# Russell's Road to Logicism

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One of the most significant events in the emergence of analytic philosophy was Bertrand Russell's rejection of the idealist philosophy of mathematics contained in his *An Essay on the Foundations of Geometry* [*EFG*] (1897) in favor of the logicism that he developed and defended in his *Principles of Mathematics* [*POM*] (1903). Russell's language in *POM*—and in the sharp and provocative paper, "Recent Work on the Principles of Mathematics" (1901a), which is the first public presentation of many of the doctrines of *POM*—is deliberately revolutionary. "The proof that all pure mathematics, including Geometry, is nothing but formal logic, is a fatal blow to Kantian philosophy" (1901a, 379)—including, of course, the Kantian philosophy that he defended just four years earlier in *EFG*. "The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age" (*POM*, §4), a discovery that will initiate a new era in philosophy: "[T]here is every reason to hope that the near

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future will be as great an epoch in pure philosophy as the immediate past has been in the principles of mathematics" ("on a level with the greatest age of Greece"!).

But what exactly did this revolution consist in? That is, which philosophical turns did Russell need to make in order to move him down the revolutionary road from his idealist philosophy of geometry to logicism? What was his road to logicism? In this paper, I will argue that this road was much shorter than Russell's language suggests, and that the number of substantive philosophical changes that Russell made to get from *EFG* to logicism was surprisingly small. (Indeed, I will argue that there is a sense in which Russell was already a "logicist" in *EFG*.)

In *POM*, Russell argues that Kant's philosophy of mathematics stands or falls with the contention that mathematics requires non-logical reasoning, employing a drawn figure:

> What is essential, from the logical point of view, is, that the a priori intuitions supply methods of reasoning and inference which formal logic does not admit; and these methods, we are told, make the figure (which may of course be merely imagined) essential to all geometrical proofs. (*POM*, §433)

Modern logic, however, makes this contention "capable of final refutation" (*POM*, §4). One might naturally infer from passages like this one that the inferential necessity of intuition in mathematics was part of the fundamental core of Russell's earlier Kantian philosophy of geometry. But, as I'll argue, this is not the case. In fact, Russell's road from Kantian philosophy of geometry to logicism is not well described as "a crisis of intuition," as Nicholas Griffin does in his pathbreaking book, *Russell's Idealist Apprenticeship*:

> One of the things which makes Russell's development in the 1890s so interesting is that, within the space of seven years, he moves from a full-blooded Kantian position, such as might have been widely accepted at the beginning of the century, to a complete rejection of Kant, a position which was not common even among the advanced mathematicians of the

> time. In short Russell's intellectual development between 1893 and 1899 encapsulates the declining fortunes of Kantian intuition among nineteenth century mathematicians. (1991, 99)

(A similar interpretation is defended in Coffa 1981, 255.)

Russell, when he later described his philosophical development, often emphasized the revolutionary effect of his encounter with Peano in August 1900.[1] It might be natural to conclude, then, that Russell adopted logicism only after his exposure to Peano. I will be arguing that this is not the case: Russell was already committed to logicism in 1898–1899, after his exposure to Whitehead's *Treatise on Algebra* (1898) and to Moore's new ideas on judgment and truth. For this reason, in this paper I will be investigating not only *EFG*, but also the transitional writings of 1898–1899, when Russell began and abandoned a series of book manuscripts on the principles of mathematics. These include "An Analysis of Mathematical Reasoning" (April–July 1898) [*AMR*] (1898a), "On the Principles of Arithmetic" (late 1898), "Fundamental Ideas and Axioms of Mathematics" (1899), and "Principles of Mathematics" (August 1899–June 1900). I will also be looking at two published papers, "Are Euclid's Axioms Empirical?" (August 1898, 1898b) and "The Axioms of Geometry" (August 1899, 1899b), which specifically address the question whether pure intuition grounds the axioms of geometry.[2] In describing Russell's writings during this transitional period as logicist, I am disagreeing with both Ian Proops (2006, §4, esp. note 24) and Nicholas Griffin (1991, 98, 274–275), who argue that Russell did not endorse logicism until after his encounter with Peano in August 1900.

My question in this paper is related to, but distinct from the question "What was Russell's 'philosophical motivation' for logicism?" (Proops 2006, 267) This question has received a variety of answers. Proops has argued that the philosophical motivation for logicism was to provide an account of the fundamental nature of mathematics that respects its certainty and exactness. For Peter Hylton, logicism played an essential role in refuting idealism, by refuting the alleged contradictions in space and time and providing a clear case of absolute truth (Hylton 1990).

Griffin has claimed that logicism is motivated by Russell's desire to secure the certainty and necessity of mathematics (Griffin 1980). The interpretations of Proops, Hylton, and Griffin are all posed from the point of view of Russell's developed, published versions of logicism in *POM* and later. They ask: What philosophical role does logicism play in *POM* and subsequent writings? But my question is posed prospectively, not retrospectively: from the point of view of Russell in *EFG* in 1897, what changes would have to be made to arrive at logicism? For this reason, my discussion in this paper will not directly address these interpretations. (Still, though, my question does dovetail with theirs, and some of what I say will impinge on these interpretations.)

This chapter has three sections. In Sect. 1, I clarify my question—"What was Russell's road to logicism?"—by explaining what I take logicism to entail, and I distinguish the general idea of logicism from the specific way that Russell articulates logicism in *POM*. In Sect. 2 ("'Logicism' in *An Essay on the Foundations of Geometry*"), I argue that Russell's road to logicism began much closer to the destination than is often recognized. In particular, I argue that intuition has a very modest role in the philosophy of mathematics in *EFG*, and that the position there is in fact a nonstandard sort of "logicism," which employs a conception of logic as transcendental. In the last section, I then articulate three turns that Russell had to make to move from *EFG* to genuine logicism (without the scare quotes), and show that all three were made already in 1898–1899.

