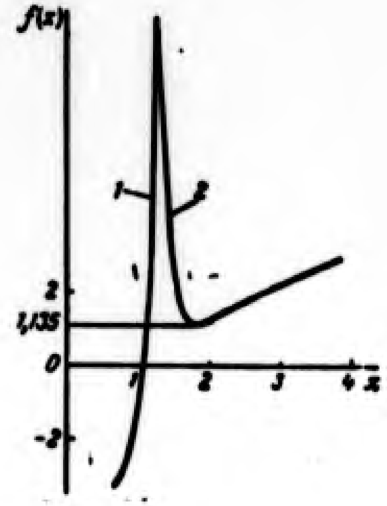


Fig. 207.

In order to take into account the force N_x^0 in an axial direction from the internal pressure, one should make use of the formula

$$\frac{6P}{5E\delta}(Rk)^3 = \frac{4}{x(-12+4x+3x^2)} + \frac{-12+4x+3x^2}{4x} + \frac{qR^2}{E\delta^2}\left(4{,}95+\frac{1{,}65}{x}\right).$$

With the solution of this problem in the second approximation for function of stress ϕ one could use the expression

$$\varphi = B_1 e^{-kx}(\sin kx + \cos kx)\cos n\theta + B_2 e^{-2kx}(\sin 2kx + \cos 2kx)\cos n\theta.$$

In this case the previous expression (13.1) was used for w. For the critical value of force P in this instance

$$\frac{6P}{5E\delta}(Rk)^3 = \frac{\frac{16}{15}f_1 - f_2 - \frac{256}{225}f_3 + \frac{16}{15}f_4}{x(f_1 f_4 - f_2 f_3)} + f_3 + \frac{3qR^3k^2}{E\delta} + \frac{2N_x^0 R^2 k^2}{E\delta x},$$

where

$$f_1 = -\frac{28}{15} + \frac{16x}{15} + \frac{7x^2}{5}, \quad f_2 = -24 + 2x + \frac{3}{8}x^2,$$
$$f_3 = -3 + x + \frac{3}{4}x^2, \quad f_4 = -\frac{448}{15} + \frac{16x}{15} + \frac{7x^2}{15},$$
$$x = \left(\frac{n}{Rk}\right)^2;$$

N_x^0 — axial force in the shell. If this force is created by internal pressure, then $N_x^0 = \frac{1}{2}qR$.

By inserting $N_x^0 = 0$ in the given expression and by determining the minimum expression obtained in this case according to parameter x, let us find the value $P_{кр}$ in the second approximation:

$$P_{кр} = 0{,}415E\delta\sqrt{\left(\frac{\delta}{R}\right)^3} + 1{,}94qR\sqrt{\frac{\delta}{R}} \text{ when } n = 1{,}72\sqrt{\frac{R}{\delta}}.$$

From the comparison of the two approximations it is evident that with a sufficient degree of accuracy it is possible to only limit the first approximation. Therefore, the solutions of the problem, given below, apply only to the first approximation.

2. Let us examine the *case of a loss in rigidity*, when the deflection of all holes are turned to the center of the curvature of the shell. One can obtain such a form of loss in resistance, if an infinitely long cylinder is fitted on the outside of the cover without clearance. Then the waves, which are generated after the loss in rigidity, will be turned towards the inside of the shell.

One can express the form of the deformed surface of the shell approximately in the expression

$$w = Ae^{-kx}(\sin kx + \cos kx)\cos^2 n\theta.$$

Although this expression does not completely satisfy the imposed requirements of uniqueness of the deflection along the axis of the shell, because function $e^{-kx}(\sin kx + \cos kx)$ bears vibrational character, the branch of this curve, containing the greatest amplitude then, will be turned towards the inside of the shell.

Let us take for function of stresses the expression

$$\varphi = Be^{-kx}(\sin kx + \cos kx)\cos^2 n\theta,$$

which reverts tangential stresses to zero when $x = 0$. The uniqueness of the stresses at an angle θ is considered to be the square of the cosine. By eliminating all intermediate computations, analogous to the above examined case, let us present the final results for the critical value of force P (in this case, the axial force N_x^0 equal to zero):

$$P_{кр}=2{,}5E\delta\sqrt{\left(\frac{\delta}{R}\right)^3}+1{,}94qR\sqrt{\frac{\delta}{R}}\ \text{when}\ n=1{,}29\sqrt{\frac{R}{\delta}}. \qquad (13.3)$$

By comparing the formulas (13.2) and (13.3), we will see that the imposition of constraint on the form of the deformed surface has resulted in an increase in the coefficient in the first member (13.3) by 5.5 times.

3. Let us examine the problem dealing with *the rigidity of a cylindrical shell,* loaded according to Fig. 208, with an internal pressure q and an axial tension N_x^0 in its presence.

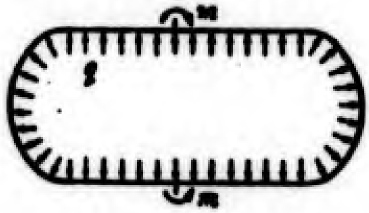


Fig. 208.

In the subcritical state of the shell we will have the following expressions for the internal forces:

$$N_x^0=-N_x^0,$$
$$N_y^0=mRk^2e^{-kx}\sin kx-qR.$$

For the functions of deflection and stresses we will take the expressions

$$w=Ae^{-kx}\sin kx\cos n\theta,$$
$$\varphi=e^{-kx}(B_1\sin kx+B_2\cos kx)\cos n\theta.$$

Considering the character of the load and the expected form of deformed surface of the shell after the loss of rigidity from moment m, we will see that when $x = 0$ the additional normal stresses σ_x should pass through zero, and so forth.

$$\sigma_x=\left(\frac{\partial^2\varphi}{\partial y^2}\right)_{x=0}=0.$$

From this condition we will find that $B_2 = 0$. By eliminating all the intermediate computations, connected with the integration of Vlasov's equations according to the method of Bubnov-Galerkin within the limits $0-\infty, 0-2\pi$, let us write down the final result:

$$\frac{4m}{15E\delta R}(Rk)^4 = \frac{4}{x(-4+4x+x^2)} + \frac{-4+4x+x^2}{4x} + \frac{qR^3k^2}{E\delta} + \frac{2N_x^0R^4k^4}{E\delta x},$$

where

$$x = \left(\frac{n}{Rk}\right)^2.$$

The minimum of the given experssion when $N_x^0 = 0$, $\mu = 0,3$

$$m_{кр} = 1,7E\delta R\left(\frac{\delta}{R}\right)^2 + 2,28qR\delta \text{ when } n = 1,72\sqrt{\frac{R}{\delta}}.$$

If the axial force N_x^0 is not equal to zero, then the critical value of moment m can be determined from the above given complete expression by means of its minimization according to parameter x.

4. *From the expression for* $m_{кр}$ it is easy to obtain the formula for the critical value of angle θ_0 (Fig. 209) taking into account the internal pressure q and the axial tensile force N_x^0.

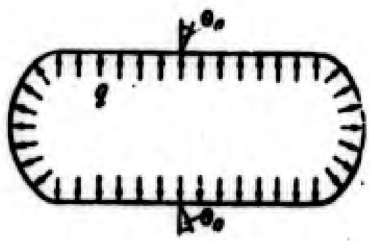


Fig. 209.

In this instance the expressions for force N_y^0 and the deflection w in the subcritical state coincide with the analogous expression under a load of the shell by a distributed moment m. From the comparison of these expressions it follows that $m = 4Dk\theta_0$, whence

$$\theta_0 = \frac{m}{4Dk}.$$

By substituting $m_{кр}$ in this expression when $N_x^0=0$ we will obtain

$$\theta_{0кр}=3,62\sqrt{\frac{\delta}{R}}+\frac{4,85qR}{E\delta}\sqrt{\left(\frac{R}{\delta}\right)^3}.$$

From this formula it is evident that when $q = 0$ the critical value of angle θ_0 does not depend on the elastic constants of the material of the shell.

§ 64. Rigidity of a Spherical Shell Under Loading by its Axisymmetrical Linear Load and by Int rnal Pressure

Under loading of a spherical shell according to Fig. 210 in subcritical state the following expressions for the internal forces are:

$$N_x^0=-N_x^0+\frac{P}{2}e^{-\beta\psi}\cos\beta\psi\,\mathrm{tg}\,\psi,$$

$$N_y^0=-\frac{qR}{2}+\frac{P\beta}{2}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi),$$

where

$$N_x^0=-\frac{1}{2}qR.$$

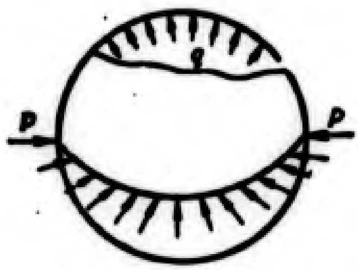


Fig. 210.

Here, the minus sign indicates the presence of stress. For the solution of the problem let us assume that

$$w=Ae^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi)\cos n\theta,$$

$$\varphi=Be^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi)\cos n\theta.$$

The expression for w based on the structure coincides with the expressions for ε_θ in the subcritiaal state. The expression for ϕ satisfies the condition

$$\sigma_{xy} = -\left(\frac{\partial^2 \varphi}{\partial x \, \partial y}\right)_{\psi=0} = 0.$$

By substituting the corresponding derivatives from the expressions of ϕ and w in Vlasov's equations and by integrating them according to Bubnov-Galerkin's method within the limits through ψ from zero to ∞, through θ - from zero to 2π, we will obtain

$$B = -\frac{E\delta R A(\beta^2 + 1,5n^2)}{2\left(-3\beta^4 + \beta^2 n^2 + \frac{3}{4} n^4\right)},$$

$$-\frac{B}{2R}(\beta^2 + 1,5n^2) + \frac{DA}{R^2} \times$$
$$\times\left(-3\beta^4 + \beta^2 n^2 + \frac{3}{4} n^4\right) -$$
$$-\left(\frac{3P\beta}{10} - \frac{3qR}{8}\right) An^2 +$$
$$+\left(Pf + \frac{qR}{4\beta}\right) A\beta^3 = 0.$$

By excluding parameters A and B from these equations, we will have

$$\frac{P\beta^3}{E\delta} = \frac{1}{0,3x - f}\left[\frac{(1 + 1,5x)^2}{-12 + 4x + 3x^2} + \frac{-12 + 4x + 3x^2}{16} + \right.$$
$$\left. + \frac{qR\beta^2}{4E\delta}(1 + 1,5x)\right],$$

where

$$x = \left(\frac{n}{\beta}\right)^2,$$

$$f = -\int_0^{\pi/2} e^{-3\beta\psi} \cos\beta\psi \cos 2\beta\psi \, \mathrm{tg}\, \psi \, d\psi.$$

According to Simpson's formula the value of the last integral will be

$$f = -\frac{\pi}{24}\left(1,66e^{-\frac{3\pi\beta}{8}} \cos\frac{\pi\beta}{4} \cos\frac{\pi\beta}{8} + 2e^{-\frac{3\pi\beta}{4}} \cos\frac{\pi\beta}{2} \cos\frac{\pi\beta}{4} + \right.$$
$$\left. + 9,66e^{-\frac{9\pi\beta}{8}} \cos\frac{3\pi\beta}{4} \cos\frac{3\pi\beta}{8} + 1255,8\, e^{-\frac{3\pi\beta}{2}} \cos\pi\beta \cos\frac{\pi\beta}{2}\right).$$

The meridional force N_φ, which appears in the shell from the action of load $P_{кр}$ is considered in this expression. The calculations of the values of f are presented in Table 14.

Table 14.

β	10	30	50	90
f	$0{,}5\cdot 10^{-6}$	$0{,}7\cdot 10^{-50}$	$0{,}9\cdot 10^{-98}$	$0{,}13\cdot 10^{-181}$

The table shows that component f in the denominator of the fraction in front of the squared bracket can be ignored in comparison with $0.3x$, because the latter will be considerably greater, than f (by several orders). Then

$$\frac{P\beta^2}{E\delta}=\frac{(1+1{,}5x)^2}{0{,}3x(-12+4x+3x^2)}+\frac{-12+4x+3x^2}{4{,}8x}+\frac{qR\beta^2}{1{,}2E\delta}\left(1{,}5+\frac{1}{x}\right).$$

If in this expression we consider only the annular stresses from the internal pressure, then, by eliminating the component $1/x$ in the last member, we will obtain

$$\frac{P\beta^2}{E\delta}=\frac{(1+1{,}5x)^2}{0{,}3x(-12+4x+3x^2)}+\frac{-12+4x+3x^2}{4{,}8x}+\frac{1{,}2qR\beta^2}{E\delta}.$$

The minimum of this expression in terms of parameter x is

$$P_{кр}=1{,}45E\delta\sqrt{\left(\frac{\delta}{R}\right)^3}+0{,}97qR\sqrt{\frac{\delta}{R}}$$

when

$$n=2{,}15\sqrt{\frac{R}{\delta}}.$$

If it is necessary to also take into account the meridional component of stresses from the internal pressure, then on should make use of the complete original expression, by determining its minimum according to parameter x.

Under a loading of the sphere by a distributed moment m according to Fig. 211 for the internal forces we will have the following expressions:

Fig. 211.

$$N_x^0 = \frac{m\beta}{2R} e^{-\beta\psi} (\sin\beta\psi + \cos\beta\psi) \operatorname{tg}\psi - \frac{qR}{2},$$

$$N_\psi^0 = \frac{m\beta^2}{R} e^{-\beta\psi} \sin\beta\psi - \frac{qR}{2}.$$

For the approximate solution of the problem let us assume that

$$w = Ae^{-\beta\psi} \sin\beta\psi \cos n\theta,$$

$$\varphi = Be^{-\beta\psi} \sin\beta\psi \cos n\theta.$$

These expressions satisfy all the necessary kinematic and statical boundary conditions along the contour of pits and bulges:

$$\left.\begin{aligned} & w = 0, \\ & \varepsilon_x = \frac{\partial^2\varphi}{\partial y^2} = 0. \end{aligned}\right\} \psi = 0$$

After the substitution of the accepted expressions for deflection and function of stresses in Vlasov's equation and their integration according to the Bubnov-Galerkin method within the limits 0-∞ and 0-2π we will obtain the following value for the moment m:

$$\frac{m\beta^4}{2E\delta R} = \frac{1}{\frac{x}{15} + f} \left[\frac{(2+x)^2}{8(-4+4x+x^2)} + \frac{-4+4x+x^2}{32} + \frac{qR\beta^2}{16E\delta}(2+x) \right],$$

where

$$x = \left(\frac{n}{\beta}\right)^2,$$

$$f = \int_0^{\pi/2} e^{-3\beta\psi} (\sin^2\beta\psi \cos\beta\psi + \cos^2\beta\psi \sin\beta\psi) \operatorname{tg}\psi \, d\psi.$$

After the calculation of this integral according to Simpson's formula one obtains

$$f = \frac{\pi}{24\sqrt{2}} \Big[1{,}66 e^{-3\pi\beta} \times \sin\left(\frac{\pi\beta}{8} + \frac{\pi}{4}\right) \sin\frac{\pi\beta}{4} + 2e^{-\frac{3\pi\beta}{4}} \sin\left(\frac{\pi\beta}{4} + \frac{\pi}{4}\right) \sin\frac{\pi\beta}{2} + 9{,}66 e^{-\frac{3\pi\beta}{8}} \sin\left(\frac{3\pi\beta}{8} + \frac{\pi}{4}\right) \sin\frac{3\pi\beta}{4} + 1255{,}8 e^{-\frac{3\pi\beta}{2}} \sin\left(\frac{\pi\beta}{2} + \frac{\pi}{4}\right) \sin\pi\beta \Big].$$

The meridional force N_φ, appearing in the shell due to action of moment $m_{кр}$ is considered in this expression, just as in the case of the action of distributed force P. The calculations of the value f for the various β shows that just as in the preceding case, they are by far less than the expected value of parameter x. Therefore, the value f can be ignored in the comparison with $\frac{1}{15}x$. Then

$$\frac{m\beta^4}{2E\delta R} = \frac{15(2+x)^2}{8x(-4+4x+x^2)} + \frac{15(-4+4x+x^2)}{32x} + \frac{15qR\beta^2}{16E\delta}\left(1+\frac{2}{x}\right). \quad (13.4)$$

If in this expression we discard the meridional component of stress from the internal pressure, then one can obtain the following formula for the critical moment

$$\frac{m\beta^4}{2E\delta R} = \frac{15(2+x)^2}{8x(-4+4x+x^2)} + \frac{15(-4+4x+x^2)}{32x} + \frac{15qR\beta^2}{16E\delta}.$$

The minimum of this expression in terms of parameter x will be equal to

$$m_{кр} = 2{,}57\frac{E\delta^3}{R} + 1{,}13qR\delta \quad (13.5)$$

when

$$n = 2{,}04\sqrt{\frac{R}{\delta}}.$$

For the determination $m_{кр}$ taking into account meridional component of stress from the internal pressure one must use the complete expression (13.4), by determining it minimum according to parameter x.

If the loading of a spherical shell is carried out according to Fig. 212, then the formula for the critical value of angle θ_0 can be obtained from the results of the example, given on page 407 by using the dependence

$$\theta_0 = \frac{mR}{4D\beta^3}.$$

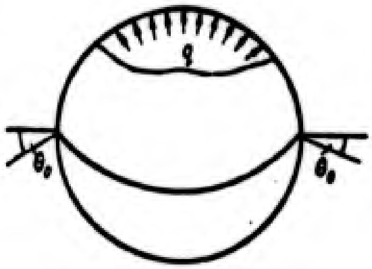


Fig. 212.

Then when using, for example, the formula (13.5) we will obtain the expression for $\theta_{0_{кр}}$

$$\theta_{0кр} = 5,4\sqrt{\frac{\delta}{R} + \frac{2.4qR}{E\delta}}\sqrt{\frac{R}{\delta}}.$$

§ 65. Rigidity of a Doughnut-Shaped Shell Loading Them with an Axisymmetrical Linear Load and by Internal Pressure

A doughnut-shaped shell under a load according to Fig. 213 for forces N_x^0 and N_y^0 of the subcritical state we have the expressions

$$N_x^0 = -N_x^0 + \frac{P}{2} e^{-\beta\psi} \cos \beta\psi \, \mathrm{tg}\, \psi,$$

$$N_y^0 = -N_y^0 + \frac{PR_2\beta}{2R_1} e^{-\beta\psi} (\sin \beta\psi + \cos \beta\psi).$$

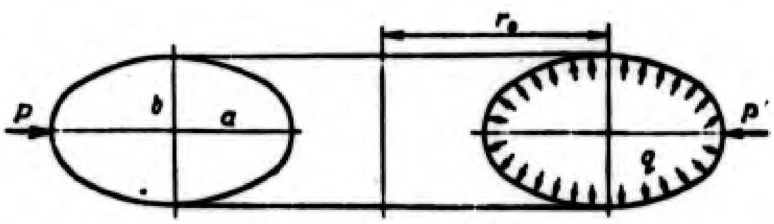


Fig. 213.

As can be seen from the previous paragraphs of this chapter, the method of the solution of the investigated problems are very uniform. Therefore in this case let us limit ourselves to the summary of final results.

For the investigated case the expression for P_{kp} has the form

$$\frac{3P_{kp}\beta^2R_2}{10E\delta R_1}=\frac{(1+1.5\gamma x)^2}{4x(-3+\gamma^2x+0.75\gamma^4x^2)}+$$
$$+\frac{-3+\gamma^2x+0.75\gamma^4x^2}{4x}+\frac{N_x^0R_2^2\beta^2}{2E\delta R_1^2x}+\frac{3N_y^0\beta^2}{4E\delta},$$

where

$$x=\left(\frac{\alpha}{\beta}\right)^2.$$

In the case of an elliptical torus

$$R_1=\frac{b^2}{a},\ R_2=r_0+a,\ \gamma=\frac{R_1}{R_2}=\frac{b^2}{a(r_0+a)},$$
$$N_x^0=\frac{qa(2r_0+a)}{2(r_0+a)},\qquad N_y^0=\frac{q[(b^2-a^2)(2r_0+a)^2+ab^2]}{2b^2}.$$

If the torus is circular in cross section, then

$$R_1=a,\ R_2=r_0+a,\ \gamma=\frac{a}{r_0+a},\quad N_x^0=\frac{qa(2r_0+a)}{2(r_0+a)},\quad N_y^0=\frac{qa}{2}.$$

In the case of a sphere

$$r_0=0,\ R_1=R_2=a=b=R,\ \gamma=1,\ N_x^0=N_y^0=\frac{qR}{2}.$$

A torus under a load according to Fig. 214

$$N_x^0=-N_x^0+\frac{m\beta}{2R_1}e^{-\beta\psi}(\sin\beta\psi+\cos\beta\psi),$$
$$N_y^0=-N_y^0+\frac{mR_2\beta^2}{R_1^2}e^{-\beta\psi}\sin\beta\psi.$$

Then

$$\frac{4(1-\mu^2)m_{kp}R_1^2}{3E\delta^3R_2}=\frac{(2+\gamma x)^2}{x(-4+4\gamma^2x+\gamma^4x^2)}+$$
$$+\frac{-4+4\gamma^2x+\gamma^4x^2}{4x}+\frac{2N_x^0R_2^2\beta^2}{E\delta R_1^2x}+\frac{N_y^0\beta^2}{E\delta}.$$

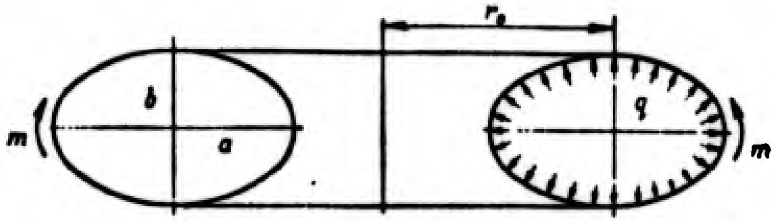


Fig. 214.

