SUB-HELMHOLTZIAN VIBRATIONS OF A RUBBED STRING.

By Harry Clark.

I. Statement of the Problem and Review of Previous Investigations.

THE problem of the longitudinal vibration of a rubbed string has been extensively studied by Davis,[1] to whom it was suggested by its usefulness in the study of the effects of mechanical stress upon the magnetic properties of materials. The writer has undertaken a further study of the subject with a two-fold purpose: first, of determining more carefully the conditions necessary for the production and accurate duplication of the various wave forms with unfailing regularity; and second, of studying in detail particular conditions of rubbing hitherto uninvestigated.

A satisfactory solution of the first of these problems is due largely to the adoption of a new rubbing device whose action is extremely uniform and reliable. The second problem refers to the so-called sub-Helmholtzian vibrations, which under favorable circumstances may be produced by rubbing with slight pressure. These waves were investigated by means of an improved photographic method of study. Several features of interest are presented by them.

In addition to the study of sub-Helmholtzian waves, the experiments to be described have verified nicely some of the previous work on Helmholtzian forms, and have brought out a new criticism of the laws governing them. A complete understanding of the work will be facilitated by reviewing briefly at this point some of the studies of transverse vibration.

The problem of the bowed string cannot be successfully attacked by the use of analysis without a complete knowledge of the nature of the bowing process. Much of this knowledge is embodied in the following laws:

Young's Law.[2]—No overtone is present which would have a node at the point of excitation.

Helmholtz's Law.[3]—When a string is bowed at an aliquot point $(1/q)$, the part of the string immediately under the bow moves to and fro with

[1] Proc. of the Am. Acad. of Arts and Sci., Vol. XLI., No. 32, May, 1906.

[2] Phil. Trans. of the Roy. Soc. of London; 901, 106, 1800.

[3] Proc. Glasgow Phil. Soc., Dec. 19, 1860; reprinted in Phil. Mag., Ser. 4, 21, 393, 1861.

constant velocities whose ratio is equal to the ratio $(1/(q-1))$ of the segments into which the string is divided by the point in question.

Helmholtz's Velocity Law.—The smaller of these two velocities has the same direction as that of the bow and is equal to it.

From these facts Helmholtz concluded that the action of the bow is accounted for by something analogous to the difference between the coefficients of static and moving friction. That wave will be developed which allows the element of string under the bow to travel as much of the time as possible, in the direction of the bow's motion, without slipping.

On the basis of these laws Helmholtz solved completely all aliquot cases of the problem. His solution is given by

$$u = u_1 - u_2,$$

where u_1 and u_2 are defined by

$$u_1 = A \sum_n \frac{1}{n^2} \sin \frac{n\pi y}{l} \sin \frac{n\pi t}{T}, \tag{1}$$

$$u_2 = A \sum_m \frac{1}{q^2 n^2} \sin \frac{qm\pi y}{l} \sin \frac{qm\pi t}{T}, \tag{2}$$

u represents the displacement of a point, y, on the string at time, t. The length of the string is l, and the period of a complete vibration is $2T$, m and n are integers, A is constant for a given speed of bowing, and q has been defined above. Equation (1) is called Helmholtz's *major* solution. The value of u, determined by it, is seen to be independent of q, which defines the place of bowing. Equation (2), the *minor* solution, serves, for any particular value of q, to exclude those partials whose presence would violate the law of Young. It becomes relatively unimportant for large values of q.

Rational bowing points, which include the aliquot cases of course, may be solved if the law of Helmholtz be replaced by

Krigar-Menzel's Law.[1]—When a string is bowed at any rational point (p/q), where p and q are prime to each other, the part of the string immediately under the bow moves to and fro with constant velocities whose ratio depends only on q and is $1/(q-1)$.

These four laws constitute the established theory of rational cases of strings transversely bowed. No investigations of other cases have been published. It is to be noted that the solutions which have been obtained are considered to be independent of the nature of the bow and of the force with which it is pressed against the string. All waves obeying the above laws, whether transverse or longitudinal, are called *Helmholtzian waves.*

[1] Inaugural Dissertation, Berlin, 1888.

Although the theories of transverse and longitudinal waves are very much alike, it is desirable to note some of the differences between the two classes of phenomena in order to see for what reasons a particular interest attaches to the study of longitudinal forms.

Sharp corners of transverse waves are quickly rounded off by the quality of "stiffness" in the string. In the case of longitudinal vibrations, however, these sharp corners are replaced by discontinuities of density. Such discontinuities are not difficult of conception, and there is no evidence to show that they exercise any distorting influence on the waves. On the theoretical side therefore the longitudinal is the simpler type. Since the frequency of longitudinal vibrations involves only the nature of the material in the string and the length of the vibrating portion, it remains very nearly constant. The operation of longitudinal "bowing" may by mechanical means be made continuous, so that a given state of vibration may be maintained indefinitely.

The very complete investigation by Davis of longitudinal Helmholtzian waves is briefly reviewed below.

He used for the "string" a steel wire 4 mm. in diameter and somewhat less than 7 meters in length, with ends firmly clamped to massive iron blocks. A knife edge, pressed lightly against the wire at the middle, divided it into two equal segments, one of which was reserved for observation. In the other was placed the "bow," consisting of a motor-driven wheel, of which the face was covered with well-resined chamois skin.

The methods of observation were two. In the first, a very small glass ball, perhaps 1 mm. in diameter, was cemented to the wire, brilliantly illuminated from a point source, and observed through a microscope. From the limits of its motion the amplitude of vibration was obtained. Such amplitudes, measured at many points along the wire, and plotted as though the motions were transverse, gave the envelope of the wave form. In the second method, a short piece of a needle was attached to the wire, and the point was allowed to trace a record of its motion upon a smoked glass drawn transversely underneath. Although the amplitudes employed were very small, only a few tenths of a millimeter, very perfect tracings were obtained.

By the experiments it was shown that all of the laws of transverse vibration apply without change to the longitudinal forms. A large number of cases were investigated, and the verification is quite complete.