## 1 What Is Logicism?

Answering the question "What was Russell's road to logicism?" requires first getting clear on what logicism is. In *POM*, Russell expresses his logicism as the claim that "all mathematics follows from symbolic logic" (§10), though he also says instead that pure mathematics is nothing but "formal" logic (*POM*, §434; 1901a, 379) or "general" logic (*POM*, §4; 1901a, 366–367). More determinately, Russell claims:

> [A]ll pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles … All mathematics is deduction by logical principles from logical principles. (*POM*, xv, §5; cf. 1901a, 366–367; 1901b, 187)

Logicism thus requires that all the *indefinable concepts* of pure mathematics are logical concepts, that all the *indemonstrable propositions* are logical principles, and that all mathematical *reasoning* is deduction by logical principles. That all indemonstrable propositions are logical principles requires that they contain no constants except logical constants.[3] The logicist view of mathematical reasoning is opposed to the view that "a priori intuitions supply methods of reasoning and inference which formal logic does not admit" and that mathematical reasoning requires these intuitional methods (*POM*, §433). In the same way, logicism about indefinables and indemonstrables is opposed to the view that some mathematical indefinables or indemonstrables are knowable only through intuition, where intuition is opposed to logical ways of knowing.[4]

Russell's position in *POM* is clearly intended to be logicist in this sense. I will be arguing, in Sect. 3 below, that Russell's position in 1898–1899 (even before he developed the logic of relations in Fall 1900) is also logicist in this sense. To understand this historical thesis, it will be necessary to separate these three requirements for the general notion of logicism from the particular way that Russell satisfies them in *POM*. Not surprisingly, there are many features of the *POM* version of logicism that depend on the system of logic that Russell developed after his encounter with Peano in Fall 1900. There are three features of the *POM* position that I will discuss in this section: cardinal numbers are defined as equinumerous *classes*; *rigorous proofs* of all mathematical propositions are alleged to follow from the primitive principles of the theories of propositions, classes, and relations; and the generality of logic is articulated in terms of *variables*. None of these three features appear in Russell's philosophy before Fall 1900. But, I maintain, none of these specific features are essential components of the general notion of logicism. I discuss each of these three features in turn.

First, in *POM* Russell defines cardinal numbers as classes of equinumerous classes. Russell adopted this definition around March 1901, and first stated this definition in his Peano-inspired paper "The Logic of Relations" (1901c, 320).[5] The general notion of logicism does not require this definition. Indeed, there is very convincing evidence that Russell's own commitment to logicism predates it: Russell's paper 1901a (written in January 1901) is incontestably logicist, but does not contain the definition and in fact was written while Russell still held *cardinal number* to be indefinable.[6]

Second, a fully satisfying defense of logicism requires rigorous *proofs* from logical axioms using logical modes of inference of all of the fundamental propositions of mathematics (along with demonstrations that these fundamental propositions suffice to prove all the theorems of pure mathematics). In *POM*, Russell claims that such proofs could be given using only the primitive principles of the theory of propositions, classes, and relations (see *POM*, Chap. II) that he developed after his encounter with Peano. But a philosopher can be a logicist even if she has not carried out these proofs, and even if she has no determinate idea how these proofs could be carried out. Leibniz and Wolff in the eighteenth century were committed to the possibility of proving all the propositions of mathematics logically from definitions alone, even though Leibniz at least was not able to formulate the requisite definitions and proofs even to his own satisfaction.[7] Frege gave an informal argument for logicism in *Grundlagen*—based on the generality of arithmetic (Frege 1884, §14)—that might reasonably convince someone that mathematics is reducible to logic, even if she has no idea how to execute that reduction. Even in *POM*, Russell claims only that rigorous proofs could be given (in a promised second volume, which became *Principia* seven years later)—though he admitted that he did not really know how these proofs would go, since he did not yet have a solution to the paradoxes. An answer to the question "What did Russell need to do to execute his proof of logicism?" would obviously cite Russell's adoption of his new Peano-inspired symbolism (and much else besides). But an answer to the question "What turns did Russell need to make in order to adopt logicism?" need not.

Third, Russell's notion of *generality* in 1901–1903 essentially involves the notion of a variable, which he derived from Peano and did not appear in his writings before August 1900. In *POM,* a proposition is made more general when constants are replaced with variables; completely general propositions are those where all (non-logical) constants are replaced with variables of universal scope (§9). This conception of generality is put to use in 1901b to define logic: "logic may be defined as … the study of what can be said of everything" (1901b, 187). If logicism required the conception of logic as general, and of generality as containing variables of universal scope, then Russell's road to logicism would have required an adoption of Peano's notation. But logicism clearly does not require this, and in any case, this way of characterizing the generality of logic had been dropped by the time Russell writes *POM*.[8]

In fact, the general notion of logicism does not imply any particular views on what logic is. Even in *POM*, Russell variously described logic as "symbolic," "formal," and "general," though there are no sustained analyses of these three notions, nor an argument to show that these three are equivalent. Moreover, Russell's conception of logic continued to shift in the decades after *POM*, even as he continued to maintain logicism. Logicism surely then needs to be independent of any particular conception of logic, and it is possible (though obviously not ideal) to defend logicism even while being unclear about the nature of logic itself. I emphasize this because before 1901 Russell does not seem to identify logic with symbolic logic; in fact, after he abandons transcendental logic in early 1898, he seems not to have attempted until 1901 to develop a determinate conception of logic. But for the reasons I have just given, I do not believe that this disqualifies his view as logicist, even if it is quite obviously an ill-formed version of logicism.

Though I do not think that logicism is committed to any particular conception of logic, I do believe that there is one constraint that logicism puts on the philosophy of logic—if at least logicism is to be even recognizably akin to the position Russell defends in 1901a, 1901b, and 1903. In these writings, Russell characterizes logic as "formal," "general," and "symbolic." "Formal" or "[pure] general" logic was opposed in Kant's *Critique of Pure Reason* to "transcendental" logic.

> General logic abstracts, as we have seen, from all content of cognition, i.e. from any relation of it to the objects, and considers only the logical form in the relation of cognitions to one another, i.e. the form of thinking in general. But now since there are pure as well as empirical intuitions (as the Transcendental Aesthetic proved), a distinction between pure and empirical thinking of objects could also well be found. In this case, there would be a logic in which one did not abstract from all content of cognition; for that logic that contained merely the rules of the pure thinking of an object would exclude all those cognitions that were of empirical content. It would therefore concern the origin of our cognition of objects insofar as that cannot be ascribed to the objects. (Kant 1781/1787, A55-6/B79-80)

Because Russell employs these Kantian terms in *EFG*, *POM*, and the writings from 1898–1899, it will be helpful to get clear on the difference between "pure general," "formal," and "transcendental" logic. "Pure general" logic, according to Kant, gives the rules of thinking that are both a priori and that apply to all thinking whatsoever. "Formal" logic gives the rules that apply to thinking in abstraction from its content. Kant holds that pure general logic is formal logic. He then contrasted this formal logic with transcendental logic, which "concerns the origin of our cognition of objects insofar as that cannot be ascribed to the objects." It gives the conditions for knowing an object. In order to understand Russell's development (and indeed the whole development of logic from Kant through the nineteenth century), it is important to note that, though Kant held that pure general logic is identical to formal logic, many philosophers (including Russell in *EFG*) have held that pure general logic is in fact identical to transcendental logic, not formal logic.[9]

In determining whether a candidate philosophy of mathematics counts as "logicist" (and in what sense), it is necessary to determine whether or not the logic that is meant to ground mathematics is pure general logic, and, if so, whether this pure general logic is taken to be formal or transcendental. I will argue in the following sections that Russell's position on these questions shifts. In *EFG*, geometry is grounded in pure, general logic, which is understood as transcendental logic. In 1898–1899, Russell grounds mathematics in pure, general logic, but no longer thinks of this logic as transcendental. In *POM*, mathematics is grounded in pure, general logic, which is now identified explicitly as "formal" (though Russell's characterization of "formal" logic is not worked out philosophically).