A shell under a load according to Fig. 215; the expression for the critical value of angle θ_0 can be obtained from the relationship

$$\theta_{0\text{кр}} = \frac{m_{\text{кр}} R_1}{4D\beta}.$$

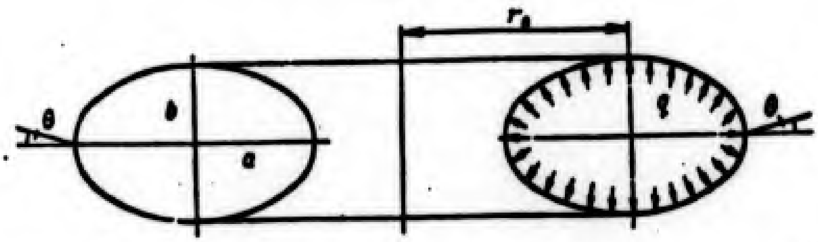


Fig. 215.

CHAPTER XIV

RIGIDITY OF FRAMES, ATTACHED TO THE SHELL BY LOADING THEM WITH A LINEAR UNIFORM LOAD

During the construction of tanks and other containers, consisting of a shell of various geometrical form at the places of their overlap rigid elements - frames are usually placed; these absorb the bracing forces which appear in the section of the joint of the shell during the action of internal pressure. For a certain value of this pressure the bracing forces can attain a magnitude at which the loss in the rigidity of the frame is possible.

In this chapter problems based on the determination of a critical value of the internal pressure in the tanks, at which the loss in the rigidity of the reinforced frame is possible are examined.

The obtained formulas can also be used in the case when frame is loaded by a distributed load of some other origin.

§ 66. Rigidity of a Frame of a Lenticular Container

Let us examine a linticular container, loaded by an internal excess pressure q (Fig. 216).

From this figure it is evident that the linear forces S_1 and S_2, applied to the frame from the bottom side, compress it, and at a certain value of internal pressure q the frame can lose rigidity. Subsequently, the loss in rigidity of the frame only in its plane will be considered.

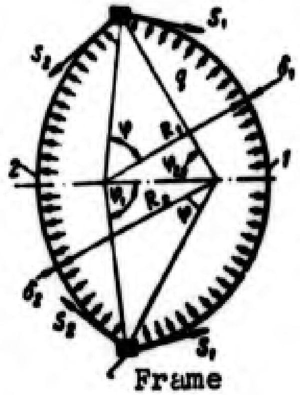


Fig. 216.

For certainty let us consider that the frame has rectangular form in cross section, and line of intersection of the average surfaces of shells 1 and 2 coincides with line of the centers of gravity of this frame.

Problem on hand can be expediently solved by the energy method. The total potential energy of the investigated system is

$$Э=V_{ср}^{(1)}+V_{изг}^{(1)}+V_{ср}^{(2)}+V_{изг}^{(2)}+V_{кол}+T,$$

where V - energy of deformation of the shells $V_{ср}$, $V_{изг}$ and ring $V_{кол}$, and T - work of the external forces. During the computing of the work of the external forces let us consider only the work of radial components of forces S_1 and S_2, and disregard the work of pressure q on the deflection of the shell.

Furthermore, during the solution of all the problems considered below let us base it on the fact that the loss in force of the shell attached to the frame, bears a local character. The pits and bulges are generated only in the narrow zone, which is adjacent to the frame. Hence, let us consider that in this case the apparatus of the theory of gently sloping shells is applicable.

The results obtained under the above shown assumptions provide values for the critical forces as a reserve of force that is especially important during the analysis of the bearing capacity of the investigated complex systems.

In the developed form the expression for the total energy is written so:

$$
\begin{aligned}
Э = {} & \frac{1}{2E\delta_1}\int_0^{\infty}\int_0^{2\pi R_1}\left[\left(\frac{\partial^2\varphi_1}{\partial x^2}\right)^2+\left(\frac{\partial^2\varphi_1}{\partial y^2}\right)^2-2\mu\frac{\partial^2\varphi_1}{\partial x^2}\frac{\partial^2\varphi_1}{\partial y^2}+\right.\\
& \left.+2(1+\mu)\left(\frac{\partial^2\varphi_1}{\partial x\,\partial y}\right)^2\right]dx\,dy+\frac{D_1}{2}\int_0^{\infty}\int_0^{2\pi R_1}\left[\left(\frac{\partial^2 w_1}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_1}{\partial y^2}\right)^2+\right.\\
& \left.+2\mu\frac{\partial^2 w_1}{\partial x^2}\frac{\partial^2 w_1}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w_1}{\partial x\,\partial y}\right)^2\right]dx\,dy+\\
& +\frac{1}{2E\delta_2}\int_0^{\infty}\int_0^{2\pi R_1}\left[\left(\frac{\partial^2\varphi_2}{\partial x^2}\right)^2+\left(\frac{\partial^2\varphi_2}{\partial y^2}\right)^2-2\mu\frac{\partial^2\varphi_2}{\partial x^2}\frac{\partial^2\varphi_2}{\partial y^2}+2(1+\mu)\left(\frac{\partial^2\varphi_2}{\partial x\,\partial y}\right)^2\right]dx\,dy+\\
& +\frac{D_2}{2}\int_0^{\infty}\int_0^{2\pi R_1}\left[\left(\frac{\partial^2 w_2}{\partial x^2}\right)^2+\left(\frac{\partial^2 w_2}{\partial y^2}\right)^2+2\mu\frac{\partial^2 w_2}{\partial x^2}\frac{\partial^2 w_2}{\partial y^2}+2(1-\mu)\left(\frac{\partial^2 w_2}{\partial x\,\partial y}\right)^2\right]dx\,dy+\\
& +\frac{EJ}{2}\int_0^{2\pi R_k}\left(\frac{\partial^2 w_k}{\partial y^2}+\frac{w_k}{R_k^2}\right)^2 dy+\frac{S}{2}\int_0^{2\pi R_k}\left(\frac{\partial^2 w_k}{\partial y^2}+\frac{w_k}{R_k^2}\right)w_k\,dy.
\end{aligned}
$$

In this expression the first two integrals determines the strain energy of the shell 1, third and fourth - strain energy of the shell 2, fifth - strain energy of the frame. The last integral expresses the work of compressive force S, equal to the projection of the linear forces S_1 and S_2 on the plane of the frame.

For the connection of the functions of stresses ϕ_1 and ϕ_2 with the deflections of the shells w_1 and w_2 the equations of compatibility of deformations are used:

$$\nabla^2\nabla^2\varphi_1=\frac{E\delta_1}{R_1}\left(\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}\right),$$

$$\nabla^2\nabla^2\varphi_2=\frac{E\delta_2}{R_2}\left(\frac{\partial^2 w_2}{\partial x^2}+\frac{\partial^2 w_2}{\partial y^2}\right).$$

Because the deflection of pits and bulges diminishes in proportion to the distance from the frame, then in the expressions for the deflections and functions of stress of covers 1 and 2, this must also be considered and by limiting ourselves to only the first approximation, let us assume that

$$w_1 = Ae^{-\beta_1\varphi}(\sin\beta_1\varphi + \cos\beta_1\varphi)\cos n\theta,$$
$$w_2 = Be^{-\beta_2\varphi}(\sin\beta_2\varphi + \cos\beta_2\varphi)\cos n\theta,$$
$$\varphi_1 = Ce^{-\beta_1\varphi}(\sin\beta_1\varphi + \cos\beta_1\varphi)\cos n\theta,$$
$$\varphi_2 = De^{-\beta_2\varphi}(\sin\beta_2\varphi + \cos\beta_2\varphi)\cos n\theta.$$

In these expressions, the coordinate ϕ is counted off just as shown in Fig. 216. When $\phi = 0$ the projection of deflections w_1 and w_2 on the plane of the frame should be equal to one another:

$$A\sin\varphi_1 = B\sin\varphi_2. \qquad (14.1)$$

Furthermore, designated here is:

$$\beta_1 = \sqrt[4]{\frac{3(1-\mu^2)R_1^2}{\delta_1^2}}, \qquad \beta_2 = \sqrt[4]{\frac{3(1-\mu^2)R_2^2}{\delta_2^2}}.$$

The expressions, accepted for the functions w and ϕ, agree with the analogous expressions, obtained with the solution of the problem of the edge effect. Analogous expressions were used in § 64 during the determination of the critical value of the linear loads P and m for a spherical shell. After the substitution of the corresponding derivatives from the functions w and ϕ in the expression of total energy and after the calculation of the integrals we will obtain

$$Э = \frac{\pi C^2}{2E\delta_1 R_1^2\beta_1}(\beta_1^4 + n^2\beta_1^2 + 0{,}75n^4) + \frac{\pi D_1 A^2}{2R_1^2\beta_1}(\beta_1^4 + n^2\beta_1^2 + 0{,}75n^4) +$$
$$+ \frac{\pi D^2}{2E\delta_2 R_2^2\beta_2}(\beta_2^4 + n^2\beta_2^2 + 0{,}75n^4) + \frac{\pi D_2 B^2}{2R_2^2\beta_2}(\beta_2^4 + n^2\beta_2^2 + 0{,}75n^4) +$$
$$+ \frac{\pi EJA^2(n^2-1)^2\sin^2\varphi_1}{2R_к^3} - \frac{\pi SA^2(n^2-1)\sin^2\varphi_1}{2R_к}.$$

Let us obtain the connection between the parameters C and A, D and B from the equations of compatibility of the deformations after their integration according to Bubnov-Galerkin method:

$$C = -\frac{E\delta_1 R_1 A(2\beta_1^2 + 3n^2)}{-12\beta_1^4 + 4\beta_1^2 n^2 + 3n^4},$$
$$D = -\frac{E\delta_2 R_2 B(2\beta_2^2 + 3n^2)}{-12\beta_2^4 + 4\beta_2^2 n^2 + 3n^4}. \qquad (14.2)$$

After using the relationship (14.1) and (14.2), the expression for the total energy assumes the form

$$\frac{2\text{Э}}{\pi E\delta_1}=B^2\left\{\frac{N_1}{\beta_1}\left(\frac{\sin\varphi_2}{\sin\varphi_1}\right)^2\left[Q_1^2+\frac{\left(\frac{\delta_1}{R_1}\right)^2}{12\,(1-\mu^2)}\right]+\right.$$

$$\left.+\frac{N_2\delta_2}{\beta_2\delta_1}\left[Q_2^2+\frac{\left(\frac{\delta_2}{R_2}\right)^2}{12\,(1-\mu^2)}\right]+\frac{J\,(n^2-1)^2\sin^2\varphi_2}{R_\text{к}^3\delta_1}-\frac{S\,(n^2-1)\sin^2\varphi_2}{E_1\delta_1R_\text{к}}\right\}.$$

Designated here is:

$$N_1=\beta_1^4+n^2\beta_1^2+0{,}75n^4,\quad N_2=\beta_2^4+n^2\beta_2^2+0{,}75n^4,$$

$$Q_1=\frac{2\beta_1^2+3n^2}{-12\beta_1^4+4n^2\beta_1^2+3n^4},$$

$$Q_2=\frac{2\beta_2^2+3n^2}{-12\beta_2^4+4n^2\beta_2^2+3n^4}.$$

Let us first apply the expression to the obtained energy of virtual displacements, according to which under conditions of equilibrium the sum of the work of all forces, applied to the given body for virtual possible movements, should be equal to zero, i.e. $\delta\text{Э}=0$.

From this condition we will find that

$$P_\text{кр}=\frac{EJ\,(n^2-1)}{R_\text{к}^3}+\frac{E\delta_1}{4\beta_1\,(n^2-1)\sin^2\varphi_2}\left\{\left(\frac{\sin\varphi_2}{\sin\varphi_1}\right)^2\left(1+\frac{n^2}{\beta_1^2}+\right.\right.$$

$$\left.+0{,}75\frac{n^4}{\beta_1^4}\right)\left[1+\frac{\left(1+1{,}5\frac{n^2}{\beta_1^2}\right)^2}{\left(-3+\frac{n^2}{\beta_1^2}+0{,}75\frac{n^4}{\beta_1^4}\right)^2}\right]+$$

$$\left.+\frac{\beta_1\delta_2}{\beta_2\delta_1}\left(1+\frac{n^2}{\beta_2^2}+0{,}75\frac{n^4}{\beta_2^4}\right)\left[1+\frac{\left(1+1{,}5\frac{n^2}{\beta_2^2}\right)^2}{\left(-3+\frac{n^2}{\beta_2^2}+0{,}75\frac{n^4}{\beta_2^4}\right)^2}\right]\right\}. \qquad (14.3)$$

Here

$$P_\text{кр}=\frac{S_\text{кр}}{R_\text{к}};$$

$R_к$ — radius of the frame (ring).

If the linear load P, acting on the frame, is made up from the projections of forces S_1 and S_2, appearing in the bottom due to internal pressure, then

$$P=S_1\cos\varphi_1+S_2\cos\varphi_2=\frac{qR_1\cos\varphi_1}{2}\left(1+\frac{R_2\cos\varphi_2}{R_1\cos\varphi_1}\right).$$

Here q - intensity of the internal pressure.

Then for the determination of critical internal pressure we will obtain the formula of the form

$$\frac{q_{кр}R_1\cos\varphi_1}{2}\left(1+\frac{R_2\cos\varphi_2}{R_1\cos\varphi_1}\right)=\frac{EJ(n^2-1)}{R_к^3}+\frac{E\delta_1}{4\beta_1(n^2-1)\sin^2\varphi_2}\times$$
$$\times\left\{\left(\frac{\sin\varphi_2}{\sin\varphi_1}\right)^2\left(1+\frac{n^2}{\beta_1^2}+0{,}75\frac{n^4}{\beta_1^4}\right)\left[1+\frac{\left(1+1{,}5\frac{n^2}{\beta_1^2}\right)^2}{\left(-3+\frac{n^2}{\beta_1^2}+0{,}75\frac{n^4}{\beta_1^4}\right)^2}\right]+\right.$$
$$\left.+\frac{\beta_1\delta_2}{\beta_2\delta_1}\left(1+\frac{n^2}{\beta_2^2}+0{,}75\frac{n^4}{\beta_2^4}\right)\left[1+\frac{\left(1+1{,}5\frac{n^2}{\beta_2^2}\right)^2}{\left(-3+\frac{n^2}{\beta_2^2}+0{,}75\frac{n^4}{\beta_2^4}\right)^2}\right]\right\}. \qquad (14.4)$$

The entire set of problems based on calculation for the rigidity of the frames, which reinforce lenticular container is embraced by formulas (14.3) and (14.4). These frames can be loaded by braced forces from internal pressure or by a distributed linear load of some other origin.

In those and other cases the given formulas allow for the determination of critical values of the shown loadings inasmuch as these loads will be somewhat under rated for those containers, which differ from a complete sphere by virture of the safety factor.

§ 67. Rigidity of the Frames of Cylindrical Containers having Spherical Bottoms

Figure 217 represents a diagram of a cylindrical container with a spherical bottom. Because the container is under the action of

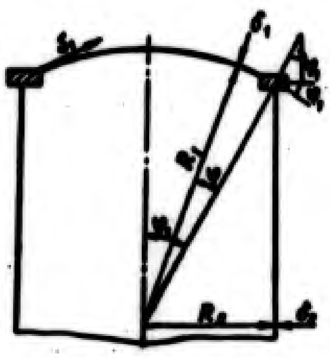


Fig. 217.

internal pressure, then the frame will be loaded by a distributed compressive force from the bottom side, equal to $S_1 \cos\varphi_1$, and by forces of the fringe effect, which we can subsequently ignore.

For the solution of this problem let us employ the energy method, used in the previous paragraph.

For the components of deflection and for the function of stresses in this case it is possible to assume

$$w_1 = Ae^{-\beta_1\varphi}(\sin\beta_1\varphi + \cos\beta_1\varphi)\cos n\theta,$$
$$w_2 = Be^{-k_2x}(\sin k_2x + \cos k_2x)\cos n\theta,$$
$$\varphi_1 = Ce^{-\beta_1\varphi}(\sin\beta_1\varphi + \cos\beta_1\varphi)\cos n\theta,$$
$$\varphi_2 = De^{-k_2x}(\sin k_2x + \cos k_2x)\cos n\theta.$$

Here

$$k_2 = \sqrt[4]{\frac{3(1-\mu^2)}{R_2^2 h_2^2}}.$$

The remaining designations are the same as those in the previous paragraph.

Let us eliminate the intermediate computations and limit ourselves to the presentation of the final result:

$$\frac{S_{кр}}{R_2} = \frac{EJ(n^2-1)}{R_2^3} + \frac{Eh_1}{4\beta_1(n^2-1)}\left\{\frac{1}{\sin^2\varphi_1}\left(1+\frac{n^2}{\beta_1^2}+0,75\frac{n^4}{\beta_1^4}\right)\times\right.$$
$$\left.\times\left[1+\frac{\left(1+1,5\frac{n^2}{\beta_1^2}\right)^2}{\left(-3+\frac{n^2}{\beta_1^2}+0,75\frac{n^4}{\beta_1^4}\right)^2}\right]+\frac{\beta_2 h_2}{R_2 k_2 h_1}\times\left(1+\frac{n^2}{R_2^2k_2^2}+0,75\frac{n^4}{R_2^4k_2^4}\right)\left[1+\frac{1}{\left(-3+\frac{n^2}{R_2^2k^2}+0,75\frac{n^4}{R_2^4k_2^4}\right)^2}\right]\right\}.$$

(14.5)

Here S - the force in the transverse section of the frame.

If the frame is under the action of braced forces, caused by internal pressure, then in the left part of this formula one ought to substitute

$$\frac{S_{\text{кр}}}{R_2} = \frac{q_{\text{кр}} R_1 \cos \varphi_1}{2}.$$

In this instance we will obtain the formula for the determination of the critical pressure in the tank, at which the frame loses rigidity.

§ 68. Rigidity of a Frame which Reinforces the Cylindrical Part of the Tank

In certain cases the cylindrical tanks are fitted with frames, spaced from the bottom, which during operational use can be loaded by a uniformly distributed compressive radial load. Let us determine the critical value of this load.

Figure 218 represents a diagram of such a container. For the solution of the problem let us employ the same methods as those in the previous paragraphs. For the functions of deflections and for the function of stresses let us use the expressions

$$w_1 = Ae^{-k_1 x}(\sin k_1 x + \cos k_1 x)\cos n\theta,$$

$$w_2 = Be^{-k_2 x}(\sin k_2 x + \cos k_2 x)\cos n\theta,$$

$$\varphi_1 = Ce^{-k_1 x}(\sin k_1 x + \cos k_1 x)\cos n\theta,$$

$$\varphi_2 = De^{-k_1 x}(\sin k_2 x + \cos k_2 x)\cos n\theta.$$

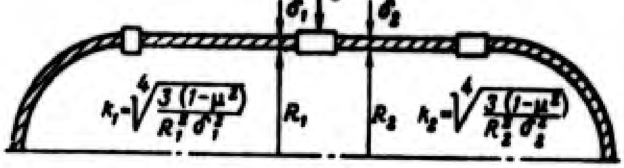


Fig. 218.

When $x = 0$, w_1 should be equal to w_2. Hence, it follows that $A = B$. The result of the solution of this problem can be written based on the analog in the previous paragraph, where the coupling of a cylindrical shell with a spherical one was considered. For this purpose in the formula (14.5) one ought to substitute k_1R_1 in place of β_1 and set $\phi_1 = 90°$. Then

$$\frac{S_{\text{кр}}}{R_2} = \frac{EJ(n^2-1)}{R_2^3} + \frac{E\delta_1}{4R_1k_1(n^2-1)}\left\{\left(1 + \frac{n^2}{R_1^2k_1^2} + 0{,}75\frac{n^4}{R_1^4k_1^4}\right) \times \left[1 + \frac{1}{\left(-3 + \frac{n^2}{R_1^2k_1^2} + 0{,}75\frac{n^4}{R_1^4k_1^4}\right)^2}\right] + \frac{R_1k_1\delta_2}{R_2k_2\delta_1}\left(1 + \frac{n^2}{R_2^2k_2^2} + 0{,}75\frac{n^4}{R_2^4k_2^4}\right)\left[1 + \frac{1}{\left(-3 + \frac{n^2}{R_2^2k_2^2} + 0{,}75\frac{n^4}{R_2^4k_2^4}\right)^2}\right]\right\}.$$

Value of a linear load will be found from the relationship

$$q_{\text{кr}} = \frac{S_{\text{кр}}}{R_2}.$$

§ 69. Determination of the Effective Width of a Shell During the Calculation for the Strength of Isolated Frames

Formulas (14.3)-(14.5) are valid only within the limits of proportionality. If the critical stresses in the frame will be higher than the limit of proportionality, then such a frame can be considered an isolated ring.

