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ANOVA USING COMMUTATIVE JORDAN ALGEBRAS, AN APPLICATION

Paulo Canas Rodrigues and João Tiago Mexia

Mathematics Department
Faculty of Science and Technology
New University of Lisbon
Monte da Caparica 2829–516 Caparica

e-mail: paulocanas@gmail.com

Abstract

Binary operations on commutative Jordan algebras are used to carry out the ANOVA of a two layer model. The treatments in the first layer nests those in the second layer, that being a sub-model for each treatment in the first layer. We present an application with data retried from agricultural experiments.

Keywords: commutative Jordan algebras; variance components; orthogonal models; ANOVA.

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1. Introduction

Jordan algebras can be used to study estimation problems in normal orthogonal models, namely to obtain minimum variance unbiased estimators (UMVUE).

So, a commutative Jordan algebra is a vector space constituted by symmetric matrices that commute these algebras. Each commutative Jordan algebras has, see Seely *et al.* (1971), has an unique basis, the principal basis, constituted by orthogonal projection matrices that are mutually orthogonal.

Our goal is the estimation of variance components in a two layers random effects model. In the first layer we have two factors that cross. Each level combination of these factors nests the level combinations of the factors en the second layer. In the second layer we have three factors: the first crosses with second which nests the third.

Using the binary operations defined in Fonseca *et al.* (2006) we obtained the principal basis for the commutative Jordan algebra associated to the model, as well as complete sufficient statistics.

After the analysis of the first and second layers, the external and internal factors, respectively we will present an application about castes and clones in the wine production.

2. Binary operations

We now present the binary operations defined in Fonseca *et al.* (2006).

Consider the following matrices: $\boldsymbol{I}_s$, the $p \times s$ identity matrix, $\boldsymbol{J}_s = \boldsymbol{1^s 1^{s\prime}}$, $\bar{\boldsymbol{J}}_s = \boldsymbol{I}_s - \frac{1}{s}\boldsymbol{J}_s$ and $\boldsymbol{T}_s$ obtained deleting the first line equal to $\frac{1}{\sqrt{s}}\boldsymbol{1^s}$ of an orthogonal $s \times s$ matrix.

Let $g_1, \ldots, g_w$ be the ranks of the matrices $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_w$ in the principal basis of a commutative Jordan algebra. The matrices in the principal basis of $\mathcal{A}_1 \otimes \mathcal{A}_2$ will be the $\boldsymbol{Q}_{1,j_1} \otimes \boldsymbol{Q}_{2,j_2}, j_1 = 1, \ldots, w_1, j_2 = 1, \ldots, w_2$, with $\{\boldsymbol{Q}_{l,1}, \ldots, \boldsymbol{Q}_{l,w_l}\}$ the principal basis of $\mathcal{A}_l, l = 1, 2$.

If $\boldsymbol{Q}_{j,1}, \ldots, \boldsymbol{Q}_{j,w_j}$ is the principal basis of $\boldsymbol{A}_j, j = 1, 2$ and $\boldsymbol{Q}_{2,1} = \frac{1}{n_2}\boldsymbol{J}_{n_2}$, the principal basis of the restricted product $\boldsymbol{A}_1 * \boldsymbol{A}_2$ will be

$$(1) \qquad \{\boldsymbol{Q}_{1,1} \otimes \boldsymbol{Q}_{2,1}, \ldots, \boldsymbol{Q}_{1,w_1} \otimes \boldsymbol{Q}_{2,1}\} \cup \{\boldsymbol{I}_1 \otimes \boldsymbol{Q}_{2,2}, \ldots, \boldsymbol{I}_1 \otimes \boldsymbol{Q}_{2,w_2}\}.$$

These operations use as building blocks the very simple commutative Jordan algebras $\mathcal{A}(s)$ with principal basis $\{\frac{1}{s}\boldsymbol{J}_s, \bar{\boldsymbol{J}}_s\}$.