Clearly, when Russell claims in 1901–1903 that pure mathematics follows from "general" or "formal" logic, he is self-consciously rejecting the project of grounding mathematics in the "origin of our cognition of objects insofar as that cannot be ascribed to the objects"—a project that Russell had attempted in *EFG* (see Sect. 2 below). Thus, if the general notion of logicism is to be even recognizably akin to the position Russell defends in 1901a, 1901b, and 1903, I take it that the indefinable concepts, indemonstrable propositions, and rules of inference of pure mathematics must all be grounded in pure, general logic, *understood in a non-transcendental way*.[10] Russell's road to logicism thus required rejecting transcendental logic—at least as part of the origin of mathematics—a rejection that (as we will see in Sect. 3) he carried out in 1898 under the influence of G. E. Moore.

## 2 "Logicism" in *An Essay on the Foundations of Geometry*

In *EFG*, Russell clearly was not a logicist. Nevertheless, I will argue in this section that he did think that pure mathematics could be deduced from "logic." In this way, Russell's road to logicism was much shorter than it might at first appear.

The main argument of *EFG* is a "transcendental proof" (74) of the axioms of projective geometry and the axioms common to all metric geometries of constant curvature and finite dimensions (or "general metric geometry," for short), on the grounds that they are necessary conditions for making empirical judgments about a world of diverse physical objects. This argument can be divided into three steps.[11] First, Russell argues that experience is possible only if there is some "form of externality." Second, he argues that if there is a form of externality, then the axioms of projective geometry and general metric geometry need to be true of it. Third, he argues that these axioms are sufficient to derive all of the theorems of projective geometry and of general metric geometry.

Russell sketches his argument for his first step in the following way:

> [I]n any world in which perception presents us with various things, with discriminated and differentiated contents, there must be, in perception, at least one 'principle of differentiation,' [here Russell cites (Bradley 1883, 63)] an element, that is, by which the things presented are distinguished as various. This element, taken in isolation, and abstracted from the content which it differentiates, we may call the form of externality. (*EFG*, 136)

To say that there is a form of externality, then, amounts to claiming that it is possible to be conscious—through experience and not through inference, and not through any intrinsic differences—that there are numerically distinct things that stand in some relation. In our experience, the form of externality is space, but Russell emphasizes that his transcendental proof establishes only that there is something in experience that performs the function of a form of externality (*EFG*, 179 *et passim*). Following Bradley (1883) and Bosanquet 1888, Russell claims that experience, as empirical knowledge, depends on judgment, which is essentially the "recognition of diversity in relation, or, if we prefer it, of identity in difference" (184). But there could be no route from perception to judgment unless there already exists in perception (prior to any inference, or any recognition of conceptual differences), the consciousness of numerically distinct things standing in relations.

Russell presents his argument as a revision of Kant's Transcendental Aesthetic in light of non-Euclidean geometry, and he concludes that the

> Kantian argument—which was correct, if our reasoning has been sound, in asserting that real diversity, in our actual world, could only be known by the help of space—was only mistaken, so far as its purely logical scope extends, in overlooking the possibility of other forms of externality. (186)

I believe that Russell's conciliatory remarks toward Kant can be highly misleading. In particular, given Russell's later claim that Kant's philosophy of geometry stands or falls with the doctrine that mathematical reasoning is not strictly deductive, but depends on methods of reasoning supplied by a priori intuitions and thus on an intuition of a figure (*POM*, §433), it is tempting to read this doctrine back into *EFG*, as Coffa (1981, 255) does. But Russell in fact asserts that all three steps of his argument are fully deductive:

> I wish to point out that projective geometry is wholly a priori; that it deals with an object whose properties are logically deduced from its definition, not empirically discovered from data; that its definition, again, is founded on the possibility of experiencing diversity in relation, or multiplicity in unity; and that our whole science, therefore, is logically implied in, and deducible from, the possibility of such experience. (146, emphasis added)

> [T]hese three axioms [of metric geometry] can be deduced from the conception of a form of externality, and owe nothing to the evidence of intuition. They are, therefore, like their equivalents the axioms of projective Geometry, a priori, and deducible from the conditions of spatial experience. (149, emphasis added)

Whatever role is played by intuition in *EFG*, then, it is not in deducing the axioms of geometry, nor in demonstrating its theorems on their basis.

Clearly, Russell cannot successfully substantiate his claim that the theorems of projective and general metric geometry can be deduced from axioms that can themselves be deduced from the existence of a form of externality. Both the notion of "form of externality" and his axioms of projective and general metric geometry are beset by vagueness. Russell's argument that his axioms of projective geometry are sufficient to derive all of the requisite theorems is very sketchy and not rigorous (121ff). As Poincaré showed devastatingly in his review of *EFG* (Poincaré 1899), Russell's axioms are certainly not sufficient to prove the theorems of projective geometry, even if they were not hopelessly vague. In fact, until Russell's response to Poincaré (Russell 1899b), which drew heavily on (Whitehead 1898), he did not have a remotely adequate axiomatization of projective geometry. His various axiomatizations of projective, descriptive, and metric geometries in *POM*, which all differed from the axiomatization in (Russell 1899b), depended on a close study of the works of Pasch, Pieri, and Peano, which he had not done before 1900.[12] And, of course, without a developed theory of deduction, such as the one he developed after his encounter with Peano, Russell could not show conclusively that all the theorems of projective and general metric geometry follow from his axiom candidates. Still, though, Russell could (and did) argue that projective and general metric geometry can be deduced from the existence of a form of externality, even if his logic was too underdeveloped to decisively support his conclusion.

Russell is also, surprisingly, insistent that the axioms of projective geometry are "purely intellectual," derivable from "the laws of thought" of "general logic." Describing the axioms of projective geometry, he writes:

> Projective geometry, in so far as it deals only with the properties common to all spaces, will be found, if I am not mistaken, to be wholly a priori, to take nothing from experience, and to have, like Arithmetic, a creature of the pure intellect for its object. (*EFG*, 118)

Later, he takes up Grassmann's project of finding a branch of pure mathematics that would consider extension in a purely intellectual way. Russell argues that projective geometry fits Grassmann's description:

> [W]hat is merely intuitional can change, without upsetting the laws of thought, without making knowledge formally impossible: but what is purely intellectual cannot change, unless the laws of thought should change, and all our knowledge simultaneously collapse. I shall therefore follow Grassmann's distinction in constructing an a priori and purely conceptual form of externality … Projective Geometry, abstractly interpreted, is the science which he foresaw, and deals with a matter which can be constructed by the pure intellect alone. (135)

In this way, projective geometry contrasts with metric geometry, which concerns a new indefinable, "quantity," which is not intellectual but empirical (147).[13] It is not surprising, then, that Russell also characterizes the projective axioms as derivable from "general logic." When discussing the proper notion of the a priori, he writes:

> [W]e can retain the term a priori, for those assumptions, or those postulates, from which alone the possibility of experience follows. Whatever can be deduced from these postulates, without the aid of the matter of experience, will also, of course, be a priori. From the standpoint of general logic, the laws of thought and the categories, with the indispensable conditions of their applicability, will be alone a priori. (60)

Of course, the existence of a form of externality is just such an indispensable condition that general logic recognizes for the applicability of the laws of thought. So its existence, along with the axioms of geometry that it implies, follows from the "laws of thought"—though the existence of space (the particular form of externality found in our world) does not (186).