For such a calculation it is necessary to know, which part of the adjoining shell will operate in conjunction with the frame. One ought to take into account this part of the sheathings during the calculation of the isolated frame. First, let us examine the work of the cylindrical shell which is adjacent to the frame. Let us assume that the permissible deformation ε_0 are accepted for the frame. Then, for the deflection of the cover we will have the expression (as Fig. 219)

$$w = R\varepsilon_0 e^{-kx}(\sin kx + \cos kx),$$

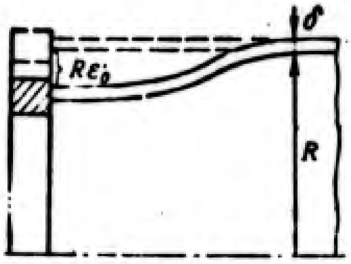


Fig. 219.

which is obtained from the general formula for the deflection of an infinitely long cylindrical cover, loaded with edge by force M_0 and Q_0 (§ 26):

$$w = \frac{2R^2k}{E\delta}\left[Q_0 \cos kx + M_0 k (\cos kx - \sin kx)\right],$$

if Q_0 and M_0 are determined from the conditions

$$(w)_{x=0} = R\varepsilon_0, \quad \left(\frac{dw}{dx}\right)_{x=0} = 0.$$

We will regard the effective width from the condition of equality of the energy of deformation of the shell, which should be adjacent to the frame, and regard the energy of deformation of the effective width of the cover, which should be included in the work of the frame during its calculation as an isolated frame (Fig. 220).

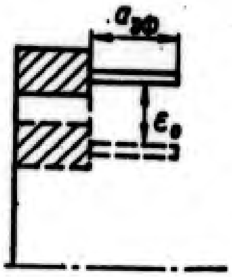


Fig. 220.

The strain energy of the shell, adjacent to the frame (Fig. 219), will be equal to

$$Э = \frac{ER\delta}{2(1-\mu^2)} \int_0^\infty \int_0^{2\pi} \varepsilon_y^2 dx\, d\theta +$$

$$+ \frac{E\delta^3 R}{24(1-\mu^2)} \int_0^\infty \int_0^{2\pi} (\chi_x^2 + \chi_y^2 + 2\mu\chi_x\chi_y)\, dx\, d\theta,$$

where

$$\varepsilon_\theta=\frac{w}{R},\quad \chi_x=\frac{d^2w}{dx^2},\quad \gamma_\theta=\frac{w}{R^2}.$$

Having substituted here the accepted expression for w, after integration we will obtain

$$Э_1=\frac{3\pi ER\delta\varepsilon_0^2}{4(1-\mu^2)k}+\frac{\pi E\delta^3\varepsilon_0^2}{12(1-\mu^2)Rk}\left(R^2k^4+\frac{3}{4}-\mu R^2k^2\right).$$

The strain energy of the shell in the section of effective width $a_{эф}$ (see Fig. 220) is

$$Э_2=\frac{E\delta R}{2(1-\mu^2)}\int_0^{a_{эф}}\int_0^{2\pi}\varepsilon_\theta^2\,dx\,d\theta+\frac{E\delta^3R}{24(1-\mu^2)}\int_0^{a_{эф}}\int_0^{2\pi}\gamma_\theta^2\,dx\,d\theta,$$

where

$$\varepsilon_\theta=\varepsilon_0,\quad \gamma_\theta=\frac{\varepsilon_0}{R}.$$

Then

$$Э_2=\frac{\pi E\delta R\varepsilon_0^2a_{эф}}{1-\mu^2}+\frac{\pi E\delta^3\varepsilon_0^2a_{эф}}{12(1-\mu^2)R}.$$

From the condition $Э_1=Э_2$ we will obtain when $\mu = 0.3$

$$a_{эф}\approx 0{,}76\sqrt{R\delta}.$$

The expression for $a_{эф}$ can also be obtained, if we base it on other prerequisites. Let us select, for example, such an area of a part of the shell attached to the frame, so that it could be transferred the same as the circumferential force, acting in the shell:

$$N_\theta=E\delta\frac{w}{R}=E\delta\varepsilon_0e^{-kx}(\sin kx+\cos kx).$$

The total circumferential force

$$Q_1=\int_0^\infty N_\theta\,dx=\frac{E\delta\varepsilon_0}{k}.$$

On the other hand, for the connected part of the shell one can write the relationship in the form

$$Q_2 = EF\varepsilon_0 = E\delta a_{\text{эф}}\varepsilon_0.$$

Because according to the condition

$$Q_1 = Q_2,$$

then

$$a_{\text{эф}} \approx 0{,}785\sqrt{R\delta}.$$

Now let us examine the determination of the effective width of the shell for the case of coupling the frame with the spherical shell. In this instance the expression for the circumferential force has the form

$$N_\theta = E\delta\varepsilon_0 e^{-\beta\varphi}(\sin\beta\varphi + \cos\beta\varphi).$$

The total force in the section of the shell is

$$Q_1 = \int_0^\infty N_\theta dx = \frac{ER_1\delta\varepsilon_0}{\beta}.$$

On the other hand, the total force, acting in the section of the connected part of the cover, is determined in the form

$$Q_2 = EF\varepsilon_0 = E\delta a_{\text{эф}}\varepsilon_0.$$

From the condition that $Q_1 = Q_2$, we will obtain

$$a_{\text{эф}} = 0{,}785\sqrt{R_1\delta}.$$

C H A P T E R   XV

RIGIDITY OF REINFORCED SHELLS

Reinforced shells are widely used in various engineering construction.

Subsequently we will consider only those shells, in which the superstructure is located along the lines of the main curvatures. Such shells are called orthotropic-structural shells. The methods of calculating such shells are based on the well worked-out methods of calculating isotropic shells. For this purpose one usually substitutes the reinforced shell with a certain equivalent, having a smooth shell with different hardnesses along the lines of the main curvatures. After introducing this for the calculation of an orthotropic-structural shell one can use the calculation apparatus of the theory of smooth shells.

§ 70. Derivation of the Expressions for the Given Rigidity. Hooke's Law for Reinforced Shells

Let us examine a cylindrical shell, reinforced in axial and circumferential directions by stringers and frames. Let us replace this shell with a certain equivalent such as a smooth shell. By the word "equivalent" is meant the equality in rigidity of the reinforced and smooth shells.

Rigidity under tension-compression in an axial direction. Force P acting on a smooth shell under compression in an axial direction does work, equal to

$$A_1 = P\Delta l,$$

where Δl - shortening of the shell. But

$$\Delta l = \varepsilon l = \frac{Pl}{FE_x} = \frac{Pl}{2\pi R\delta E_x}.$$

Then

$$A_1 = \frac{P^2 l}{2\pi R\delta E_x}.$$

With the compression of the reinforced shell by force P, a part of this force will be absorbed by the stringers $P_{стр}$ and a part - by the sheathing $P_{об}$. In this case force P does work, equal to

$$A_2 = nP_{стр}\Delta l_{стр} + P_{об}\Delta l_{об}.$$

But

$$\Delta l_{стр} = \varepsilon_{стр} l = \frac{P_{стр} l}{F_{стр} E_{стр}},$$

$$\Delta l_{об} = \varepsilon_{об} l = \frac{P_{об} l}{F_{об} E_{об}}.$$

Then

$$A_2 = \left(\frac{nP^2_{стр}}{F_{стр}E_{стр}} + \frac{P^2_{об}}{F_{об}E_{об}}\right) l.$$

A_1 should be equal to A_2, or

$$\frac{P^2}{2\pi R\delta E_x} = \frac{nP^2_{стр}}{F_{стр}E_{стр}} + \frac{P^2_{об}}{F_{об}E_{об}}. \qquad (15.1)$$

Having in view that

$$nP_{стр} + P_{об} = P, \quad \Delta l_{стр} = \Delta l_{об},$$

from these conditions

$$P_{стр} = \frac{F_{стр}E_{стр}}{F_{об}E_{об}} \frac{P}{1 + \frac{nF_{стр}E_{стр}}{F_{об}E_{об}}},$$

$$P_{об} = \frac{P}{1 + \frac{nF_{стр}E_{стр}}{F_{об}E_{об}}}.$$

Having substituted the values $P_{стр}$ and $P_{об}$ in (15.1), we obtain

$$E_x = F_{об}\left(1 + \frac{E_{стр}}{E_{об}} \frac{nF_{стр}}{2\pi R\delta}\right). \qquad (15.2)$$

In most cases $E_{об} = E_{стр} = E$, and the expression (15.2) is simplified.

Determination of the rigidity under tension-compression in the circumferential direction. In the determination of rigidity in a circumferential direction it is possible to make the assumption that the shell is under the action of a uniform internal pressure q, which produces work

$$A_1 = q\Delta V,$$

where ΔV - increase in the volume of the shell, equal to

$$\Delta V = [\pi(R+w)^2 - \pi R^2]\, l \approx 2\pi R l w.$$

But

$$\varepsilon = \frac{w}{R} = \frac{\sigma_y}{E_y} = \frac{qR}{E_y\delta},$$

whence

$$w = \frac{qR^2}{E_y\delta}.$$

Then

$$\Delta V = \frac{2\pi R^3 l q}{E_y\delta}.$$

Consequently

$$A_1 = \frac{2\pi R^3 l q^2}{E_y\delta}.$$

In the case of a reinforced shell, pressure q accomplishes the work

$$A_2 = q_{об}\Delta V_{об} + mS_{ш}\Delta l_{ш},$$

where $S_{ш} = q_{ш}a_{ш}R$ - the force being exerted on one frame; $a_{ш}$ - distance between frames;

$$\Delta V_{\text{об}}=[\pi(R+w)^2-\pi R^2]l=\frac{2\pi R^2 lq}{E_y\delta},$$

$$\Delta l_{\text{ш}}=\frac{q_{\text{ш}}a_{\text{ш}}R}{F_{\text{ш}}E_{\text{ш}}}\,2\pi R.$$

Then

$$A_2=\frac{2\pi R^3 lq_{\text{об}}^2}{E_{\text{об}}\delta}+\frac{2\pi mR^3a_{\text{ш}}^2q_{\text{ш}}^2}{E_{\text{ш}}F_{\text{ш}}}.$$

Just as in the first case, A_1 should be equal to A_2. Furthermore,

$$q_{\text{ш}}+q_{\text{об}}=q,\quad \varepsilon_{\text{об}}=\varepsilon_{\text{ш}}.$$

By eliminating all the intermediate computations, let us present the final result

$$E_y=E_{\text{об}}\left(1+\frac{E_{\text{ш}}F_{\text{ш}}}{E_{\text{об}}a_{\text{ш}}\delta}\right). \qquad (15.3)$$

If material of the sheathing and of the frames is identical, then $E_{\text{об}} = E_{\text{ш}} = E$.

Determination of the rigidity to shear. During the calculation of orthotropic-structural shells it is considered that only the sheathing takes part in the work of shear. Therefore, let us assume that

$$G=\frac{E_{\text{об}}}{2(1+\mu)}. \qquad (15.4)$$

Determination of the specified rigidity to bending and torsion. Flexural rigidity for a reinforced shell can be determined in the following form:

$$D_x=\frac{E_{\text{об}}\delta^3}{12(1-\mu_x\mu_y)}+\frac{E_{\text{стр}}J_{\text{стр}}}{a_{\text{стр}}},$$

$$D_y=\frac{E_{\text{об}}\delta^3}{12(1-\mu_x\mu_y)}+\frac{E_{\text{ш}}J_{\text{ш}}}{a_{\text{ш}}},$$

where $J_{\text{стр}}$ and $J_{\text{ш}}$ - moments of inertia of the stringer and frame; $a_{\text{стр}}$, $a_{\text{ш}}$ - distances between the stringers and frames.

As a safety factor the moments of inertia of the stringers and frames are commonly calculated relative to their own axis.

The specified rigidity under the twisting of a reinforced thin-walled shell is usually assumed without allowing for the effects of the superstructure according to the expression

$$D_k = \frac{G\delta^3}{12}.$$

By having the expressions (15.2)-(15.4) for Young's modulus and for the modulus of normal elasticity, it is possible to write the relationships for the deformations and stresses according to Hooke's law

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \mu_y \frac{\sigma_y}{E_y}, \qquad \varepsilon_y = \frac{\sigma_y}{E_y} - \mu_x \frac{\sigma_x}{E_x},$$
$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G},$$
$$\sigma_x = \frac{E_x}{1-\mu_x\mu_y}(\varepsilon_x + \mu_y\varepsilon_y), \quad \sigma_y = \frac{E_y}{1-\mu_x\mu_y}(\varepsilon_y + \mu_x\varepsilon_x), \quad \sigma_{xy} = G\varepsilon_{xy}.$$

The expressions for the linear forces and moments will have the form

$$N_x = \frac{E_x\delta}{1-\mu_x\mu_y}(\varepsilon_x + \mu_y\varepsilon_y), \quad N_y = \frac{E_y\delta}{1-\mu_x\mu_y}(\varepsilon_y + \mu_x\varepsilon_x),$$
$$N_{xy} = G\varepsilon_{xy}\delta, \quad M_x = -D_x(\chi_x + \mu_y\chi_y), \qquad (15.5)$$
$$M_y = -D_y(\chi_y + \mu_x\chi_x), \qquad M_{xy} = -2D_k\chi_{xy}.$$

The components of deformation and change in the curvatures are determined by the formula

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{w}{R_1}, \qquad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R_2},$$
$$\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \chi_x = -\frac{\partial^2 w}{\partial x^2} - \frac{w}{R_1^2}, \qquad (15.5')$$
$$\chi_y = -\frac{\partial^2 w}{\partial y^2} - \frac{w}{R_2^2}, \qquad \chi_{xy} = -\frac{\partial^2 w}{\partial x\,\partial y}.$$

Furthermore, the following relationships are valid:

$$E_x\mu_y = E_y\mu_x, \quad D_x\mu_y = D_y\mu_x. \qquad (15.6)$$

These relationships can be derived on the basis of the following reasonings.

Hooke's law in the biaxial strained state of an orthotropic body has the form

$$\sigma_x = C_{11}\varepsilon_x + C_{12}\varepsilon_y, \quad \sigma_y = C_{22}\varepsilon_y + C_{12}\varepsilon_x.$$

Let us examine the case, when $\sigma_y = 0$:

$$\frac{\varepsilon_y}{\varepsilon_x} = \nu_x = -\frac{C_{12}}{C_{22}}$$

and for stress σ_x we will have the expression

$$\sigma_x = E_x\varepsilon_x,$$

where

$$E_x = C_{11} - \frac{C_{12}^2}{C_{22}}.$$

If $\sigma_x = 0$, then

$$\frac{\varepsilon_x}{\varepsilon_y} = \nu_y = -\frac{C_{12}}{C_{11}}$$

and the stress σ_y will be determined in the form

$$\sigma_y = E_y\varepsilon_y,$$

where

$$E_y = C_{22} - \frac{C_{12}^2}{C_{11}}.$$

Let us set up the products

$$E_x\nu_y = -C_{12} + \frac{C_{12}^3}{C_{11}C_{22}},$$

$$E_y\nu_x = -C_{12} + \frac{C_{12}^3}{C_{11}C_{22}}.$$

Because the right sides of these expressions are equal to each other, then hence the equality (15.6) follows.

§ 71. Differential Equations and Boundary Conditions for the Calculation of Reinforced Shells

For the derivation of equations of the equilibrium of reinforced shells let us employ the energy method.

The expression of the total potential energy of the shell has the form

$$Э=\frac{1}{2}\int_0^a\int_0^b(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_{xy}\varepsilon_{xy})\,dx\,dy+$$
$$+\frac{1}{2}\int_0^a\int_0^b(M_x\chi_x+M_y\chi_y+2M_{xy}\chi_{xy})\,dx\,dy-\int_0^a\int_0^b qw\,dx\,dy.$$

Here through a and b the sizes of the shell are designated.

By representing the expression for $Э$ through the components of displacements u, v and w, we obtain

$$Э=\int_0^a\int_0^b F(u_x, u_y, v_x, v_y, w, w_{xx}, w_{yy}, w_{xy})\,dx\,dy,$$

where

$$F=\frac{1}{2}\left[\frac{E_x}{1-\mu_x\mu_y}\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)^2+\frac{E_y}{1-\mu_x\mu_y}\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)^2+\right.$$
$$+\frac{E_x\mu_y+E_y\mu_x}{1-\mu_x\mu_y}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{w}{R_2}\frac{\partial u}{\partial x}+\frac{w}{R_1}\frac{\partial v}{\partial y}+\frac{w^2}{R_1R_2}\right)+$$
$$\left.+G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]+\frac{1}{2}\left[D_x\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}\right)^2+\right.$$
$$+D_y\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}\right)^2+(D_x\mu_y+D_y\mu_x)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}+\right.$$
$$\left.\left.+\frac{w}{R_1^2}\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}\frac{\partial^2 w}{\partial x^2}+\frac{w^2}{R_1^2R_2^2}\right)+4Dk\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]-qw,$$
$$u_x=\frac{\partial u}{\partial x},\ \ldots\ w_{xy}=\frac{\partial^2 w}{\partial x\,\partial y}.$$

Because the equilibrium conditions are considered, then the sum of work of all forces, acting on the shell, on the possible concordants with relationships to displacements, should be equal to zero, i.e., $\delta Э=0$. By satisfying the condition, we obtain

ced

$$\delta Э=\int_0^a\int_0^b\left(\frac{\partial F}{\partial u_x}\delta u_x+\frac{\partial F}{\partial u_y}\delta u_y+\frac{\partial F}{\partial v_x}\delta v_x+\frac{\partial F}{\partial v_y}\delta v_y+\frac{\partial F}{\partial w}\delta w+\right.$$
$$\left.+\frac{\partial F}{\partial w_{xx}}\delta w_{xx}+\frac{\partial F}{\partial w_{yy}}\delta w_{yy}+\frac{\partial F}{\partial w_{xy}}\delta w_{xy}\right)dx\,dy=0.$$

After the integration of each component of this expression in parts, using the relationships of Hooke's law (15.5), we will have

$$\delta Э=-[(2M_{xy})_0^a\delta w]_0^b-\int_0^a\left[\left(\frac{\partial M_x}{\partial x}+2\frac{\partial M_{xy}}{\partial y}\right)\delta w\right]_0^b dy-$$
$$-\int_0^b\left[\left(\frac{\partial M_y}{\partial y}+2\frac{\partial M_{xy}}{\partial x}\right)\delta w\right]_0^a dx+\int_0^b[N_x\delta u]_0^a dy+\int_0^b[N_{xy}\delta v]_0^a dy+$$
$$+\int_0^b\left[M_x\delta\left(\frac{\partial w}{\partial x}\right)\right]_0^a dy+\int_0^a[N_y\delta v]_0^b dx+\int_0^a[N_{xy}\delta u]_0^b dx+$$
$$+\int_0^a\left[M_y\delta\left(\frac{\partial w}{\partial y}\right)\right]_0^b dx-\int_0^a\int_0^b\left(\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}\right)\delta u\,dx\,dy-$$
$$-\int_0^a\int_0^b\left(\frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}\right)\delta v\,dx\,dy+\int_0^a\int_0^b\left(\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\,\partial y}+\right.$$
$$\left.+\frac{\partial^2 M_y}{\partial y^2}+\frac{M_x}{R_1^2}+\frac{M_y}{R_2^2}+\frac{N_x}{R_1}+\frac{N_y}{R_2}-q\right)\delta w\,dx\,dy=0.$$

From this expression, as is known, one can also obtain the differential equations and the boundary conditions of the problem.

Inasmuch as variations δu, δv, δw are arbitrary, then from the condition of equality of the variations $\delta Э=0$ to zero it follows that

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\qquad \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0,$$
$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\,\partial y}+\frac{\partial^2 M_y}{\partial y^2}+\frac{M_x}{R_1^2}+\frac{M_y}{R_2^2}+\frac{N_x}{R_1}+\frac{N_y}{R_2}-q=0.$$

Each of the remaining nine components of the variation of total energy should be converted to zero at the points of the contour of the shell. They determine the boundary conditions of the problem.