Among other contributions of Davis is the *integral surface*, a graphical construction designed to facilitate the study of both longitudinal and transverse Helmholtzian waves. It gives the motion of all points on the string if the motion of one point be completely known. Since any curve may be represented approximately by a broken line of straight segments, the usefulness of the integral surface may be extended to types of waves other than Helmholtzian. The original paper[1] should be consulted for a complete description of the method.

Davis discovered that when, under certain circumstances not well understood, the pressure of the rubbing wheel against the wire is decreased, a point is reached below which the waves, though quite stable, cease to be Helmholtzian. Their amplitudes (observed by his first method only) were found to be too small to fit Helmholtzian theory. In some instances there were maintained waves with amplitudes scarcely more than six-tenths as great as those of high-pressure waves. To distinguish waves of this type they are called *sub-Helmholtzian*.

II. Apparatus and Methods of Use.

The string used in these experiments was a steel wire .101 cm. in diameter, of the quality commonly known as "music wire." It was carefully straightened and nickel-plated. To secure consistent results it was found that the use of a simple vibrating segment is preferable, although the difficulties of observation are thereby increased.

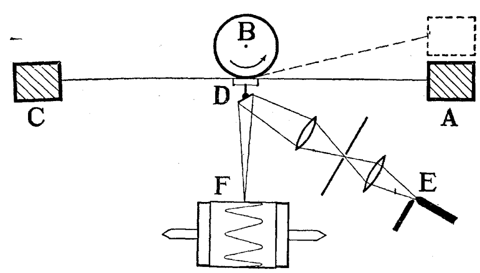

Fig. 1.

Thus arranged the vibrating portion of the wire is ten meters long. This gives a frequency of 254 vibrations per second corresponding to a pitch very nearly that of C_3. The low frequency of vibration allows the recording mechanism to work under more favorable conditions.

The wire was stretched horizontally, as shown in Fig. 1. One end is held firmly in the jaws of a vise, *A*, weighing about 6 kg., which can be moved by a feed-screw horizontally in a direction perpendicular to that of the wire. By means of this adjustment the wire can be pressed against the rubbing wheel, *B*, with any desired pressure. At the other end the wire, after passing between the jaws of a second vise, *C*, and over a pulley, supports a load which serves to hold the wire under tension. This second vise, weighing also 6 kg., is suspended by a two-point support, so that although by its inertia it furnishes an effective end for the vibrating segment, it does not interfere with the slight longitudinal motion of the wire necessary to maintain a constant tension under varying conditions of temperature and adjustment. Although the tension has no direct effect on longitudinal waves, its exact value must be known for the computation of rubbing pressures from the settings of the feed screw.

Many different rubbing wheels were tried, some of which will be considered in detail later. They rotate about a vertical axis on centers which are mounted on the slide-rest of a small lathe-bed. This mounting makes possible the necessary adjustments of the wheels with respect to the wire. Power for rotation is furnished by a 1/8 H.P. induction motor, of which the speed under favorable conditions has been found to remain very nearly constant.

A transmission unit, consisting of a pair of counter shafts with cones of pulleys, reduces properly the motor speed, so that by shifting the belts the rubbing wheel can be given a peripheral speed having any one of thirty-three values between 6 and 50 cm. per second. Greater variety of speeds is obtained by changing the pulley of the motor shaft. An essential part of the transmission is a small fly-wheel mounted on centers and nicely balanced. This is placed close to the rubbing wheel and influences it through the single belt which furnishes power to both. All pulleys are of rolled brass, carefully turned and balanced. Endless braided linen belts, such as are used on dental engines, transmit the motions, and the design of the transmission prevents slipping and automatically regulates the tension of these belts. The constant positive motion of the rubbing wheel thus made possible has contributed much to the success of the experiments.

To examine the motion of a point on the wire a photographic method is used. A small plane mirror suitably mounted to allow free rotation about a fixed, vertical axis communicates with the wire and indicates by its angular position the wire's longitudinal displacement. The device is shown at *D* in Fig. 1. Light from a fixed "point-source," *E*, having passed through a suitable system of lenses, is reflected by this mirror and comes to a focus at a point on the surface of a photographic film wrapped around the cylinder, *F*. If the exposure be made while the cylinder rotates with uniform speed a record of the displacements of the wire plotted against *time* is obtained. A detailed description of the recording mechanism follows.

The small mirror is attached by beeswax to the flat side of a steel staff shown at *A* in Fig. 2. The staff is carefully fitted to rotate freely but without shake in cylindrical jeweled bearings with end-stones. A radial arm, B, fits into a conical hole bored through the thick middle part of the staff. Two of these staffs were used, to each of which were fitted radial arms of various lengths. The larger, used for waves of greater

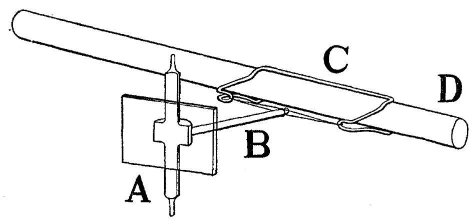

Fig. 2.

amplitude, measures 7 mm. in length and weighs 20 mg., while the smaller is but 4 mm. long and weighs 9 mg.[1]

The motions of the wire are communicated to the arm B through the spring clamp shown at C, and a fine silk thread. Such clamps are bent from natural hard music wire (no. 0000, diameter .215 mm.), are about 6 mm. long and weigh from 4 to 5 mg. each. Although they grip the vibrating string, D, with sufficient force to prevent slipping, their design permits them to loose their hold in case of accident before harm comes to the staff or jewels. Midway between the horns of the clamp the end of the staff's radial arm is held rigidly by fine silk thread.

This mechanism differs radically from those commonly used for similar purposes in that motions are produced in the staff positively and without the use of a "restoring force," usually derived from a spring. Springs, on account of their inertia, free period, and uncertain reactions, have proved to be a source of much trouble. The natural frequency of the new device is so great that no evidence of its existence has been found. The photographs of Helmholtzian waves agree with theory to a degree of approximation represented by 150 terms of a harmonic series; yet there appears no evidence of natural period so great as the smallest of these. Therefore the frequency must exceed 37,500 vibrations per second.

