ORDER THEORETIC EQUATIONS, MAXIMALITY, AND INCOMPLETENESS

by

Harvey M. Friedman*
Distinguished University Professor of Mathematics, Philosophy, and Computer Science Emeritus
Ohio State University

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ABSTRACT. We work with binary relations R on an ambient space X. We first consider the set equations R[A] = A and R[A] = A^c, with known R and unknown A, for arbitrary R ⊆ X^2. We give a necessary and sufficient condition for solvability of the first equation (requiring A ≠ ∅) and sufficient conditions for solvability of the second. We then focus on X = $Q[0,1]^k$, where Q[0,1] = Q ∩ [0,1], with very concrete R. In particular, we assume R ⊆ $Q[0,1]^{2k}$ is order theoretic. We seek to determine the order theoretic R,S such that the set equations R[A] = A, S[A] = A^c have a common solution. By Gödel's completeness theorem, each such common solvability statement is provably equivalent to a universal arithmetic sentence, representing the lowest level of complexity of a mathematical statement involving infinitely many objects. We show that there are specific order theoretic R,S such that ZFC is insufficient to decide whether the equations R[A] = A, S[A] = A^c have a common solution. We conjecture that all such common solvability statements can be decided using some well studied standard large cardinal hypotheses. In fact, we use specific natural order theoretic equivalence relations EQR_k ⊆ $Q[0,1]^{2k}$ for these independence results. In particular, we show that "R[A] = A, S[A] = A^c have a common solution if R = EQR_k and S is a purely order theoretic graph" can be proved using standard large cardinal hypotheses but not in ZFC (Theorem 7.6). We convert these results to maximal independent sets and maximal cliques as "in every purely order theoretic graph on $Q[0,1]^k$, some union of cosets of EQR_k is maximally independent (is a maximal clique)" (Propositions 8.1,8.2). We also convert these results to maximal squares as "in

every purely order theoretic subset of $Q[0,1]^{2k}$, some union of cosets of EQR_{2k} is a maximal square" (Proposition 8.3). Again, these are provable in SRP but not in ZFC (assuming ZFC is consistent). In this way, significant information about the common solvability of equations $R[A] = A$, $S[A] = A^c$ for order theoretic R,S, maximal independent sets cliques in purely order theoretic graphs, and maximal squares in purely order theoretic sets can be obtained using standard large cardinal hypotheses but not in ZFC alone.

1. R[A] = A.

DEFINITION 1.1. A binary relation is a pair (X,R), where $R \subseteq X^2$. $R[A] = \{y: (\exists x \in A)(x R y)\}$ is the forward image of A under R. Whenever we write $R[A]$, it is understood that R is a binary relation coming with an ambient space X, with $R \subseteq X^2$ and $A \subseteq X$.

We consider fixed point equations $R[A] = A$, where R is known, and A is unknown. Solutions A are required to be subsets of the ambient space X.

THEOREM 1.1. The solutions to $R[A] = A$ are the unions of infinite backward chains $\ldots R\ x_{-2}\ R\ x_{-1}\ R\ x_0$, where repetitions are allowed. The union of any set of solutions is a solution, and there is a largest solution. There is a nonempty solution if and only if there is an infinite backward chain (repetitions allowed).

THEOREM 1.2. If R is an equivalence relation then the solutions to $R[A] = A$ are the unions of cosets of R.

DEFINITION 1.2. Let R_u, $u \in I$, be an indexed family of binary relations on X. An infinite backward tree is a function f from finite sequences of elements of I into X such that for all $(x_1,\ldots,x_r,u)$,
$R_u(f(x_1,\ldots,x_r,u),f(x_1,\ldots,x_r))$, $r \geq 0$.

THEOREM 1.3. The common solutions to $R_u[A] = A$, $u \in I$, $R_u \subseteq X^2$, are the unions of ranges of the infinite backward trees. The union of any set of common solutions is a common solution, and there is a largest common solution. There is a nonempty common solution if and only if there is an infinite backward tree.

THEOREM 1.4. Let $R = \{(0,2),(1,2)(0,0),(1,1)\}$ with ambient space $\{0,1,2\}$. $\{0,2\}$ and $\{1,2\}$ are solutions to $R[A] = A$, but their intersection, $\{2\}$, is not a solution to $R[A] = A$.

THEOREM 1.5. Not every family of subsets of Z, closed under arbitrary unions, is the set of solutions to some equation $R[A] = A$, $R \subseteq Z^2$.

