Lab 3: Coupled systems

Theresa Ong
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Coupled Human-Natural Systems

ENVS 80.10/EEES 181 Winter 2020

Due: Thursday, 1/30/2020 at 2:25pm to Canvas

Your submission should be a single PDF or Word file (no need to submit your code). A printed copy of a partial answer key will be made available before the deadline. You will self-assess your assignment and post your edited version of the lab exercises to Canvas by the due date.

Please consult the syllabus for any accessibility concerns.

Getting Started

This is a Mathematica Notebook. It works similarly to R Markdown but uses a different language. We are using Mathematica because some models are easier to code and quicker to run in this platform. We can also use Mathematica to create interactive plots that are much more difficult and slower to implement in R.

To begin, install Mathematica on your machine (if you don't already have it) using Dartmouth's license:

https://caligari.dartmouth.edu/public/downloads/mathematica/

Then open this Lab3.nb file from Canvas.
**To save your file as a PDF in Mathematica all you have to do is navigate to File>Save As and select PDF under Format.

Mathematica

On the right hand side of this notebook you'll notice a series of brackets. These indicate each section of the notebook. If you double click on the larger bracket encompassing this text and the subsection title, "Mathematica" you will cause the section to collapse. Double clicking again will reveal the full text. Try this for a number of brackets to the right to get a sense of the navigation.

I. Predator-prey theory

This week we have been discussing coupled - human natural systems and how management choices can have dramatic consequences for ecological systems.

The predator-prey equations from ecology provide some of the simplest examples of coupled systems. They describe the changes in the populations of victims (V) and predators (P) over time. Let’s look at the equations in detail:

$$\frac{dV}{dt} = rV - aVP$$
$$\frac{dP}{dt} = aVP - mP \quad (1)$$

where r is the per capita growth rate, a is the attack rate of the predator and m is the mortality rate of the predator.

To solve these equations we will consider what values of V and P have to be so that neither population changes, ie. the system is at equilibrium.

Exercise 1: Graphic relationships and isoclines

a.) The predator-prey equations is a formula that couples predators to prey. Draw a diagram on a piece of paper with two circles labeled “V” and “P.” Connect the circles with arrows indicating effects of one species on the other or itself and label the arrows with -/+ signs to indicate whether the effects are negative or positive. You will notice that Equation 1 has positive and negative parts for each dynamic variable (V/P). Label each arrow in your diagram with the portion of Equation 1 that it corresponds to.

After you are done, take a photo of your **flow diagram and insert it into the document. All you have to do is drag your photo file into this cell. Once your photo is in the Mathematica notebook, you can click on it and resize it just like in a word-processor.

b.) By hand, take the set of equations (1) above, set them equal to 0 and solve for when V and P should be 0. You should end up with 2 equations for the prey and 2 equations for the predator. These are your prey isoclines and predator isoclines . Write your sets of equations for V,P=0 (V* and P* for short) down here in the next cell below. You can change the format of the cell by clicking into the cell and navigating in the menu from Format>Style>DisplayFormulaNumbered.

**To format the formula nicely go to Palettes>Basic Math Assistant and expand the section labeled “Typesetting.” This will give you the appropriate formats for different math functions as you need them.

THIS IS YOUR NEW CELL,
CONVERT ME TO DISPLAYFORMULANUMBERED THEN ERASE THIS TEXT AND PUT IN YOUR EQUATIONS

c.) Next on a piece of paper, plot your isoclines on a graph where V is on the x axis and P on the y axis. Let’s assume that all of your parameters (a,m,r) are greater than 0 and only consider populations that are equal or greater to 0. You don’t need specific values of the parameters to make your graph, but mark the x and y intercepts of your isoclines with the appropriate values/formula. Make your prey isoclines dashed and your predator isocline solid and insert your hand-drawn graph below:

Mathematica is great at math, so we can check our answer using the Solve function. You can find help on functions the same way we do in R:

In[∘]:=
```
? Solve
```

Out[∘]=

Symbol

Solve[*expr*, *vars*] attempts to solve the system *expr* of equations or inequalities for the variables *vars*.
Solve[*expr*, *vars*, *dom*] solves over the domain
dom. Common choices of *dom* are Reals, Integers, and Complexes.