The mounting of the jeweled bearings proved to be a difficult problem. At first they were held rigidly in a fixed position, but the slight lateral motions of the wire, resulting from unavoidable irregularities in the rubbing surface, caused the staff to bind in its bearings. The resulting frictional losses interfered seriously with sub-Helmholtzian waves of slight stability. Worse than this was the danger to the staff from greater

[1] For the admirable workmanship on these pieces the writer is indebted to Mr. W. B. Hughes, in the employ of the B. C. Ames Co. of Waltham, Mass.

motions of the wire of an accidental nature. That the staff can endure rough usage is demonstrated by the fact that in responding to Helmholtzian waves its angular acceleration at the "corners" often exceeds 7,000,000 radians per second per second.

The mounting finally adopted is shown in Fig. 4. The bearings with the larger staff in place are shown at *A*. Together with necessary means of adjustment they are attached to the end of a steel rod supported near the middle in a universal joint, *B*, and balanced by a counterweight, *C*. This arrangement permits the staff to follow the wire in all of its transverse movements.

The bearings at *A* can be moved parallel to the wire through any desired distance by a turn of the micrometer head, *E*. Their movement produces a rotation of the staff equal to that caused by a longitudinal displacement of the wire through the same distance. In this way the film may be calibrated for magnification of the vibrations. The calibration, which requires a slight correction, is represented by the parallel straight lines in many of the photographs, for example, numbers *1*, *2* and *3* in Fig. 5. Magnifications of from 80 to 140 diameters were commonly used.

The entire unit shown in Fig. 4 rests upon a shelf which communicates with nothing except the brick wall to which it is attached. Since even this precaution did not suffice to prevent response to vibrations of the building, nearly all of the photographs were made at night.

A small self-feeding arc-lamp furnishes the necessary light. In front of it is mounted a screen of colored celluloid, upon which an image of the arc is formed by a lens. In order that holes of various sizes may be brought to the brightest part of the image, the screen is movable, an arrangement which provides "point-sources" of various sizes, circular in shape, evenly illuminated, and with sharp edges.

The cylindrical drum at *F*, Fig. 1, is shown again at *A* in Fig. 3, which is a vertical section of the photographic unit. The drum is of wood and measures about 15 cm. in diameter. It rotates on centers at speeds ranging from zero to 4,800 R.P.M. The corresponding velocities of the film vary from zero to 38 meters per second. Power is supplied by a small direct-current motor, direct-connected to the drum through a spring clutch.

[1] In many oscillographic experiments it is a difficult problem to get sufficient illumination. Granted a plane mirror, it is easy to show that the problem is solvable. Let it be assumed that the size of the mirror and the frequency and angular amplitude of its motion have been arbitrarily determined; also that in the photographic record is desired a certain "definition," or limiting ratio between the width of the line and its length per cycle. By adjustment of distances the amplitude in the record can be varied without changing the definition and the amount of "exposure" of the film varies inversely as the square of this amplitude.

The cylinder is enclosed in a light-tight box, *B*, provided with a simple hand-operated shutter, *C*, in front and a "loading sleeve," *D*, of black cloth behind. A method of attaching the specially adapted films has been devised, whereby they can be changed easily in a few seconds, yet are held with sufficient firmness to prevent tearing, even at the highest speeds. On account of the simplicity of the shutter it is difficult to restrict the duration of exposure to a single revolution of the film, and as a result many of the records show more than one curve.

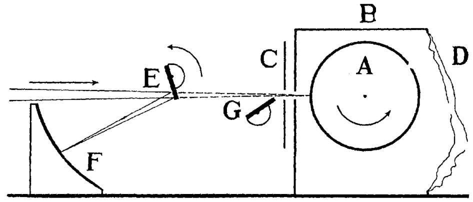

Fig. 3.

In front of the box and parallel to the axis of the cylinder, a shaft carrying a long narrow mirror, *E*, and a fly-wheel is mounted on centers and connected to the motor by a rubber belt. This mirror, by intercepting the beam of light on its way to the film and returning it to a curved screen, *F*, of white cardboard, makes possible visual observation of waves. The mirror is attached eccentrically to the shaft, which in turn is placed just above the path of the light. The shaft has an angular speed of rotation about one third as great as that of the film. The mirror in its rotation intercepts the light for visual use during an interval corresponding to one revolution of the film, leaving it free for photographic use during the two following consecutive revolutions. By proper government of speed successive visual images may be made to coincide and to produce a persistent stationary curve which need not be interrupted during exposure of the film. Transient phenomena are thus easily studied. It should be noted, however, that no attempt is made to superpose images on the film.

This entire unit is mounted on leveling screws, which rest on the table of a heavy adjustable iron stand. It is not connected to any of the other pieces of apparatus. The precaution is necessary to prevent transmission of the slight vibrations caused by the rotation of the drum.

If observations are to be accurate, the axis of rotation of the film must be strictly parallel to the motion of the vibrating spot of light. To secure this relation a straight-edge, *G*, of white celluloid is carried on a shaft mounted in bearings, which are attached to box, *B*, in front of and

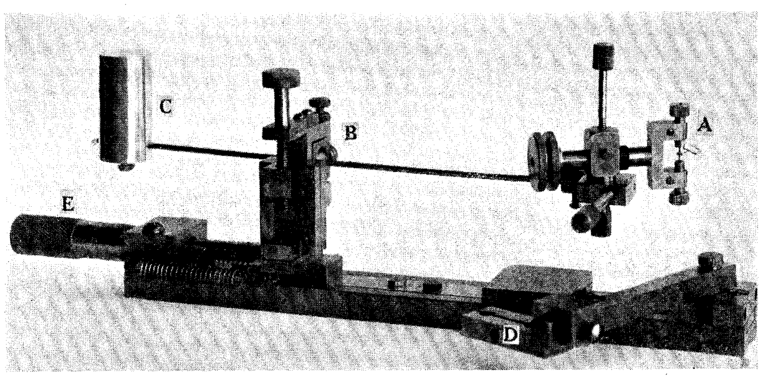

Fig. 4.