To a factor with a levels we associate the algebra $\mathcal{A}(a)$. When two factors with a_1 and a_2 levels crosses, they define a model to which we associate the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$. If the first of these two factors nests the second, the model is associate to the algebra $\mathcal{A}_1 * \mathcal{A}_2$. More generally when the treatments of two models cross we get a model associated to $\mathcal{A}_1 \otimes \mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ associated to both initial models. If each treatment of the first models nests the treatments in the second model we get a model associated to $\mathcal{A}_1 * \mathcal{A}_2$.

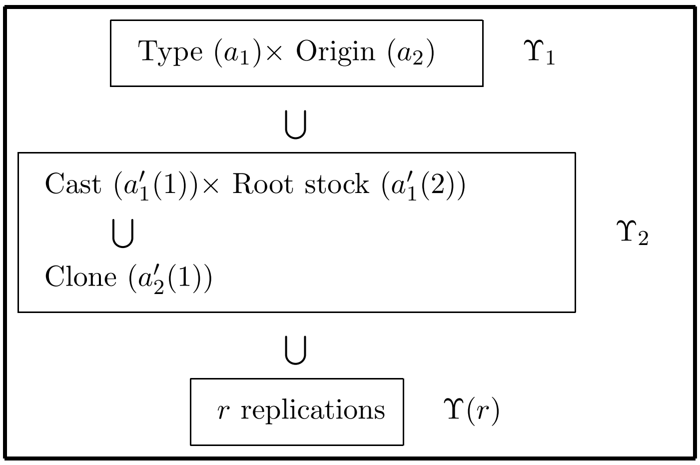
In models with two strata we may associate a sub-model to each strata. Then the treatments of the sub-model corresponding to the first strata nest the treatment of the other sub-model.

If for every treatment in a model associated with algebra $\mathcal{A}$ we take r observations we get a model associated to $\mathcal{A} * \mathcal{A}(r)$.

For more details on these operations see Fonseca *et al.* (2005).

3. The model

As stated above we have a two-stata model. In the first strata two factors, type of wine and origin, cross. The treatments defined by these two factors nest the treatments defined by the factors in the second strata: root-stock that crosses with caste which nests the clone factor. This model may be represented by:



The factores in the first strata have two levels each. There are two types of wine: white wine and red wine and two origins were considered: Douro and Dão.

In the second strata we had four root-stocks and two castes. From each caste three clones were used. Lastly for each treatment we had four replications.

Thus the algebra associated to this model was

$$\mathcal{A} = (\mathcal{A}(2) \otimes \mathcal{A}(2)) * (\mathcal{A}(4) \otimes (\mathcal{A}(2) * \mathcal{A}(3))) * \mathcal{A}(4). \tag{2}$$