By means if the introduction of the function of stress ϕ according to the formula $N_x = \frac{\partial^2\varphi}{\partial y^2}$, $N_y = \frac{\partial^2\varphi}{\partial x^2}$, $N_{xy} = -\frac{\partial^2\varphi}{\partial x\,\partial y}$ the first two equations will be satisfied identically, and the third assumes the form

$$D_x \frac{\partial^4 w}{\partial x^4} + 2(D_y\mu_x + 2D_k)\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} + 2D_x\left(\frac{1}{R_1^2}+\frac{\mu_y}{R_2^2}\right)\frac{\partial^2 w}{\partial x^2} + 2D_y\left(\frac{1}{R_2^2}+\frac{\mu_x}{R_1^2}\right)\frac{\partial^2 w}{\partial y^2} + \left[D_x\left(\frac{1}{R_1^4}+\frac{\mu_y}{R_1^2R_2^2}\right) + D_y\left(\frac{1}{R_2^4}+\frac{\mu_x}{R_1^2R_2^2}\right)\right]w - \frac{1}{R_1}\frac{\partial^2\varphi}{\partial y^2} + \frac{1}{R_2}\frac{\partial^2\varphi}{\partial x^2} - q = 0. \qquad (15.7)$$

For the relationship of the function of stresses ϕ with the function of deflection the following equation can be obtained.

According to Hooke's law (15.5) and to the relationships (15.5') we obtain

$$\frac{\partial u}{\partial x} = \frac{1}{E_x\delta}\frac{\partial^2\varphi}{\partial y^2} - \frac{\mu_y}{E_y\delta}\frac{\partial^2\varphi}{\partial x^2} - \frac{w}{R_1},$$
$$\frac{\partial v}{\partial y} = \frac{1}{E_y\delta}\frac{\partial^2\varphi}{\partial x^2} - \frac{\mu_x}{E_x\delta}\frac{\partial^2\varphi}{\partial y^2} - \frac{w}{R_2},$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -\frac{1}{G\delta}\frac{\partial^2\varphi}{\partial x\,\partial y}.$$

After excluding u and v from these expressions we obtain

$$\frac{1}{E_y}\frac{\partial^4\varphi}{\partial x^4} + \left(\frac{1}{G} - \frac{\mu_x}{E_x} - \frac{\mu_y}{E_y}\right)\frac{\partial^4\varphi}{\partial x^2\,\partial y^2} + \frac{1}{E_x}\frac{\partial^4\varphi}{\partial y^4} = -\frac{\delta}{R_1}\frac{\partial^2 w}{\partial y^2} + \frac{\delta}{R_2}\frac{\partial^2 w}{\partial x^2}. \qquad (15.8)$$

On the basis of equations (15.7) and (15.8) using the expression (p. 322) for a lateral load, the solution of wide range of problems of the rigidity of reinforced shells can be obtained.

§ 72. Rigidity of a Reinforced Cylindrical Shell Under Axial Compression and with Internal Pressure

The critical stress of axial compression for a reinforced cylindrical shell can be determined, if for the function of deflection and for the function of stresses one assumes expressions, which were used in the solution of an analogous problem for a smooth shell (p. 329). By limiting the first approximation, let us select functions of w and ϕ in the form

$$w = A \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \qquad (15.9)$$
$$\varphi = B \sin^2\frac{\pi x}{a} \sin^2\frac{\pi y}{b},$$

where a and b - sizes of the sides of pits and bulges.

The accepted functions satisfy the following conditions along the contour of pits and bulges:

$$M_x = 0, \quad \sigma_x = 0,$$
$$M_y = 0, \quad \sigma_y = 0,$$
$$w = 0, \quad \tau_{xy} = 0.$$

Because an axial compressive force N_x^0 acts on the shell along with an internal pressure q, then

$$q = -N_x^0 \frac{\partial^2 w}{\partial x^2} + N_y^0 \left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right). \qquad (15.10)$$

where $N_y^0 = qR$; q - intensity of the internal pressure.

By substituting the expressions (15.9) and (15.10) in equations (15.7) and (15.8) of the previous paragraph, and by applying

Bubnov-Galerkin's procedure when $E_{об} = E_{стр} = E_{ш} = E;\ \mu_x = \mu_y = \mu$ we obtain the expression for the critical stress

$$\left(\frac{\sigma R}{E\delta}\right)_{кр} = \frac{a_7(m^4 + a_1 n^4 + a_2 m^2 n^2 - a_3 m^2 - 2a_1 n^2 + a_1)}{m^2} + \\ + \frac{a_4 m^2}{0.75 m^4 + a_5 n^4 + a_6 m^2 n^2} + \frac{qR^2}{E\delta^2}\,\frac{n^2 - 1}{m^4}\,\frac{1}{\left(\frac{\pi R}{l}\right)^2}, \qquad (15.11)$$

where

$$m = \frac{l}{a}, \qquad a_3 = \frac{2\mu}{\left(\frac{\pi R}{l}\right)^2},$$

$$n = \frac{\pi R}{b}, \qquad a_4 = \frac{0.13\left(1 + \frac{F_{ш}}{a_{ш}\delta}\right)\frac{R}{\delta}}{\left(\frac{\pi R}{l}\right)^2},$$

$$a_1 = \frac{D_y}{D_x\left(\frac{\pi R}{l}\right)^4}, \qquad a_5 = \frac{0.75 E_y}{E_x\left(\frac{\pi R}{l}\right)^4},$$

$$a_2 = \frac{2\left(\mu + 2\frac{D_k}{D_x}\right)}{\left(\frac{\pi R}{l}\right)^2}, \qquad a_6 = \frac{0.25\left(\frac{E_y}{G} - 2\mu\right)}{\left(\frac{\pi R}{l}\right)^2},$$

$$a_7 = \frac{\frac{\delta}{R}\left(\frac{\pi R}{l}\right)^2}{12(1-\mu^2)}\left[1 + \frac{12(1-\mu^2)J_{стр}}{a_{стр}\delta^3}\right], \quad D_k = \frac{E\delta^3}{24(1+\mu)},$$

$$E_x = E\left(1 + \frac{F_{стр}}{a_{стр}\delta}\right), \quad D_y = \frac{E\delta^3}{12(1-\mu^2)}\left[1 + \frac{12(1-\mu^2)J_{ш}}{a_{ш}\delta^3}\right],$$

$$E_y = E\left(1 + \frac{F_{ш}}{a_{ш}\delta}\right),$$

$$D_x = \frac{E\delta^3}{12(1-\mu^2)}\left[1 + \frac{12(1-\mu^2)J_{стр}}{a_{стр}\delta^3}\right].$$

By knowing the amount of critical stress, one can determine the limiting value of the compressive force:

$$P_{кр} = (2\pi R\delta + nF_{стр})\sigma_{кр}.$$

Formula (15.11) can be used for the determination of the critical stress when the sheathing loses its rigidity simultaneously with the superstructure. Most advantageous construction will be, obviously, that which at the minimum weight will sustain the greatest axial load, i.e., the shell will be optimum, if $\left(\frac{P_{кр}}{G}\right)_{max}$, where G - weight of the entire shell.

§ 73. Rigidity of a Reinforced Cylindrical Panel Under Axial Compression and with Uniform Diametrical Pressure

Let us examine a cylindrical panel, reinforced by longitudinal and lateral supports and under compression in an axial direction during the simultaneous action of pressure, evenly distributed on its surface. For certainty we can consider that the pressure is applied from the concave side of the panel. Then, for the critical stress of compression we can make use of the formula also in the case when the pressure acts on the convex side of the panel. In this case it is proposed that the pressure is less than its critical value for the specified panel.

From the general solution a formula can be obtained both as the particular case and for a nonreinforced panel.

The solving equations of the problem remain in the form (15.7), (15.8).

Let us apply these equations to the investigation of a reinforced panel (Fig. 221). We will consider the extreme longitudinal profiles as being rather rigid upon bending as well as upon twisting.

Fig. 221.

Let us realize the following boundary conditions along the contour of the panel

$$\left.\begin{array}{r} w=0 \\ M_x=0 \end{array}\right\} \begin{array}{l} x=0 \\ x=a \end{array} \qquad \left.\begin{array}{r} \sigma_x=0 \\ \sigma_{xy}=0 \end{array}\right\} \begin{array}{l} x=0 \\ x=a \end{array}$$

$$\left.\begin{array}{r} w=0 \\ \frac{\partial w}{\partial y}=0 \end{array}\right\} \begin{array}{l} y=0 \\ y=b \end{array} \qquad \left.\begin{array}{r} \sigma_y=0 \\ u=0 \end{array}\right\} \begin{array}{l} y=0 \\ y=b, \end{array}$$

where a - size of the half-wave in an axial direction.

These boundary conditions can be satisfied, if we take the functions of w and ϕ in the form of the following approximations:

$$w=A\sin\frac{m\pi x}{l}\sin^2\frac{\pi y}{b}=A\sin\frac{\pi x}{a}\sin^2\frac{\pi y}{b},$$

$$\varphi=B\sin^2\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

Having substituted these functions in the original equations and having integrated them according to the Bubnov-Galerkin method, we obtain

$$\frac{3\sigma_{\text{кр}}}{16E_y}=\frac{256\left(\frac{l}{\pi R}\right)^2 m^2}{81\pi^4\left[m^4+\frac{1}{4}a_2\left(\frac{l}{b}\right)^2 m^2+\frac{3}{16}a_1\left(\frac{l}{b}\right)^4\right]}+$$

$$+\frac{a_3}{\left(\frac{l}{\pi R}\right)^2 m^2}\left\{\frac{3}{16}m^4+\frac{1}{2}\left[a_5\left(\frac{l}{b}\right)^2-\frac{3\mu}{4}\left(\frac{l}{\pi R}\right)^2\right]m^2+\right.$$

$$\left.+a_4\left[\left(\frac{l}{b}\right)^4+\frac{1}{2}\left(\frac{l}{b}\right)^2\left(\frac{l}{\pi R}\right)^2+\frac{3}{16}\left(\frac{l}{\pi R}\right)^4\right]\right\}+$$

$$+\frac{3qR\left(\frac{l}{\pi R}\right)^2\left[\frac{4}{3}\left(\frac{\pi R}{b}\right)^2-1\right]}{16E_y\delta m^2},$$

where it is designated

$$a_1=\frac{E_y}{E_x},\quad a_2=\frac{E_y}{G}-\mu\left(1+\frac{E_y}{E_x}\right),\quad a_3=\frac{D_x}{E_y\delta R^2},$$

$$a_4=\frac{D_y}{D_x},\quad a_5=\mu\frac{D_y}{D_x}+2\frac{D_k}{D_x},$$

$$E_x=E\left(1+\frac{F_{стр}}{a_{стр}\delta}\right),\quad E_y=E\left(1+\frac{F_{ш}}{a_{ш}\delta}\right),$$

$$D_x=\frac{E\delta^3}{12(1-\mu^2)}\left[1+\frac{12(1-\mu^2)J_{стр}}{a_{стр}\delta^3}\right],$$

$$D_y=\frac{E\delta^3}{12(1-\mu^2)}\left[1+\frac{12(1-\mu^2)J_{ш}}{a_{ш}\delta^3}\right],$$

$$D_k=\frac{E\delta^3}{24(1+\mu)},\quad m=\frac{l}{a}.$$

The critical force for a panel, which loses rigidity together with the support, will be

$$P_{кр}=\left(b\delta+F_{стр}\frac{b}{a_{стр}}\right)\sigma_{кр}.$$

One ought to make use of the obtained expression for $\sigma_{кр}$ of the reinforced panel when the size of the panel b is less than the size of the half-wave, which is generated in a circumferential direction in an enclosed circular shell under compression, reinforced just as is the investigated panel. Therefore, at first it is necessary to determine the size of the half-wave in a circumferential direction for an enclosed circular shell. If this size should be greater than size b of the investigated panel, then the above obtained formula for $\sigma_{кр}$ will be applicable.

If however, this size should be less than the size of panel b, then the critical stress for the panel can be assumed to be equal to the critical stress of the enclosed shell.

Under the assumption of the absence of a reinforced support we will obtain

$$a_1=1,\quad a_2=2,\quad a_3=\frac{(\delta/R)^2}{12(1-\mu^2)}=a,$$

$$a_4=1,\quad a_5=1,\quad E_x=E_y=E.$$

Then

$$\frac{3\sigma_{\text{кр}}}{16E}=\frac{256\left(\frac{l}{\pi R}\right)^2 m^2}{81\pi^4\left[m^4+\frac{1}{2}\left(\frac{l}{b}\right)^2 m^2+\frac{3}{16}\left(\frac{l}{b}\right)^4\right]}+$$
$$+\frac{\alpha}{\left(\frac{l}{\pi R}\right)^2 m^2}\left\{\frac{3}{16}m^4+\frac{1}{2}\left[\left(\frac{l}{b}\right)^2-\frac{3\mu}{4}\left(\frac{l}{\pi R}\right)^2\right]m^2+\left(\frac{l}{b}\right)^4+\right.$$
$$\left.+\frac{1}{2}\left(\frac{l}{b}\right)^2\left(\frac{l}{\pi R}\right)^2+\frac{3}{16}\left(\frac{l}{\pi R}\right)^4\right\}+\frac{3qR\left(\frac{l}{\pi R}\right)^2\left[\frac{4}{3}\left(\frac{\pi R}{b}\right)^2-1\right]}{16E\delta m^2}. \qquad (15.12)$$

The formula (15.12) is determined by the critical stress of compression of the sheathing of a nonreinforced cylindrical panel (in the cages between the superstructure). For the optimum cylindrical shell it is necessary that the critical stress, according to the formula (15.11) should be equal to critical stress, determined by the formula (15.12), in the absence of a local loss of rigidity of the shelf of the superstructure.

§ 74. Rigidity of a Cylindrical Shell, Reinforced by Equidistant Elastic Frames, Under External Pressure

In this case for the solution of the problem let us use the simplified equations of rigidity in the form

$$\begin{aligned}&\frac{1}{R}\frac{\partial^2\varphi}{\partial x^2}+D_x\frac{\partial^4 w}{\partial x^4}+2(D_y\mu+2D_k)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_y\frac{\partial^4 w}{\partial y^4}+N_y^0\frac{\partial^2 w}{\partial y^2}=0,\\&\frac{1}{E_y}\frac{\partial^4\varphi}{\partial x^4}+\left(\frac{1}{G}-\frac{\mu}{E_x}-\frac{\mu}{E_y}\right)\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{1}{E_x}\frac{\partial^4\varphi}{\partial y^4}=-\frac{\delta}{R}\frac{\partial^2 w}{\partial x^2},\end{aligned} \qquad (15.13)$$

where

$$E_x=E,\quad E_y=E\left(1+\frac{F_{\text{ш}}}{a_{\text{ш}}}\right),\quad D_x=\frac{E\delta^3}{12(1-\mu^2)},$$
$$D_y=\frac{E\delta^3}{12(1-\mu^2)}+\frac{EJ_{\text{ш}}}{a_{\text{ш}}},\quad D_k=\frac{G\delta^3}{12}.$$

The equations (15.13) allow for the determining of the critical forces $N_y^0 = qR$. Inasmuch as the method of solution of this problem is analogous to that used for the solution of the corresponding problem for a smooth shell, let us limit ourselves by presenting the final results. Additionally let us assume that $E_y \approx E$. Then

$$q_{\text{кр}}=0{,}92E\frac{\delta^2}{RL}\sqrt{D_0\frac{\delta}{R}\sqrt{D_0}}, \qquad (15.14)$$

where L - length of the shell;

$$D_0=1+\frac{12(1-\mu^2)J_{\text{ш}}}{\delta^3 a_{\text{ш}}}; \qquad (15.15)$$

$J_{\text{ш}}$ - moment of inertia of the frame; $a_{\text{ш}}$ - distance between the frames; δ - thickness of the shell.

If the shell is long, then the equation for the determination of the critical pressure can be obtained from (15.7):

$$D_y\left(\frac{d^4w}{dy^4}+\frac{2}{R^2}\frac{d^2w}{dy^2}+\frac{w}{R^4}\right)+N_y^0\left(\frac{d^2w}{dy^2}+\frac{w}{R^2}\right)=0,$$

where

$$D_y=\frac{E\delta^3}{12(1-\mu^2)}+\frac{12(1-\mu^2)J_{\text{ш}}}{\delta^3 a_{\text{ш}}}.$$

By approximating function w with the expression

$$w=A\cos n\theta,$$

for the determination of the external critical pressure we obtain the formula

$$q_{\text{кр}}=\frac{3DD_0}{R^3},$$

where

$$D=\frac{E\delta^3}{12(1-\mu^2)},$$

D_0 - given according to the formula (15.15).

The limit of applicability of the formula (15.14) is determined by the condition

$$\frac{3DD_0}{R^3} = 0{,}92E\frac{\delta^2}{RL}\sqrt{D_0\frac{\delta}{R}\sqrt{D_0}}.$$

Hence, we will find that within the limits of elasticity, the formula (15.14) will be applicable to those shells whose length satisfies the condition

$$L \leqslant 3{,}35R\frac{\sqrt{\frac{R}{\delta}}}{\sqrt[4]{D_0}}.$$

§ 75. Rigidity of a Reinforced Cylindrical Shell with Frames Under an External Pressure and Axial Tension

In the solution of the given problem a simplified equation of equilibrium (15.7) is used, in which the components, containing $w, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}$ are cancelled out. In this case the equation of compatibility of the deformations remains unchanged.

For the solution of a system of solving equations let us use the method of division of variables in conjunction with the method of finite differences. By approximating the functions of w and ϕ

$$w = W(x)\cos n\theta, \quad \varphi = F(x)\cos n\theta$$

and by substituting them in the system of solving equations of the problem, let us set up these equations in the from

$$\left.\begin{aligned} &F_k a_1 - a_2(F_i + F_l) + F_t + F_s - \frac{Eh^2}{R}(W'_i + W'_l - 2W_k) = 0,\\ &F_t + F_l - 2F_k + \frac{DR}{h^2\delta}[W'_k a_3 - a_2(W'_i + W_l) + W_t + W_s] -\\ &\quad - \frac{N_y^0 n^2 h^2}{R\delta} W_k + \frac{N_x^0 R}{\delta}(W_i + W_l - 2W_k) = 0. \end{aligned}\right\} \qquad (15.16)$$

Designated here is:

$$\left.\begin{aligned}
a_1 &= 6+4n^2\left(\frac{h}{R}\right)^2+n^4\left(\frac{h}{R}\right)^4,\\
a_2 &= 4+2n^2\left(\frac{h}{R}\right)^2,\\
a_3 &= 6+4n^2\left(\frac{h}{R}\right)^2+D_0n^4\left(\frac{h}{R}\right)^4,\\
D_0 &= 1+\frac{12(1-\mu^2)J}{l\delta^3},\\
D &= \frac{E\delta^3}{12(1-\mu^2)},
\end{aligned}\right\} \qquad (15.17)$$

J - moment of inertia of the frame; l - distance between the frames; R, δ - radius and the thickness of the wall of the shell; $2n$ - number of waves in the circumferential direction.

The equations (15.16) are used under the assumption that after the loss in rigidity from the action of external pressure in an axial direction of the shell only one half-wave can be generated. Such an assumption will be realized, for example, in the case of the action of a tensile axial force N_x^0. If this force were compressive, then the accepted assumption would be proportionable when the absolute value of the compressive force N_x^0 is small in comparison with its critical value. Such a situation appears with the creation of a deep vacuum in an enclosed cylindrical tank.

Subsequently, in the given equations it is considered that $N_y^0 = qR$, where q - intensity of the external pressure.

Boundary conditions at the ends of the shell

$$w=0,\quad M_x=0,\quad N_x=0,\quad N_{xy}=0.$$

These conditions, expressed through finite differences, have the form (see § 50):

$$W'_k=0,\quad W_l=-W'_l,\quad F_k=0,\quad F_l=F_l.$$

The first approximation: $h = \frac{1}{2}L$ (Fig. 222).

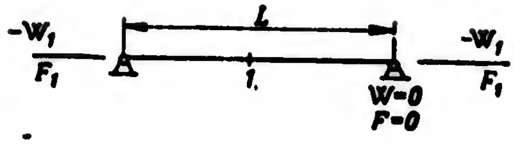


Fig. 222.

In this instance the given equations assume the form

$$F_1a_1+F_1+F_1+\frac{2Eh^2}{R}W_1=0,$$

$$-2F_1+\frac{DR}{h^2\delta}(W'_1a_3-W_1-W_1)-\frac{qRn^2h^2}{R\delta}W_1-\frac{2N_x^0R}{\delta}W_1=0$$

or

$$F_1(a_1+2)=-\frac{2Eh^2}{R}W_1,$$

$$2F_1=\frac{DR}{h^2\delta}(a_3-2)W'_1-\frac{qRn^2h^2}{R\delta}W'_1-\frac{2N_x^0R}{\delta}W_1,$$

whence after the exclusion of W_1 and F_1 we obtain

$$\frac{q^{I}Rn^2h^2}{R\delta}+\frac{2N_x^0R}{\delta}=\frac{4Eh^2}{R(a_1+2)}+\frac{DR(a_3-2)}{h^2\delta}. \qquad (15.18)$$

The second approximation: $h=\frac{1}{3}L$ (Fig. 223). In this instance we will have

$$\frac{q^{II}Rn^2h^2}{R\delta}+\frac{N_x^0R}{\delta}=\frac{Eh^2}{R(a_1-a_2+1)}+\frac{DR(a_3-a_2-1)}{h^2\delta}. \qquad (15.19)$$

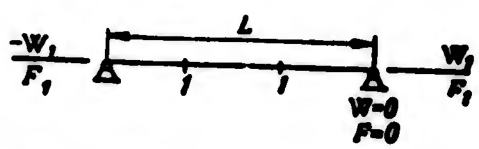


Fig. 223.