Proof: There are 2^c subsets of $\wp(Z)$ closed under arbitrary unions, but only c binary relations on Z. Hence being closed under arbitrary unions is not sufficient for being the solution set to some equation $R[A] = A$, $R \subseteq Z^2$. QED

One should be able to obtain deeper information concerning the possible sets of solutions and sets of common solutions to these equations, for both infinite X and finite X. We will not pursue this here.

2. $R[A] = A^c$.

DEFINITION 2.1. A is R independent if and only if $A \subseteq X$, and for all $x,y \in A$, $x \neg R\ y$. A is maximally R independent if and only if A is R independent and not a proper subset of any R independent set. The reflexive part of R is $\{x: x\ R\ x\}$. The irreflexive part of R is $\{x \in X: x \neg R\ x\}$.

DEFINITION 2.2. The solutions to $R[A] = A^c$ are the A such that $R[A] = X\backslash A$.

Just like $R[A] = A$, we can view $R[A] = A^c$ as a fixed point equation, as it is equivalent to $X\backslash R[A] = A$.

THEOREM 2.1. A is a solution to $R[A] = A^c$ if and only if $A \cup. R[A] = X$ if and only if A is R independent and $A \cup R[A] = X$. If A is a solution to $R[A] = A^c$ then A is maximally R independent. No solution to $R[A] = A^c$ is properly contained in a solution to $R[A] = A^c$. (Here $\cup.$ indicates disjoint union).

Proof: Let R,A be as given. Suppose $R[A] = A^c$. Then $R[A] = X\backslash A$ and so $A \cap R[A] = \emptyset$ and $A \cup R[A] = X$. I.e., $A \cup. R[A] = X$.

Suppose $A \cup. R[A] = X$. Then $A \cap R[A] = \emptyset$, and so A is R independent.

Suppose A is R independent and $A \cup R[A] = X$. By the first, $R[A] \subseteq X\backslash A$. By the second $R[A] \supseteq X\backslash A$. Hence $R[A] = X\backslash A$.

For the second claim, suppose $A \cup \{x\}$ is R independent. Then $x \notin R[A]$, $x \in X\backslash A$, $x \in A$. The third claim follows immediately from the second claim. QED

From Theorem 2.1, we can view solutions to $R[A] = A^c$ as a kind of basis for R. I.e., A is independent and generates X in the sense that X is $A \cup R[A]$. We shall see that there may not be solutions. However, there is an important case where the solutions are exactly the maximal R independent sets. These always exist by Zorn's Lemma.

DEFINITION 2.3. R is a graph if and only if R is irreflexive and symmetric. R is called the edge relation, and X is called the vertex set.

THEOREM 2.2. Let R be a graph. The solutions to $R[A] = A^c$ are exactly the maximally R independent sets. In particular, if R is a graph then $R[A] = A^c$ has a solution. If the ambient space is nonempty then all solutions are nonempty.

Proof: The first claim follows immediately from Theorem 2.3 below. The second claim follows immediately from the first. For the third claim, let $x \in A$. Then $\{x\}$ is R independent, and so can be extended to a maximally R independent set. QED

THEOREM 2.3. Let R be symmetric. The solutions to $R[A] = A^c$ are exactly the maximally R independent sets whose forward image under R includes the reflexive part of R. In particular, if $R[A] = A^c$ has a solution then the forward image of the irreflexive part includes the reflexive part. However, this condition is not sufficient.

Proof: Let R be as given. By Theorem 2.1, we have only to check the equivalence of

$$A \text{ is maximally R independent and } A \cup R[A] = X$$

A is maximally R independent and R[A] includes {x: x R x}

Suppose the former. Let x R x. If $x \in A$ then $x \in R[A]$, $x \notin A$. Hence $x \notin A$, $x \in R[A]$. Hence the latter.

Suppose the latter. Let $x \in X\backslash A$. Then $A \cup \{x\}$ is not R independent. Hence $x R x \vee (\exists y \in A)(y R x) \vee (\exists y \in A)(x R y)$. By irreflexivity, the first is impossible. By symmetry, $x \in R[A]$. Hence $X\backslash A \subseteq R[A]$, and so the former holds.

The second claim is now immediate. For the final claim, let R = {(1,2),(2,1),(1,3),(3,1),(2,4),(4,2),(3,3),(4,4)} with ambient space {1,2,3,4}. The maximally R independent sets are {1} and {2}. The irreflexive part is {1,2}. The reflexive part is {3,4}. Thus the forward image of the irreflexive part includes the reflexive part. On the other hand, R[{1}] and R[{2}] do not include {3,4}. Thus $R[A] = A^c$ has no solution. QED

THEOREM 2.4. Let R be a graph. $R[A] = A^c$ has a unique solution if and only if $R = \emptyset$.

