Inter-transformability II

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Abstract

Nature consists of endlessly transforming energy! Is there a complex polyhedral model which is capable of transformation within itself without the addition or subtraction of further elements? Below we describe a two-block-system in which a single sixty-four-block cube is transformed into a similar but different sixty-four-block cube. We will show how this phenomenon of "inter-transformability" is the direct result of the cubes constituent parts, how its parts can combine and recombine to form many other geometric structures and the implications of this two-block-system for education.

Introduction

In my 1993 United States Patent I described a geometric building block system "employing sixteen blocks, eight each of only two tetrahedral shapes, for constructing a regular rhombic dodecahedron [1]." In a 2010 paper for Bridges Mathematics and Art [2] I briefly described this block system and explained that its unique feature of "inter-transformability" is the direct result of the division of the tetrahedron into fourths and the octahedron into eighths, as illustrated in Figure 1 and Figure 2 below.

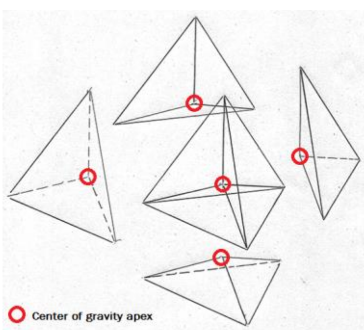

Figure 1: *Four 1/4 Tetrahedrons.*

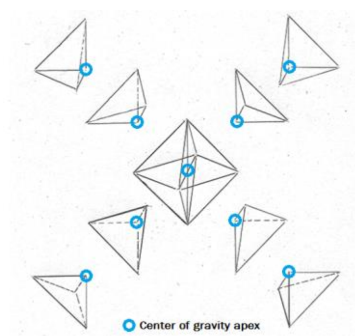

Figure 2: *Eight 1/8 Octahedrons.*

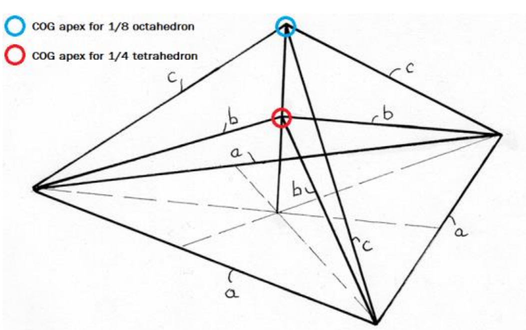

Figure 3: *1/4 Tetrahedron inside 1/8 Octahedron.*

Critical to the simplicity and versatility of the present system is that the geometric structural forms employed are center-of-gravity (COG) apexed. COG apexing means that the apex of the blocks in question (the 1/4 tetrahedron or the 1/8 octahedron) are located at the geometrical center of the regular tetrahedron from which four identical COG building blocks are formed (see Figure 1) or at the geometrical center of the regular octahedron from which eight identical COG apexed 1/8 octahedron building blocks are formed (see Figure 2).

The 1/4 tetrahedron and the 1/8 octahedron are both COG apex tetrahedrons. As demonstrated in Figure 3 above, the 1/8 octahedron has twice the height of the 1/4 tetrahedron. Since they have the same equilateral triangular base it can be seen that the former has twice the volume of the latter. The 1/4 tetrahedron is composed of edges labeled “a” and “b”; the 1/8 octahedron is composed of edges labeled “a” and “c”. Length “a” is the system vector, common to both blocks.

The block system includes the integration of a color system and a numerical system intended to assist the advanced block builder. It is coordinated with the faces of the tetrahedron and the octahedron and their sub-components, the 1/4 tetrahedron and the 1/8 octahedron. These systems account for the rotation of positive and negative tetrahedrons around the four pairs of diametrically opposed octahedron faces.