The principal basis of the commutative Jordan algebra $\mathcal{A}$ is

$$
\begin{aligned}
\mathcal{A} =& \mathcal{A}_1 * \mathcal{A}_2 * \mathcal{A}(r) \\
=& [\mathcal{A}(a_1) \otimes \mathcal{A}(a_2)] * \left\{ \left[\mathcal{A}(a_1'(1)) * \mathcal{A}(a_2'(1))\right] \otimes \mathcal{A}(a'1(2)) \right\} * \mathcal{A}(r) \\
=& \Bigg\{ \frac{1}{a_1 a_2 a_1'(1) a_2'(1) a_1'(2) r} \boldsymbol{J}_{a_1 a_2 a_1'(1) a_2'(1) a_1'(2) r} \ ; \\
& \frac{1}{a_1} \boldsymbol{J}_{a_1} \otimes \bar{\boldsymbol{J}}_{a_2} \otimes \frac{1}{a_1'(1) a_2'(1) a_1'(2) r} \boldsymbol{J}_{a_1'(1) a_2'(1) a_1'(2) r}; \\
& \bar{\boldsymbol{J}}_{a_1} \otimes \frac{1}{a_2 a_1'(1) a_2'(1) a_1'(2) r} \boldsymbol{J}_{a_2 a_1'(1) a_2'(1) a_1'(2) r}; \\
& \bar{\boldsymbol{J}}_{a_1} \otimes \bar{\boldsymbol{J}}_{a_2} \otimes \frac{1}{a_1'(1) a_2'(1) a_1'(2) r} \boldsymbol{J}_{a_1'(1) a_2'(1) a_1'(2) r}; \\
(3) \qquad & \boldsymbol{I}_{a_1 a_2} \otimes \frac{1}{a_1'(1) a_2'(1)} \boldsymbol{J}_{a_1'(1) a_2'(1)} \otimes \bar{\boldsymbol{J}}_{a_1'(2)} \otimes \frac{1}{r} \boldsymbol{J}_r; \\
& \boldsymbol{I}_{a_1 a_2} \otimes \bar{\boldsymbol{J}}_{a_1'(1)} \otimes \frac{1}{a_2'(1) a_1'(2) r} \boldsymbol{J}_{a_2'(1) a_1'(2) r}; \\
& \boldsymbol{I}_{a_1 a_2} \otimes \bar{\boldsymbol{J}}_{a_1'(1)} \otimes \frac{1}{a_2'(1)} \boldsymbol{J}_{a_2'(1)} \otimes \bar{\boldsymbol{J}}_{a_1'(2)} \otimes \frac{1}{r} \boldsymbol{J}_r; \\
& \boldsymbol{I}_{a_1 a_2 a_1'(1)} \otimes \bar{\boldsymbol{J}}_{a_2'(1)} \otimes \frac{1}{a_1'(2) r} \boldsymbol{J}_{a_1'(2) r}; \\
& \boldsymbol{I}_{a_1 a_2 a_1'(1)} \otimes \bar{\boldsymbol{J}}_{a_2'(1)} \otimes \bar{\boldsymbol{J}}_{a_1'(2)} \otimes \frac{1}{r} \boldsymbol{J}_r; \\
& \boldsymbol{I}_{a_1 a_2 a_1'(1) a_2'(1) a_1'(2)} \otimes \bar{\boldsymbol{J}}_r \Bigg\}.
\end{aligned}
$$

4. Complete sufficient statistics

Now the model may be written as

$$\begin{aligned} \boldsymbol{y}^n = & \sum_{h_1=0}^{1}\sum_{h_2=0}^{1} \boldsymbol{X}_1(h_1,h_2)\boldsymbol{\beta}_1(h_1,h_2 \\ & + \underbrace{\sum_{h_1=0}^{2}\sum_{h_2=0}^{1} \boldsymbol{X}_2(h_1,h_2)\boldsymbol{\beta}_2(h_1,h_2)}_{\substack{(h_1,h_2)\neq(0,0)\\ (h_1,h_2)\neq(0,2)}} + \boldsymbol{e}^n, \end{aligned} \tag{4}$$

where

$$\boldsymbol{X}_1(0,0) = \mathbf{1}^{a_1} \otimes \mathbf{1}^{a_2} \otimes \mathbf{1}^{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \mathbf{1}^{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_1(1,0) = \boldsymbol{I}_{a_1} \otimes \mathbf{1}^{a_2} \otimes \mathbf{1}^{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \mathbf{1}^{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_1(0,1) = \mathbf{1}^{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \mathbf{1}^{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \mathbf{1}^{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_1(1,1) = \boldsymbol{I}_{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \mathbf{1}^{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \mathbf{1}^{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_2(1,0) = \boldsymbol{I}_{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \boldsymbol{I}_{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \mathbf{1}^{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_2(2,0) = \boldsymbol{I}_{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \boldsymbol{I}_{a_1'(1)} \otimes \boldsymbol{I}_{a_2'(1)} \otimes \mathbf{1}^{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_2(0,1) = \boldsymbol{I}_{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \mathbf{1}^{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \boldsymbol{I}_{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_2(1,1) = \boldsymbol{I}_{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \boldsymbol{I}_{a_1'(1)} \otimes \mathbf{1}^{a_2'(1)} \otimes \boldsymbol{I}_{a_1'(2)} \otimes \mathbf{1}^{r}$$

$$\boldsymbol{X}_2(2,1) = \boldsymbol{I}_{a_1} \otimes \boldsymbol{I}_{a_2} \otimes \boldsymbol{I}_{a_1'(1)} \otimes \boldsymbol{I}_{a_2'(1)} \otimes \boldsymbol{I}_{a_1'(2)} \otimes \mathbf{1}^{r}$$