HARRY CLARK.

slightly below the level of the shutter. The parts are so adjusted that the edge in all of its possible angular positions remains parallel to the axis of the drum. When raised by hand the celluloid intercepts a small portion of the oscillating beam of light, and with adjustments well made the resulting illumination of the edge should be uniform.

III. Qualitative Description of Observed Phenomena.

It is proposed to devote this section to a qualitative presentation of longitudinal wave phenomena as observed with the aid of the apparatus just described, and to a brief discussion of some of the factors which enter into the successful production of these waves. In what follows it is to be assumed that both the rubbing wheel and the observing apparatus are located at the middle point of the wire.

In the earlier part of the work solids only were employed as friction agents, among them the various resins used on stringed instruments; also beeswax, and several mixtures. The surfaces of the wheels on which these agents were used were of various leathers, and of all thicknesses from two tenths of a millimeter to two millimeters. They varied greatly in quality. One was extremely hard, having been turned from discs of leather steeped in molten resin and tightly clamped together while hot. Another was extremely soft, consisting of a ring of thin chamois skin stretched over a wheel faced with thick gum-rubber. None of these devices, however, gave consistent results. Occasionally good sub-Helmholtzian waves were produced, but they could seldom be maintained long, and could not be duplicated at will. Although Helmholtzian forms could be produced, they often exhibited peculiarities of which something will be said later.

The influence of temperature was noticeable. Somewhat better results could be obtained at high temperatures than at low ones. When, in a cold room, a flame was applied to the wheel operating lightly with resin, the waves were appreciably modified. A considerable sharpening usually occurred, though this at times gave way to distortions.

The desired end was attained by the use of alcohol on a silk-covered wheel. This wheel, which was of brass, measured 42.2 cm. in diameter, and in its face was cut a shallow groove which served to guide the wire. Different thicknesses of silk were tried, and the best results were obtained by using perhaps ten layers of thinnest taffeta ribbon with a total thickness of less than a millimeter. The irregularity in the surface, caused by the free end of the ribbon, soon disappeared with use; the surface became hard and polished, and the accuracy of wave-forms increased. It may be of interest to note that this wheel, when nearly dry, gave the waves characteristic of resin.

Light pressure of the wire against the wheel, well saturated with alcohol, built up a wave which produced after a few minutes an audible tone of great purity. Continuing to grow, it assumed a permanent sub-Helmholtzian form, characterized by rounded "corners" and smooth symmetrical curves, and could be maintained indefinitely. Little could be learned about minimum building-up pressures. At low pressures the rate of building up was at first very small, and the necessary time varied, under fixed conditions, from twenty seconds to as many minutes.

As the pressure was increased, the wave gradually assumed the Helmholtzian form, the resulting tone closely resembling that of the violoncello. The corners of these waves, as shown by photographs, are surprisingly sharp. From the photographs it was estimated that the change of velocity is completed in less than .00001 second. When the pressure was then decreased, sub-Helmholtzian forms reappeared, which, by careful manipulation of pressure, could be carried down to very small amplitudes without loss of stability. In this way sub-Helmholtzian waves could be maintained, the amplitudes of which were not more than one-tenth or two-tenths as great as those built up directly from a state of rest under minimum pressure.

Some of the normal stable forms observed at the middle point are shown in Fig. 5, nos. *1*, *2* and *3*. These records, which are about one-half as large as the original photographs, were made at the same bowing speed and with equal magnifications. Nos. 2 and 3 are sine curves. This is true of all records of normal sub-Helmholtzian waves the amplitudes of which are not much greater than one-half that of the corresponding Helmholtzian. In Fig. 5 the time axis, in each case, extends from *left* to *right*, and the positive direction of the displacement axis, corresponding to the motion of the bow, is from *top* to *bottom* of the page.

It will be recalled that pressure was varied by a transverse movement of the vise at the end of the wire. The operation seemed to cause changes in the vibration system within, for it was accompanied by a marked increase in the dissipation of energy. To get the smallest amplitudes the pressure had to be decreased by small steps. It was moreover necessary to make each adjustment quickly and to allow time for the wave to recuperate before repeating the process.

All of these phenomena, described for a single bowing speed, were produced also at others. The forms of vibration suffered no change thereby, although the amplitudes of corresponding waves seemed to vary directly as the speed. At extremely low speeds it was found difficult to produce any waves whatsoever.

In addition to the experiments described above, a few trials were

PHYSICAL REVIEW, Vol. VII., SECOND SERIES,
May, 1916.