Documentation Local » | Web »
Options › Cubics → Automatic … (9 total)
Attributes {Protected}
Full Name System`Solve

Click on the *i* to get more information on the Solve function.
Nullclines are solutions for dynamic variables (V,P) when they are at rest, so we can solve:

In[8]:=
```
nullV = Solve[r * V - a * V * P == 0, V]
```

Out[8]= {{V → 0}}

In[7]:=
```
nullV2 = Solve[r * V - a * V * P == 0, P]
```

Out[7]= $\{\{P \to \frac{r}{a}\}\}$

In[∘]:=
```
nullP = Solve[a * V * P - m * P == 0, V]
```

Out[∘]= $\{\{V \to \frac{m}{a}\}\}$

In[9]:=
```
nullP2 = Solve[a * V * P - m * P == 0, P]
```

Out[9]= {{P → 0}}

Solving for both at the same time gives us equilibrium points where the prey and predator isoclines cross:

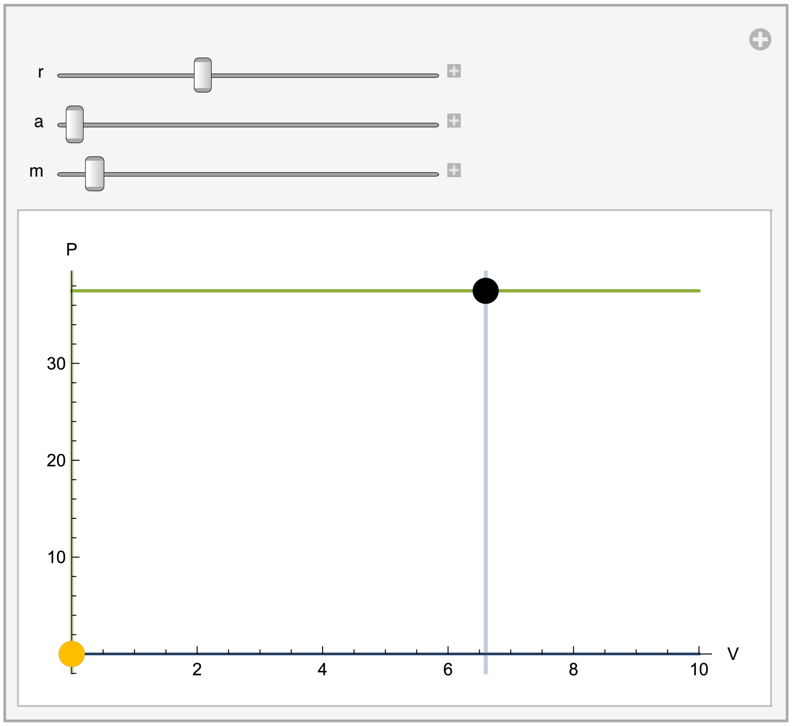
In[∘]:=
```
eqs = Solve[{r * V - a * V * P == 0, a * V * P - m * P == 0}, {V, P}]
```

Out[∘]= $\{\{V \to 0, P \to 0\}, \{V \to \frac{m}{a}, P \to \frac{r}{a}\}\}$

Mathematica has provided 2 nullcline equations where V or P is not expected to change for both the prey and predators, giving us 4 equations total. We also have 2 equilibrium points. We can plot these here, with the predator isoclines in blue and the prey isoclines in green:

In[38]:=
```
Manipulate[
 Plot[{0, r/a}, {v, 0, 10}, GridLines -> {{{0, Directive[ColorData[97, 3], Thick]},
     {m/a, Directive[ColorData[97, 1], Thick]}}, None}, Epilog → {PointSize[0.04],
    Point[{{0, 0}, {m/a, r/a}}, VertexColors → {ColorData[97, 8], Black}]},
   AxesLabel → {"V", "P"}, PlotStyle → {ColorData[97, 1], ColorData[97, 3]}],
 {r, 0.01, 1}, {a, 0.01, 1}, {m, 0.01, 1}]
```

Out[38]=



In[24]:=
```
? Point
```

Out[24]=

Symbol

Point[p] is a graphics and geometry primitive that represents a point at p.
Point[{p_1, p_2, …}] represents a collection of points.

Mathematica uses a Manipulate function to allow easy manipulation of parameters in models. You can move the cursor bars for each parameter to see changes in the graph. Clicking on the + symbol next to the scroll bars will reveal an input box where you can see and specify values for the parameters.