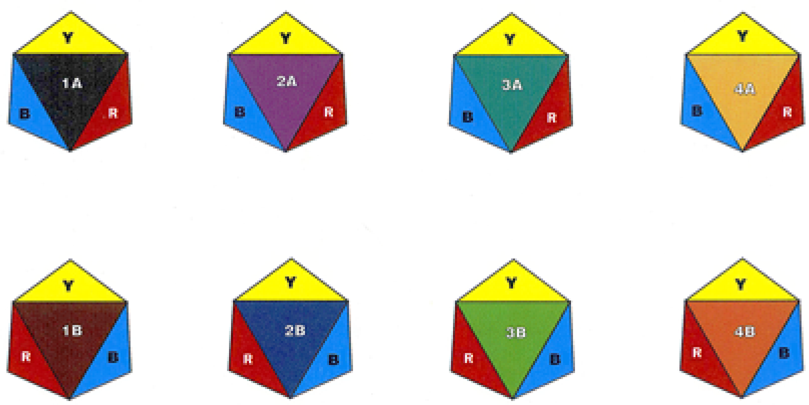

Figure 4*:* *Four positive and four negative 1/4 tetrahedrons.*

Figure 4 shows eight 1/4 tetrahedrons (unfolded). The 1/4 tetrahedrons in the top row are used to structure a “positive” tetrahedron; the 1/4 tetrahedrons in the bottom row are used to structure a “negative" tetrahedron. Colors are used to identify positive and negative structures and to make geometrical and structural relationships clear and easier to use. The four equilateral faces of the “positive” tetrahedron in the top row have been assigned the colors black, red-violet, blue-green and yellow-orange, and the numbers 1A, 2A, 3A and 4A. The four equilateral faces of a “negative”, reverse-oriented tetrahedron in the bottom row are assigned the colors brown, blue-violet, yellow-green and red-orange and the numbers 1B, 2B, 3B and 4B. The non-equilateral faces of the eight 1/4 tetrahedrons are assigned the colors blue, yellow and red (clockwise rotation for the positive 1/4 tetrahedrons; counter-clockwise for the negative 1/4 tetrahedrons).

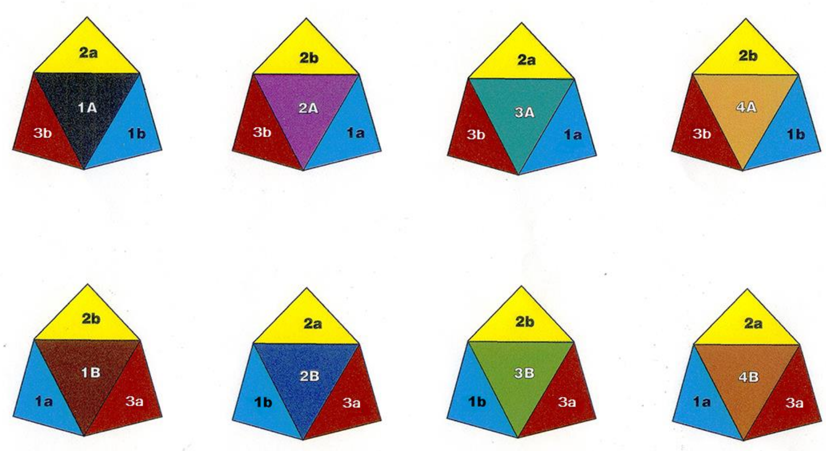

Figure 5*:* *Four positive and four negative 1/8 octahedrons.*

In the case of the 1/8 octahedrons shown in Figure 5 the same colors are used to define the upper and lower hemisphere of the octahedron. Eight equilateral triangular faces of the octahedron are assigned four "positive" tetrahedron positions 1A, 2A, 3A, 4A (top row of Figure 4), corresponding to black, red-violet, blue-green, yellow-orange and four "negative" tetrahedron positions 1B, 2B, 3B and 4B (bottom row of Figure 5), corresponding to brown, blue-violet, yellow-green and red-orange. The non-equilateral faces of the 1/8 octahedrons whose equilateral faces occupy "positive" tetrahedron positions are assigned the colors blue, yellow and red counter-clockwise rotation (1b, 2a, 3b, 1a, 2b, 3b, 1a, 2a, 3a, 1b, 2b, 3a). The non-equilateral faces of the 1/8 octahedrons whose equilateral faces occupy "negative" tetrahedron positions are assigned the colors blue, yellow, red clockwise rotation (1a, 2b, 3a, 1b, 2a, 3a, 1b, 2b, 3b, 1a, 2a, 3b).