The project of deducing, purely logically, all of projective geometry from the laws of thought naturally fits with the "logical" characterization of the a priori that Russell holds to in *EFG*.

> My test of a priority will be strictly logical: Would experience be impossible, if a certain axiom or postulate were denied? or: Would experience as to the subject-matter of that science be impossible, without a certain axiom or postulate? My results also, therefore, will be purely logical. (3)

To assert that the axioms of projective geometry are a priori is to assert that they are "logically presupposed in experience" (2)—that they can be deduced purely logically from the fact that experience is possible. Nothing follows from this concerning the subjectivity of the axioms, or their origin in the mind. This contrasts with a faulty "inspectionist" view of the a priori, which Russell finds in Kant:

> Kant would seem to have supposed himself immediately aware, by inspection, that some knowledge was apodeictic, and its subject matter, therefore, a priori: but he did not always deduce its apriority from any further principle. (60)

Not surprisingly, Russell makes clear that he speaks of "form" in a purely logical way as well. To show that some element in knowledge is "formal" does not amount to locating it in some faculty of the mind;

it need not be subjective or grounded in any way in the mind. What is formal is just the "postulates which are required to make knowledge possible at all, and of all that can be deduced from these postulates" (3).

The role for intuition in *EFG* is surprisingly slim. We can deduce, purely logically from the laws of thought, that it must be possible to be conscious *in perception* of numerically distinct things.[14] These numerically distinct things must be objects of perception and not just objects of thought, and their numerical distinction must be perceptible and not just a result of inference (146, 180). In short: from logic and by logic alone we deduce the existence of something that is not itself purely conceptual. But those features of the numerically distinct things that are perceived and not deduced (other than their being different) are in fact irrelevant to pure mathematics (135). Moreover, Russell is indifferent as to whether the numerical diversity that must exist in perception is perceived through pure intuition or merely through sensation. It is a necessary truth that we can so perceive such diversity; the faculty by means of which we do it is an irrelevant matter of psychology (180). It should be obvious, then, that in *EFG* intuition has absolutely no role in providing a priori truths as it were by inspection. The axioms of projective and general metric geometry "owe nothing to the evidence of intuition" (149).

Despite all this, the "logicism" in *EFG* is not proper logicism, since the "general logic" from whose laws projective and general metric geometry are deduced is "transcendental logic." The "laws of thought" express the most general conditions of experience. Russell alludes to Kant's notion of "formal logic" in *EFG* (59) in the course of a criticism of "Kant's logical position" (58–60). Russell's remarks, though sketchy, suggest that he takes Kantian "formal logic" as dependent on the principle of contradiction alone, independent of the possibility of experience. He criticizes this move, on the grounds, that the "law of contradiction, without a given whole or a given hypothesis, is powerless," since no two terms taken in isolation can be contradictories, but are only so in a wider context. For this reason, the project of formal logic, which applies the principle of contradiction in abstraction from all content, is impossible. Russell connects this point to Kant's distinction between analytic and synthetic judgments, which Russell also rejects.[15] As we saw in Russell's argument for the form of externality from the possibility of empirical judgments, Russell in *EFG* views a judgment as the recognition of unity in diversity. For this reason, "[e]very judgment—so modern logic contends—is both synthetic and analytic; it combines parts into whole, and analyzes a whole into parts" (58). Both formal logic and the analytic/synthetic distinction depend on the possibility of applying the principle of contradiction to two terms in isolation from all content. "Modern logic," by which Russell means the idealist logic of Bradley and Bosanquet,[16] shows us that both are impossible. Thus, it would be misleading to say that Russell in *EFG* derives pure geometry from transcendental, *as opposed to* formal logic. Russell derives pure geometry from "general logic," which gives the most general conditions of knowing an object of experience, and is thus identical to transcendental logic.[17] Transcendental logic is not opposed to formal logic, because there is no such thing as formal logic—"modern logic" shows us that transcendental logic is in fact fully general, because it is the only kind of logic there is.

## 3 Logicism in 1898–1899

Given the results of the previous two sections, it is clear what turns Russell had to make on his road to logicism. First, he had to eliminate even the last vestiges of "pure intuition" from his theory of pure mathematics. Second, he had to adopt a new conception of logic that separates pure general logic from the principles of the possibility of experience. Third, he had to develop a new conception of projective geometry grounded in indefinables that are logical in this new, non-transcendental sense. In this section, I argue that Russell made all of these turns in 1898–1899 (and indeed, I believe) by the end of 1898.

In 1898–1899, Russell began and abandoned a series of four book manuscripts on the principles of mathematics: "An Analysis of Mathematical Reasoning" [*AMR*] (April–July 1898), "On the Principles of Arithmetic" (late 1898), "Fundamental Ideas and Axioms of Mathematics" (1899), and "Principles of Mathematics" (August 1899–June 1900). In each of these four manuscripts, Russell discusses "pure number," and in each work, he argues that pure arithmetic owes nothing to pure intuition (e.g., *AMR*, 194–195; Russell 1983, 270). In particular, in "Principles of Arithmetic," he argues that even ordinal numbers depend on "an abstract relation of order" which "abstracts from everything essentially spatial" (Russell 1983, 252). Similarly, in 1899 Russell is adamant that even geometry requires no assistance from pure intuition, since the mathematical study of space and time is in "no need of intuition any more than in Arithmetic" (Russell 1983, 270; cf. Russell 1899b, 409). In the 1899–1900 draft of "Principles of Mathematics," Russell argues that the infinitesimal calculus does not require intuition, and he in fact ridicules appeals to intuition as "that lazy limbo of mystery" (Russell 1993, 106).[18]

Indeed, even in 1898 nothing of pure intuition remained in Russell's theory of pure mathematics but the name. Russell addressed head-on the question "What is an a priori intuition?" in his paper, "Are Euclid's Axioms Empirical?" This paper, completed in August 1898, was a response to Couturat's review of *EFG*, in which Couturat argued that the axioms of specifically Euclidean geometry, which Russell had claimed to be empirical, were in fact knowable by an a priori intuition. On May 12 of that year, Russell wrote to Couturat:

> At the moment I do not know how to uphold the empirical character of the axioms; that depends on the question—perhaps the most difficult in philosophy—as to what it means to have an a priori intuition, and whether such an intuition, supposing that it exists, can have only some of the properties of space. At the moment I am thinking a great deal about such questions, and it would give me much pleasure to reply to you—if that is possible—either by a discussion or by an article in depth; I shall not know which it will be until I begin to write. (Russell 1983, 322–323)

In the published reply that he completed three months later, Russell argued for the a priority of the axioms of projective geometry in the following way:

> Certain mathematical propositions, for instance that, if $A = B$, then $B = A$, or that if $A > B$, then $B < A$, or the axioms concerning order, seem to be necessary and synthetic. … [A]ll these judgments depend upon a

> diversity of logical subjects: they are not restricted to affirming a necessary connection of the contents; they affirm that, if A has an adjective, B must have another, or other more complicated assertions of the same type. In brief, they all depend upon relations which imply material diversity, i.e. a plurality of existent beings. If, then, these judgments are truly necessary, the possibility of several beings is also necessary; and this condition seems satisfied, in our present real world, by space and time. But we cannot say for this reason that space and time are a priori; we can only declare that some form of externality, sufficient for the a priori judgments of Mathematics, is a priori. (Russell 1898b, 334)

The starting point of the argument for the a priority of the projective axioms is the necessity of some other judgments, concerning equality or order. The necessity of these axioms, Russell claims, is just as immediately perceived as the "blue color of the sky" (334). But the necessity of the axioms of geometry is not so perceived; it is a derived necessity, inasmuch as the axioms of projective geometry are provable from the existence of a form of externality, and the existence of the form of externality is provable from the axioms of equality or order.

The conclusion of this argument is reminiscent of the argument in *EFG*: what is a priori is the existence of some form of externality or other, and geometry is a priori only to the extent that it is deducible from the existence of such a form. But the way Russell reaches the conclusion here differs from *EFG* in two significant ways. First, the premise of the argument is not the existence of experience: in fact, the conditions of the possibility of experience are never mentioned. Rather, the necessity of the axioms of order and magnitude, from which Russell derives the form of externality, is a brute fact, which can only be immediately perceived. Russell thus does not attempt to argue that these axioms are necessary *because* they are preconditions of experience. Second, the "form of externality" is now completely abstract. In *EFG,* the existence of a form of externality amounted to the claim that is possible to be conscious *in perception* of numerically distinct things. The references to perception in (Russell 1898b) have all been dropped: the form of externality now amounts simply to "the possibility of some material diversity" (338). To say, then, that mathematics is grounded in "pure intuition" amounts only to claiming that one of the axioms of geometry is "It is possible that there are materially distinct beings": that is, it is possible that there are two numerically distinct things whose diversity is not grounded in a distinction of intrinsic qualities. Whether this numerical diversity can be perceived or sensed is simply not relevant to the argument.

Russell admits that this theory of the a priori is "rational," but argues that there are two senses in which it is "intuitive." First, the argument for the necessity of a form of externality starts with the necessity of the axioms of equality or order—a necessity which is just as immediately perceived as the blueness of the sky is. Second, the axioms of mathematics are not "susceptible of proof by the principle of contradiction" (338). But these admissions amount to much less than they seem. Even in *POM* our knowledge of the logical constants and their relations is just as receptive as it is in the case of immediate perception (*POM*, xv, §37, §124), and the laws of logic themselves (save the principle of contradiction) are synthetic (*POM*, §434; Russell 1900, §11). Thus, on these two issues, the distinction between the "intuitionism" of Russell (1898b) and the logicism of *POM* is merely verbal.

If this is all "pure intuition" amounts to, then there is nothing here that would make a logicist balk—granted of course that the notion of "numerical diversity" is itself a logical notion (which Russell had argued for already in *AMR*, 219).[19] This brings us to the second turn that Russell needed to make on his road to logicism: the adoption of a conception of logic that does not concern the conditions of the possibility of experience. I believe that this turn was made already in early 1898, in *AMR*. Initially, Russell's rhetoric suggested otherwise. In a letter to Couturat describing the project, Russell writes:

> I am asking the question from the Prolegomena, "Wie ist reine Mathematik möglich?" ["How is pure mathematics possible?"] I am preparing a work of which this question could be the title, and in which the results will, I think, be for the most part purely Kantian. (Russell 1983, 157)

In fact, though, *AMR* addresses the presuppositions of pure mathematics in a way that bears little resemblance to Kant's critical project.

> It is the purpose of the present work to discover those conceptions, and those judgments, which are necessarily presupposed in pure mathematics. It is the habit of mathematicians to begin with definitions—to which axioms are sometimes added—and to assume that definitions, in so far as they are relevant, are always possible. It is, however, sufficiently evident that some conceptions, at least, must be indefinable. For a conception can only be defined in terms of other conceptions, and this process, if it is not to be a vicious circle, must end somewhere. In order that it may be possible to use a conception thus left undefined, the conception must carry an unanalyzable and intuitively apprehended meaning. … But besides the fundamental conceptions, we must have fundamental judgments, or axioms, which form the rules of inference—or, in a certain formal sense, the major premisses—of arguments which use the fundamental conceptions. (163)

The task, then, is to identify the indefinables and indemonstrables of pure mathematics—a task that is just as much a part of logicism as it is of transcendental logic. But there is no discussion anywhere in *AMR* of the conditions of the possibility of experience, just as we saw already in the discussion of necessity in (Russell 1898b).

Russell's turn from investigating the conditions of experience to identifying the indefinables and indemonstrables was part of his rejection of transcendentalism, a rejection that he carried out in 1898 under the influence of Moore. The truth of a principle of mathematics never depends on our apprehension of it (*AMR*, 163). The conditions of our knowledge are a priori only when they are also "conditions of truth" (1898b, 333). Even true propositions about space are "true independently of the human mind"—to believe otherwise is to confuse philosophy and psychology (1898b, 336).

In *AMR*, Russell identified the manifold, addition, number, and order as indefinables of pure mathematics (165).[20] Russell calls these indefinables "pure categories." Given Russell's rejection of the transcendental project in *AMR*, this use of Kantian language brings with it no reference to our minds, subjectivity, or the conditions of knowledge. Russell seems to mean only that they are indefinables of general logic, now understood in a non-transcendental way.[21] The argumentative strategy of *AMR* will make this more evident. Russell begins by canvassing different forms of judgments and argues that for each primitive form of judgment, there is a distinct indefinable. He then argues that these indefinables form the subject matter of pure mathematics (167). Consider the forms of judgment (168ff):

This is $P_1$ and $P_2$.

$S_1$ and $S_2$ are *P*.