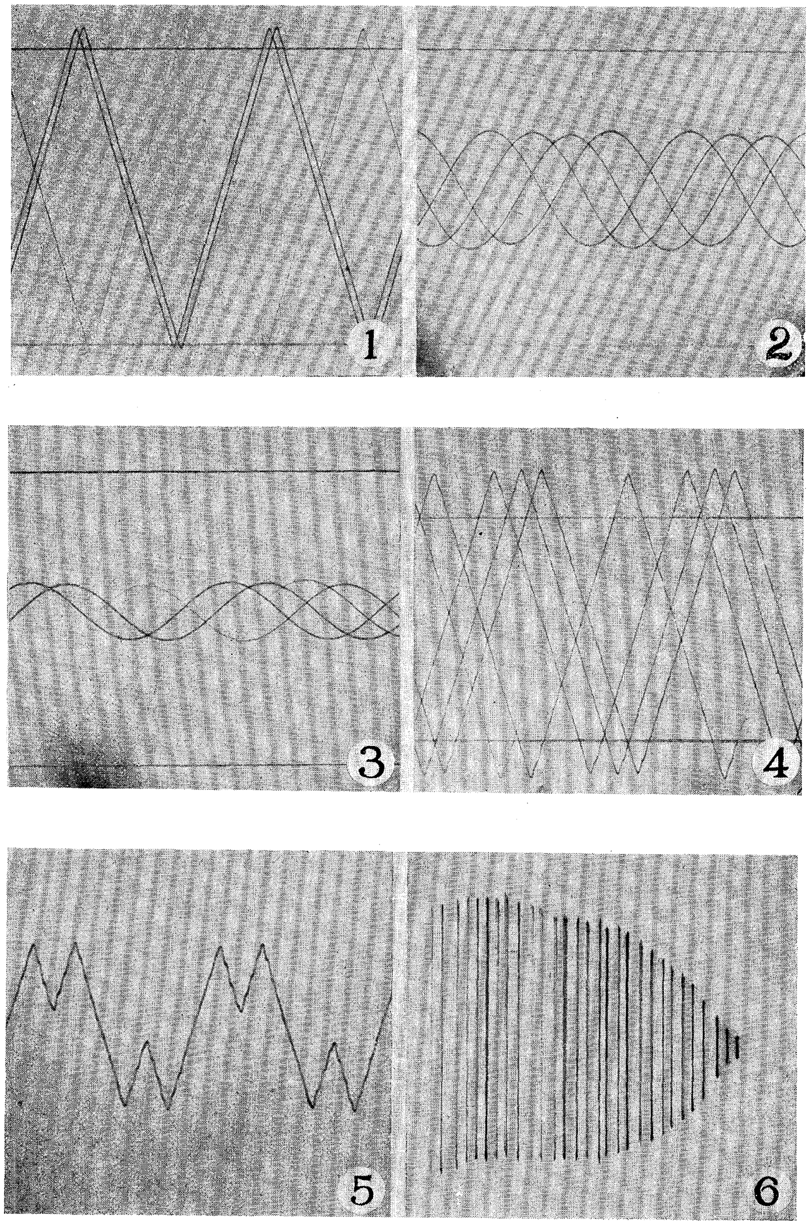

FIG. 5.

HARRY CLARK.

made in which the wire was excited by drawing the finger wet with alcohol along it. This stimulus produced rather sharp waves with sides curved in a characteristic fashion. When the finger was quickly withdrawn from the vibrating wire, the amplitude fell in about four seconds to one-half its original value. From this the rate of energy loss may be estimated, and it is found to be so small that even during rapid changes of wave form the wave equation is not violated.

It has already been seen that alcohol provides a far greater certainty of action than resin. It offers besides many other advantages. The wheel suitable to its use may be made almost truly circular, and its surface conditions, which are constant, may be readily duplicated. Unlike resin, alcohol operates with no observable temperature coefficient. Furthermore there are no incidental noises such as resin gives at the rubbed point. Finally, it is easier to handle. It is cleaner and requires less attention than resin.

Most of the conditions necessary for successful work have been pointed out in the description of apparatus. The question of tension in the wire remains. Although tension does not enter directly into simple longitudinal wave theory, it controls to a considerable extent the production of those transverse waves of slight amplitudes which have at times given trouble. Such transverse waves, it was found, could be prevented by the use of a rather high tension whose value was such as to give to the wire free transverse periods farthest removed from the period of the longitudinal wave. This tension was determined by experiment, and once learned did not need to be changed. Twenty-five kilograms gave good results.

IV. Quantitative Study of Sub-Helmholtzian Vibrations.

In this section will be presented the results of the experimental study of sub-Helmholtzian waves with special reference to the relations which connect speed, pressure, amplitude, and wave-form. For convenience the photographic records were studied by the following indirect method of procedure. The films were placed in a stereopticon, which projected their curves upon a screen of coördinate paper. When the screen had been adjusted to the proper distance and position, the curves were carefully traced with a pencil. In order to reduce them to the same degree of magnification, and also for other special purposes, all of the ordinates of the tracings had to be multiplied by the proper constant number and replotted. In much that follows it will be necessary to consider changes of the ordinates of curves whose abscissas remain unchanged. To avoid confusion, it will be found convenient to accept the following

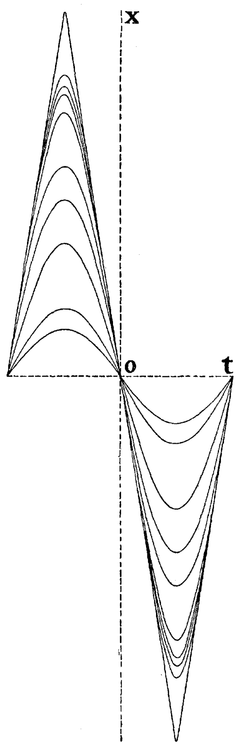

Fig. 6.

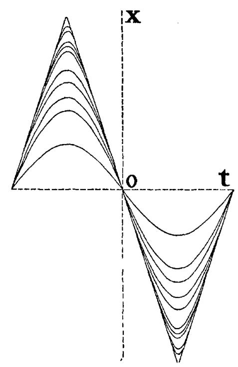

Fig. 7.

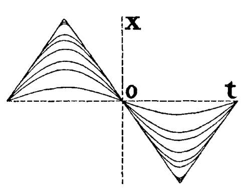

Fig. 8.

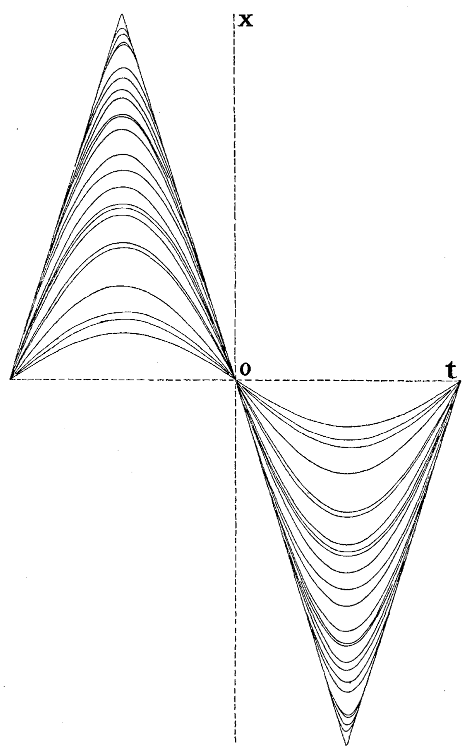

Fig. 9.

Definition.—Two curves are said to be *similar* if one of them may be produced by multiplying all of the ordinates of the other by a constant number. Figs. 6, 7 and 8 are reductions of curves treated as described above. Each of these figures represents the effect of different pressures at a single speed. The speeds, in cm. per sec., and corresponding amplitudes, in centimeters, used in each case are given in the following table.