A familiar theme is beginning to assert itself through the description of these unique geometrical structures. What we see emerging in the above set of figures is a rather extraordinary pattern of consistency through self-similarity. The regular tetrahedron of Figure 1 has six vector edges of equal length, length “a” and four identical equilateral triangular faces. The regular octahedron of Figure 2 has eight identical equilateral triangular faces with sides of length “a”. And the quarter tetrahedrons and eighth octahedrons share this same length “a”, the system vector. As we shall soon see, these only two, self-similar blocks, the 1/4 tetrahedron and the 1/8 octahedron combine to form two distinct sixty-four block cubes, each with its own unique set of polyhedrons. The blocks are held together with red and black (positive and negative) Velcro tabs but other connectors, such as magnets, can be used. Below we describe and illustrate in greater detail a “duo-tetrahedron cube” and its underlying structures, a “cube octahedron cube” and its underlying structures and how each of these cubes can “inter-transform” from one state to another. Let us begin with the cube octahedron cube.

The 64-Block Cubes

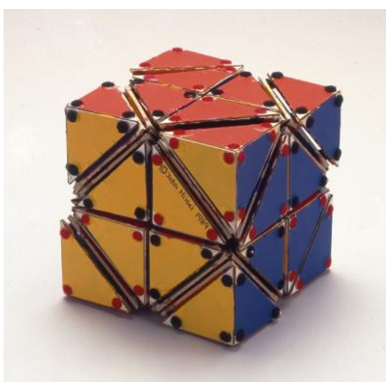

Figure 6: 64-Block Cube Octahedron Cube.

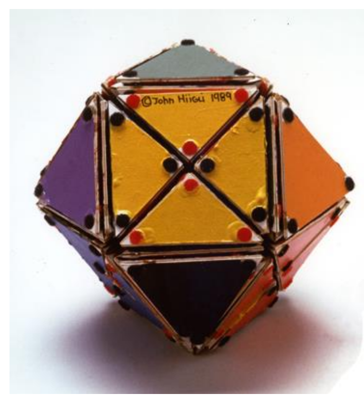

Figure 7: 56-Block Cube Octahedron.

Eight cube assemblies similar to the two in Figure 8 below can be brought together into the 64-block complex cube in two different ways that obey the symmetries of the cube. One procedure for constructing the complex cube of Figure 6 is to envision it as made by assembling an underlying cube octahedron, such as that seen in Figure 7, thereby providing eight exposed equilateral triangular faces. Then eight 1/8 octahedron blocks are placed flush onto the eight faces in Figure 7, such that their center-of-gravity apexes (cg’s) become the respective eight cube corners, as in Figure 6. Another way to approach construction of the complex cube in Figure 6 is to see that it consists of eight cubes, which serve as eight sub-assemblies, similar to the two in Figure 8 below.

Each cube in Figure 8 below has four corner points (cg), which are center-of-gravity apexes of 1/8 of an octahedron blocks and also four other corner points (jp), which are junction points where tetrahedron and 1/2 octahedron vertices meet. Note that while there are four center-of-gravity points in Figure 8 only three are labeled in the figure. The center vertex in the back is obscured, therefore not labeled.