Exercise 2: Manipulating parameters

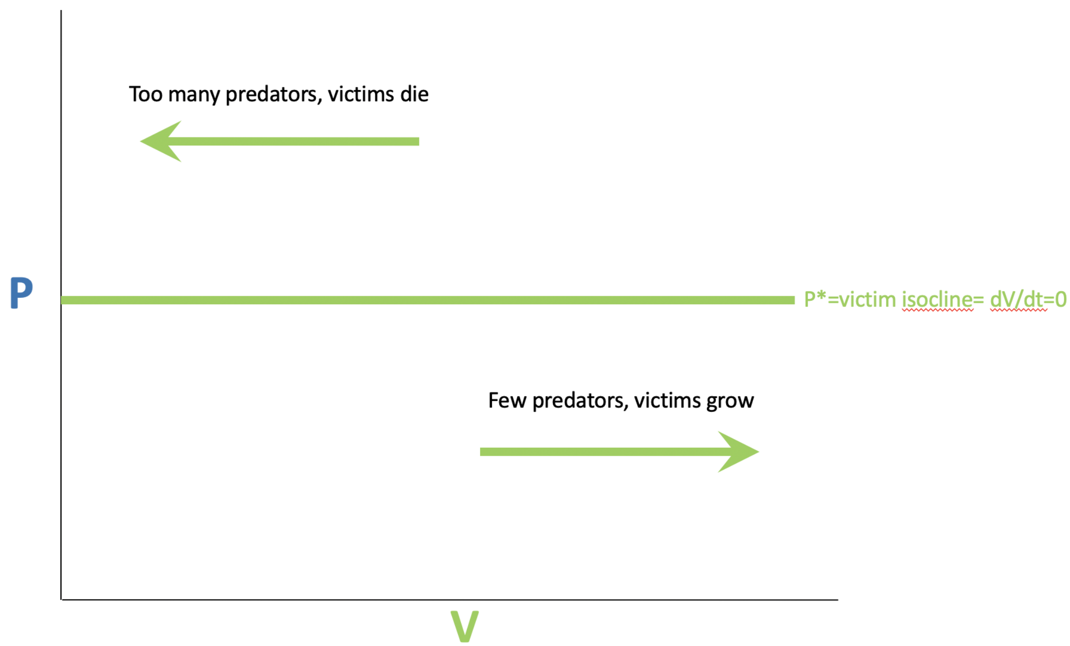
a.) In general terms, what does the equilibrium at (0,0) represent?
b.) For the second equilibrium point, toggle the cursors to find parameters of r,a,m where the predator is dominant at equilibrium.

c.) For the second equilibrium point, toggle the cursors to find parameters of r,a,m where the prey is dominant at equilibrium.
d.) In general, how does increasing values of r, a and m differentially affect the non-zero equilibrium point? Does this make ecological sense?
e.) One of these equilibria can be described as a type of coexistence between predators and prey. Which one and why?

In order to understand the stability of the two equilibrium points in our model, we will have to take a look at how the system moves around them. Let's start by just examining the non-zero predator and prey isoclines. We can first plot the prey/victim isocline where the victim population is static, this is the equation:

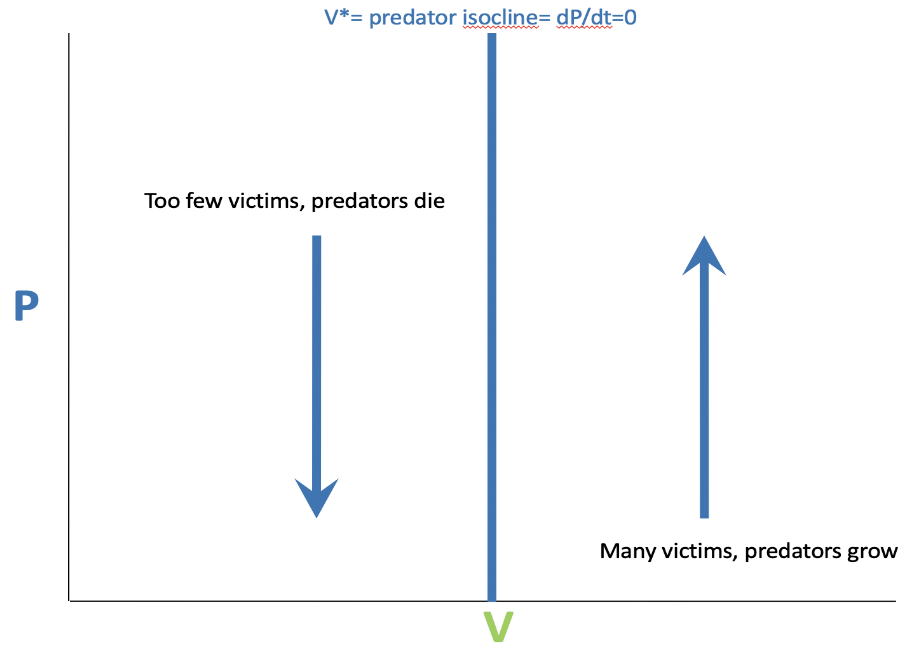
$$P^* = \frac{r}{a} \tag{2}$$

If the predator population is above the victim isocline, then there should be too many predators and the victim population should decrease. However, if the predator population is below this line then we would expect the prey/victims to increase. These dynamics are represented by flows horizontally along the x-axis since this represents changes in the number of victims in the population:

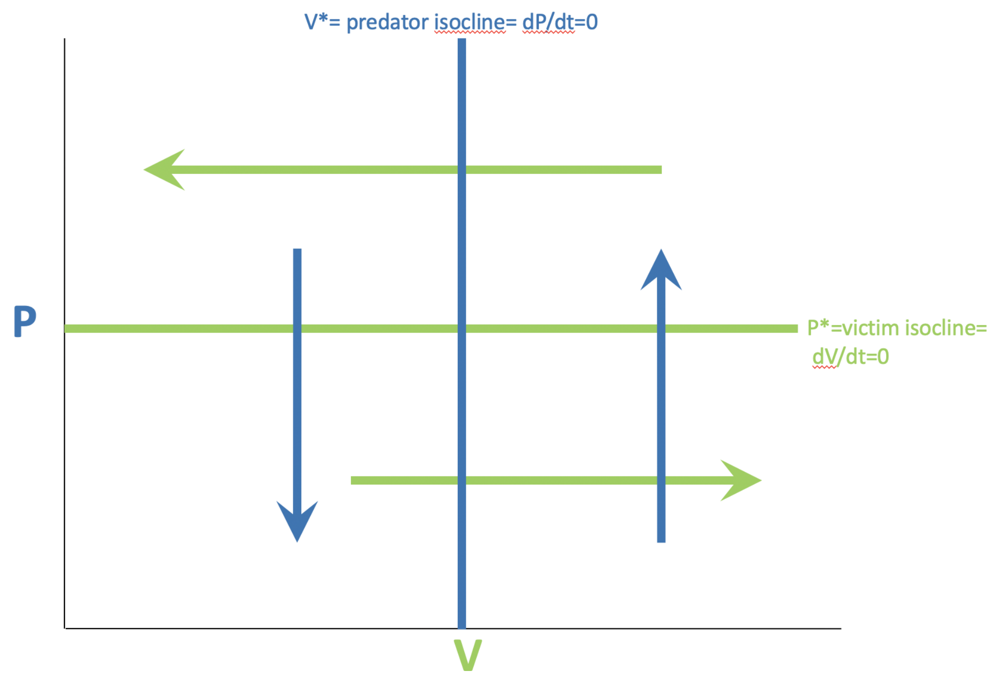


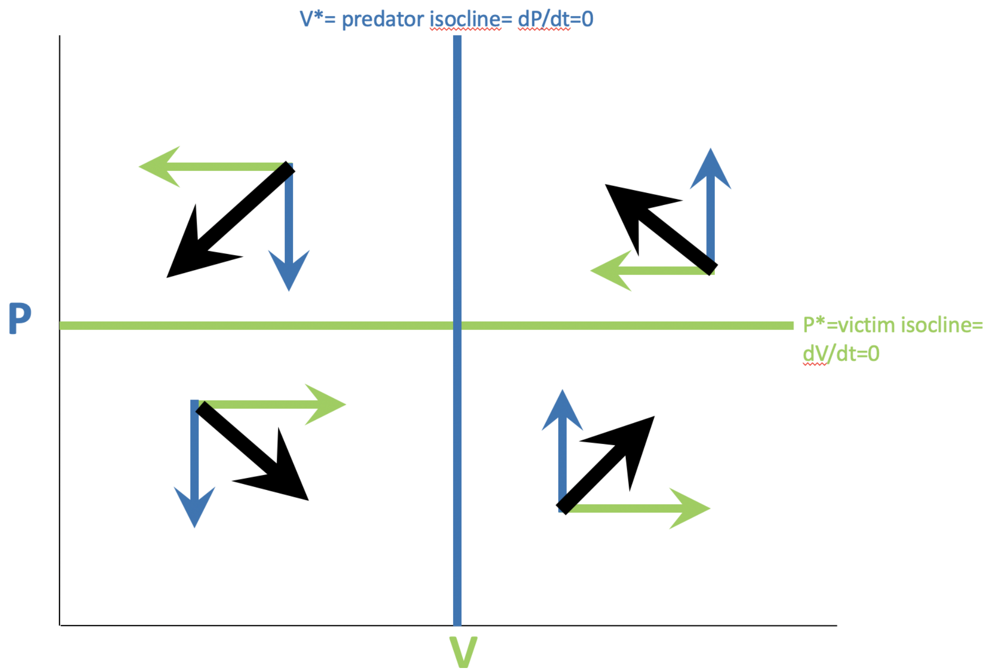
We can do a similar analysis for the predator's non-zero isocline:

$$V^* = \frac{m}{a} \tag{3}$$

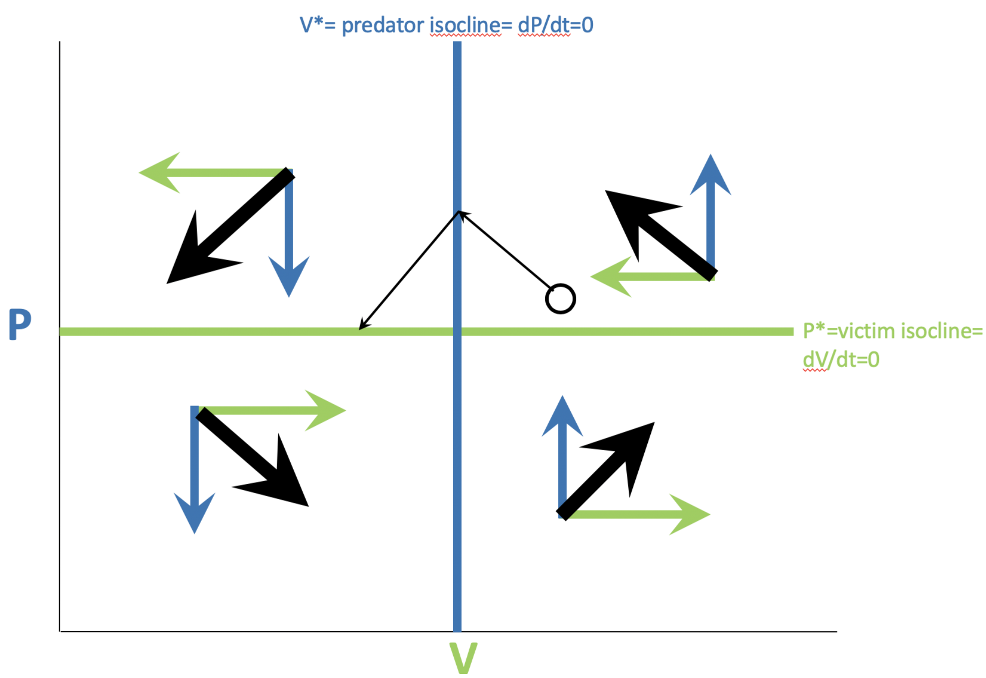


When we put these dynamics together we can sum the vectors to understand how dynamics change in each quadrat of the graph:





To visualize the flow in an intuitive way we will start at at some initial condition marked by the point and move in the direction of the flow for that quadrat until we reach an isocline where the flow shifts:



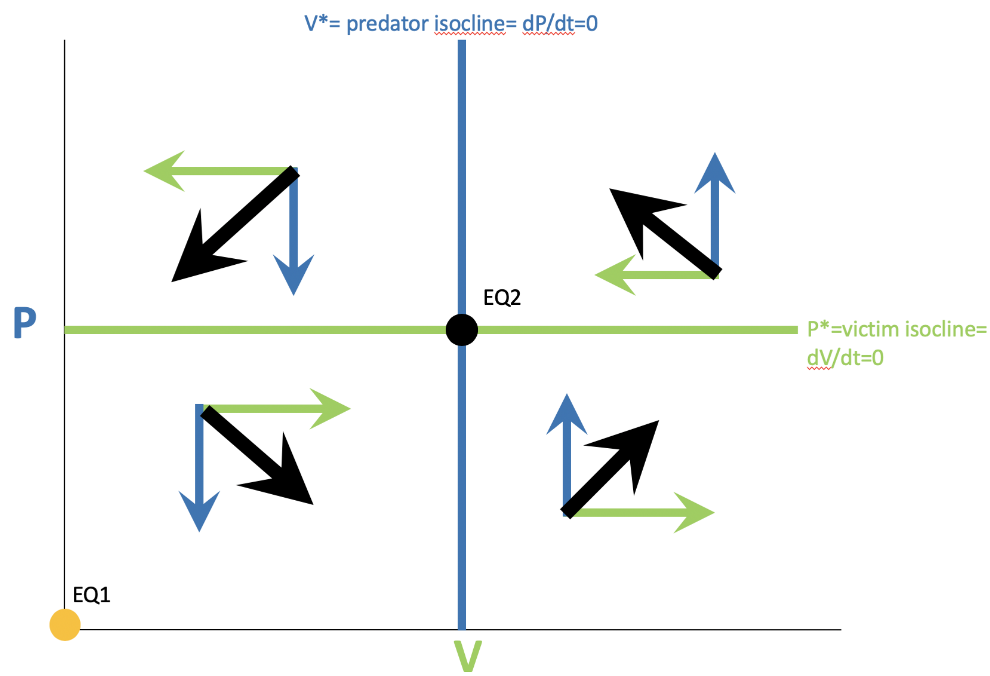
Exercise 3 : Understanding dynamic flow

a.) On a piece of paper, continue to draw the flow in the last diagram, moving through each of the quadrants as appropriate or possible. Take a photo of your graph an insert it here:

b.) Start at a few different initial condition and repeat the exercise. How would you describe the changes in predators and prey over time in this system? Is it sensitive to initial conditions?