The first judgment presupposes the prior judgment that the two contents $P_1$ and $P_2$ form a (compound) content, while the latter presupposes the judgment that $S_1$ and $S_2$ together make a new subject. Russell calls these judgments "judgments of intension" and "extension," and they introduce two kinds of identity, diversity, and unity: numerical and conceptual. Both of these kinds of unity depend on the indefinables *manifold* and *addition*, where a manifold is some collection of terms and addition is an operation that takes any two elements of a manifold and returns another element of the manifold. The most general kind of manifold is the manifold of classes or intensions, which Russell thinks is the subject matter of the logical calculus. He derives this claim from Whitehead 1898, who had derived it from Boole and Grassmann.

Another form of judgment that Russell considers (196ff) is

There are *n* Fs.

This judgment depends on the judgments of conceptual identity and numerical diversity (since all the *Fs* share the same content, *F*, but are nevertheless numerically distinct from one another). So it presupposes the manifold, addition, and the indefinables of the logical calculus. But is also contains a new indefinable, the conception of a particular cardinal number (say, 7), and the indefinable concept *cardinal number* (cf. also "On the Principles of Arithmetic," Russell 1983, 256). Numbers, he argues, are qualities of extensions of concepts, or more particularly, they are relations of the whole extension to a unit, where a unit is just any element of the extension. This relation, Russell thinks, is a logical *whole/part* relation. Judgments of pure arithmetic, such as $2 + 3 = 5$, then express necessary connections between these distinct indefinables, which are grounded (in some way not entirely worked out) in the whole/part relations of the extensions of concepts.

The inclusion of "symbolic logic" or the "logical calculus" in Russell's theory of pure mathematics indicates another important event in

Russell's development: his intensive study in the first few months of 1898 of Whitehead's *Treatise on Universal Algebra*. Universal algebra in Whitehead's book concerns various kinds of manifolds, which are distinguished from one another by the properties of their addition operator. (Some manifolds also have a multiplication operator, which takes elements from one or more manifolds and returns elements that may be of a different manifold. Examples are scalar multiplication of vectors or the intersection of classes in the logical calculus.) The most basic kind of manifold Whitehead considered was the logical calculus, though his ambition was encyclopedic, and he included in his book other areas of mathematics, including projective geometry, understood as the study of manifolds defined in terms of their operators. In *AMR*, Russell considers symbolic logic to be a "branch of mathematics" (*AMR*, 190), and he does not identify logic with "symbolic logic." The logical calculus concerns the indefinables that are presupposed in certain primitive kinds of judgments, such as the judgments of extension and intension, and the judgments that a certain term is an element of a class. However, "the vast majority of mathematical judgments, though sometimes capable of this form, are essentially of various other kinds" (167). Russell, like many Boolean logicians in Britain (Heis 2012, §5), sees the logical calculus as a generalization of the syllogism (188). But he does not believe that all reasoning is syllogistic, and he does not believe that all judgments can be put in a form amenable to syllogistic.

In *AMR*, Russell makes no attempt to provide an exhaustive list of forms of judgment. He made some progress in identifying forms of judgment in his January 1899 paper, "The Classification of Relations" (Russell 1899a). This paper is important in Russell's development because he argues here for the first time that there are relations not reducible to identity and diversity (a doctrine that he still held in *AMR*). The theory of relations, he claims, has been the "most faulty part of Logic" (Russell 1899a, 138). This gap in logic has led to an incomplete analysis of the forms of judgments and the "categories," the indefinables of logic:

> In every proposition, some relation is asserted as regards the terms of the proposition. The classification of relations is, therefore, the classification

> of the types of proposition. This fact brings to light the importance of a classification of relations, and the relation of such a classification to Kant's deduction of the categories. (145)

Later in the paper, he argues that there are transitive relations not reducible to identity and diversity, such as whole and part, and greater and less. These are the relations on which mathematics depends. But even in this paper, Russell admits that he does not have an exhaustive list of the forms of judgment and so does not know whether there are other forms of judgment necessary for mathematics. And here still the classification of relations is not symbolic.

Thus, Russell's conception of logic in 1898–1899 was still inchoate. It concerned the forms of judgment, which corresponded to various indefinables. These logical forms and logical indefinables have their being independently of being thought, and their status as logical has nothing to do with our minds and their conditions. I argued in Sect. 1 that logicism requires that the indefinables, indemonstrables, and reasoning of pure mathematics be logical, where the "logical" is pure and general but not "transcendental." I believe that Russell was committed to this position in 1898–1899, even though he was very far from having worked out the details, and even though this logic was not yet identified with "symbolic logic."[22] (The identification of logic with symbolic logic appears for the first time in Russell 1901a, after his exposure to Peano.)

The last step on Russell's road to logicism is the development of a new conception of projective geometry grounded in indefinables that are logical in this new, non-transcendental sense. Soon after reading Whitehead 1898, Russell borrowed from Whitehead the idea of reconceiving projective geometry within the theory of manifolds (see, e.g., "On Quantity and Allied Conceptions," Russell 1983, 134–135, written in early 1898). As in *EFG*, Russell argued that projective geometry is independent of quantity, and he came to believe that it does not concern order or series, nor even addition in Whitehead's sense. In "Notes on Geometry" (1899), he writes

> projective Geometry deals only with multiplication, that lines and planes are not sums of points, and that lines have the same relation to planes as to points. When, in projective Geometry, a point is said to lie on a plane,

> this does not mean, as in metrical Geometry, that it is part of the plane, i.e. that it added to other points make up the plane. It means that the point is the product of the plane with two other planes, or that the plane is the product of the point with two other points. (Russell 1983, 377)

The operation that takes two points and returns a line is multiplication in Whitehead's sense, since the result of the operation (the line) belongs to a different manifold from the elements multiplied (the points). It is not an addition operation, because the points are not parts of the line as the elements of a class are part of the whole class. With this primitive operation of multiplication (which again is a logical indefinable, since multiplication is used already in the logical calculus, for the intersection of classes or disjunction of contents), Russell wants to provide a complete axiomatization of projective geometry.