In the expressions of the first and second approximations in each concrete case it is necessary to substitute the value which is supplied from the concrete conditions of the problem instead of force N_x^0.

Stability of the internal shell of an annular tank (Fig. 224). By setting up the sum of projections of all forces in the direction of the axis of the tank (Fig. 224a), we obtain

$$2\pi R N_x^0 = \pi[(R+r)^2 - R^2]q,$$

whence

$$N_x^0 = \left(1 + \frac{r}{2R}\right)qr.$$

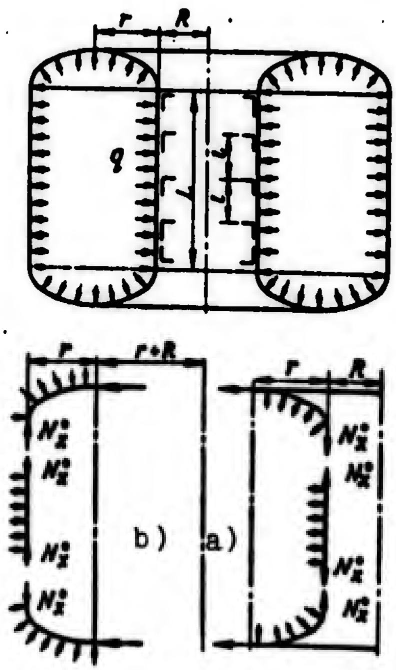


Fig. 224.

Having substituted the obtained expression for N_x^0 in formulas (15.18)-(15.19) with the minus sign (tension), after simple conversions we will have

$$\left(\frac{qr}{E\delta}\right)^{I}=\frac{\dfrac{4\left(\dfrac{L}{2R}\right)^{2}}{A_{1}}+\dfrac{A_{2}\left(\dfrac{\delta}{R}\right)^{2}}{12(1-\mu^{2})}\left(\dfrac{2R}{L}\right)^{2}}{\dfrac{R}{r}\left(\dfrac{L}{2R}\right)^{2}n^{2}-\left(2+\dfrac{r}{R}\right)},$$

$$\left(\frac{qr}{E\delta}\right)^{II}=\frac{\dfrac{1}{B_{1}}\left(\dfrac{L}{3R}\right)^{2}+\dfrac{B_{2}\left(\dfrac{\delta}{R}\right)^{2}}{12(1-\mu^{2})}\left(\dfrac{3R}{L}\right)^{2}}{\dfrac{R}{r}\left(\dfrac{L}{3R}\right)^{2}n^{2}-\left(1+\dfrac{r}{2R}\right)},$$

where

$$\begin{aligned}
A_{1}&=8+4n^{2}\left(\frac{L}{2R}\right)^{2}+n^{4}\left(\frac{L}{2R}\right)^{4},\\
A_{2}&=4+4n^{2}\left(\frac{L}{2R}\right)^{2}+D_{0}n^{4}\left(\frac{L}{2R}\right)^{4},\\
B_{1}&=3+2n^{2}\left(\frac{L}{3R}\right)^{2}+n^{4}\left(\frac{L}{3R}\right)^{4},\\
B_{2}&=1+2n^{2}\left(\frac{L}{3R}\right)^{2}+D_{0}n^{4}\left(\frac{L}{3R}\right)^{4},\\
D_{0}&=1+\frac{12(1-\mu^{2})J}{l\delta^{3}}.
\end{aligned} \qquad (15.20)$$

After the determination of $q^{I}_{\text{кр}}$ and $q^{II}_{\text{кр}}$ a more accurate value of the critical pressure can be determined by means of extrapolation:

$$q_{\text{кр.экстр}}=-0{,}8\,q^{I}_{\text{кр}}+1{,}8q^{II}_{\text{кр}}.$$

Rigidity of an external cover of an annular tank under external pressure (Fig. 224b). Let us set up the sum of projections of all the forces in the direction of the axis of the tank. From the condition of equilibrium we obtain

$$2\pi(2r+R)N^{0}_{x}=\pi\left[(2r+R)^{2}-(r+R)^{2}\right]q,$$

whence

$$N^{0}_{x}=\left(\frac{1+1{,}5\dfrac{r}{R}}{1+2\dfrac{r}{R}}\right)qr.$$

After the substitution of the expression N_x^0 in the formulas (15.18), (15.19) with a plus sign (compression) these formulas will assume the form

$$\left(\frac{qr}{E\delta}\right)^{I}=\frac{\frac{4}{A_1}\left(\frac{L}{2R}\right)^2+\frac{A_2\left(\frac{\delta}{R}\right)^2}{12(1-\mu^2)}\left(\frac{2R}{L}\right)^2}{\frac{R}{r}\left(\frac{L}{2R}\right)^2 n^2+\frac{2+3\frac{r}{R}}{1+2\frac{r}{R}}},$$

$$\left(\frac{qr}{E\delta}\right)^{II}=\frac{\frac{1}{B_1}\left(\frac{L}{3R}\right)^2+\frac{B_2\left(\frac{\delta}{R}\right)^2}{12(1-\mu^2)}\left(\frac{3R}{L}\right)^2}{\frac{R}{r}\left(\frac{L}{3R}\right)^2 n^2+\frac{1+1,5\frac{r}{R}}{1+2\frac{r}{R}}}.$$

The values of quantities A_i and B_i remain the same as in the first case.

Rigidity of a cylindrical shell reinforced by frames under a three-dimensional external pressure. The force N_x^0 in this case is determined by the expression

$$N_x^0=\frac{qR}{2}.$$

By substituting this value in the equations (15.18), (15.19), we obtain

$$\left(\frac{qR}{E\delta}\right)^{I}=\frac{\frac{4}{A_1}\left(\frac{L}{2R}\right)^2+\frac{A_2\left(\frac{\delta}{R}\right)^2}{12(1-\mu^2)}\left(\frac{2R}{L}\right)^2}{1+\left(\frac{L}{2R}\right)^2 n^2},$$

$$\left(\frac{qR}{E\delta}\right)^{II}=\frac{\frac{1}{B_1}\left(\frac{L}{3R}\right)^2+\frac{B_2\left(\frac{\delta}{R}\right)^2}{12(1-\mu^2)}\left(\frac{3R}{L}\right)^2}{0,5+\left(\frac{L}{3R}\right)^2 n^2},$$

where the values are A_i, B_i see (15.20).

If the frames are lacking, then in the expressions D_0 one ought to set $J = 0$, and then these formulas can be used for the calculation of smooth shells. We can find the refined value of the critical pressure by extrapolation:

$$q_{\text{кр.экстр}} = -0{,}8\,q^{\text{I}}_{\text{кр}} + 1{,}8\,q^{\text{II}}_{\text{кр}}.$$

§ 76. Rigidity of a Spherical Reinforced Shell Under External Pressure

In the case of a reinforced spherical shell, the sizes of the boxes between the superstructural elements, generally speaking, are obtained by variables; therefore, the integration of the equations of equilibrium becomes rather complex. In order to simplify the solution and obtain the result as a safety factor, we will consider the sizes of the boxes approximately as constants and equal to their greatest values. Then, the differential equations of the rigidity will have the form

$$\frac{1}{E_{0y}}\frac{\partial^4\varphi}{\partial x^4} + \left[2(1+\mu) - \frac{\mu}{E_{0x}} - \frac{\mu}{E_{0y}}\right]\frac{\partial^4\varphi}{\partial x^2\,\partial y^2} + \frac{1}{E_{0x}}\frac{\partial^4\varphi}{\partial y^4} = = \frac{E\delta}{R}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

$$\frac{1}{R}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2}\right) + D\left[D_{0x}\frac{\partial^4 w}{\partial x^4} + 2(1-\mu+\mu D_{0y})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \right. + D_{0y}\frac{\partial^4 w}{\partial y^4} + \frac{2(1+\mu)D_{0x}}{R^2}\frac{\partial^2 w}{\partial x^2} + \frac{2(1+\mu)D_{0y}}{R^2}\frac{\partial^2 w}{\partial y^2} + \left. + \frac{1+\mu}{R^4}(D_{0x}+D_{0y})w\right] + N\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{2w}{R^2}\right) = 0.$$

Designated here is:

$$D = \frac{E\delta^3}{12(1-\mu^2)}, \quad D_{0x} = 1 + \frac{12(1-\mu^2)J_{\text{стр}}}{a_{\text{стр}}\delta^3},$$

$$D_{0y} = 1 + \frac{12(1-\mu^2)J_{\text{ш}}}{a_{\text{ш}}\delta^3}, \quad E_{0x} = 1 + \frac{F_{\text{стр}}}{a_{\text{стр}}\delta},$$

$$E_{0y} = 1 + \frac{F_{\text{ш}}}{a_{\text{ш}}\delta}, \quad N = \frac{qR}{2}.$$

Let us consider that after the loss of rigidity the shell is covered with pits and bulges (see § 59), close to being square.

We obtain the least value of load, if for the functions w and Φ we assume that the expressions are

$$w = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{a},$$

$$\varphi = B\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a}.$$

Subsequently, let us limit ourselves only by the first approximation, because in this case the result of the solution will differ little from the higher approximations (see § 49).

As a result of the integration of the solving equations according to the Bubnov-Galerkin method we will have

$$\frac{qR}{2E\delta} = \frac{2048}{81\pi^4\left[(0,75-0,25\mu)\left(\frac{1}{E_{0x}}+\frac{1}{E_{0y}}\right)+\frac{1+\mu}{2}\right](m^2-1)} + \frac{\alpha m^4}{m^2-1}\Big[D_{0x}+(1+2\mu)D_{0y}+2(1-\mu)- -(1+\mu)(D_{0x}+D_{0y})\left(\frac{2}{m^2}-\frac{1}{m^4}\right)\Big],$$

where

$$m = \frac{\pi R}{a}, \quad \alpha = \frac{\left(\frac{\delta}{R}\right)^2}{24(1-\mu^2)}.$$

One should expect that the surface of the shell after the loss of rigidity is covered by large number of pits and bulges. In this case $m^2 \gg 1$. Then, one can approximately assume that

$$\frac{qR}{2E\delta} = \frac{2048}{81\pi^4\left[(0,75-0,25\mu)\left(\frac{1}{E_{0x}}+\frac{1}{E_{0y}}\right)+\frac{1+\mu}{2}\right]m^2} + \alpha m^2[D_{0x}+(1+2\mu)D_{0y}+2(1+\mu)].$$

The minimum of this expression through parameter m^2 will be

$$q_{кр}=0{,}438E\left(\frac{\delta}{R}\right)^2\sqrt{\frac{D_{0x}+(1+2\mu)D_{0y}+2(1-\mu)}{(0{,}75-0{,}25\mu)\left(\frac{1}{E_{0x}}+\frac{1}{E_{0y}}\right)+\frac{1+\mu}{2}}}.$$

For a smooth shell with $D_{0x}=1$, $D_{0y}=1$, $E_{0x}=1$, $E_{0y}=1$, we obtain

$$q_{кр}=0{,}62E\left(\frac{\delta}{R}\right)^2 \quad \text{or} \quad \sigma_{кр}=0{,}31E\frac{\delta}{R}.$$

§ 77. Rigidity of a Reinforced Spherical Shell Under a Load of Its Rapidly Increasing External Uniform Pressure

If the rate of application of an external load causes the accelerated motion of particles of the body, then in the equations of equilibrium it is necessary to add members, containing forces of inertia. During the examination of the rigidity of a spherical shell one can be limited by the addition of inertial load only in the direction of radius of the shell. Then, the equation of equilibrium assumes the form

$$\frac{1}{R}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+D\left[D_{0x}\frac{\partial^4 w}{\partial x^4}+2(1-\mu+\mu D_{0y})\frac{\partial^4 w}{\partial x^2\partial y^2}+\right.$$
$$+D_{0y}\frac{\partial^4 w}{\partial y^4}+\frac{2(1+\mu)D_{0x}}{R^2}\frac{\partial^2 w}{\partial x^2}+\frac{2(1+\mu)D_{0y}}{R^2}\frac{\partial^2 w}{\partial y^2}+$$
$$\left.+\frac{1+\mu}{R^4}(D_{0x}+D_{0y})w\right]+\frac{\gamma\delta_{пр}}{g}\frac{\partial^2 w}{\partial t^2}+N\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{2w}{R^2}\right)=0,$$

where $\delta_{пр}$ - the given thickness of the shell.

The equation of compatibility without changes is used in the form

$$\frac{1}{E_{0y}}\frac{\partial^4\varphi}{\partial x^4}+\left[2(1+\mu)-\frac{\mu}{E_{0x}}-\frac{\mu}{E_{0y}}\right]\frac{\partial^4\varphi}{\partial x^2\partial y^2}+$$
$$+\frac{1}{E_{0x}}\frac{\partial^4\varphi}{\partial y^4}=\frac{E\delta}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right).$$

The solution of the posed problem will be taken in the following form:

$$w = W(t) \sin\frac{\pi x}{a} \sin\frac{\pi y}{a},$$

$$\varphi = F(t) \sin^2\frac{\pi x}{a} \sin^2\frac{\pi y}{a},$$

where

$$W(t) = W_n(t) - W_0;$$

W_0 - initial deflection; $W_n(t)$ - complete deflection.

Having substituted these functions in the original equations of the problem and having integrated them according to the Bubnov-Galerkin method, we obtain

$$D\left\{D_{0x} + (1+2\mu) D_{0y} + 2(1-\mu) - (1+\mu)(D_{0x}+D_{0y}) \times \left[\frac{2}{\left(\frac{\pi R}{a}\right)^2} - \frac{1}{\left(\frac{\pi R}{a}\right)^4}\right]\right\}\left(\frac{\pi}{a}\right)^4 W - \frac{128}{9\pi^2 R}\left(\frac{\pi}{a}\right)^2 F -$$

$$-\frac{2N}{R^2}(W + W_0)\left[\left(\frac{\pi R}{a}\right)^2 - 1\right] + \frac{\gamma \delta_{np}}{g}\frac{d^2 W_n}{dt^2} = 0,$$

$$F\left(\frac{\pi}{a}\right)^2 = -\frac{32 E \delta W}{9\pi^2 R\left[(0{,}75 - 0{,}25\mu)\left(\frac{1}{E_{0x}} + \frac{1}{E_{0y}}\right) + \frac{1-\mu}{2}\right]}.$$

Designated here is

$$\delta_{np} = \delta + \frac{F_{стр}}{a_{стр}} + \frac{F_{ш}}{a_{ш}};$$

δ - thickness of the shell, and

$$N = \frac{1}{2} qR.$$

After the exclusion of function F from these expressions we find that

$$q^* = q_в^* \frac{Y_n - Y_0}{Y_n} + \frac{a_0}{Y_n}\frac{d^2 Y_n}{dt^2}, \tag{15.21}$$

where it is designated

$$Y_0=\frac{W_0}{\delta},\quad Y_n=\frac{W_n}{\delta},\quad q^*=\frac{q}{E},$$

$$a_0=\frac{\gamma R\delta_{np}}{(m^2-1)Eq},\quad m=\frac{\pi R}{a},$$

$$q_n^*=\frac{4096\,\frac{\delta}{R}}{81\pi^4(m^2-1)\left[(0{,}73-0{,}25\mu)\left(\frac{1}{E_{0x}}+\frac{1}{E_{0y}}\right)+\frac{1+\mu}{2}\right]}+$$
$$+\frac{\left(\frac{\delta}{R}\right)^3 m^4}{12(1-\mu^2)(m^2-1)}\left\{D_{0x}+(1+2\mu)D_{0y}+2(1-\mu)-\right.$$
$$\left.-(1+\mu)(D_{0x}+D_{0y})\left[\frac{2}{m^2}+\frac{1}{m^4}\right]\right\}.$$

Let us rewrite the equation (15.21) in the following form:

$$b^2\frac{d^2Y_n}{dt^2}+\left(1-\frac{q^*}{q_n^*}\right)Y_n=Y_0, \tag{15.22}$$

where

$$b^2=a_0^2/q_n^*.$$

Let us, for example, allow the external pressure q to increase linearly according to the law, $q = ct$. Then

$$\frac{q^*}{q_n^*}=\frac{q}{Eq_n^*}=\frac{ct}{Eq_n^*}=\tau.$$

Hence

$$t=\frac{Eq_n^*}{c}\tau. \tag{15.23}$$

Let us assume τ as the new independent variable, connected to t by the relationship (15.23).

Then, the equation (15.22) can be written in the following form:

$$\frac{d^2Y_n}{d\tau^2}+S(1-\tau)Y_n=Y_0S, \tag{15.24}$$

where

$$S=\frac{qE^{3}\left(q_{n}^{*}\right)^{3}(m^{2}-1)}{C^{2}\gamma R\delta_{np}}.$$

The obtained equation (15.24) is a solving heterogeneous linear equation of the second order with variable coefficients. If we in this equation set $Y_0 = 0$, then the solution of the given equation may merely be trivial. The presence in the right part of this equation of a member, containing an amplitude of the initial deflection, offers possibility to obtain the solution of the equation, different from zero.

For the determination of the amount of critical pressure of the given shell, as shown further on, the amount of initial deflection is unessential. This amount can be assigned arbitrarily.

For the solution of the equation (15.24) let us apply the method of Bubnov-Galerkin.

Let us regard the function Y_n in the form of a power series

$$Y_n=a_0+a_1\tau+a_2\tau^2+a_3\tau^3+\ldots$$

with initial conditions

$$Y_n=Y_0,\quad \frac{dY_n}{d\tau}=0 \quad \text{when} \quad \tau=0.$$

This yields $Y_0 = a_0$, $a_1 = 0$. By limiting ourselves to the first three members, let us use subsequently the truncated series

$$Y_n=Y_0+a_2\tau^2+a_3\tau^3.$$

Let us substitute this series in the equation (15.24) and integrate it according to the Bubnov-Galerkin method. As a result we obtain

$$a_2=\frac{Y_0S\tau_0(0{,}25B_2-0{,}2A_2)}{A_1B_2-A_2B_1};$$

$$a_3=-\frac{Y_0S\tau_0(0{,}25B_1-0{,}2A_1)}{A_1B_2-A_2B_1}.$$

Then

$$Y_n = Y_0 \left\{1 + \frac{S\tau_0}{A_1B_2 - A_2B_1}\left[(0{,}25B_2 - 0{,}2A_2)\tau^2 - (0{,}25B_1 - 0{,}2A_1)\tau^3\right]\right\}. \qquad (15.25)$$

Designated here is:

$$A_1 = \frac{2}{3} + \frac{S\tau_0^2}{5} - \frac{S\tau_0^3}{6},$$

$$A_2 = \frac{3}{2}\tau_0 + \frac{S\tau_0^3}{6} - \frac{S\tau_0^4}{7},$$

$$B_1 = \frac{1}{2} + \frac{S\tau_0^2}{6} - \frac{S\tau_0^3}{7},$$

$$B_2 = \frac{6}{5}\tau_0 + \frac{S\tau_0^3}{7} - \frac{S\tau_0^4}{8},$$

$$\tau_0 = \frac{C(t_0 + \Delta t_0)}{Eq_n^*}, \qquad \tau = \frac{Ct}{Eq_n^*}.$$

With the integration of the equation (15.24) the value τ_0 corresponding to the time $t_0 + \Delta t_0$, is taken as the upper limit, where t_0 - time of accretion of pressure q from zero up to the greatest value, and Δt_0 - a small quantity, selected arbitrarily, and satisfying the condition $\Delta t_0 \ll t_0$.

Having the expression for Y_n one can determine the limiting value of pressure q. For this purpose for each value of the number of half-waves m it is necessary to plot the graph $Y_n = f(\tau)$. The number of half-waves m, at which there is an intensive increase in deflection of Y_n, determines the approximately limiting value of load q. The character of curves of $Y_n = f(\tau)$ is presented in Fig. 225, where it is clear that time τ_1 can be taken as the limiting value. Then, the limiting value of the load will be

$$q_{кр} = Eq_n^*\tau_1.$$

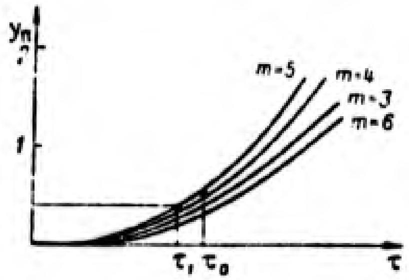


Fig. 225.