Figure.	Bow Speed.	Helm. Amplitude.
6	27.85	.0274
7	13.30	.0131
8	6.33	.00625

By careful measurements the limiting Helmholtzian amplitudes were determined and it was found, in verification of the law of Helmholtz, that their values are given by the ratio of *speed* to *four times the frequency*. Comparative studies of sub-Helmholtzian forms could not so readily be made, but some useful suggestions were obtained by a variation of the treatment of tracings, made clear by the following illustration: The ordinates of all of the curves of Fig. 7 were multiplied by a constant number whose value was so chosen that the resulting Helmholtzian curve in particular would be identical with that of Fig. 6. The curves of Fig. 8 were similarly treated with another proper constant. The three nests of curves were then superposed. All of the abscissas of the composite nest thus obtained were then multiplied by a constant, simply for convenience in illustration, with the result shown in Fig. 9.

A consideration of the figures above referred to, and of the methods by which they were produced, brings out several interesting facts. It is evident that *similar* curves may be produced at various speed by the application of suitable pressures. The waves are seen to be in a restricted sense symmetrical, since each positive displacement is followed, half a period later, by a negative one of equal magnitude. Symmetry about vertical axes is also indicated. Finally, it is evident that the amplitudes, not only of Helmholtzian waves, but also of the members of *any* family of *similar* waves, are directly proportional to the bowing speeds.

Since it had thus been shown that a normal sub-Helmholtzian wave is completely determined by its amplitude and bowing speed, certain simplifications of the work were possible. A slight variation of the photographic method, therefore, was introduced. In the original method all records were made on rapidly rotating films. The variation consisted in allowing the film to remain motionless to receive a very short exposure. The record, under these circumstances, is a narrow straight line, from the length of which the amplitude of vibration can be determined. The device offered the advantages of saving both time and material, for many exposures could be made upon a single film. Fig. 5, no. *6*, which illustrates the result, shows decrease of amplitude with decreasing pressure, the bow speed having been kept constant. Under favorable conditions it was found possible to determine amplitude by direct measurement of the illuminated region of the screen without the aid of photography.

About six hundred determinations were made, some visually, others photographically, and the work covered a great variety of speeds and pressures. A single "run" consisted in determining the amplitudes corresponding at a single speed to a great many different pressures, and the results were embodied in a single plotted curve.

The manner of plotting requires some explanation. All of the amplitudes were modified by the same constant factor, the value of which was so determined that the modified Helmholtzian amplitude was equal to unity. All such curves representing the same speed should be identical. Most of the observed differences may be accounted for on the supposition that an inexact zero setting of the wire against the wheel introduced into all of the pressure readings errors constant for a given curve, but differing from curve to curve. Owing to the nature of the rubbing surface, and to the fact that the wire lies against it in a groove, it is difficult to determine the exact adjustment for zero pressure. An error of .01 cm. at this point introduces a pressure error of approximately one gram. The above explanation is therefore reasonable, inasmuch as the errors to be accounted for did not usually exceed one or two grams.

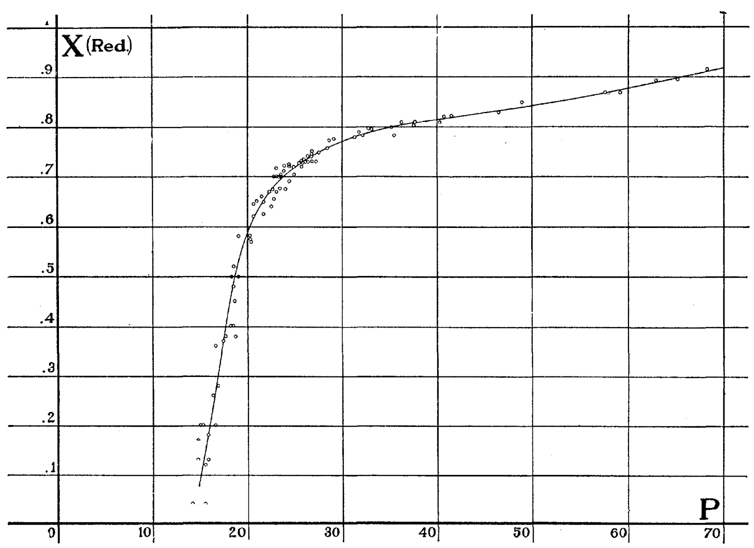

Fig. 10.

The method of averaging these amplitude-pressure curves may be briefly mentioned. They were superposed and if necessary displaced along the pressure axis to give a best compromise curve. This new curve was then replotted with a proper displacement to represent the mean zero position. Such a curve, representing *all* of the determinations at one speed, is shown in Fig. 10. The distribution of points here shown

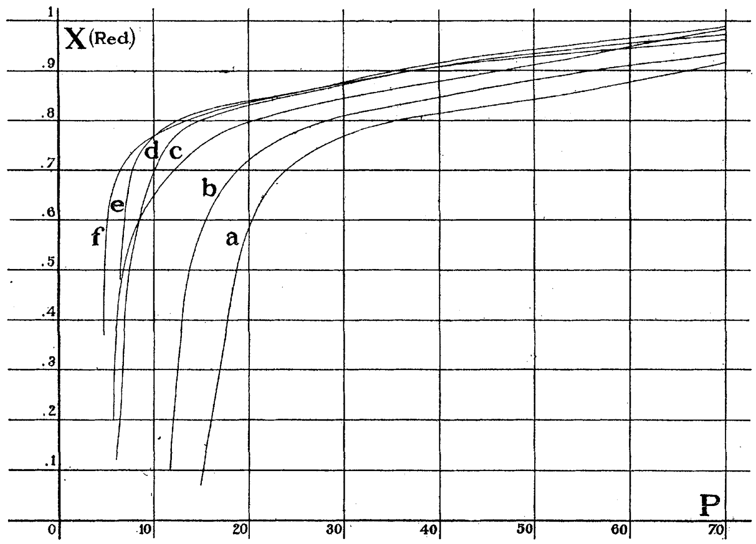

Fig. 11.

gives an estimate of the consistency of the data. Several of these compromise curves taken at widely different speeds are shown in Fig. 11, in which the pressures are given in grams. The following table gives the speed and corresponding Helmholtzian amplitude for each.