The only worked-out version of this approach appears in Russell (1899b), which was Russell's response to Poincaré's review of his book. In that review, Poincaré justly criticized Russell's axiomatization of projective geometry in *EFG*, and Russell worked up an axiomatization using a modified version of Whitehead's notation to meet Poincaré's challenge. (See Gandon 2004 for details). What's important for our purposes is that projective geometry under this interpretation becomes the general theory of the reciprocal determination of different subject matters and applies to space or potentially any other subject matter with a similar multiplication operation:

> "The whole assemblage of straight lines being given, the mention of two points will determine one out of all these lines as specially connected with the two points." The connection is like that of a book in a library with its mark in the Catalogue. That the straight line happens to pass through the two points is, I think, irrelevant to projective Geometry. What is essential, and what I omitted duly to emphasize in my *Essay*, is the reciprocal determination of lines and planes by points, and of lines and points by planes. If the propositions of projective geometry be put into symbolic language, they will apply to any subject-matter in which the same kind of reciprocal determination exists. I have not been able to think of any such subject-matter except space, but that is not the important point. (Russell 1899b, 403)

Projective geometry has now been logicized: Its indefinables (multiplication of terms) are logical, being derived from certain fundamental forms of judgment. Its indemonstrable propositions contain only those indefinable terms, and all of its proofs are rigidly deductive. It is independent of any pure intuition, of space, time, or any other non-logical "form of externality," and its truth is in no way grounded in the constitution of our mind. Of course, Russell's execution of the logicist project has a very long way to go, and he has not yet made the identification of logic with symbolic logic. But he has taken all the requisite turns on his surprisingly short road to logicism.[23]

## Notes

1. Russell (1967, 144–145): "The [International Philosophy] Congress was a turning point in my intellectual life, because I there met Peano." See also the 1910 letter to Jourdain (Grattan-Guinness 1977, 133).
2. The dates of the unpublished works, and the dates when published works were completed, are supplied by the editors of Volumes 2 and 3 of Russell's *Collected Papers*: Nicholas Griffin and Albert C. Lewis, and Gregory Moore. As will be obvious throughout this paper, my work on Russell's development crucially depends on their truly extraordinary editorial work.
3. Logical constants for Russell are non-linguistic: they are the simple (i.e., indefinable) constituents of the propositions of logic.

   In *POM*, §9, Russell claims that mathematical premises are "concerned exclusively with logical constants" (§9, §10; cf. 1901b, 187). Nowhere in his writings from 1901–1903 does he consider the possibility that an axiom might contain only logical constants, be true, and yet not be a logical truth (as later he recognizes the axiom of infinity might be). For this reason, Russell's statements of logicism in 1901–1903 specifically mention only that the indefinable concepts and reasoning of mathematics are logical. That is, he effectively takes himself to have shown that an indemonstrable is a logical principle as soon as he has shown that it is unprovable and that it contains only logical constants. (This is significant for my argument, since—as far as I am aware—before August 1900, Russell did not make any attempt to list the fundamental axioms of logic, or indeed of pure mathematics—except for isolated branches of mathematics, such as projective geometry. Even in *POM*, Russell considered it the essential task of a logicist to show that the indefinables of mathematics are logical.)

4. I have not included in my characterization of logicism the requirement that the indemonstrable propositions and proofs of pure mathematics be expressed in a logic that is *symbolic*: that is, in a notation that differs from natural languages. As the quotations from *POM* show, Russell often expressed logicism in terms of "symbolic" logic. However, Russell's considered view even in *POM* is that the symbolic nature of logic is philosophically irrelevant: "The word *symbolic* designates the subject by an accidental characteristic, for the employment of mathematical symbols, here as elsewhere, is merely a theoretically irrelevant convenience" (*POM*, §11).
5. See Gregory Moore's introduction to Russell (1993, xxvii).
6. This observation undermines Griffin's claim that Russell could not have been a logicist in (1898a), since there he held number to be indefinable (Griffin 1991, 274). Griffin (2013) also maintains that Russell could not have been a logicist in October 1900, since he had not yet developed his definition of cardinal number as equinumerous classes. Again, this thesis conflicts with the fact that Russell 1901a is incontestably and stridently logicist, even though it does not contain the definition of cardinal number.
7. On Leibniz's various attempts, see De Risi (2007). On Wolff's failed attempts, see Sutherland (2010).
8. There are good reasons for Russell to drop this notion of the generality of logic in *POM*. There, all variables range over everything (even the variable in *if x is human, then x is mortal*). Furthermore, some of Russell's laws of logic (such as "implication is a relation": *POM*, §4) neither contain variables, nor constants whose extension includes everything (*is a relation* is not true of everything).

   In *POM,* Russell thus holds that the notion *logical constant* is indefinable, and that no account of logic can be given. I agree with Peter Hylton that Russell's theoretical modesty is inevitable, since "no theoretical account of logic or of necessity is available within Platonic Atomism" (Hylton 1990, p. 148; cf. p. 199).
9. Please note that one can consistently believe that there is a science of transcendental logic without being committed to transcendental

idealism: the view that our knowledge extends only to things as they appear, not as they are in themselves. Transcendental idealism follows only when one distinguishes between transcendental and pure general logic in the specific way that Kant does. See note 17 below.

10. There are figures in the history of philosophy who describe their views as "logicism," while conceiving of pure general logic as transcendental and not formal. The Marburg Neo-Kantians Paul Natorp and Ernst Cassirer, for instance, were "transcendental" logicists: see Natorp (1910), Chap. 1. As I am characterizing logicism in this chapter, the Marburg view would not be logicist (even though they followed Russell's *POM* in rejecting a role for intuition in pure mathematics). Readers are free, of course, to use the word in a more permissive sense if they prefer. If so, my question in this chapter would be "What was Russell's road to *non-transcendental* logicism?"
11. These are not the only arguments that Russell makes in *EFG*: he also gives an argument from the possibility of measurement and a regressive argument.

    He argues that if measurement is possible, then the axioms of general metric geometry must be true. However, he believes that ultimately the condition of measurement just is the "qualitative similarity" of the things measured, and that the possibility of qualitative similarity simply follows from the existence of a form of externality. Thus, the argument for the axioms of general metric geometry from the possibility of measurement in the end reduces to the argument from the form of externality. See *EFG*, 147–148.

    His regressive argument is the converse of step two of his main argument: he argues regressively from the truth of the axioms of projective and general metric geometry to the necessity of a form of externality.

    Though a complete discussion of *EFG* would address these as well, my conclusions in this section apply equally to them.
12. On the axiomatizations in *POM*, see Chaps. 1 and 2 in Gandon (2012). On the axiomatization in Russell 1899b, see Gandon (2004).
13. Quantities are what can be greater or less than each other. Russell continued to hold that quantity is not purely intellectual, but empirical, in both his transitional period (1898a, 165) and *POM* (§150).

    Metric geometry, then, is empirical in two senses in *EFG*. First, it concerns an empirical notion, quantity. Second, though all of its axioms concern this empirical indefinable, some of its axioms are only knowable on the basis of experience. These are axioms, such as the three dimensionality of space or the parallel postulate, that are not common to the rival metric geometries. The axioms of general metric geometry, on the other hand, are a priori, despite concerning an empirical indefinable, because they follow from the a priori projective axioms: they are their instantiations, as it were, when the empirical concept of quantity is introduced.

14. The conception of logic in *EFG* is therefore peculiar: Logic itself demonstrates the necessity of there being some non-logical element in our knowledge. For Russell, this point generalizes to all sciences. Each science demonstrates the necessity of some further science that resolves some tension or contradiction that would be present if our knowledge included only the first science. In particular, the science of geometry, without the science of mechanics, is contradictory, since it characterizes points as numerically distinct despite being qualitatively identical (see *EFG*, §§194–199). This dialectical ascent through the sciences has come to be known as the "Tiergarten program," and is sketched in *EFG,* §209.