§ 78. Rigidity of Square Spherical Panel in a Plan Under a Load with a Rapidly Increasing External Pressure

The most rational construction of a reinforced spherical shell will be that in which a loss in rigidity of the entire construction on the whole (sheathing with the superstructure), as well as in the sheathings in the boxes between the superstructure elements, occurs simultaneously.

If the superstructure is rather rigid, then for the determination of the time which intensively increases with the beginning of the deflection of the sheathing, contained between the superstructure elements, one can take the expressions

$$w = W'(t)\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a},$$

$$\varphi = F(t)\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a},$$

where

$$W'(t) = W_n(t) - W_0.$$

The accepted approximations of functions w and ϕ satisfy the following boundary conditions along the contour of the panel:

$$w = 0, \quad \sigma_x = 0,$$

$$\frac{\partial w}{\partial x} = 0, \quad \sigma_y = 0,$$

$$\frac{\partial w}{\partial y} = 0, \quad \sigma_{xy} = 0.$$

The original equations of problem in this case take the form

$$D\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{2(1+\mu)}{R^2}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+\right.$$
$$\left.+\frac{2(1+\mu)w}{R^4}\right]+\frac{1}{R}\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)+\frac{\gamma\delta}{g}\frac{\partial^2 w}{\partial t^2}+$$
$$+N\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{2w}{R^2}\right)=0,$$
$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=\frac{E\delta}{R}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right).$$

By not repeating all the intermediate computations, analogous to the computations of the previous paragraph, we can write the solving equation

$$\frac{d^2Y_n}{d\tau^2}+S(1-\tau)Y_n=Y_0S, \qquad (15.26)$$

where

$$\tau=\frac{q}{Eq_n^*}=\frac{Ct}{Eq_n^*},\quad Y_0=\frac{w_0}{\delta},\quad Y_n=\frac{w_n}{\delta},$$

$$S=\frac{4(Eq_n^*)^3 g\left[\left(\frac{\pi R}{a}\right)^2-\frac{3}{4}\right]}{3C^2\gamma R\delta},$$

$$q_n^*=\frac{3\frac{\delta}{R}}{8\left[\left(\frac{\pi R}{a}\right)^2-\frac{3}{4}\right]}+$$
$$+\frac{4\left(\frac{\delta}{R}\right)^3\left(\frac{\pi R}{a}\right)^4\left[2-\frac{3(1+\mu)}{4}\left(\frac{a}{\pi R}\right)^2+\frac{9(1+\mu)}{32}\left(\frac{a}{\pi R}\right)^4\right]}{9(1-\mu^2)\left[\left(\frac{\pi R}{a}\right)^2-\frac{3}{4}\right]}.$$

The expression (15.25), in which it is necessary only to substitute the quantities S and q_B^* will be solution of the equation (15.26). The procedure further on of the solution remains as before: for each value of the size of the box $\pi R/a$ a graph is plotted, $Y_n = f(\tau)$. The curve of this chart, will climb steeply sooner from the axis τ than from the others, and will determine the limiting value of external load.

If the superstructure should be comparatively weak, then one can assume in terms of a safety factor that the panels of boxes have a hinged support. Then

$$w = W'(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{a},$$

$$\varphi = F(t)\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{a},$$

where

$$W'(t) = W'_{н}(t) - W_0.$$

For plotting the graph $Y_{п} = f(\tau)$ one should make use of the expression (15.25) of the previous paragraph, having preliminarily substituted in it the expressions S and q, having form for this instance

$$S = \frac{q\,(Eq_{н}^{*})^{2}\left[\left(\frac{\pi R}{a}\right)^{2} - 1\right]}{C^{2}\gamma R^{3}},$$

$$q_{н}^{*} = \frac{2048\,\frac{\delta}{R}}{81\pi^{4}\left[\left(\frac{\pi R}{a}\right)^{2} - 1\right]} +$$

$$+ \frac{\left(\frac{\pi R}{a}\right)^{4}\left(\frac{\delta}{R}\right)^{2}}{6(1-\mu^{2})\left[\left(\frac{\pi R}{a}\right)^{2} - 1\right]}\left\{2 - (1+\mu)\left[\frac{2}{\left(\frac{\pi R}{a}\right)^{2}} - \frac{1}{\left(\frac{\pi R}{a}\right)^{4}}\right]\right\}.$$

Remaining designations are given above.

C H A P T E R XVI

RIGIDITY OF THREE-PLY SHELLS WITH FILLER IN THE FORM OF HONEYCOMBS

Three-ply layered constructions of shells with filler in the form of honeycombs have a number of advantages in comparison with a single-layer, the main one being their lightness. At a substantially less weight the honeycomb constructions are able to absorb large loads.

Recently these constructions made of steel have found utilization in aviation technology [3].

In this chapter the problem dealing with the calculation of shells with honeycomb filler for rigidity is examined. In this case let us limit ourselves to the examination of only such covers, in which the external layers have an identical thickness and are made from the same isotropic material. Let us assume that the honeycombs are likewise made from isotropic material, but are different from the material of the external layers.

As accepted symmetry of construction of the wall, one can consider that the hypothesis of straight standards is applicable to it. In this case we can ignore the bending of the external layers, because their thickness is assumed to be small in comparison with the height of the honeycombs. Finally, let us assume that the rigidity of the honeycombs under tension in a tangential plane is insignificantly small in comparison with the rigidity of the external layers. Honeycombs are able to absorb only shear loads.

Having made these stipulations, let us work toward a derivation of equations of equilibrium.

§ 79. Differential Equations of Equilibrium and Boundary Conditions for Three-Ply Honeycomb Covers

Equations of equilibrium of three-ply honeycomb covers can be obtained by the variation method. First, let us write down the expression for the strain energy of the bending of a shell.

For the element of the shell having an area $dxdy$ the potential energy of flexural strain will be

$$d\text{Э}_{и}=\left(\frac{1}{2}M_{x}\chi_{x}+\frac{1}{2}M_{y}\chi_{y}+M_{xy}\chi_{xy}\right)dx\,dy,$$

where M_x, M_y, M_{xy} – internal moments, acting in the shell (Fig. 226);

χ_x, χ_y, χ_{xy} – changes in the curvatures, corresponding to these moments.

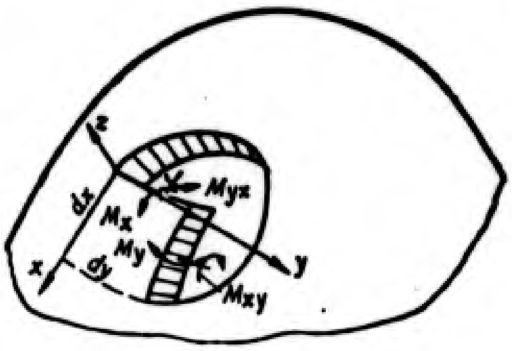


Fig. 226.

This energy basically can accumulate in the external layers of the shell, because the honeycomb filler, having low rigidity, does not absorb bending and tortional moments. At the same time the filler can absorb the shear loads. For the calculation of energy, produced by these loads, let us use Fig. 227.

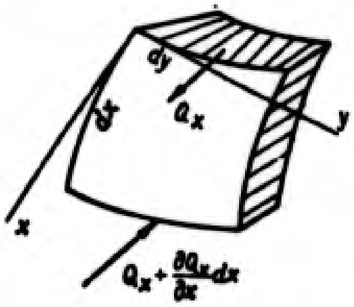


Fig. 227.

The work of lateral force Q_x will be equal to

$$-\frac{1}{2} Q_x \frac{\partial w_1}{\partial x} dx\, dy,$$

where w_1 – deflection of the shell from the shearing forces.

By considering force Q_x as an external one with respect to element $dxdy$, it is possible to show that the work of this force is numerically equal to the strain energy of the element. For the determination of the angle of inclination to the tangent $\partial w_1/\partial x$, let us assume that the height of the filler after deformation does not change and is equal to its height before deformation. From this assumption it follows that the angle of inclination to the tangent $\partial w_1/\partial x$, numerically equal to the angle of shear of the filler, is constant along the entire height of the filler.

As a constancy of the angle of shear the shear stress in terms of the height of the filler will also be constant. Therefore, accordingly to Hookes' law

$$\frac{\partial w_1}{\partial x} = \frac{Q_x dy}{FG} = \frac{Q_x dy}{2HG dy} = \frac{Q_x}{2HG},$$

where $2H$ – height of the filler; G – shear modulus of the filler.

Then, the potential energy in the filler from the shearing force Q_x will be

$$\frac{1}{2} Q_x \frac{\partial w_1}{\partial x} dx\, dy = \frac{1}{2} \frac{Q_x^2}{2HG} dx\, dy.$$

Similarly from force Q_y

$$\frac{1}{2} Q_y \frac{\partial w_1}{\partial y} dx\, dy = \frac{1}{2} \cdot \frac{Q_y^2}{2HG} dx\, dy.$$

The total energy of bending and shear will be

$$d Э_u = \left(\frac{1}{2} M_x \chi_x + \frac{1}{2} M_y \chi_y + M_{xy} \chi_{xy} + \frac{1}{2} \frac{Q_x^2}{2HG} + \frac{1}{2} \frac{Q_y^2}{2HG} \right) dx\, dy.$$

For the relationship of the shearing forces Q_x and Q_y with the bending and tortional moments, let us use the equations

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y}, \qquad Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x},$$

which express the conditions of equilibrium of the element $dxdy$ with respect to axes x, y.

The bending and tortional moments (Fig. 228) can be expressed through the corresponding normal and tangential stresses in the following manner:

$$M_x dy = 2\sigma_x \delta_0 dy \left(H + \frac{\delta_0}{2} \right),$$

$$M_y dx = 2\sigma_y \delta_0 dx \left(H + \frac{\delta_0}{2} \right),$$

$$M_{xy} dx = 2\sigma_{xy} \delta_0 dy \left(H + \frac{\delta_0}{2} \right),$$

$$M_{yx} dy = 2\sigma_{yx} \delta_0 dx \left(H + \frac{\delta_0}{2} \right).$$

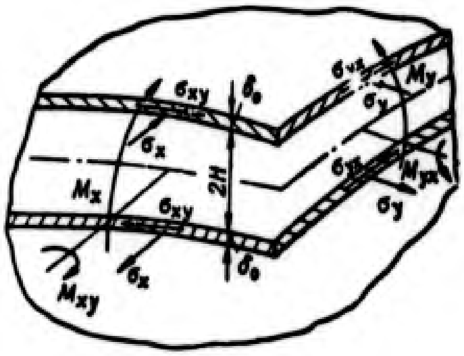


Fig. 228.

During the formulation of these expressions it was assumed that the external layers do not operate under bending, and that the bending and tortional moments in the sections of the shell will be realized through normal and tangential stresses, evenly distributed through the thickness of the external layers.

Furthermore, accordingly to Hooke's law

$$\sigma_x = \frac{E_0}{1-\mu_0^2}\left(\frac{z}{\varrho_x} + \mu\frac{z}{\varrho_y}\right) = \frac{E_0}{1-\mu_0^2}\left(H + \frac{\delta_0}{2}\right)(\chi_x + \mu_0\chi_y).$$

Analogously one can obtain

$$\sigma_y = \frac{E_0}{1-\mu_0^2}\left(H + \frac{\delta_0}{2}\right)(\chi_y + \mu\chi_x),$$

$$\sigma_{xy} = \frac{E_0}{1+\mu_0}\left(H + \frac{\delta_0}{2}\right)\chi_{xy}, \quad \sigma_{yx} = \sigma_{xy}.$$

Then, it is possible to write

$$M_x = D_0(\chi_x + \mu\chi_y),$$
$$M_y = D_0(\chi_y + \mu\chi_x),$$
$$M_{xy} = (1-\mu_0)D_0\chi_{xy},$$

where

$$D_0 = \frac{2E_0\delta_0\left(H + \frac{\delta_0}{2}\right)^2}{1-\mu_0^2},$$

E_0, μ_0 - Young's modulus and Poisson's ratio of the material of the external layers.

For the change in curvatures in this case we will have the following formulas:

$$\chi_x = -\frac{\partial^2 w}{\partial x^2} - \frac{w}{R_1^2} - \frac{\partial^2 w_1}{\partial x^2}, \quad \chi_y = -\frac{\partial^2 w}{\partial y^2} - \frac{w}{R_2^2} - \frac{\partial^2 w_1}{\partial y^2},$$

$$\chi_{xy} = -\frac{\partial^2 w}{\partial x^2\,\partial y^2},$$

where $\frac{\partial^2 w_1}{\partial x^2}$, $\frac{\partial^2 w_1}{\partial y^2}$ – additional components, taking into account the effect of the shear forces Q_x and Q_y.

The tension-compression energy of the external layers is determined by the known expression

$$\text{Э}_0 = 2\frac{E_0\delta_0}{2(1-\mu_0^2)}\iint_F\left(\varepsilon_x^2+\varepsilon_y^2+2\mu_0\varepsilon_x\varepsilon_y+\frac{1-\mu_0}{2}\varepsilon_{xy}^2\right)dx\,dy,$$

where coefficient 2 takes into account the number of external layers.

Components of deformation of the average surface are supplied by the following formulas:

$$\varepsilon_x=\frac{\partial u}{\partial x}+\frac{w}{R_1},\qquad \varepsilon_y=\frac{\partial v}{\partial y}+\frac{w}{R_2},$$
$$\varepsilon_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}.$$

The total potential energy of the shell is

$$\text{Э}=\text{Э}_0+\text{Э}_н+T,$$

where T – work of the external forces.

In the developed form this expression will have the form

$$\begin{aligned}\text{Э}=\iint_F\Bigg\{&\frac{E_0\delta_0}{1-\mu_0^2}\left[\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)^2+\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)^2+2\mu_0\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)+\right.\\ &\left.+\frac{1-\mu_0}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]+\frac{D_0}{2}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}\right)^2+\right.\\ &+\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}\right)^2+2\mu_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}\right)+\\ &\left.+2(1-\mu_0)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]+\frac{2HG}{2}\left[\left(\frac{\partial w_1}{\partial x}\right)^2+\left(\frac{\partial w_1}{\partial y}\right)^2\right]-qw\Bigg\}dx\,dy,\end{aligned}$$

where q – the intensity of the external distributed pressure.

During the formulation of the expression for the total energy it was assumed that the radii of curvature on the small surface of the shell can be taken as constants. Let us use the condition subsequently that

$$\delta Э=0,$$

which offers possibility to obtain both the differential equations of problem, and the boundary conditions. If the expression of the total energy pertains to the separately taken pit or bulge, being formed on the surface of the shell after the loss of rigidity, then the boundary conditions obtained in this case will be natural for it.

By not repeating all the intermediate computations, analogous to computations in § 34, let us present only the final results.

The balance equations are

$$\begin{gathered}
\frac{\partial}{\partial x}\frac{\partial F}{\partial u_x}+\frac{\partial}{\partial y}\frac{\partial F}{\partial u_y}=0,\\
\frac{\partial}{\partial x}\frac{\partial F}{\partial v_x}+\frac{\partial}{\partial y}\frac{\partial F}{\partial v_y}=0,\\
\frac{\partial F}{\partial w}+\frac{\partial^2}{\partial x\,\partial y}\frac{\partial F}{\partial w_{xy}}+\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{xx}}+\frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial w_{yy}}=0,\\
\frac{\partial^2}{\partial x^2}\frac{\partial F}{\partial w_{1xx}}+\frac{\partial^2}{\partial y^2}\frac{\partial F}{\partial w_{1yy}}-\frac{\partial}{\partial x}\frac{\partial F}{\partial w_{1x}}-\frac{\partial}{\partial y}\frac{\partial F}{\partial w_{1y}}=0.
\end{gathered} \qquad (16.1)$$

The boundary conditions are

$$\begin{gathered}
\int_0^b\left[\frac{\partial F}{\partial w_{xx}}\delta\left(\frac{\partial w}{\partial x}\right)\right]_0^a dy=0,\\
\int_0^a\left[\frac{\partial F}{\partial w_{yy}}\delta\left(\frac{\partial w}{\partial y}\right)\right]_0^b dx=0,\\
\int_0^b\left[\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial w_{xx}}+\frac{\partial}{\partial y}\frac{\partial F}{\partial w_{xy}}\right)\delta w\right]_0^a dy=0,\\
\int_0^a\left[\left(\frac{\partial}{\partial y}\frac{\partial F}{\partial w_{yy}}+\frac{\partial}{\partial x}\frac{\partial F}{\partial w_{xy}}\right)\delta w\right]_0^b dx=0,
\end{gathered}$$

t.

$$\left[\left(\frac{\partial F}{\partial w_{xy}}\delta w\right)_0^a\right]_0^b = 0,$$

$$\int_0^b\left(\frac{\partial F}{\partial u_x}\delta u\right)_0^a dy = 0, \qquad \int_0^a\left(\frac{\partial F}{\partial u_y}\delta u\right)_0^b dx = 0,$$

$$\int_0^b\left(\frac{\partial F}{\partial v_x}\delta v\right)_0^a dy = 0, \qquad \int_0^a\left(\frac{\partial F}{\partial v_y}\delta v\right)_0^b dx = 0,$$

$$\int_0^b\left[\frac{\partial F}{\partial w_{1xx}}\delta\left(\frac{\partial w_1}{\partial x}\right)\right]_0^a dy = 0,$$

$$\int_0^a\left[\frac{\partial F}{\partial w_{1yy}}\delta\left(\frac{\partial w_1}{\partial y}\right)\right]_0^b dx = 0,$$

$$\int_0^b\left[\left(\frac{\partial}{\partial x}\frac{\partial F}{\partial w_{1xx}} + \frac{\partial F}{\partial w_{1x}}\right)\delta w_1\right]_0^a dy = 0,$$

$$\int_0^a\left[\left(\frac{\partial}{\partial y}\frac{\partial F}{\partial w_{1yy}} + \frac{\partial F}{\partial w_{1y}}\right)\delta w_1\right]_0^b dx = 0.$$

Here F – the subintegral function in the expression for $Э$; $u_x = \frac{\partial u}{\partial x}, \ldots, w_{1y} = \frac{\partial w_1}{\partial y}$ – designations of the derivatives from the components of displacements u, v, w, w_1.