Curve.	Bow Speed.	Helm. Amplitude.
a	42.2	.0415
b	27.85	.0274
c	20.17	.0198
d	13.30	.0131
e	9.50	.00935
f	6.33	.00625

The curves are rendered comparable by the scale to which all of their ordinates have been reduced. Similar waves are represented in the figure by points having equal ordinates. At higher pressures, not here plotted, all the curves rise to within one or two per cent. of the limiting Helmholtzian. The speeds for *b*, *d* and *f* are the same as those illustrated in Figs. 6, 7 and 8.

Although the curves of Fig. 11 are very irregular, a few inferences may safely be drawn from them. They show that for the production of similar curves the necessary pressure increases with the speed. At speeds *a*, *b* and *c* the one is almost directly proportional to the other. Curves *d*, *e* and *f* on the other hand suggest a convergence of pressure upon a finite value as speed decreases indefinitely, and this apparent limiting value is far too great to be accounted for by errors in adjustment. In all of these curves, moreover, there is a certain similarity of shape which suggests similar laws of pressure and speed for all of the various waveforms.

V. Minor Phenomena Peculiar to Helmholtzian Vibrations.

In connection with the experiments upon Helmholtzian waves it was found that any attempt to increase the sharpness of the corners by greatly increased pressure was usually accompanied by the production of ripples in the photographic curve. The phenomenon is shown in Fig. 5, no. *4*. Although these ripples are not in themselves particularly interesting, a consideration of them is rendered necessary by the possible inference that they do not exist in the wire's vibration, but instead are due to an imperfection of the observing apparatus. Their occurrence immediately after a sudden change in the direction of the string's motion points to a natural period of the oscillating mirror-system brought out by the very great acceleration which the system is at that instant undergoing.

It is not difficult to show, however, that this inference is untrue, and that the ripples constitute real phenomena of vibration in the string. In the first place their period could not be changed either by using staffs of different weights and dimensions, or by attaching considerable masses to the radial arms; and again, extremely sharp corners without ripples were sometimes observed. When resin was used ripples were very readily produced and were evident at times when the corners were far from sharp. On a few occasions they were seen superposed upon sinusoidal waves, which afterwards disappeared without destruction of the ripples. When very marked their pitch was easily recognized by the ear both before and after removal of the staff.

Their occasional occurrence in the straight parts of the Helmholtzian curve attests the innocence of the mirror, as does also the following consideration. A disturbance at any point of the wire is propagated in both directions and suffers reflection at the ends of the vibrating segment. The exact times of recurrence of these waves at its origin may be determined by an integral-surface construction. A photograph taken at $\frac{3}{4}$, the bow being at $\frac{1}{4}$, not only exhibited ripples at the corners, but also showed their recurrence later at exactly the points indicated by theory. Ripples are therefore real vibrations of the string, caused doubtless by the bow, whose soft covering may have a highly damped natural period.

Besides ripples there were observed other unexpected phenomena peculiar to Helmholtzian waves. The following theoretical treatment will make clear their meaning.

It has been stated[1] that the wave-form is completely determined by the laws of Young, Helmholtz, and Krigar-Menzel. That this is not strictly true is made reasonable at least by a consideration of the following special case. With the bow at the middle point, let it be assumed that the whole string is divided into $(2n + 1)$ equal segments, n being a positive integer. The bow, being at the middle point of the middle segment may set up in this segment a complete Helmholtzian wave similar to the fundamental one and with amplitude and period equal to $[1/(2n + 1)]$ of those of the fundamental. If the same type of vibrations exist, with proper phase-difference, in all of the other segments, the wave equation, end conditions, and Young's law are satisfied, and the motion of the bowed part obeys perfectly the laws of Helmholtz. In other words these laws do not exclude the possibility of producing certain *Helmholtzian harmonics* as well as the fundamental Helmholtzian.

Since these laws show that the fundamental type of vibration is determined in all aliquot and rational cases by the position p/q of the bowed point, this fractional form may for present purposes be utilized to indicate the type of vibration even when applied to harmonics. For example, all of the harmonics described in the preceding paragraph may be said to be of the $(\frac{1}{2})$ type.

The impossibility of producing certain Helmholtzian harmonics is evident from two illustrations: first—the third harmonic, with the bow at $(1/3)$, violates Young's law; second—the second harmonic, with the bow at $(\frac{1}{4})$, violates the spirit if not the letter of Helmholtz's velocity law, because the type is so changed that the bowed point does not travel *with* the bow during so great a part of the entire period as the fundamental form would allow. The following statement defines completely all *possible* cases save those at $(\frac{1}{2})$. If the string be rubbed at any aliquot or rational point (p/q), where p and q are prime to each other, the production of the nth Helmholtzian harmonic is consistent with the laws of Young and Helmholtz if n and q are prime to each other. Its type of vibration is given by $(np/q - m)$ where m is a positive integer so chosen that

[1] Davis; see note 1.

$$0 < \left(\frac{np}{q} - m\right) < 1.$$

It is to be noted that although the wave equation and end conditions permit Helmholtzian harmonics to be superposed, the laws of the bow are thereby violated.

Special interest attaches to the ($\frac{1}{2}$) case, in which the motion of the middle point is so slightly restricted by the wave-equation and end conditions that it may be determined arbitrarily over an entire half-period. An infinity of complex Helmholtzian solutions exists therefore, and all necessary restrictions upon the forms of these solutions are defined by the following construction: Lay off on the time axis an interval of one half-period (of either fundamental or any odd harmonic), and connect its ends by any broken line which represents x as a single-valued function of t, and the segments of which are straight and have either one of two slopes corresponding to $\pm v$, where v is the bowing speed. When these forms are interpreted by means of an integral surface it is found that the displacement is zero along all sides of the square, so that all parts of the string pass through the position of equilibrium at the same instant. It has been pointed out by Rayleigh[1] that this is a necessary property of all natural free vibrations of strings. Since the periods of a string are not altered by the operation of rubbing,[2] the resulting vibration may be considered free and therefore, in one respect, the complex forms are *natural*.