Proof: Let R be a graph, and nonempty. Let x R y. Let B be a maximally R independent set containing x, and C be a maximally R independent set containing y. Then $A \neq B$, and according to Theorem 2.2, B,C are solutions to $R[A] = A^c$. QED

THEOREM 2.5. Let R,S be graphs. Suppose $R[A] = A^c$ and $S[A] = A^c$ have the same solutions. Then R = S.

Proof: Let R,S be as given, and let x R y but $x \neg S y$. Then {x,y} is R independent, and so extends to a solution of $S[A] = A^c$. But no solution to $R[A] = A^c$ contains {x,y}. QED

What can we say about common solutions to $R[A] = A^c$ and $S[A] = A^c$? This appears to be an involved topic even for graphs R,S. We do not pursue this important topic here.

DEFINITION 2.4. R is a partial ordering if and only if R is irreflexive and transitive. x is R minimal if and only if $(\forall y)(y \neg R x)$.

THEOREM 2.6. Let R be a partial ordering. Every solution to $R[A] = A^c$ is the set of R minimal elements. There is a solution to $R[A] = A^c$ if and only if $(\forall y)(\exists x)$(y is R minimal $\vee$ (x R y $\wedge$ x is R minimal)).

Proof: Let R be a partial ordering. Suppose R[A] = X\A. Suppose x R y, y ∈ A. Then x ∉ A, x ∈ R[A]. By transitivity, y ∈ R[A], which is impossible. Hence every y ∈ A is R minimal.

Suppose x is R minimal, x ∉ A. Then x ∈ R[A], which is impossible. Hence every R minimal x lies in A.

The second claim follows from the first and Theorem 2.1. QED

THEOREM 2.7. Let R be reflexive. There is a solution to $R[A] = A^c$ if and only if $X = \emptyset$.

Proof: Let R be reflexive. Suppose $R[A] = A^c$. Then A is R independent, and so $A = \emptyset$, R[A] = X, $X = \emptyset$. QED

3. (PURELY) ORDER THEORETIC SETS.

DEFINITION 3.1. Q is the set of rationals with its usual ordering <. $Q[0,1] = Q \cap [0,1]$.

Henceforth, we use only the ambient spaces $X = Q[0,1]^k$, $k \geq 1$. The presence of endpoints is significant for the proofs of our main results. It is open whether some of our claims hold if we instead use the Q^k, $k \geq 1$. See section 9.

DEFINITION 3.2. The purely order theoretic subsets are the finite Boolean combinations of inequalities $v_i < v_j$, $1 \leq i,j \leq n$, on $Q[0,1]^k$. More formally, they are the subsets of $Q[0,1]^k$ that are defined in terms of variables $v_1,\ldots,v_k$ over Q[0,1], inequalities $v_i < v_j$, and connectives not, and, or $(\neg,\wedge,\vee)$. Thus the purely order theoretic subsets of $Q[0,1]^k$ form a finite Boolean algebra of subsets of $Q[0,1]^k$.

DEFINITION 3.3. The order theoretic subsets of $Q[0,1]^k$ are the finite Boolean combinations of inequalities $v_i < v_j$, $p < v_i$, $v_i < p$, $1 \leq i,j \leq n$, where the $p \in Q[0,1]$ act as constants. More formally, they are the subsets of $Q[0,1]^k$ that are defined in terms of variables $v_1,\ldots,v_k$ over Q[0,1], inequalities $v_i < v_j$, $v_i < p$, $p < v_i$, and connectives $\neg,\wedge,\vee$, where the p's lie in Q[0,1]. Thus the order theoretic subsets of $Q[0,1]^k$ form an infinite Boolean algebra of $Q[0,1]^k$.

THEOREM 3.1. The purely order theoretic subsets of Q[0,1] are just $\emptyset$,Q[0,1]. The order theoretic subsets of Q[0,1] are the finite unions of subintervals of $Q[0,1]$ with rational endpoints.

THEOREM 3.2. Let $E \subseteq Q[0,1]^k$ be order theoretic. There is an inclusion least set C such that E can be defined as in Definition 3.3 using only constants from C.

DEFINITION 3.3. Let $E \subseteq Q[0,1]^k$ be order theoretic. The constants of E are the elements of the least set C such that E can be defined as in Definition 3.2 using only constants from C.

Obviously $E \subseteq Q[0,1]^k$ is purely order theoretic if and only if it is order theoretic with constant set $C = \emptyset$.