)

The partial derivatives from function F has the form

$$\begin{aligned}
\frac{\partial F}{\partial w} = {} & \frac{2E_0\delta_0}{1-\mu_0^2}\left[\left(\frac{1}{R_1}+\frac{\mu_0}{R_2}\right)\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)+\left(\frac{1}{R_2}+\frac{\mu_0}{R_1}\right)\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)\right]+ \\
& + D_0\left[\left(\frac{1}{R_1^2}+\frac{\mu_0}{R_2^2}\right)\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}\right)+\right. \\
& \left.+\left(\frac{1}{R_2^2}+\frac{\mu_0}{R_1^2}\right)\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}\right)\right]-q, \\
\frac{\partial F}{\partial w_{xx}} = {} & D_0\left[\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}+\mu_0\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}\right)\right]; \\
\frac{\partial F}{\partial w_{yy}} = {} & D_0\left[\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}+\mu_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}\right)\right]; \\
\frac{\partial F}{\partial w_{xy}} = {} & 2(1-\mu_0)D_0\frac{\partial^2 w}{\partial x\,\partial y}.
\end{aligned} \tag{16.2}$$

$$\frac{\partial F}{\partial u_x}=\frac{2E_0h_0}{1-\mu_0^2}\left[\frac{\partial u}{\partial x}+\frac{w}{R_1}+\mu_0\left(\frac{\partial v}{\partial y}+\frac{w}{R_2}\right)\right],$$

$$\frac{\partial F}{\partial u_y}=\frac{E_0h_0}{1+\mu_0}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),$$

$$\frac{\partial F}{\partial v_y}=\frac{2E_0h_0}{1-\mu_0^2}\left[\frac{\partial v}{\partial y}+\frac{w}{R_2}+\mu_0\left(\frac{\partial u}{\partial x}+\frac{w}{R_1}\right)\right],$$

$$\frac{\partial F}{\partial v_x}=\frac{E_0h_0}{1+\mu_0}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right);$$

$$\frac{\partial F}{\partial w_{1xx}}=D_0\left[\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}+\mu_0\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}\right)\right],$$

$$\frac{\partial F}{\partial w_{1yy}}=D_0\left[\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2^2}+\frac{\partial^2 w_1}{\partial y^2}+\mu_0\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1^2}+\frac{\partial^2 w_1}{\partial x^2}\right)\right],$$

$$\frac{\partial F}{\partial w_{1x}}=\frac{4HG}{2}\frac{\partial w_1}{\partial x},\quad \frac{\partial F}{\partial w_{1y}}=\frac{4HG}{2}\frac{\partial w_1}{\partial y}.$$

If we take into account the values of the partial derivatives from function F, then the first two equations of equilibrium (16.1) can be written in the form

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0,\quad \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0. \tag{16.3}$$

In this case these relationships were used

$$\frac{\partial F}{\partial u_x}=2N_x,\quad \frac{\partial F}{\partial u_y}=2N_{xy},$$
$$\frac{\partial F}{\partial v_y}=2N_y,\quad \frac{\partial F}{\partial v_x}=2N_{xy}.$$

After the introduction of the function of stresses ϕ according to the formula

$$N_x=\frac{\partial^2\varphi}{\partial y^2},\quad N_y=\frac{\partial^2\varphi}{\partial x^2},\quad N_{xy}=-\frac{\partial^2\varphi}{\partial x\,\partial y}$$

equations (16.3) will be identically satisfied, and the third equation (16.1) will assume the form

$$\frac{2}{R_1}\frac{\partial^2\varphi}{\partial y^2}+\frac{2}{R_2}\frac{\partial^2\varphi}{\partial x^2}+D_0\left[\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\right.$$
$$+2\left(\frac{1}{R_1^2}+\frac{\mu_0}{R_2^2}\right)\frac{\partial^2 w}{\partial x^2}+2\left(\frac{1}{R_2^2}+\frac{\mu_0}{R_1^2}\right)\frac{\partial^2 w}{\partial y^2}+\left(\frac{1}{R_1^4}+\frac{2\mu_0}{R_1^2R_2^2}+\frac{1}{R_2^4}\right)w+$$

$$+\frac{\partial^4 w_1}{\partial x^4}+2\mu_0\frac{\partial^4 w_1}{\partial x^2\partial y^2}+\frac{\partial^4 w_1}{\partial y^4}+\left(\frac{1}{R_1^2}+\frac{\mu_0}{R_2^2}\right)\frac{\partial^2 w_1}{\partial x^2}+$$

$$+\left(\frac{1}{R_2^2}+\frac{\mu_0}{R_1^2}\right)\frac{\partial^2 w_1}{\partial y^2}\Bigg]=q. \qquad (16.4)$$

In this case the fourth equation of equilibrium (16.1) in developed form will be

$$D_0\Bigg[\frac{\partial^4 w}{\partial x^4}+2\mu_0\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\left(\frac{1}{R_1^2}+\frac{\mu_0}{R_2^2}\right)\frac{\partial^2 w}{\partial x^2}+$$

$$+\left(\frac{1}{R_2^2}+\frac{\mu_0}{R_1^2}\right)\frac{\partial^2 w}{\partial y^2}+\frac{\partial^4 w_1}{\partial x^4}+2\mu_0\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w_1}{\partial y^4}\Bigg]-$$

$$-2HG\left(\frac{\partial^2 w_1}{\partial x^2}+\frac{\partial^2 w_1}{\partial y^2}\right)=0. \qquad (16.5)$$

The equation, which connects the function of stresses ϕ with the deflection w, can be obtained from the expressions for the components of deformations using Hooke's law

$$\frac{\partial^4\varphi}{\partial x^4}+2\frac{\partial^4\varphi}{\partial x^2\partial y^2}+\frac{\partial^4\varphi}{\partial y^4}=E_0h_0\left(\frac{1}{R_1}\frac{\partial^2 w}{\partial x^2}+\frac{1}{R_2}\frac{\partial^2 w}{\partial y^2}\right). \qquad (16.6)$$

The strained and deformed state of the shell with honeycomb filler is completely described by equations (16.4)-(16.6). From these equations just as in special cases the equations of the bending of beams and plates made from honeycombs, can be received.

In problems of dynamics and of rigidity the distributed load q in these equations is replaced by the expression of the form

$$q-\frac{2(H+h_0)\gamma}{g}\frac{\partial^2 w}{\partial t^2}+N_x^0\left(\frac{\partial^2 w}{\partial x^2}+\frac{w}{R_1}\right)+$$

$$+N_y^0\left(\frac{\partial^2 w}{\partial y^2}+\frac{w}{R_2}\right)+2N_{xy}^0\frac{\partial^2 w}{\partial x\,\partial y},$$

where the first component determines the value of the inertial load, and the remaining three – the value of the projections of the internal compressive and shear force normal to the shell.

During the solution of problems of the rigidity or of the vibrations of the shell it is necessary to select that combination of given boundary conditions for the functions of ϕ, w, w_1 along the contour of pits and bulges, which conform with the character of the expected wave formation and which result in a minimum value of frequency or critical load.

As an illustration let us solve the problem dealing with the rigidity of a hinged supported rod, loaded along the axis. In this instance the original equations will assume the form

$$EJ\left(\frac{d^4w}{dx^4}+\frac{d^4w_1}{dx^4}\right)=-P\frac{d^2w}{dx^2},$$

$$EJ\left(\frac{d^4w}{dx^4}+\frac{d^4w_1}{dx^4}\right)-FG\frac{d^2w_1}{dx^2}=0.$$

The boundary conditions of the problem will be satisfied by the following expressions for w and w_1:

$$w=A\sin\frac{\pi x}{l},\qquad w_1=B\sin\frac{\pi x}{l}.$$

By substituting the necessary derivatives in these equations, we obtain

$$A\left[EJ\left(\frac{\pi}{l}\right)^2-P\right]+BEJ\left(\frac{\pi}{l}\right)^2=0,$$

$$AEJ\left(\frac{\pi}{l}\right)^2+B\left[EJ\left(\frac{\pi}{l}\right)^2+FG\right]=0.$$

By equating the determinant of these equations to zero, we obtain

$$P_{\text{кр}}=\frac{EJ\left(\frac{\pi}{l}\right)^2}{1+\frac{EJ\left(\frac{\pi}{l}\right)^2}{FG}}.$$

This formula during the course of resistance of the material can be derived by another means.

Let us apply the above obtained equations to the solution of certain problems of the rigidity of shells.

§ 80. Rigidity of a Cylindrical Shell with Honeycomb Filler Under Axial Compression

In this instance $R_1 = \infty$, $R_2 = R$. Furthermore, let us assume $\mu_0 = 0$. Then, the original equations will have the form

$$\begin{gathered}
\frac{2}{R}\frac{\partial^2\varphi}{\partial x^2} + D_0\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{2}{R^2}\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^4} + \right. \\
\left. + \frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^4 w_1}{\partial y^4} + \frac{1}{R^2}\frac{\partial^2 w_1}{\partial y^2}\right) = -N_x^0 \frac{\partial^2 w}{\partial x^2}, \\
D_0\left(\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + \frac{1}{R^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^4 w_1}{\partial x^4} + \frac{\partial^4 w_1}{\partial y^4}\right) - 2HG\left(\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2}\right) = 0, \\
\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2 \partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = \frac{E_0 h_0}{R}\frac{\partial^2 w}{\partial x^2}.
\end{gathered} \qquad (16.7)$$

For the functions w, w_1, ϕ let us assume that the following expressions are:

$$\begin{gathered}
w = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \\
w_1 = B\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \\
\varphi = C\sin^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b}.
\end{gathered}$$

These functions satisfy the condition of a hinged support of the contour of pits and bulges and the additional (self-balanced) normal and tangential stresses revert to zero along their contour.

After the substitution of the accepted expressions in the original equations and after their integration we obtain

$$\begin{gathered}
-\frac{64C}{9\pi^2 R}\left(\frac{\pi}{a}\right)^2 + D_0 A\left(\frac{\pi}{a}\right)^4\left[1 + \left(\frac{a}{b}\right)^2\right]^4 + D_0 B\left(\frac{\pi}{a}\right)^4\left[1 + \left(\frac{a}{b}\right)^4\right] - \\
- N_x^0 A\left(\frac{\pi}{a}\right)^2 = 0,
\end{gathered}$$

$$C\left(\frac{\pi}{a}\right)^2 = -\frac{16E_0\delta_0 A}{9\pi^2 R\left[0{,}75+0{,}5\left(\frac{a}{b}\right)^2+0{,}75\left(\frac{a}{b}\right)^4\right]},$$

$$B = -\frac{D_0 A\left(\frac{\pi}{a}\right)^2\left[1+\left(\frac{a}{b}\right)^4\right]}{D_0\left(\frac{\pi}{a}\right)^2\left[1+\left(\frac{a}{b}\right)^4\right]+2HG\left[1+\left(\frac{a}{b}\right)^2\right]}.$$

Having excluded parameters B and C from the first equation, we obtain the following expression for critical stress when $A \neq 0$:

$$\left(\frac{N_x^0}{E_0\delta_0}\right)_{\text{кр}} = \frac{1024}{81\pi^4\left[0{,}75+0{,}5\left(\frac{l}{\pi R}\right)^2\left(\frac{n}{m}\right)^2+0{,}75\left(\frac{l}{\pi R}\right)^4\left(\frac{n}{m}\right)^4\right]} +$$
$$+\frac{D_0}{E_0\delta_0 R^2}\left(\frac{\pi R}{l}\right)^2 m^2 -$$
$$-\frac{D_0^2\left(\frac{\pi R}{l}\right)^4 m^4\left[1+\left(\frac{l}{\pi R}\right)^4 m^4\right]^2}{2HGE_0\delta_0 R^4\left\{1+\left(\frac{l}{\pi R}\right)^2\left(\frac{n}{m}\right)^2+\frac{D_0\left(\frac{\pi}{l}\right)^2}{2HG}m^2\left[1+\left(\frac{l}{\pi R}\right)^4\left(\frac{n}{m}\right)^4\right]\right\}}. \qquad (16.8)$$

Designated here is

$$m = \frac{l}{a}, \qquad n = \frac{\pi R}{b},$$

where l – length of the shell; a, b – sizes of pits and bulges in the axial and circumferential directions.

The remaining designations are given above.

By assigning parameters of wave formation m, n, the various integral values, based on the given expression, can determine the least value $N_{x\text{кр}}^0$.

The critical force of axial compression of the shell

$$P_{\text{кр}} = 2\pi R N_{x\text{кр}}^0.$$

To get a more simple approximate formula one can discard the last component in the expression (16.8), considering the work of the honeycombs from lateral forces. By such an approximation a somewhat overrated value of the critical stress will be determined. Furthermore, in the elaborated expression it is possible to discard the components, containing the parameter of wave formation n in comparison with m. This additional assumption, as the comparative calculations show, somewhat lowers magnitude of critical force.

With the allowance for the shown assumptions comparatively simple expression can be obtained, from which we can determine the analytical minimum of critical stress:

$$N^0_{x\,\text{кр}} \simeq 1{,}67 E_0 \delta_0 \frac{H + \frac{t_0}{R}}{R} \quad \text{при} \quad m = 0{,}66 \sqrt{\frac{R}{H + \frac{t_0}{2}}}.$$

§ 81. Rigidity of a Cylindrical Shell with Honeycomb Filler Under the Action of External Pressure

For the solution of this problem it is necessary to make a replacement of the given load in the equations (16.7)

$$N^0_x \frac{\partial^2 w}{\partial x^2} \text{ in place of } N^0_y \left(\frac{\partial^2 w}{\partial y^2} + \frac{w}{R^2}\right).$$

As a solution of these equations one can assume that

$$w = A \sin \frac{\pi x}{l} \sin n\theta,$$
$$w_1 = B \sin \frac{\pi x}{l} \sin n\theta,$$
$$\varphi = C \sin \frac{\pi x}{l} \sin n\theta.$$

Following the substitution of the expressions for w, w_1 and ϕ in the solving of equations (16.7), we obtain

$$-\frac{2C}{R}\left(\frac{\pi}{l}\right)^2+D_0A\left[\left(\frac{\pi}{l}\right)^2+\left(\frac{n}{R}\right)^2\right]^2+D_0B\left[\left(\frac{\pi}{l}\right)^4+\left(\frac{n}{R}\right)^4\right]-$$

$$-AN_y^0\left[\left(\frac{n}{R}\right)^2-\left(\frac{1}{R}\right)^2\right]; \qquad (16.9)$$

$$D_0A\left[\left(\frac{\pi}{l}\right)^4+\left(\frac{n}{R}\right)^4\right]+D_0B\left[\left(\frac{\pi}{l}\right)^4+\left(\frac{n}{R}\right)^4\right]+$$

$$+2HGR\left[\left(\frac{\pi}{l}\right)^2+\left(\frac{n}{R}\right)^2\right]=0,$$

$$C\left[\left(\frac{\pi}{l}\right)^2+\left(\frac{n}{R}\right)^2\right]^2=-\frac{E_0\delta_0A\left(\frac{\pi}{l}\right)^2}{R}.$$

For a shell of average length one can expect that as a result of the loss of its rigidity in the circumferential direction many half-waves are generated. Then, one can assume that

$$\left(\frac{n}{R}\right)^2\gg\left(\frac{\pi}{l}\right)^2, \quad n^2\gg 1.$$

On this basis of equation (16.9), one can substitute more simply

$$-\frac{2C}{R}\left(\frac{\pi}{l}\right)^2+D_0A\left(\frac{n}{R}\right)^4+D_0B\left(\frac{n}{R}\right)^4=AN_y^0\left(\frac{n}{R}\right)^2,$$

$$C=-\frac{E_0\delta_0A\left(\frac{\pi}{l}\right)^2}{R}\left(\frac{R}{n}\right)^4,$$

$$B=-\frac{D_0A\left(\frac{n}{R}\right)^2}{D_0\left(\frac{n}{R}\right)^2+2HG}.$$

After excluding parameters A, B, C from these equations we obtain the following expression for the critical external pressure:

$$q_{kp}=\frac{2E_0\delta_0\left(\frac{\pi}{l}\right)^4R^3}{n^6}+\frac{D_0}{R^3}n^2-\frac{\frac{D_0^2}{2HG}n^4}{R^5\left[1+\frac{D_0}{2HG}\left(\frac{n}{R}\right)^2\right]}. \qquad (16.10)$$

For the determination of the least value of pressure it is necessary to subsequently set $n = 2, 3, \ldots$

To get the approximate formula for the critical load in the expression (16.10) one can discard the component from D_0^2, associated with the work of the honeycombs to shear. Then, after the determination of the minimum through parameter n we obtain

$$q_{кр} = 11 E_0 \delta_0 \frac{H + \frac{t_0}{2}}{Rl} \sqrt{\frac{H + \frac{t_0}{2}}{R}} \quad \text{when} \quad n^2 = \frac{1.7\pi R}{l} \sqrt[4]{\frac{E_0 \delta_0 R^2}{D_0}}.$$

In the case of a very long shell the wave formation does not depend upon the longitudinal coordinate x. Then, equations (16.7) will assume the form

$$D_0\left(\frac{d^4 w}{dy^4} + \frac{2}{R^2}\frac{d^2 w}{dy^2} + \frac{w}{R^4} + \frac{d^4 w_1}{dy^4} + \frac{1}{R^2}\frac{d^2 w_1}{dy^2}\right) = -N_y^0\left(\frac{d^2 w}{dy^2} + \frac{w}{R^2}\right). \tag{16.11}$$

$$D_0\left(\frac{d^4 w}{dy^4} + \frac{1}{R^2}\frac{d^2 w}{dy^2} + \frac{d^4 w_1}{dy^4}\right) - 2HG\frac{d^2 w_1}{dy^2} = 0. \tag{16.12}$$

In this case for the solution of the problem one can assume that

$$w = A\cos n\theta, \quad w_1 = B\cos n\theta.$$

After the substitution of these expressions in equations (16.11)-(16.12) for the critical pressure we obtain

$$q_{кр} = \frac{D_0(n^2 - 1)}{R^3}\left(1 - \frac{D_0}{2HGR^2}\,\frac{1}{1 + \frac{D_0 n^2}{2HGR^2}}\right).$$

One can be certain that the least value of critical pressure will be when $n = 2$. Then

$$q_{кр} = \frac{3D_0}{R^3}\left(1 - \frac{\frac{D_0}{2HGR^2}}{1 + \frac{2D_0}{HGR^2}}\right).$$

§ 82. Determination of Rigidity of a Honeycomb Construction to Shear

In the practical calculations of honeycomb constructions, it is necessary to determine the rigidity of the honeycombs to shear at a rated value of shear moduli of the material made of honeycomb.

At the assigned value of shear moduli G_1 of the material made of honeycomb, the given shear moduli G_2, which will have a honeycomb construction, as a whole, made from this material, can be determined from the condition of equality of the angles of shear during the loading of the samples, cut from the material with a known modulus G_1 and from the honeycomb construction with an unknown modulus G_2 (Fig. 229).

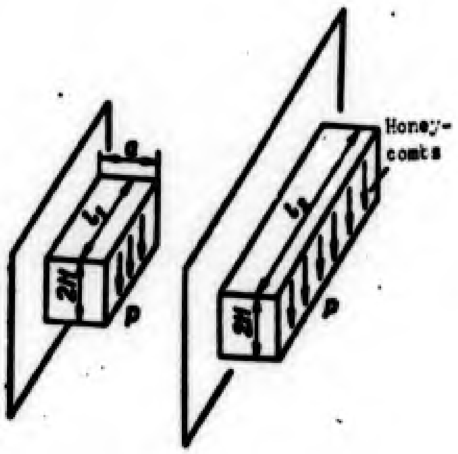


Fig. 229.

Thus, $\gamma_1 = \gamma_2$,

where

$$\gamma_1 = \frac{P}{F_1 G_1};$$

$$\gamma_2 = \frac{P}{F_2 G_2}; \quad F_1 = 2Hl_1; \quad F_2 = 2Hl_2.$$

Then

$$F_1 G_1 = F_2 G_2.$$

Therefore

$$G_2 = G_1 \frac{l_1}{l_2}.$$

The length of the sample made from honeycomb construction l_2 can be expressed by the number of sheared honeycombs in the following manner. The number of walls, which can be made from the length l_1, will be

$$n = \frac{l_1}{\delta_c},$$

where δ_c – thickness of the wall made of honeycombs.

In this case, the quantity of complete honeycombs is

$$m = n - 1 = \frac{l_1}{\delta_c} - 1.$$

Then for the length l_2

$$l_2 \approx m d_{an} = \left(\frac{l_1}{\delta_c} - 1\right) d_{an},$$

where d_{an} – diameter of the inscribed circumference in the honeycomb.

Therefore, it is possible to write

$$G_2 = G_1 \frac{l_1}{\left(\frac{l_1}{\delta_c} - 1\right) d_{an}}.$$

Let us rewrite this expression in the following form:

$$G_2 = G_1 \frac{\delta_c}{d_{an}} \frac{\frac{l_1}{\delta_c}}{\frac{l_1}{\delta_c} - 1}.$$

Inasmuch as the size of the sample l_1 has been taken randomly, then from the last expression it is necessary to obtain that value of G_2, at which the critical force will be the least. From this requirement we obtain

$$G_2 = G_1 \frac{h_0}{d_{gr}} \text{ wr.er. } \left(\frac{\frac{l_1}{h_0}}{\frac{l_1}{h_0} - 1} \right)_{\frac{l_1}{h_0} \to \infty} - 1.$$

C H A P T E R XVII

CALCULATION OF SHELLS FOR RIGIDITY UNDER THE SIMULTANEOUS ACTION OF SEVERAL LOADS

If several different loads act on a shell, then the determination of their critical value from the solution of equations of stability. presents a problem of extreme complexity, and sometimes it is almost unsolvable. Therefore, it is advisable to have estimates of the bearing capacity of such a shell, by not solving the differential equations of the problems directly and by using the results, which can be obtained during the loading of the shell of each of the acting forces separately. Such a way, then is more expedient due to the fact that many problems of rigidity of shells have already been solved.

For the solution of the posed problem let us examine V. Z. Vlasov's equations:

$$\nabla^2\nabla^2\varphi = E\delta\left(\frac{1}{R_1}\frac{\partial^2 w}{\partial x^2}+\frac{1}{R_2}\frac{\partial^2 w}{\partial y^2}\right), \tag{17.1}$$

$$\frac{1}{R_1}\frac{\partial^2\varphi}{\partial x^2}+\frac{1}{R_2}\frac{\partial^2\varphi}{\partial y^2}+D\nabla^2\Delta^2 w = N^0_x\chi_x+N^0_y\chi_y+2N^0_{xy}\chi_{xy}. \tag{17.2}$$

The right side of the equation (17.2) determines the membrane forces, which appear in the shell due to the action of external forces. Depending on the character of these loads and the geometry of the shell, forces N^0_x, N^0_y, N^0_{xy} can act either simultaneously or in some combination. For example, with the loading of an ellipsoidal doughnut-shaped shell with internal pressure on it N^0_x and N^0_y forces arise when the loading of a spherical segment by a bending moment all three components of internal forces appear simultaneously. In contrast to this, with the loading of a cylindrical shell by an axial compressive force, only the force N_x^0 appears, when a load by an external uniform pressure - force N_y^0, by a tortional moment - force N_{xy}^0. Therefore, in the general, the expression for internal forces can be expressed in the form

$$
\begin{aligned}
N_x^0 &= \alpha' P + \alpha'' Q + \alpha''' R + \ldots, \\
N_y^0 &= \beta' P + \beta'' Q + \beta''' R + \ldots, \\
N_{xy}^0 &= \gamma' P + \gamma'' Q + \gamma''' R + \ldots,
\end{aligned}
\qquad (17.3)
$$

where P, Q, R, ... - external loads, acting on the shell; α^i, β^i, γ^i - number of coefficients, which depend upon the parameters and on the current coordinates of the point on the surface of the shell.

The structure of these coefficients is determined by the form of solution of the problem of the shell in the subcritical state.