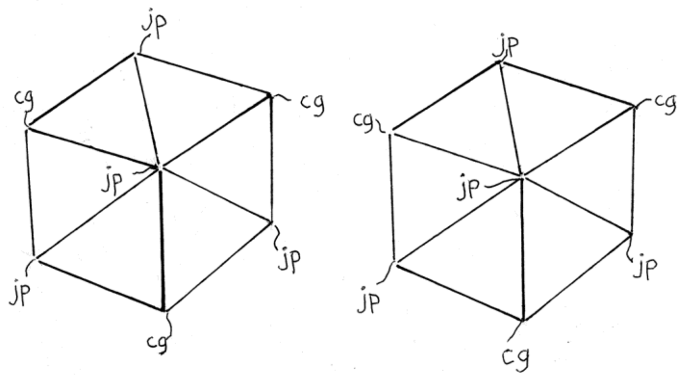

Figure 8: *Sub-assembly cubes showing center-of-gravity vertexes and junction points.*

To be more specific, eight cube assemblies similar to the two in Figure 8 can be brought together into the 64-block complex cube in two different ways, namely with their center-of-gravity vertexes (cg) located at the eight corners of the complex cube of Figure 6, or alternatively with their junction points (jp) located at the eight corners of the complex cube as in Figure 9. When the center-of-gravity apex points of eight 1/8 octahedron blocks are located at the eight corners of the complex cube octahedron cube then this complex cube is in its Cube Octahedron State (or Mode), because the underlying geometric structure is a cube octahedron. When the junction points of the eight cubes are positioned at the eight corners of the complex cube, then this complex cube is in its Duo-Tetrahedron State (or Mode), as in Figure 9. As such it has embedded within it the duo-tetrahedron of Figure 10, which has embedded within it the rhombic dodecahedron of Figure 11, which has embedded within it the octahedron of Figure 12.

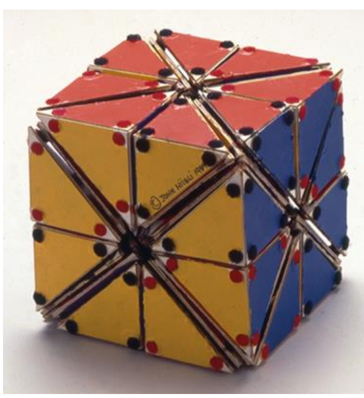

Figure 9: *64-Block Duo-Tetrahedron Cube.*

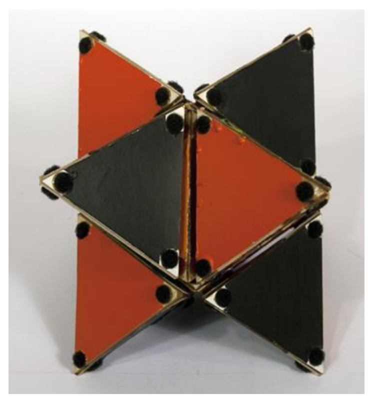

Figure 10: *40-Block Duo-Tetrahedron.*

To put it another way, if we remove twelve 2-block 1/4 octahedrons nested between adjacent tetrahedrons of Figure 9, the remainder is the 40-block duo-tetrahedron of Figure 10. If we remove twelve 3-block 1/4 tetrahedron assemblies from the tetrahedrons of Figure 10 the remainder is the rhombic dodecahedron of Figure 11. Finally, if we remove eight 1/4 tetrahedrons from the rhombic dodecahedron of Figure 11, the remainder is the 8-block octahedron of Figure 12. This characteristic of 'embedding' (the octahedron within the rhombic dodecahedron; the rhombic dodecahedron within the duo-tetrahedron; the duo-tetrahedron within the complex cube) is *a powerful metaphor for how all of nature, animate and inanimate, structures itself* (i.e. the embryo in humans and animals, seeds in fruits and nuts, etc.). Now let us look more closely at how the rhombic dodecahedron is formed from only two building blocks, that is, the 1/4 tetrahedron and the 1/8 octahedron.