Moreover, writing $\boldsymbol{Z} \sim N(\boldsymbol{\eta}, \boldsymbol{V})$ to indicate that $\boldsymbol{Z}$ is normal with mean vector $\boldsymbol{\eta}$ and variance-covariance matrices $\boldsymbol{V}$, we assume that

$$(5)\qquad \begin{cases} \boldsymbol{\beta}_1(h_1,h_2) \sim N\left(\mathbf{0}^{a_1a_2}, \sigma^2(h_1,h_2)\boldsymbol{I}_{a_1a_2}\right) \\ \\ \boldsymbol{\beta}_2(h_1,h_2) \sim N\left(\mathbf{0}^{a_1'(1)a_2'(1)a_1'(2)}, \sigma^2(h_1,h_2)\boldsymbol{I}_{a_1'(1)a_2'(1)a_1'(2)}\right) \\ \\ \quad \boldsymbol{e}^n \sim N(\mathbf{0}^n, \sigma^2\boldsymbol{I}_n), \end{cases}$$

and that these vectors are independent.

In the model equation the first group of terms relate to the factors in the first strata while the second group of terms will relate to the factors en the second strata.

Note that $\boldsymbol{\beta}_2(0,0)$ is the general mean, $\boldsymbol{\mu}$.

Writing $V \sim \gamma\chi_g^2$ to indicate that V is the product by γ of a central chi-square with g degrees of freedom we will, see Fonseca *et al.* (2006), have the independent statistics:

$$(6)\qquad S(h_1,h_2) = \|\boldsymbol{Q}(h_1,h_2)\boldsymbol{y}\|^2 \sim \gamma(h_1,h_2)\chi^2_{g(h_1,h_2)}$$

$$(7)\qquad S^0(h_1,h_2) = \|\boldsymbol{Q}^{\mathbf{0}}(h_1,h_2)\boldsymbol{y}\|^2 \sim \gamma^0(h_1,h_2)\chi^2_{g^0(h_1,h_2)}$$

$$(8)\qquad S^\perp = \|\boldsymbol{Q}^\perp\boldsymbol{y}\|^2 \sim \sigma^2\chi_g^2, \qquad g = a_1a_2a_1'(1)a_2'(1)a_1'(2)(r-1),$$

Using the factorization theorem we may show, see Fonseca *et al.* (2006), that the general mean and the sum of squares presented above are sufficient statistics. Moreover, since the normal density belongs to the exponential family, they will be complete.

5. Analysis of the second layer

According to the Blackwell- Lehman-Scheffé theorem, the estimators

$$(9)\qquad \begin{cases} \quad \widetilde{\sigma}^2 = \dfrac{S}{g} \\ \\ \widetilde{\gamma}(h_1,h_2) = \dfrac{S(h_1,h_2)}{g(h_1,h_2)} \end{cases}$$

are UMVUE as well as the estimators obtained through them.