After the substitution of the expressions (17.3) in (17.2), we obtain

$$
\begin{aligned}
&\frac{1}{R_1}\frac{\partial^2\varphi}{\partial x^2} + \frac{1}{R_2}\frac{\partial^2\varphi}{\partial y^2} + D\nabla^2\Delta^2 w = \\
&= P(\alpha'\chi_x + \beta'\chi_y + 2\gamma'\chi_{xy}) + Q(\alpha''\chi_x + \beta''\chi_y + 2\gamma''\chi_{xy}) + \\
&+ R(\alpha'''\chi_x + \beta'''\chi_y + 2\gamma'''\chi_{xy}) + \ldots
\end{aligned}
\qquad (17.4)
$$

The subsequent problem amounts to how to solve the combined system of equations (17.1) and (17.4).

Let us assume that this system has been solved and the result of this solution can be written in the form

$$
\begin{aligned}
&F_1(E, \delta, R_1, R_2, m, n) + F_2(D, R_1, R_2, m, n) = \\
&= P(\alpha_1^{\cdot}, \beta_1^{\cdot}, \gamma_1^{\cdot}, R_1, R_2, m, n) + Q(\alpha_1^{\cdot}, \beta_1^{\cdot}, \gamma_1^{\cdot}, R_1, R_2, m, n) + \\
&+ R(\alpha_1^{\cdot}, \beta_1^{\cdot}, \gamma_1^{\cdot}, R_1, R_2, m, n) + \ldots
\end{aligned}
\qquad (17.5)
$$

Designated here is: m, n – parameters, which characterize the wave formation of the shell; F, F_2 – components, which characterize the work of shell under tension-compression and bending.

The right part of the equation (17.5) depends on the forces P, Q, R applied to the shell, the least critical value of which one is required to determine.

For the future it is convenient to use these ooads as coordinate axes of n-dimensional space and to rewrite the equation (17.5) in the form

$$a_1X_1+a_2X_2+a_3X_3+\ldots+a_nX_n-a_0=0,$$

where X_i – accepted designation for forces P, Q, R, ...; a_i – coefficients, which depend both on the parameters of the shell, and on the parameters of wave formation; $a_0=F_1+F_2$.

Equation (17.6) determines a certain plane in n-dimensional space, called a hyperplane. By assigning different values to the parameters of wave formation m, n, it is possible to obtain various hyperplanes. Each such hyperplane can be broken down, with respect to the whole space, into two halfspaces, one of which includes the origin of coordinates. It is possible to prove that any hyperplane is convex. For proof let us introduce a certain function Φ into the examination, representing the left part of the equation (17.6):

$$\Phi=a_1X_1+a_2X_2+a_3X_3+\ldots+a_nX_n-a_0.$$

For the future let us introduce the concept of directional cosines in n-dimensional space.

It is known that the position of a straight line in three-dimensional space is entirely determined by the amount of directing cosines. Position of a straight line in the n-dimensional space likewise is entirely determined by its directional cosines with respect to its axes X_1, X_2, ..., X_n. Based on this, let us assume that

$$X_1 = X_1^0 + i_1 r,$$
$$X_2 = X_2^0 + i_2 r,$$
$$\cdots\cdots\cdots$$
$$X_n = X_n^0 + i_n r,$$

where i_1, i_2, ..., i_n – directional cosines of the radius vector r, projecting from a point in halfspace, including point O of the origin of coordinates and carrying its origin to a point, having the coordinates and carrying its origin to a point, having coordinates X_1^0, X_2^0, ..., X_n^0. Then, for function Φ we obtain the expression

$$\Phi = a_1(X_1^0 + i_1 r) + a_2(X_2 + i_2 r) + \ldots + a_n(X_n^0 + i_n r) - a_0.$$

If now one changes the length of the radius vector r, then the numerical value of function Φ will change.

Inasmuch as this function is the left part of equation (17.6), then at a certain value of r it will be revert to zero, and then with an increase in r, it will no longer take a zero value nor change signs.

Thus, it is possible to prove that all hyperplanes, corresponding to different values of parameters of wave formation m, n, are convex.

From this assertion it follows that the surface, formed by the intersection of the various hyperplanes, is likewise convex. For the first time this theorem was proven by P. F. Papkovic by another method.

The proposed case can be used for the calculation of covers for rigidity by loading them simultaneously with several types of loads.

From the concept of the critical force it follows that all the hyperplanes intercept on the coordinate axes X_1, X_2, ..., X_n segments, equal to corresponding critical forces. For the numerical determination of the least value of these critical forces it is sufficient to assume that all the acting loads, aside from a load of one type, are equal to zero, and to solve the problem of the rigidity of the shell only from the load of one type. In that case the hyperplane can determine only one point on the corresponding axis X_i. After this one should solve the problem of rigidity due to the action of a load of another type at zero values of remaining loads, and so forth.

Thus, the coordinates of the critical hyperplane for a given shell will be determined, since the hyperplane, corresponding to the minimum values of critical forces, will be unique. Inasmuch as the plotting of the convex hyperplane for an assigned combination of critical forces is very difficult to do, then one can approximately substitute the hyperplane simply by a plane, intercepting at the coordinate axes X_i as segments equal to corresponding critical forces.

As an illustration let us offer two examples.

Loading the Cylindrical Shell with an Axial Compressive Force and with a Lateral Uniform Pressure

In this case the hyperplane degenerates into a straight line on a plane (Fig. 230). In this figure the external pressure carries a minus sign and it is deferred to negative section of the axis of the abscissa.

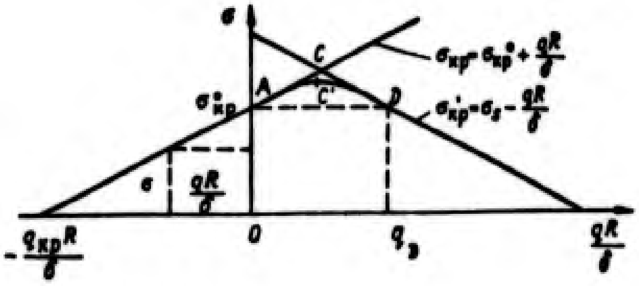


Fig. 230.

The equation of the obtained straight line will have the form

$$\frac{q}{q_{\kappa p}}+\frac{\sigma}{\sigma^{0}_{\kappa p}}=1, \qquad (17.7)$$

where $q_{\kappa p}=0{,}92E\frac{\delta^2}{Rl}\sqrt{\frac{\delta}{R}}$ - critical value of the external pressure;

$\sigma^{0}_{\kappa p}=0{,}26E\frac{\delta}{R}$ - critical stress of pure compression.

This straight line bisects the plane into two half-planes, one of which includes point O of the origin of coordinates. If we designate this half-plane as V^+, and the second half-plane V^-, then it can be said that the points of the first half-plane, excluding the points which lie on the straight line (17.7), correspond to the state of steady equilibrium of the cover. The points, arranged on the second half-plane, excluding the points on the straight line (17.7), correspond to the state of unstable equilibrium of the shell.

All the points on the straight line (17.7), including its ends, determine the critical state of the cover.

Because the equation (17.5) - linear, relative to the external forces, then, by extending the straight line, defined by equation (17.7), to the side of positive values of pressure q, the dependence of critical stress of compression from the internal pressure, can be obtained.

The equation of a straight line in this case will have the form[1]

$$\sigma_{\kappa p}=\sigma^{0}_{\kappa p}+\frac{qR}{\delta}. \qquad (17.8)$$

This formula, apparently, will be valid only up to determined values of internal pressure q, because beginning from a certain value

[1]This formula is valid under the assumption that square pits and bulges are generated on the surface of the shell after the loss of rigidity.

of internal pressure, the phenomenon of the loss of resistance, connected with the formation pits and bulges, will be accompanied by the flow of material of the shell at it ends. Therefore, the formula (17.8) will be valid only up to certain "small" values of internal pressure q, depending upon the mechanical features of the material and geometric parameters of the shell. For an estimate of the bearing capacity of the shell at large pressures it is necessary to make use of one of the theories of strength. For plastic materials it is possible to use the theory of the largest tangential stresses

$$\sigma_{max}-\sigma_{min}=\sigma_s.$$

By substituting

$$\sigma_{max}=\frac{qR}{\delta},\quad \sigma_{min}=-\sigma'_{кр},$$

we will have

$$\sigma'_{кр}=\sigma_s-\frac{qR}{\delta}. \qquad (17.9)$$

Graphically, this formula describes the equation of a straight line by segments on the axes of coordinates (Fig. 230).

Experiments show that the formulas (17.8) and (17.9) give overrated values for the critical stresses, which fall in the interval between "small" and "large" pressures. Within this range of pressures, as a rule, a mixed form of loss in rigidity, connected with the formation of pits and bulges, and corrugated folds, is obtained. With an increase in the internal pressure one observes the gradual transition from wave formation in the form of pits and bulges to the formation of folds in the form of corrugation, arranged on the circumference of the shell.

Inasmuch as formulas (17.8) and (17.9) are derived independently from one another even under different assumptions, then it can be expected that the real curve, characterizing the critical state of the shell under axial compression and internal pressure, should be inscribed as angle *ACD* so that the straight lines *AC* and *CD* would appear on it at points *A* and *D*. Experiments show that with sufficient accuracy for all practical purposes this curve can be selected in the form of a square parabola. In order to enscribe this parabola in angle *ACD*, let us select a new system of coordinates with an origin at point *A*. Then, the equation of the parabola will have the form

$$\sigma'_{\text{кр}} = \frac{qR}{\delta}\left(1 - \frac{\frac{qR}{\delta}}{\frac{q_D R}{\delta}}\right),$$

where q_D – internal pressure, corresponding to point *D*. This pressure can be determined from the condition

$$\sigma'_{\text{кр}} = \sigma_s - \frac{q_D R}{\delta} = \sigma^0_{\text{кр}},$$

whence

$$\frac{q_D R}{\delta} = \sigma_s - \sigma^0_{\text{кр}}. \qquad (17.10)$$

Then

$$\sigma'_{\text{кр}} = \frac{qR}{\delta}\left(1 - \frac{\frac{qR}{\delta}}{\sigma_s - \sigma^0_{\text{кр}}}\right).$$

The total value of the critical stress in this case

$$\sigma_{\text{кр}} = \sigma^0_{\text{кр}} + \sigma'_{\text{кр}} = \sigma^0_{\text{кр}} + \frac{qR}{\delta}\left(1 - \frac{\frac{qR}{\delta}}{\sigma_s - \sigma^0_{\text{кр}}}\right), \qquad (17.11)$$

where

$$\sigma^0_{\text{кр}} = 0{,}26E\frac{\delta}{R}.$$

This formula is valid under the condition that $q \leq q_D$. When $q \geq q_D$ one whould make use of the formula (17.9).

Let us determine the greatest value $\sigma_{кр}$ depending on the internal pressure:

$$\frac{d\sigma_{кр}}{d\left(\frac{qR}{\delta}\right)} = 1 - \frac{2\frac{qR}{\delta}}{\sigma_s - \sigma^0_{кр}} = 0,$$

whence

$$\frac{qR}{\delta} = \frac{1}{2}\left(\sigma_s - \sigma^0_{кр}\right),$$

or

$$q_{C'} = \frac{1}{2}\frac{\delta}{R}\left(\sigma_s - \sigma^0_{кр}\right) = \frac{1}{2}\frac{\delta}{R}\left(\sigma_s - 0{,}26E\frac{\delta}{R}\right). \qquad (17.12)$$

This pressure corresponds to point C' of parabola $AC'D$.

After the substitution of $\left(\frac{qR}{\delta}\right)_{C'}$ in formula (17.11), we obtain

$$\sigma_{кр.max} = 0{,}75\sigma^0_{кр} + 0{,}25\sigma_s.$$

From the expression (17.12) it is evident that pressure $q_{C'}$ is a function of ratio δ/R.

Considering this ratio as a variable, we will find that when

$$\frac{\delta}{R} = \frac{\sigma_s}{0{,}52E} \qquad (17.13)$$

the greatest value for $q_{C'}$ occurs:

$$q_{C'max} = 0{,}48\frac{\sigma_s^2}{E}. \qquad (17.14)$$

The greatest internal pressure, which can possibly be created in a shell during its combined work under axial compression and internal pressure determines this expression.

Having substituted the values δ/R and $q_{C'max}$ with (17.13) and (17.14) in the expression (17.11), we obtain the limiting value of the critical stress of compression

$$\sigma'_{\text{кр.max}} = 0{,}625\sigma_s. \qquad (17.15)$$

The circumferential stresses in the shell in this instance, are

$$\sigma_{\text{к}} = 0{,}25\sigma_s.$$

Thus for the construction of a shell having a maximum value of critical stress of compression in an axial direction, it is necessary to select such a ratio of δ/R, which satisfies the condition (17.13), and in the shell to create an internal pressure, which can be determined by the formula (17.14). Such a cover will be optimum from the viewpoint of its bearing capacity.

Let us examine the calculation of a cylindrical cover under axial compression and internal pressure taking into account the unloading due to internal pressure in an axial direction. We will have

$$\sigma'_{\text{кр}} = 0{,}26E\frac{\delta}{R} + \frac{qR}{\delta}, \quad \sigma''_{\text{кр}} = \sigma_s - \frac{qR}{\delta}.$$

By having added the values of axial stresses from internal pressure to the right parts of these expressions, we obtain the formulas for the critical stresses with the allowance for the unloading

$$\sigma'_{\text{кр}} = 0{,}26E\frac{\delta}{R} + \frac{3}{2}\frac{qR}{\delta}, \quad \sigma''_{\text{кр}} = \sigma_s - \frac{1}{2}\frac{qR}{\delta},$$

graphically presented in Fig. 231.

In this case in the angle *ACD*, a parabola is also inscribed, whose plotting is clear from Fig. 231. We obtain the values of pressure q, corresponding to points *C* and *D*, from the conditions

$$\sigma'_{кр} = \sigma'_{кр}, \quad \sigma''_{кр} = \sigma^0_{кр}.$$

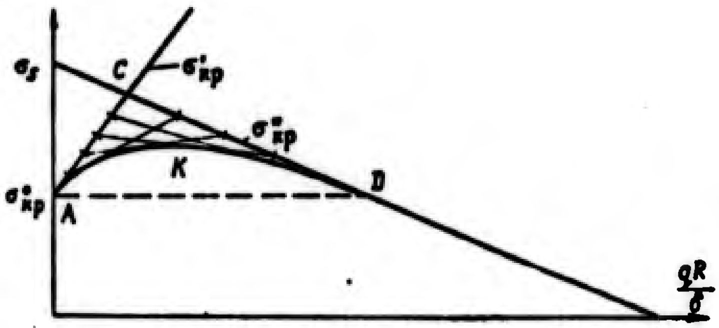


Fig. 231.

Hence, we will find that

$$q_C = \frac{1}{2}\frac{\delta}{R}\left(\sigma_s - 0{,}26E\frac{\delta}{R}\right), \quad q_D = 2\frac{\delta}{R}\left(\sigma_s - 0{,}26E\frac{\delta}{R}\right).$$

From a comparison of these expressions with expressions (17.10), (17.12) it is possible to see that point of intersection C of the straight lines in both cases is determined by the same expression for q_C. Regarding point D, in the latter case, its abscissa increased by 2 times.

Accordingly, as can be seen from Fig. 231, the maximum of critical stresses will likewise be displaced to the right. The pressure, corresponding to this stress, will be

$$q_k = \frac{3}{4}\frac{\delta}{R}\left(\sigma_s - 0{,}26E\frac{\delta}{R}\right).$$

Considering this pressure as a function of the ratio δ/R, we obtain the greatest value for q_k when $\frac{\delta}{R} = \frac{\sigma_s}{0{,}52E}$

$$q_{k\,max} = 0{,}725\frac{\sigma_s^2}{E}.$$

Having substituted the values $\delta/R = \sigma_s/0.52E$ for $\sigma'_{кр}$ and $\sigma''_{кр}$ in the original, we obtain

$$\sigma'_{кр} = 0{,}5\sigma_s + 0{,}78\frac{E}{\sigma_s}q, \quad \sigma''_{кр} = \sigma_s - 0{,}26\frac{E}{\sigma_s}q.$$

Graphically these equations are presented in Fig. 232, where the plotting of a "flattened" parabola is also presented. By knowing the position of point K on the parabola, corresponding to the position of the greatest critical stress, it is possible to evaluate approximately the limiting bearing capacity of the shell.

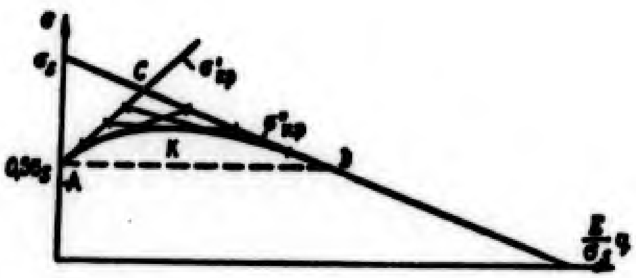


Fig. 232.

Directly from Fig. 232 we obtain

$$\sigma'_{кр.\,max} \approx 0{,}75\sigma_s. \qquad (17.16)$$

In this case taking into account the unloading by 1.12 times in comparison with that obtained in (17.15) the limiting stress is increased. The circumferential stresses at this point is

$$\sigma_k = 0{,}375\sigma_s.$$

From the formulas (17.15), (17.16) for the limiting values of critical stresses with and without allowing for unloading, it is evident that their amount differs from one another by at least 12%. Therefore, in practical calculations it is possible to make use of the formula (17.11), having supplemented its stress from the unloading:

$$\sigma_{кр} = 0{,}26E\frac{\delta}{R} + \frac{qR}{\delta}\left(1 - \frac{\frac{qR}{\delta}}{\sigma_s - 0{,}26E\frac{\delta}{R}}\right) + \frac{q_n R}{2\delta},$$

considering that $q = q_H + \gamma h$,

where q_H – pressure of the booster;

γh – pressure of the liquid column over the investigated section of the cover.

Then, critical force will be

$$P_{кр} = 2\pi R\delta\sigma_{кр}.$$

Loading a Cylindrical Cover with an Axial Compressive Force, with a Lateral Uniform Pressure and Tortional Moment

In this case we will have the following equation of the plane:

$$\frac{\sigma}{\sigma^0_{кр}} + \frac{q}{q_{кр}} + \frac{\tau}{\tau_{кр}} = 1,$$

presented graphically in Fig. 233.

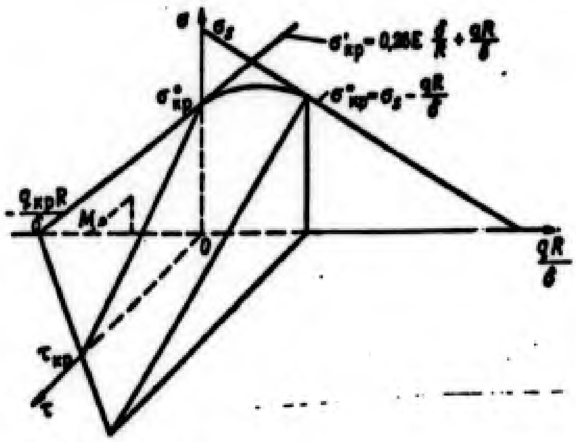


Fig. 233.

The external pressure, acting on the cover, is assumed to be negative (it is deferred in the left part of the axis of the absciasa), and the internal one – positive. In connection with this the hyperplane appeared in the left quadrant.

In this case the hyperplane divides all the space into two halfspaces: halfspace V^+, including point O of the origin of coordinates, and halfspace V^-. The first halfspace corresponds to the steady states of equilibrium of the shell, the second – to the unstable ones. The plane, which divided both these halfspaces, determines the critical states of the shell.

In the first quadrant in Fig. 233 the hyperplane is plotted just as in Fig. 230.

We can also apply the proposed method of determining the critical combination o loads to other elastic systems.

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13. ABSTRACT

In the book is examined a wide circle of problems on calculation of shells of revolution and elements of thin-walled constructions for strength, rigidity and stability under various forms of force action. Many of these problems appeared in recent years in connection with the development of new technology. Such problems include, for example, calculations of all sorts of doughnut-shaped shells, loaded by internal pressure, spherical shells, loaded by local loads, etc. Problems of stability of shells are given in the book in a new formulation, the basis of which is formed by the fact that on the contour of pits and bulges, forming as a result of loss of stability, there take place inherent boundary conditons. The conditions on the contour of half-waves are determined by loading conditions and the proposed form of loss of stability. The new approach to these problems refines and expands the concept of stability of shells and gives the possibility of solving practically important problems. The book is designed for scientific workers and engineers of aviation and other branches of industry and can be useful to college students. Orig. art. has: 14 tables, 233 illustrations.

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	ROLE	WT	ROLE	WT	ROLE	WT
Shell Structure						
Shell of Revolution						
Asymmetric Shell Structure						
Cylindric Shell Structure						
Reinforced Shell Structure						
Spheric Shell Structure						
Thin-Shell Structure						

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