Adjunctions of Higher Categories

Joseph Rennie

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Adjunctions arise very naturally in many diverse situations. Moreover, they are a primary way of studying equivalences of categories. The goal of todays talk is to discuss adjunctions in the context of higher category theories. We will give two definitions of an adjunctions and indicate how they have the same data. From there we will move on to discuss standard facts that hold in adjunctions of higher categories. Most of what we discuss can be found in section 5.2 of [Lu09], particularly subsection 5.2.2.

Adjunctions of Categories

Recall that for ordinary categories $\mathcal{C}$ and $\mathcal{D}$ we defined adjunctions as follows

Definition 1.1. *(Definition Section IV.1 Page 78 [Ml98])* Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say the pair (F, G) is an adjunction and denote it as

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

if for each object $C \in \mathcal{C}$ and $D \in \mathcal{D}$ there exists an equivalence

$$Hom_{\mathcal{D}}(FC, D) \cong Hom_{\mathcal{C}}(C, GD)$$

This definition is not quite satisfying though as the determination of the isomorphisms is not functorial here. So, let us give following equivalent definitions that are more satisfying:

Theorem 1.2. *(Parts of Theorem 2 Section IV.1 Page 81 [Ml98]) A pair (F, G) is an adjunction if and only if one of the following equivalent conditions hold:*

1. *There exists a natural transformation $c : FG \to id_{\mathcal{D}}$ (the counit map) such that the natural map*

$$Hom_{\mathcal{C}}(C, GD) \xrightarrow{F} Hom_{\mathcal{D}}(FC, FGD) \xrightarrow{(c_C)_*} Hom_{\mathcal{D}}(FC, D)$$

is an isomorphism

2. *There exists a natural transformation* $u : id_{\mathcal{C}} \to GF$ *(the unit map) such that the natural map*

$$Hom_{\mathcal{D}}(FC, D) \xrightarrow{G} Hom_{\mathcal{C}}(GFC, GD) \xrightarrow{(u_C)^*} Hom_{\mathcal{C}}(C, GD)$$

is an isomorphism

The way we get unit maps and counit maps is by simply observing that our definition implies we have following equivalences

$$Hom_{\mathcal{D}}(FC, FC) \cong Hom_{\mathcal{C}}(C, GFC)$$

$$Hom_{\mathcal{D}}(FGD, D) \cong Hom_{\mathcal{C}}(GD, GD)$$

and look at the image of the identity map.

Remark 1.3. There are several other equivalent conditions in that theorem that we will not need and so won't discuss.

Our goal is to describe adjunctions of higher categories. We will do so in two ways. One generalizes the approach we have taken for ordinary categories. But before that we will give a definition that actually employs the higher categorical machinery we have developed till now.

(co)Cartesian Fibrations over the Arrow

In [We17] we discussed the notion of a "Cartesian fibration", with the idea that it should model functors into higher categories. Roughly speaking, there is an equivalence

$$\{ \text{ Cartesian Fibrations over } S \ \} \longleftrightarrow \{ \text{ maps from } S^{op} \text{ to higher categories } \}$$

and similarly an equivalence

$$\{ \text{ coCartesian Fibrations over } S \ \} \longleftrightarrow \{ \text{ maps from } S \text{ to higher categories } \}$$

We made this equivalence a little more precise in [Ba17a] for the case of right fibrations and functors into spaces using the straightening construction and left the Cartesian case as an exercise to the reader.

Our goal is to model maps of quasi-categories as a Cartesian fibration. We can think of any map of quasi-categories $f : \mathcal{C} \to \mathcal{D}$ as a functor

$$f : \Delta^1 \to sSet$$

Using the ideas above, this will give us a Cartesian fibration $\mathcal{M}_f \to \Delta[1]$, as we discussed in the last example of [We17]. We use following convention. For a map $\mathcal{M} \to \Delta[n]$, we denote the fiber of $\mathcal{M}$ over the point i as $\mathcal{M}_{/i}$. Combining all these we get an equivalence

$$\{\ \mathcal{M}\colon \mathcal{M} \text{ a Cartesian Fibrations over } \Delta[1]\ \}$$
$$\updownarrow$$
$$\{\ \mathcal{M}_{/1} \to \mathcal{M}_{/0}\ \} \text{ maps of higher categories from the fiber over 1 to the fiber over 0}$$

Using the same argument we get

$$\{\ \mathcal{M}\colon \mathcal{M} \text{ a coCartesian Fibrations over } \Delta[1]\ \}$$
$$\updownarrow$$
$$\{\ \mathcal{M}_{/0} \to \mathcal{M}_{/1}\ \} \text{ maps of higher categories from the fiber over 0 to the fiber over 1}$$

We are now ready to discuss adjunctions of higher categories.