c.) Plot the dynamics in a time series with arbitrary increasing time on the x axis and the population size on the y axis. Draw one curve representing the prey or the predator through time. Upload your photo here:

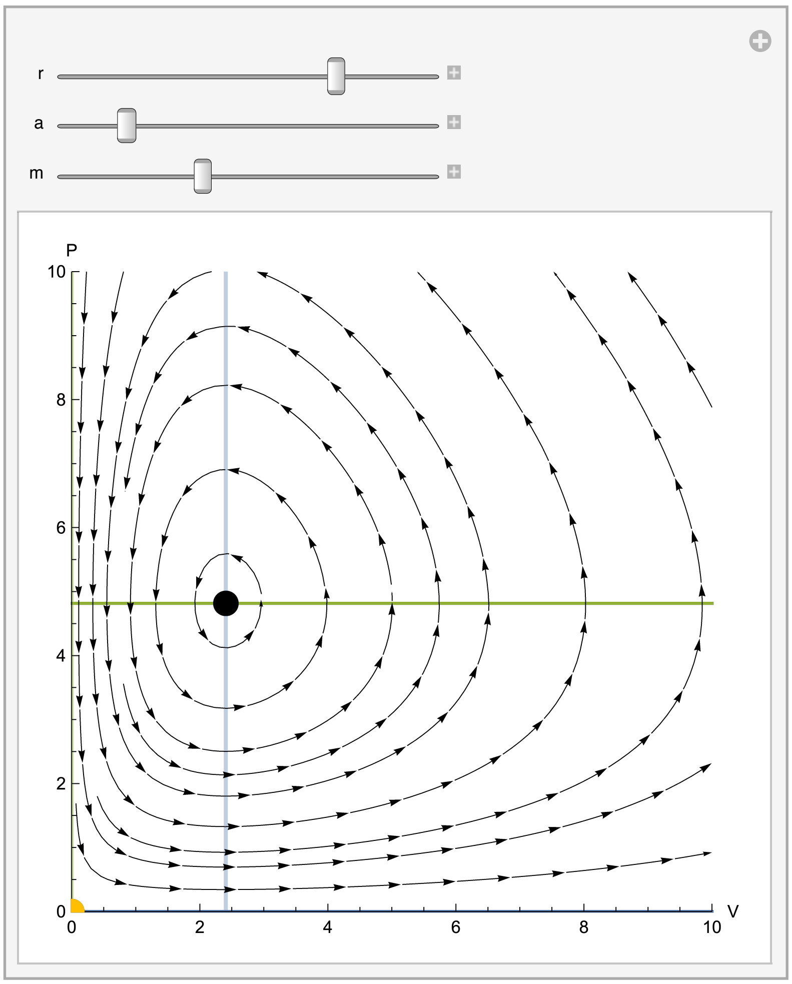
d.) Remember that there are two equilibrium points in this model, which we plot below. Can you describe whether each of them are stable (dynamics move towards it?), unstable (dynamics move away) or neither? Explain your reasoning.



e.) Use the cursors in the graph below to find parameters of r,a and m where the cycles of V are bigger in magnitude than the cycles of P.

In[49]:=
```
Manipulate[
 Show[Plot[{0, r/a}, {v, 0, 10}, GridLines -> {{{0, Directive[ColorData[97, 3], Thick]},
      {m/a, Directive[ColorData[97, 1], Thick]}}, None}, Epilog → {PointSize[0.04],
     Point[{{0, 0}, {m/a, r/a}}, VertexColors → {ColorData[97, 8], Black}]},
   AxesLabel → {"V", "P"}, PlotStyle → {ColorData[97, 1], ColorData[97, 3]},
   PlotRange → {{0, 10}, {0, 10}}, AspectRatio → 1],
  StreamPlot[{r * v - a * v * p, a * v * p - m * p}, {v, 0, 10}, {p, 0, 10},
   StreamPoints → 10, StreamStyle → Black, AxesLabel → {"V", "P"}]],
 {r, 0.01, 1}, {a, 0.01, 1}, {m, 0.01, 1}]
```

Out[49]=



**EXTRA CREDIT (+5pts): Deconstruct the Mathematica code above into functions and explain what each unique function does in the plot. Hint: Use the ? help to well.. help!