Figure 11: *16-Block Rhombic Dodecahedron.*

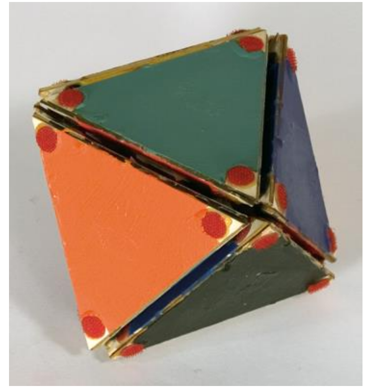

Figure 12: *8-Block Octahedron.*

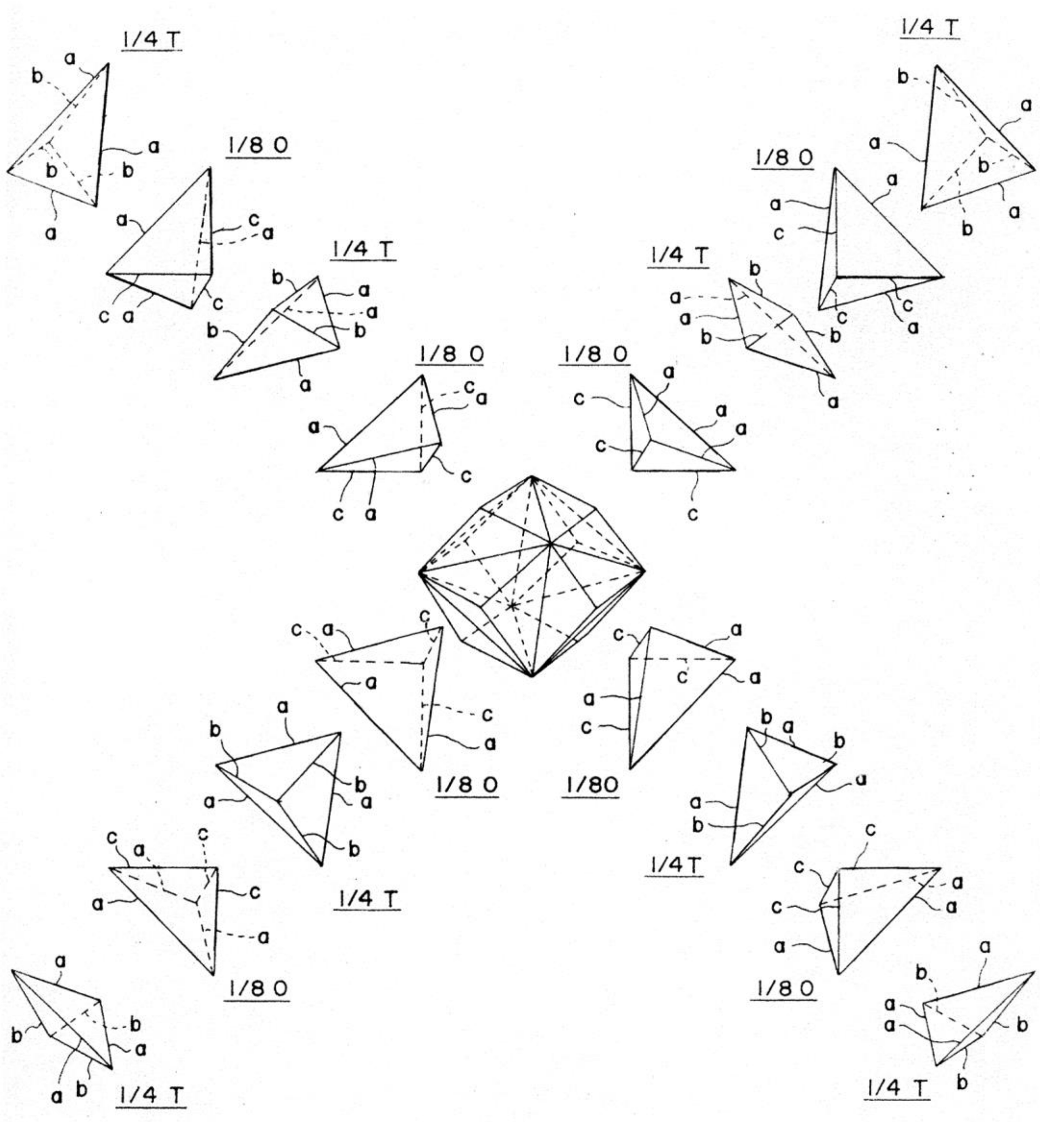

Figure 13: *Exploded Rhombic Dodecahedron.*

Figure 13 above is an exploded drawing of a rhombic dodecahedron such as that of Figure 11, also above. It shows three edge-lengths, a, b, c: that is, edge-lengths a, b for eight quarter tetrahedron blocks and edge-lengths a, c for eight 1/8 octahedron blocks. Each 1/4 tetrahedron has an apex at a point corresponding with the center-of-gravity of the regular tetrahedron, such as that in Figure 4 above. The first face is equilateral triangular with edges of length “a”; the second, third and fourth faces are isosceles triangular faces with one edge of length “a” and two edges of length “b” equal to $a/\sqrt{6}/4$.

Note that the long diagonal bisecting each of the twelve faces of the rhombic dodecahedron formed in Figure 13 is equal to “a”, the system vector.

Each 1/8 of an octahedron has an apex at a point corresponding with the center of gravity of the regular octahedron of Figure 5 above and has fifth, sixth, seventh and eighth triangular faces.