Definition of an Adjunction

We can think of an adjunction as two maps which satisfy some compatibility conditions. Using the notion of Cartesian and coCartesian maps we can make this precise in the following way

Definition 3.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two quasi-Categories. An *adjunction* from $\mathcal{C}$ to $\mathcal{D}$ is a map $p : \mathcal{M} \to \Delta^1$ which is a Cartesian and coCartesian fibration such that $p^{-1}(0) \simeq \mathcal{C}$ and $p^{-1}(1) \simeq \mathcal{D}$.

Remark 3.2. We call the lift that comes from the coCartesian fibration $f : \mathcal{C} \to \mathcal{D}$ (the "left adjoint") and the lift that comes from the Cartesian fibration $g : \mathcal{D} \to \mathcal{C}$ (the "right adjoint"). With this specification we sometimes use following common notation for the data above

$$\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathcal{D}$$

We have following basic facts about adjunctions.

Theorem 3.3. *Let $p : \mathcal{M} \to \Delta^1$ and $q : \mathcal{N} \to \Delta^1$ be two adjunctions such that $p^{-1}(0) \simeq q^{-1}(0)$ and $p^{-1}(1) \simeq q^{-1}(1)$, or, in other words, p and q are adjunctions between the same categories. Then, the left adjoints are equivalent if and only if the right adjoints are equivalent. Thus, for any given left adjoint f, the right adjoint g is unique up to equivalence.*

Theorem 3.4. *Let $p : \mathcal{M} \to \Delta^1$ and $q : \mathcal{N} \to \Delta^1$ be two adjunctions such that $p^{-1}(1) \simeq q^{-1}(0)$ or, in other words, the target of the adjunction p is equivalent to the source of the adjunction q. Then we can build an adjunction $p + q : \mathcal{M} + \mathcal{N} \to \Delta^1$ such that the left adjoint is the composition of the left adjoints of p and q.*

Note also that we have following interesting fact in case we replace our base Δ^1 by some arbitrary simplicial set S.

Remark 3.5. Let $\rho : \mathcal{M} \to S$ be a Cartesian fibration. Then ρ is coCartesian if and only if each 1-cell $g : s \to s'$ in S has a right adjoint.

Thus there is no real point in looking at the definition above in a more broader context as everything is determined by 1-simplices.

Second Definition of an Adjunction

We will now discuss a definition of an adjunction that is more along the lines of what we reviewed in the first section. Recall that one of the definitions of an adjunction involved a unit map that made following composition

$$Hom_{\mathcal{D}}(FC, D) \xrightarrow{G} Hom_{\mathcal{C}}(GFC, GD) \xrightarrow{(u_C)^*} Hom_{\mathcal{C}}(C, GD)$$

into an isomorphism.

We can exactly generalize this definition to the realm of higher categories thusly:

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two quasi-categories and $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be two maps. We say (f, g) is an adjunction if the induced map

$$Map_{\mathcal{D}}(F(c), d) \xrightarrow{G} Map_{\mathcal{D}}(GF(c), G(d)) \xrightarrow{u_c^*} Map_{\mathcal{D}}(c, G(d))$$

is an isomorphism in the homotopy category (or in other words an equivalence).

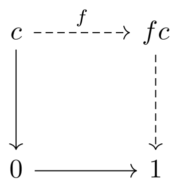
Remark 4.2. Note that this definition does not depend on any model as all models of higher categories have a notion of a mapping space.

Now that we have two definitions we need to know that these two can be reconciled. Fortunately we have following fact:

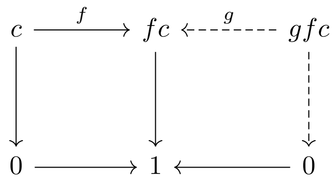
Theorem 4.3. *A map $\rho : \mathcal{M} \to \Delta^1$ is an adjunction if and only if the corresponding functors (f, g) are adjunctions.*

The actual proof of this theorem can be found in Proposition 5.2.2.8 of [Lu09]. But here we give a simple sketch of why something along these lines should be true:

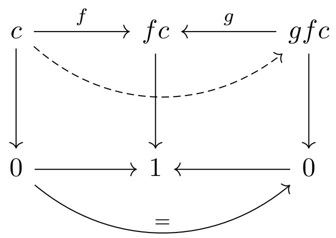
Idea of Proof. Here we will solely indicate how the Cartesian and coCartesian condition enable us to build unit maps, by indicating where it takes objects. Let $p : \mathcal{M} \to \Delta[1]$ be an adjunction from $\mathcal{C}$ to $\mathcal{D}$. Let $c \in \mathcal{C}$ be an object. Then c is in the fiber over 0. We have following picture:



Using the fact that $\mathcal{M}$ is coCartesian we can lift it to a p-coCartesian arrow $f : c \to fc$.
In the next step we look at the following diagram:



where this time we are using the fact that $\mathcal{M}$ is Cartesian and so we can lift our map to a p-Cartesian arrow $g : gfc \to fc$. Finally we have following picture:



where the lift must exist because the map f was chosen to be p-coCartesian map rather than an arbitrary arrow.
So, using the given lifting conditions I was able to construct an arrow:

$$u : c \to gfc$$

using the full power of Cartesian and coCartesian fibrations I could then prove that I get a complete natural transformation. □

Remark 4.4. Notice that similar to the case for categories the existence of units follows by looking at the lift "image" of identity maps.

Adjunctions preserve Colimits

One of the very important facts about adjunctions in ordinary category theory is their preservation of (co)limits. Fortunately, we have the analogeous statement in the realm of higher categories:

Theorem 5.1. *Right adjoints commute with limit cones and left adjoints commute with colimit cocones.*

In [Ba17b] we showed that the colimit construction is functorial, i.e. we have a "colimit functor". Using opposite categories it follows that the same is true for limits. Using the university of colimits we have following result

Theorem 5.2. *Let I be a simplicial set. We have following adjunctions*

$$\mathcal{C}^I \underset{Diag}{\overset{\varinjlim_I}{\rightleftarrows}} \mathcal{C}$$

between the "colimit map" $\varinjlim_I$ and the diagonal map.

Similarly we also have

Theorem 5.3. *Let I be a simplicial set. We have following adjunctions*

$$\mathcal{C} \underset{\varprojlim_I}{\overset{Diag}{\rightleftarrows}} \mathcal{C}^I$$

between the diagonal map and the "limit map" $\varprojlim_I$.

Now combining this we the fact we stated above this gives us following result: This implies following corollary:

Theorem 5.4. *(Co)limit functor preserves (co)limits, thus (co)limits commute with each other.*

Presentable Categories: The Adjoint Functor Theorem

Having discussed the relation between adjoint functors and (co)limits, we might wonder whether the opposite is true: If I know my map preserves colimits (resp. limits) can I then deduce that it has a right adjoint (resp. left adjoint)? Clearly, the general answer has to be no as an adjoint has strictly more structure than just preservation of (co)limits. There must be a structure on the category that allows us to construct the adjoint just from knowing the functor preserves (co)limits. Fortunately, this kind of structure exists and is captured in the notion of a *presentable category.*

Discussing presentable quasi-categories in any detail is a very difficult and tedious task. It takes Jacob Lurie around 200 pages(!) (essentially most of Chapter 5 in [Lu09]) to just give a rigorous treatment. Thus we will completely restrict ourselves to giving couple quick definitions, which allow us to give the main theorem.

Theorem 6.1. *For a small quasi-category $\mathcal{C}$ we define the quasi-category $Ind(\mathcal{C})$ as the quasi-category of ind-objects. It is the smallest sub quasi-category of $Map(\mathcal{C}^{op}, \mathcal{S})$, which is closed under small filtered diagrams.*

Definition 6.2. Let $\mathcal{C}$ be a small quasi-category and let κ be a regular cardinal. Then $Ind_\kappa(\mathcal{C})$ is the full subcategory of $Fun(\mathcal{C}^{op}, \mathcal{S})$, generated by all objects that are colimits of filtered diagrams which have size smaller than κ and is called the quasi-category of *Ind-objects*.

For more details on Ind-objects see section 5.3 and in particular 5.3.5 of [Lu09].

Definition 6.3. We say a quasi-category $\mathcal{C}$ is *accessible* if there exists a small subcategory $\mathcal{C}^0 \hookrightarrow \mathcal{C}$, such that $\mathcal{C}$ is equivalent to $Ind_\kappa(\mathcal{C}^0)$.

Definition 6.4. A quasi-category is presentable if it is accessible and admits small colimits.

Remark 6.5. There is a different way we can describe presentable quasi-categories. It turns out every presentable category is just the localization of a category of the form $Map(\mathcal{C}^{op}, \mathcal{S})$ i.e. a localization of presheaves into spaces.

Now we are in a position to state the adjoint functor theorem for presentable quasi-categories.

Theorem 6.6. *A map $F : \mathcal{C} \to \mathcal{D}$ between presentable quasi-categories is a left (right) adjoint if and only if it commutes with colimits (limits).*

References

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