The predator-prey equations above are very linear systems. They suggest that prey can grow exponentially forever and predators never stop eating/never satiate. We can add some realism to the model by adding a limit to how much the predator can consume in the following equations:

$$\frac{dV}{dt} = rV - \frac{aVP}{1+V}$$

$$\frac{dP}{dt} = \frac{aVP}{1+V} - mP \qquad (4)$$

Exercise 4: Solving equations

a.) Use Mathematica to solve the last set of equations for the predator/prey isoclines and their equilibrium points.

b.) Assuming the a>m, hand-sketch a plot of all isoclines and equilibria for V/P>0
** For extra credit (+5 pts), use the example from Ex. 1 to redo your 4b plot using Mathematica.

c.) Add vectors to show flow along the x and y directions, and draw a trajectory from an initial point near to the non-zero equilibrium throughout all quadrants of your graph. Add the graph here:

d.) Sketch a time series graph that shows how the population of prey should change through time:

e.) Based on your results in 4c, is your non-zero equilibrium stable, unstable or neither?

HOMEWORK 2: Noy-Meir's grazing model (10 pts total, 2 pts each)

**Please complete the following on your own time and submit your answers along with your lab next week.

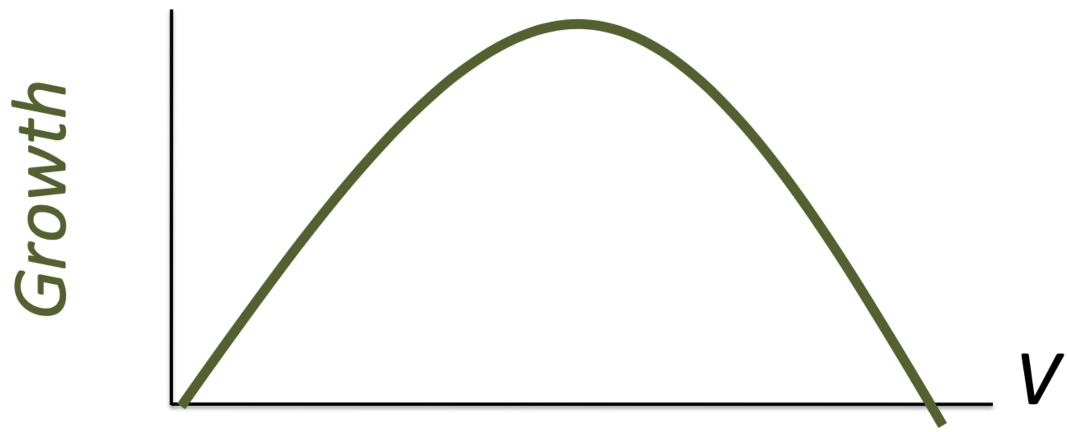
This week we have been discussing coupled-human natural systems and how management choices can have dramatic consequences for ecological systems. Today, we will take a look at cattle-grazing systems.

Back in the 1975, Imanuel Noy-Meir proposed a simple and elegant model to describe these systems. You can find the original paper here:

https://www.jstor.org/stable/2258730?seq=1#metadata _info _tab _contents

Noy-Meir described the growth of vegetation and grazers in the system as a set of coupled differential equations similar to the predator-prey equations we examined earlier. The results can be examined using a graphical approach.

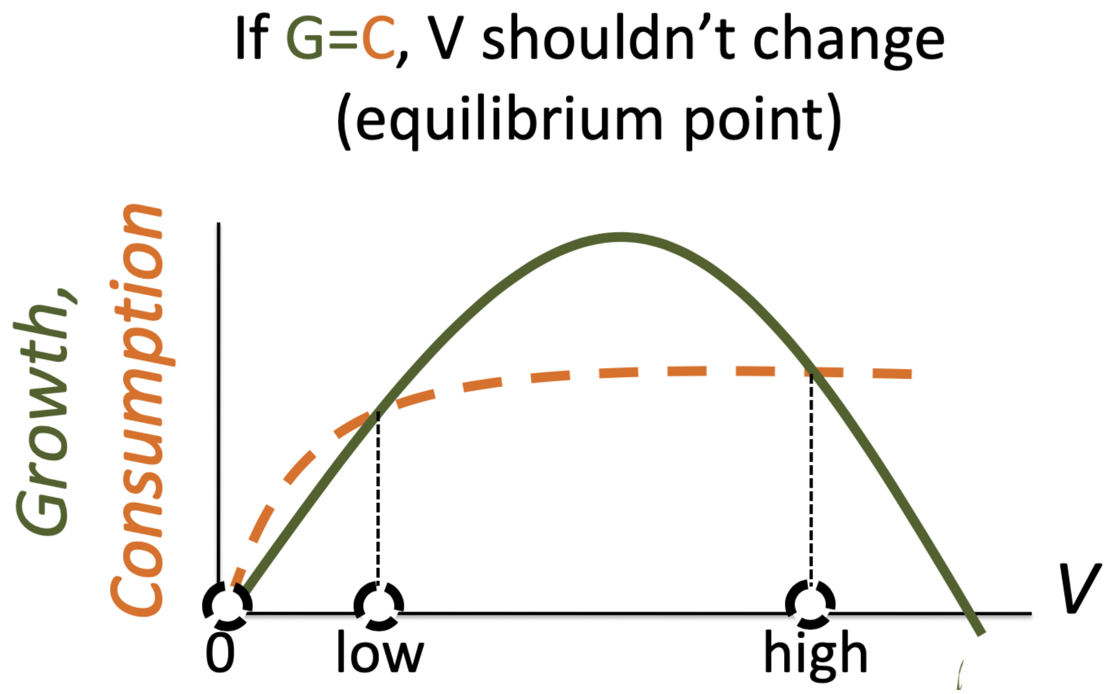
First let's graph how the growth rate of vegetation (V) changes with its density.