    The Tiergarten program introduces a dynamical element into logic and illustrates how different Russell's conception of pure general logic is in *EFG* and *POM*. It also illustrates how different the *EFG* view of logic is from Kant's own understanding of transcendental logic. A fuller account of the details of Russell's road to logicism would therefore explain, not only how he came to think of logic in a non-transcendental way, but also how he became convinced that logic and pure mathematics are not inherently contradictory when taken in isolation from their applications in natural science. Luckily, though, I believe that my thesis—that Russell's road to logicism was much shorter than often appreciated—can be proven even without following out Russell's detailed and manifold reflections on the purported antinomies of logic, space, and matter.

15. It is not obvious that Kant's notion of formal logic depends on the viability of the analytic/synthetic distinction. However, many self-styled Kantian formal logicians in Britain connected the two notions in an intimate way. For example, Mansel (1851) argued that formal logic is the science of the form of analytic (but not synthetic) judgments, and that formal logic gives the rules for maintaining consistency in thinking, and so is governed by the principle

of contradiction. Some of the original works of British idealist logic criticize this "Kantian" formal logic in terms just like Russell does in *EFG*. Adamson (1882, §36) criticizes the possibility of formal logic on the grounds that all judgments involve both synthesis and analysis. See also Green (1886, 165, 171).

16. See Bradley (1883), §14, and Bosanquet (1888), 91, on the impossibility of the analytic/synthetic distinction. On the impossibility of formal logic, see Green (1886, 161ff.), Bradley (1883, 496ff.), McTaggart (1896, §25).
17. The Russell of *EFG*, in identifying transcendental and pure general logic, differs from Kant, who distinguishes them. As I discussed at the end of Sect. 1 above, Kant's definitions of the two kinds of logic leave open the possibility that they are identical. The conditions of the possibility of knowing an object (the rules of transcendental logic) might turn out to be the most general a priori conditions of thinking (the rules of pure general logic).

    In identifying transcendental and pure general logic, Russell is thus joining a venerable tradition of post-Kantian idealism. Russell's contemporary McTaggart, for instance, argued that a science of pure thinking (i.e., pure general logic), in isolation from experience (i.e., as distinct from transcendental logic), would be "absolutely sterile, or rather impossible," since it is contrary to the very nature of thought that "a stage of thought could be conceived as existing, in which it was self-subsistent, and in which it had no reference to any data" (1896, §14). Thus, the conditions for knowing an object of experience just are the most general conditions of thinking. This view precludes transcendental idealism, since transcendental idealism requires that we be able to *think* a thing (namely, a thing-in-itself) without thereby *knowing an object of experience* (namely, an appearance). See note 10 above. On the denial of the distinction between things-in-themselves and appearances, see McTaggart (1896, §25).
18. Based on passages like this one, Griffin argues that Russell's abandonment of intuition was driven by his exposure to pathological examples in analysis such as the Weierstrass curve (Grifffin 1991, 99). I agree that Russell's appreciation of these examples may well explain the escalation of his rhetoric in 1899–1900. But given the very slim role for "pure intuition" even in *EFG*, and my contention that it has ceased to play any substantive role already in 1898, I am skeptical that it was Russell's

reflections on these examples from analysis that led him to eliminate pure intuition from his philosophy of mathematics. Moreover, these examples make special problems only for "inspectionist" models of pure intuition. But Russell had rejected these even in *EFG*.

19. The situation in *AMR*, which was written before (Russell 1898b) in April–July, is more ambiguous. In the introduction, he argues that between the indefinables of sense (such as *blue*) and the indefinables that are not derived from sense or from anything analogous to space and time, which he calls "pure categories," there is a third class, which he calls "categories of intuition." The "categories of intuition" "involve reference to a non-conceptual datum only found, in experience, within space or time" (165).

    However, the parts of *AMR* that Russell completed and that still survive do not make clear what these categories of intuition are, or what their role is. Both the logical calculus and the cardinal and ordinal numbers depend only on the pure categories, and the only extant portion of *AMR* that Russell says concerns subject matter dependent on these "categories of intuition" is the chapter on quantity (no chapter on geometry survives, if one ever existed). But Russell claims that quantity is *not* a part of pure mathematics at all (165)—a view that was a constant of Russell's from *EFG* to *POM*. Moreover, the claim that Russell in 1898b says depends on a "pure intuition"—the possibility of numerical diversity not grounded in diversity of content—is in *AMR* identified as a precondition of even the theory of assemblages (186), which is a subject of the logical calculus understood extensionally (188), and of cardinal number (198). But both of these sciences are based on pure categories alone.

    My best guess, then, is that "categories of space and time," inasmuch as they amount to more than the possibility of material diversity, concern quantities, which are the subject matter of the empirical general theory of quantities, and other empirical theories such as metric geometry. Again, that the theory of quantities and metric geometry were empirical was Russell's position in *EFG* and even in *POM*. It is clear, further, that one of Russell's goals in *AMR* (just as in *POM*) was to discuss the *applications* of pure mathematics. He intended to consider the "general nature of any subject-matter that can be dealt with mathematically" (*AMR*, 165). It is of course consistent with logicism to claim that space and time are needed when pure mathematics is *applied*.

20. He also identified quantity, the extensive continuum, and the idea of a thing as indefinables. I take it that these are not indefinables of pure mathematics, but are indefinables of applied mathematics. (See the previous note.)
21. Griffin and Lewis, in their commentary on *AMR* (Russell 1983, 160), argue that it is not a logicist work, since the indefinables are not yet identified as logical. However, all of the indefinables of pure mathematics are identified as "categories" (which in *EFG* Russell identified as parts of "general logic"). Given his rejection of transcendentalism, I cannot see what this could mean except that they are logical.
22. On the philosophical irrelevance of logic being "symbolic," see note 4. In *POM*, Russell characterizes logic as "formal." As far as I know, he nowhere in 1898–1899 characterizes logic as "formal." Does this indicate an important change in view between 1898–1899 and *POM*? This question is difficult to answer, because there is no worked-out view of logic in 1898–1899, and the "formality" of logic in *POM* is never philosophically elaborated. In *AMR* and Russell (1899a), Russell derives the mathematical indefinables from the "forms" of judgments, which may imply that logic is "formal" in some sense for Russell at that time. But again, it is difficult to find in Russell's use of the word "formal" in *POM* a substantive difference in the conception of logic that would affect whether the view there counts as logicist in a different way from the view in 1898–1898. For this reason, I have emphasized that in both 1898–1899 and *POM*, logic is pure, general, but not transcendental.
23. I would like to thank the audience at a conference on early analytic philosophy held at McMaster University and the participants in the University of California, Irvine Logic Seminar for questions and criticisms. Special thanks to Nicholas Griffin and extra thanks to the editors for their excellent suggestions, apt criticisms, and supererogatory patience.

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