The conditions for the production of all of these harmonics and other complex types are not as yet understood. Good rubbing wheels operating with alcohol commonly produced the simple "gable roof," which has always been considered the normal form. In contrast to this, however, hard resin wheels gave the more complex forms almost exclusively. These forms were apparently of infinite variety, and the vibration changed at times from one to another of them, slowly and without apparent cause. Complex forms could sometimes be produced with alcohol. To produce them it was necessary to remove the wheel along the wire to a distance of, say, three centimeters from the middle point. Once started, however, they usually persisted while the wheel was slowly drawn back to its normal position. History, then, as well as present conditions, determines type. In Fig. 5, no. 5, which shows a complex type produced with alcohol, the ripple is to be accounted for by the great bowing pressure required.

[1] Theory of Sound; Vol. 1, p. 180, 1894.

[2] F. Lippich; Mittheil. d. Deut. Math. Ges. in Prag., 1, 118, 1892.

VI. Summary.

The results of the experiments described in this paper may be summarized as follows:

1. The ordinary Helmholtzian form at ($\frac{1}{2}$) has been photographically recorded with a degree of accuracy hitherto unapproached.

2. Helmholtzian forms have been discovered which demonstrate the inadequacy of the laws, as now stated, to determine completely the form of vibration. The limitations placed by present-known laws on the number and forms of these new vibrations have been determined.

3. The study of sub-Helmholtzian vibrations has not been restricted to the forms heretofore recognized, but has been extended to include those of very much smaller relative amplitudes.

4. It has been shown that under certain well determined conditions the tone produced by the rubbed string is of exceptional purity.

5. By improved methods, standard conditions have been determined sufficient for the production and duplication of normal waves, both Helmholtzian and sub-Helmholtzian.

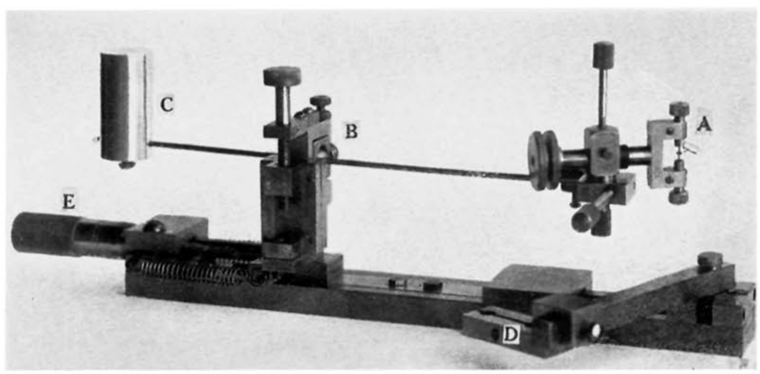

FIG. 4.

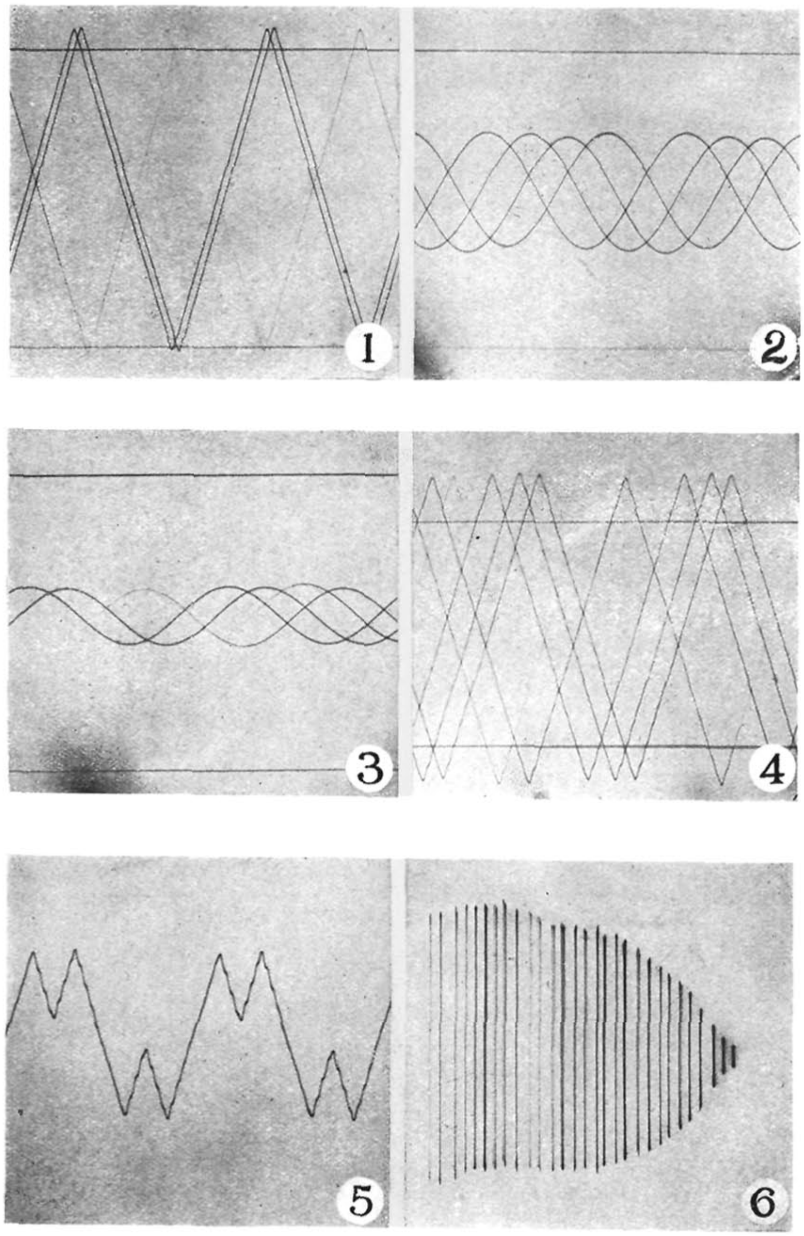

Fig. 5.