The fifth face is equilateral triangular with edges of length “a”; the sixth, seventh and eighth faces are isosceles triangular with one edge of length “a” and two edges of length “c” equal to $a/\sqrt{2}$.

The Transformation

Figure 14 below illustrates the complex volume 24 duo-tetrahedron cube, which includes the volume 12 duo-tetrahedron, the volume 6 rhombic dodecahedron and the volume 4 octahedron when the volume of the tetrahedron is equal to 1.

Note that while eight of the tetrahedron vertices rest at eight cube corners, twenty-four of the other tetrahedron vertices meet at the six (exterior) cube face centers. The eight, system vector tetrahedrons are literally busting through the seams of their single-frequency cubes, which all together constitute the double, or two-frequency cube of Figure 14.

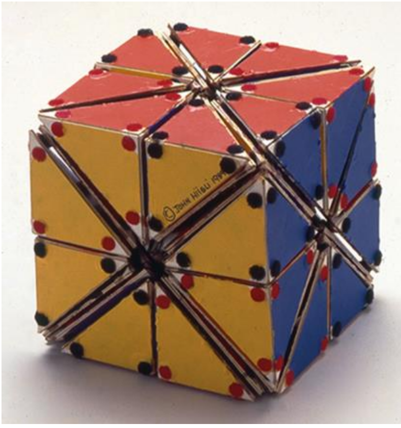

Figure 14: *Duo-Tetrahedron Cube.*

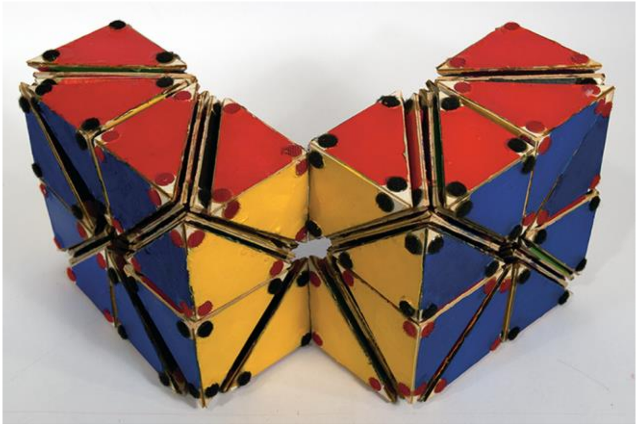

Figure 15: *90° Rotation of Split Duo-Tetrahedron Cube.*

Conversion of the complex cube of Figure 14 in its duo-tetrahedron state to a complex cube in its cube octahedron state, as in Figure 6 above and Figure 18 below, entails dividing the cube into two halves by separating (the Velcro tabs) along any one of its three orthogonal planes, as in Figure 15.