Now if we assume that the $\boldsymbol{\beta}_2(h_1,h_2) \sim N\left(\mathbf{0},\sigma^2(h_1,h_2)\boldsymbol{I}\right)$ and $\boldsymbol{e}^n \sim N(\mathbf{0},\sigma_e^2\boldsymbol{I})$ are independent we will have, see Rodrigues *et al* (2005),

$$
(10)\qquad \begin{cases}
\gamma(1,0) = \sigma^2 + r\left[b(1,0)\sigma^2(1,0) + b(2,0)\sigma^2(2,0)\right. \\
\qquad\qquad\qquad \left. + b(1,1)\sigma^2(1,1) + b(2,1)\sigma^2(2,1)\right] \\
\gamma(2,0) = \sigma^2 + r\left[b(2,0)\sigma^2(2,0) + b(2,1)\sigma^2(2,1)\right] \\
\gamma(0,1) = \sigma^2 + r\left[b(0,1)\sigma^2(0,1) + b(1,1)\sigma^2(1,1) + b(2,1)\sigma^2(2,1)\right] \\
\gamma(1,1) = \sigma^2 + r\left[b(1,1)\sigma^2(1,1) + b(2,1)\sigma^2(2,1)\right] \\
\gamma(2,1) = \sigma^2 + r\left[b(2,1)\sigma^2(2,1)\right].
\end{cases}
$$

Solving these equations for the variance components and replacing $\gamma(h_1,h_2)$ e σ^2 by their estimators we obtain the UMVUE estimators for the variance components of the second layer, see Rodrigues & Mexia (2005),

$$
(11)\qquad \begin{cases}
\widetilde{\sigma}^2(1,0) = \dfrac{\widetilde{\gamma}(1,0)+\widetilde{\gamma}(2,1)-\widetilde{\gamma}(1,1)-\widetilde{\gamma}(2,0)}{rb(1,0)} \\
\widetilde{\sigma}^2(2,0) = \dfrac{\widetilde{\gamma}(2,0)-\widetilde{\gamma}(2,1)}{rb(2,0)} \\
\widetilde{\sigma}^2(0,1) = \dfrac{\widetilde{\gamma}(0,1)-\widetilde{\gamma}(1,1)}{rb(0,1)} \\
\widetilde{\sigma}^2(1,1) = \dfrac{\widetilde{\gamma}(1,1)-\widetilde{\gamma}(2,1)}{rb(1,1)} \\
\widetilde{\sigma}^2(2,1) = \dfrac{\widetilde{\gamma}(2,1)-\widetilde{\sigma}^2}{rb(2,1)}.
\end{cases}
$$

with the degrees of freedom

$$
(12)\qquad \begin{cases}
g(1,0) = a_1'(1) - 1 = 1 \\
g(2,0) = a_1'(1) \times (a_2'(1) - 1) = 4 \\
g(0,1) = a_1'(2) - 1 = 3 \\
g(1,1) = (a_1'(1) - 1) \times (a_2'(1) - 1) = 3 \\
g(2,1) = a_1'(1) \times (a_2'(1) - 1) \times (a_1'(2) - 1) = 12.
\end{cases}
$$

6. Analysis of the first layer

Replacing σ^2 by

$$
(13)\qquad \begin{aligned}
\zeta = \sigma^2 + r \big[& b(1,0)\sigma^2(1,0) + b(2,0)\sigma^2(2,0) + b(0,1)\sigma^2(0,1) \\
& + b(1,1)\sigma^2(1,1) + b(2,1)\sigma^2(2,1) \big]
\end{aligned}
$$

we obtain, see again Rodrigues & Mexia (2005).

$$
(14)\qquad \begin{cases}
\gamma^0(1,0) = \zeta + r^0 \left[b^0(1,0)\sigma^{2^0}(1,0) + b^0(1,1)\sigma^{2^0}(1,1) \right] \\
\gamma^0(0,1) = \zeta + r^0 \left[b^0(0,1)\sigma^{2^0}(0,1) + b^0(1,1)\sigma^{2^0}(1,1) \right] \\
\gamma^0(1,1) = \zeta + r^0 \left[b^0(1,1)\sigma^{2^0}(1,1) \right]
\end{cases}
$$

where $n^0 = r a_1'(1) a_1'(2) a_2'(1)$.