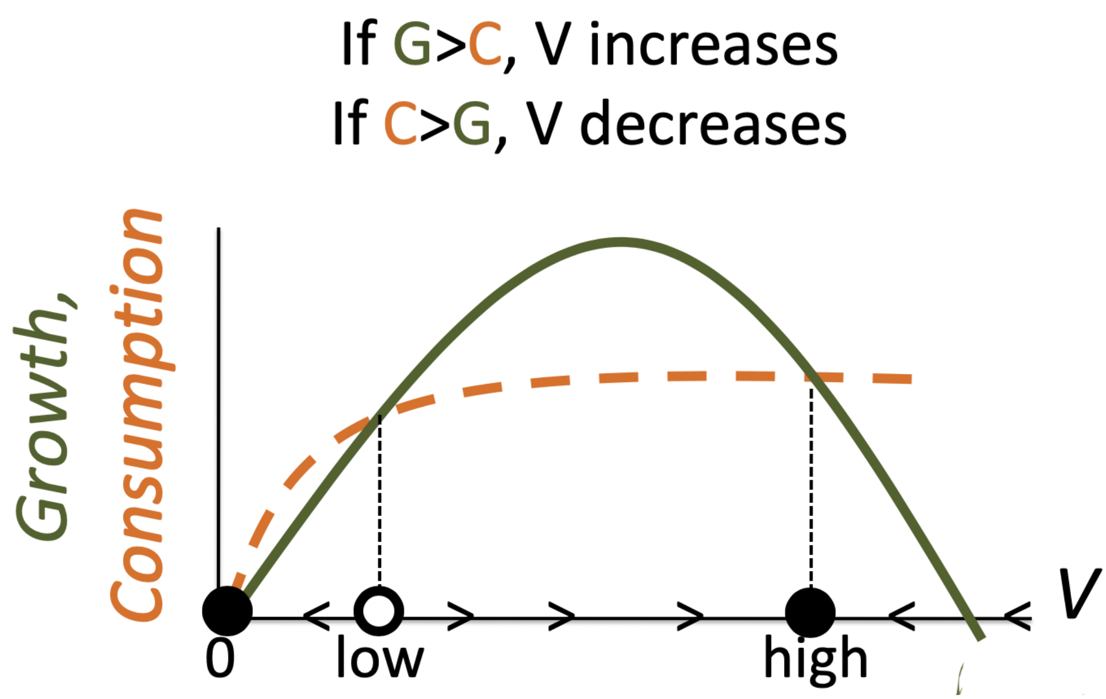
Exercise 1: Understanding the model

a.) What is a biological explanation for why growth rates of vegetation should have a hump shaped with its own density?

Now let’s plot a hypothetical curve representing the consumption rate of a grazer, say cows.



This graph shows three equilibria where G=C. We can make the logical inference that...



Here we draw flows along the x-axis that indicate how the grass is changing relative to the growth and consumption rates.

Exercise 2: Inferring stability

a.) Are the 0, low, and high equilibria stable, unstable or neither?

Exercise 3: Adjusting model assumptions

Noy-Meir’s graphical approach is easy to modify for different scenarios. Redraw the growth and consumption curves and how associated equilbria and flows along the x-axis would change assuming the following:

a.) Move only the growth curve to show what happens if the grass grows too slow for the cow to satiate.

b.) From the original graph, move only the consumption curve to show what happens if the cattle satiates at a rate much higher than the grass grows.

c.) Change the shape of the consumption curve only to show what happens if the cow doesn’t discover and consume grass until there is a critical density of it available.