In Figure 15 we see that the two halves of the 64-block duo-tetrahedron cube have been split apart, forming a kind of hinge joint, and the two halves have been rotated so that they form a 90° angle in relation to each other. As the rotation is continued, below, the position of the eight volume-three-cubes in relation to the volume twenty-four-cube will change.

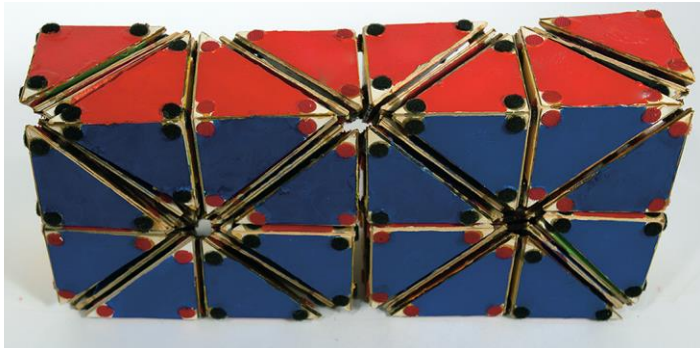

Figure 16: *180° Rotation.*

Figure 16 above shows the rotation continuing so that the two halves of the volume twenty-four-cube form a straight line. The former corner points of the eight volume-three-cubes, the ‘sub-assembly cubes’ of Figure 8, are migrating from volume twenty-four-cube vertex points to mid-edge positions while the former center-of-gravity points of the volume twenty-four-cube, formerly located at mid-edge points, are migrating to the eight vertex points of the big cube, as in Figure 18.

Figure 17 shows that the two halves of the 64-block complex cube formerly in its duo-tetrahedron state have been rotated 135° degrees each. Finally, Figure 18 shows the volume twenty-four 64-block duo-tetrahedron cube has been completely converted to the volume twenty-four 64-block cube octahedron cube with its cube octahedron core and eight 1/8 octahedrons forming the eight cube corners. Not only is the present block system novel in its ability to create, in a simple and logical way, a broad range of structural forms of increasing complexity, as well as a simple and logical system for doing so, it is commensurately capable of providing a coherent system for restructuring a given geometric form into a different and completely unique geometric form without any addition or subtraction of additional blocks. Therefore, in assembly, disassembly, and reformation this inter-transformable block system consisting of only two basic blocks, the 1/4 tetrahedron and the 1/8 octahedron is a uniquely satisfactory educational tool.