So the UMVUE estimators, in the first layer, for the variance components will be

$$
(15) \qquad \begin{cases} \widetilde{\sigma^2}^0(1,0) = \dfrac{\widetilde{\gamma}^0(1,0) - \widetilde{\gamma}^0(1,1)}{r^0 b^0(1,0)} \\[2ex] \widetilde{\sigma^2}^0(0,1) = \dfrac{\widetilde{\gamma}^0(0,1) - \widetilde{\gamma}^0(1,1)}{r^0 b^0(0,1)} \\[2ex] \widetilde{\sigma^2}^0(1,1) = \dfrac{\widetilde{\gamma}^0(1,1) - \widetilde{\zeta}}{r^0 b^0(1,1)}. \end{cases}
$$

with the degrees of freedom

$$
(16) \qquad \begin{cases} g^0(1,0) = a_1 - 1 = 1 \\[2ex] g^0(0,1) = a_2 - 1 = 1 \\[2ex] g^0(1,1) = (a_1 - 1) \times (a_2 - 1) = 1. \end{cases}
$$

7. An application

We now apply our results to an experiment on grapevines. In Table 2 in the Appendix A we presente the data of this experiment.

The aim of this application is to study the influence of the factors and interactions in that production.

In the Table 1 we present the "source of variation", the "variance components", the "estimates of the variance components" (using the results that we obtained in this work), the "F statistics", the "degrees of freedom" of the test F, and finally the p-values of each one of the variance components and interactions.

Table 1. NOVA

Factors/Interactions	Variance Components	Estimates	F Statistics	Degrees of freedom	p-values
Caste	$\sigma^2(1,0)$	0.0329	5.189	(1,12)/(3,4)	0.0470
Root stock	$\sigma^2(0,1)$	-0.0024	0.5199	3/3	0.6977
Caste × Root stock	$\sigma^2(1,1)$	0.0011	1.4664	3/12	0.2731
Clone	$\sigma^2(2,0)$	0.0016	1.9308	4/12	0.1701
Clone × Root stock	$\sigma^2(2,1)$	0.0059	6.7367	12/72	5.98×10^{-8}
Region	$\sigma^{2^0}(0,1)$	0.3913	462.06	1/1	0.0297
Type	$\sigma^{2^0}(1,0)$	0.3885	458.8	1/1	0.0296
Type × Region	$\sigma^{2^0}(1,1)$	-0.0152	0.1003	1/288	0.7517

As we observe, the p-values for factors caste, region and type are less than 5%. So we concluded that the difference between castes, regions and types of wine (white and red) are significant. Thus the hypotheses of differences between them are not rejected for the usual significance levels.

We also observe that the interaction between clone and the root stock is strong, heaving to the lowest p-value of all the factor and interactions.

The estimators of the variance components for root stock and the interaction between the region and the type of wine are negative. These results may be seen as indicating that the corresponding variance components are negligible.

For the caste we had an generalized test F and to obtain the p-value presented in the Table 1 used the program developed by Ferreira (2005).

Appendix

A. Data of this study

Table 2. Production for foot of grapevine, in Kilograms, after the codification.