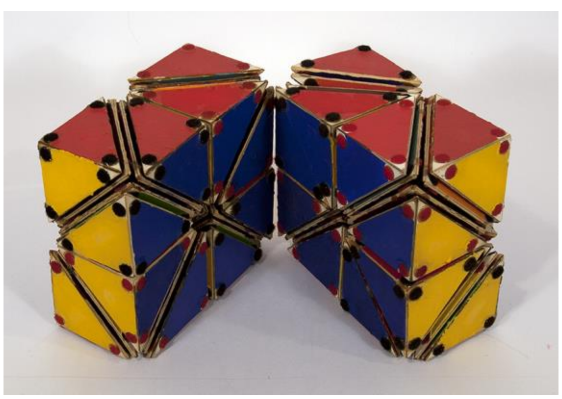

Figure 17: *270° Rotation.*

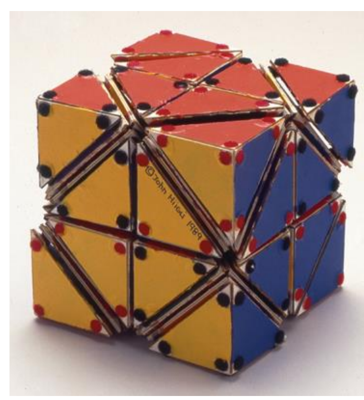

Figure 18: *64-Block Cube Octahedron Cube.*

Conclusion

The above polyhedrons structured with only two basic blocks, the 1/4 tetrahedron and the 1/8 octahedron form a system united by a common edge-length and rational, whole number volumetric values keyed to the tetrahedron as unit of volume (tetrahedron = 1, cube = 3, octahedron = 4, rhombic dodecahedron = 6, cube octahedron = 20). This system is known in Synergetic Geometry as the Isotropic Vector Matrix, meaning 'everywhere the same edge-length': "see Fuller [3]." This block system, based on the principles of Synergetic Geometry has a distinct educational advantage in respect to teaching a student how to visualize geometric or other mathematical relationships, spatial relationships, and how to manipulate or transform geometrical structures and shapes. The 1/4 tetrahedron and 1/8 octahedron share a common equilateral triangular face. The common edge length, too, helps to eliminate uncertainty in the matching and assembling of blocks. Difference of structure, faces, and edges has been eliminated, so there is less guesswork and confusion in assembling the basic blocks to arrive at more complex structures. In fact the polyhedrons inherent in these two inter-related systems form a "family of relationships united by a common edge length." As we have seen this system is inter-transformable and is the miracle of space! This extraordinary facility is entirely due to their center-of-gravity apexes and the "invariance" (regularity) of structure, faces and edge-lengths: "see Darvas [4]."

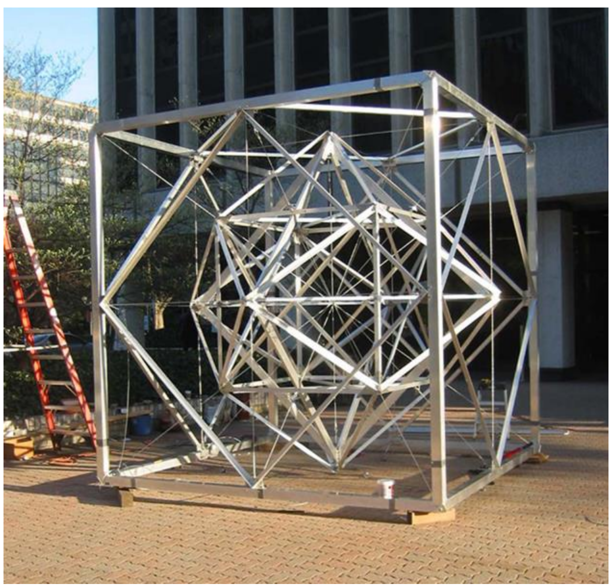

Figure 19: *Isotropic Vector Matrix*

Figure 20: *Duo-Tetrahedron*

References

[1] J.A. Hiigli, Geometric building block system employing sixteen blocks, eight each of only two tetrahedral shapes, for constructing a regular rhombic dodecahedron. *U.S. Patent 5,249,966*, filed November 16, 1992, and issued October 1993.

[2] J.A. Hiigli, Intertransformability. *Proceedings of Bridges 2010: Mathematics, Music, Art, Architecture, Culture*, pages 497-498, Tessellations Publishing. Available online at http://archive.bridgesmathart.org/2010/index.html

[3] R.B. Fuller, *Synergetics I*, Macmillan, (1975), p.594 Fig. 982.61 Synergetics Isometric of the Isotropic Vector Matrix,

[4] G. Darvas, The unreasonable effectiveness of symmetry in the sciences, *Symmetry: Culture and Science*, 26, 1, 39-82 (2015).