		Region 1					
		Cast 1			Cast 2		
		Clone 1	**Clone 2**	**Clone 3**	**Clone 1**	**Clone 2**	**Clone 3**
Type 1 ⇓ White	**Root Stock 1**	1.38	1.40	1.45	1.16	1.06	1.12
		1.43	1.38	1.46	1.25	1.15	1.21
		1.54	1.32	1.51	1.19	1.20	1.28
		1.55	1.36	1.53	1.29	1.18	1.19
	Root Stock 2	1.36	1.30	1.39	1.05	1.12	1.23
		1.34	1.26	1.41	1.16	1.19	1.28
		1.32	1.32	1.37	1.09	1.25	1.19
		1.45	1.40	1.40	1.12	1.18	1.25
	Root Stock 3	1.45	1.34	1.34	1.27	1.32	1.31
		1.53	1.36	1.43	1.34	1.33	1.27
		1.57	1.47	1.41	1.36	1.25	1.19
		1.50	1.32	1.45	1.18	1.31	1.21
	Root Stock 4	1.36	1.24	1.35	1.25	1.16	1.28
		1.40	1.32	1.34	1.23	1.12	1.21
		1.37	1.35	1.41	1.19	1.21	1.18
		1.55	1.42	1.38	1.26	1.09	1.16
Type 2 ⇓ Red	**Root Stock 1**	1.99	2.03	2.17	2.21	2.31	2.41
		1.88	2.04	2.12	2.39	2.34	2.37
		2.10	2.15	2.09	2.30	2.32	2.39
		2.05	2.18	2.14	2.21	2.36	2.43
	Root Stock 2	2.12	2.07	2.16	2.39	2.32	2.29
		2.01	2.17	2.21	2.36	2.27	2.33
		2.14	1.90	2.24	2.34	2.34	2.38
		2.14	2.12	2.19	2.26	2.26	2.35
	Root Stock 3	2.28	2.23	2.12	2.36	2.26	2.31
		2.18	2.25	2.19	1.25	2.32	2.37
		2.25	2.32	2.21	2.39	2.45	2.29
		2.19	2.12	2.09	2.27	2.21	2.33
	Root Stock 4	2.12	1.99	2.21	2.30	2.32	2.27
		2.23	2.01	2.18	2.34	2.32	2.24
		2.09	2.12	2.14	2.29	2.29	2.37
		2.32	2.08	2.19	2.38	2.31	2.34

		Region 2					
		Cast 1			**Cast 2**		
		Clone 1	**Clone 2**	**Clone 3**	**Clone 1**	**Clone 2**	**Clone 3**
	Root	0.65	0.70	0.55	0.40	0.35	0.56
T	**Stock**	0.47	0.52	0.45	0.35	0.30	0.76
y	**1**	0.52	0.44	0.51	0.22	0.27	0.75
p		0.39	0.53	0.49	0.29	0.22	0.63
e	**Root**	0.50	0.18	0.38	0.64	0.37	0.65
	Stock	0.55	0.37	0.39	0.78	0.42	0.64
1	**2**	0.48	0.27	0.41	0.69	0.53	0.57
		0.58	0.26	0.35	0.72	0.45	0.71
$\Downarrow$	**Root**	0.35	0.42	0.42	0.40	0.55	0.49
	Stock	0.42	0.53	0.52	0.39	0.37	0.46
W	**3**	0.38	0.38	0.50	0.32	0.64	0.53
h		0.39	0.49	0.47	0.36	0.54	0.51
i	**Root**	0.32	0.23	0.39	0.45	0.55	0.66
t	**Stock**	0.34	0.34	0.55	0.41	0.49	0.63
e	**4**	0.34	0.32	0.47	0.35	0.29	0.68
		0.32	0.28	0.42	0.39	0.45	0.59
	Root	1.42	1.20	1.12	1.41	1.31	1.13
	Stock	1.40	1.13	1.19	1.37	1.42	1.17
T	**1**	1.36	1.32	1.16	1.35	1.35	1.21
y		1.31	1.26	1.21	1.44	1.29	1.15
p	**Root**	1.66	1.14	1.31	1.34	1.48	1.31
e	**Stock**	1.36	1.16	1.36	1.25	1.45	1.28
	2	1.68	1.09	1.28	1.26	1.64	1.30
2		1.28	1.12	1.26	1.19	1.35	1.27
	Root	1.23	1.43	1.15	1.25	1.27	1.26
$\Downarrow$	**Stock**	1.32	1.47	1.17	1.15	1.23	1.24
	3	1.40	1.35	1.12	1.12	1.44	1.29
R		1.38	1.39	1.21	1.19	1.25	1.31
e	**Root**	1.45	1.35	1.31	1.23	1.22	1.41
d	**Stock**	1.54	1.31	1.36	1.19	1.27	1.39
	4	1.49	1.28	1.29	1.28	1.19	1.34
		1.51	1.33	1.31	1.18	1.23	1.38

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