THEOREM 3.3. Fix k and finite $C \subseteq Q[0,1]$. There are only finitely many order theoretic subsets of $Q[0,1]^k$ whose constants lie in C. They form a finite Boolean algebra of subsets of $Q[0,1]^k$.

DEFINITION 3.4. $x,y \in Q[0,1]^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$.

Note that order equivalence on $Q[0,1]^k$ is an equivalence relation with finitely many cosets.

THEOREM 3.4. The purely order theoretic subsets of $Q[0,1]^k$ are the finite unions of cosets of order equivalence on $Q[0,1]^k$.

THEOREM 3.5. The purely order theoretic subsets of $Q[0,1]$ are $\emptyset$ and $Q[0,1]$. The order theoretic subsets of $Q[0,1]$ are the finite unions of subintervals of $Q[0,1]$ with rational endpoints. The purely order theoretic subsets of $Q[0,1]^k$ are the subsets of $Q[0,1]^k$ first order definable over $(Q[0,1],<)$ without parameters. The order theoretic subsets of $Q[0,1]^k$ are the subsets of $Q[0,1]^k$ first order definable over $(Q[0,1],<)$.

Several computational complexity issues arise that may be of some interest. E.g., see can algorithmically compare two order theoretic subsets under $\subseteq$, and also algorithmically compute their sets of constants C, in addition to computing unions as in Theorem 3.4.

4. R[A] = A, R ORDER THEORETIC.

THEOREM 4.1. Let $R \subseteq Q[0,1]^{2k}$ be order theoretic with r constants. The following are equivalent.
i. R[A] = A has a nonempty solution.
ii. There exists x_1 R x_2 R ... R x_t, where t = (8kr)!!.
In particular, there is an algorithm for determining whether R[A] = A has a nonempty solution for a given order theoretic R.

Here (8kr)!! is a safe expression that will be reduced in due course.

THEOREM 4.2. The largest solution of R[A] = A, where R is order theoretic, is itself order theoretic with constants among the constants for R, and is algorithmically computable from R.

We can go further with algorithmic procedures here.

THEOREM 4.3. There is an algorithm for determining the largest common solution of $R_1[A] = A,\ldots,R_n[A] = A$, for given order theoretic $R_1,\ldots,R_n \subseteq Q[0,1]^{2k}$, n ≥ 1, s ≥ 0. The largest common solution is order theoretic with constants among the constants for the R's and 0,1, and is computable from the R's.

THEOREM 4.4. There is an algorithm for determining whether every common solution of $R_1[A] = A,\ldots,R_n[A] = A$ is a common solution of $S_1[A] = A,\ldots,S_m[A] = A$, for given order theoretic $R_1,\ldots,R_n,S_1,\ldots,S_m$.

Again, computational complexity issues arise that may be of some interest.

5. R[A] = A^c, R ORDER THEORETIC.

The study of R[A] = A^c, R order theoretic, looks to be much more difficult than that of R[A] = A, R order theoretic.

THEOREM 5.1. There is an algorithm for determining whether a given order theoretic R is reflexive, irreflexive, symmetric, transitive.

Proof: By the well known quantifier elimination for the first order theory of (Q[0,1],<). QED

THEOREM 5.2. If R is irreflexive, symmetric, and order theoretic, then R[A] = A^c has a recursive solution of low computational complexity.

Prima facie, the existence of solutions or common solutions to our equations, with order theoretic R's, is an infinitary statement, since the solution or common solution A is generally a complicated infinite subset of $Q[0,1]^k$. This is despite the fact that R's are given in finitary terms by the definition of order theoretic.

NOTE: This raises the interesting question of algorithmically determining the order theoretic R for which $R[A] = A^c$ has an order theoretic solution. This question can also be raised for $R[A] = A$, and for common solutions to systems of equations. We guess that these questions have relatively straightforward but interesting answers. As usual, there are computational complexity issues to be raised.

It is important to note that there is a general procedure for converting our solvability statements into purely universal combinatorial statements (Π^0_1 sentences). This is the lowest level of logical complexity for a mathematical statement involving infinitely many objects. E.g., FLT is normally stated as a Π^0_1 sentence, but Goldbach's Conjecture is normally stated as a Π^0_2 sentence.

DEFINITION 5.1. A Π^0_1 sentence is a sentence that asserts that a particular Turing machine does not halt at the empty input tape. A Π^0_2 sentence is a sentence that asserts that a particular Turing machine halts at every finite input tape.

These conversions do lose mathematical naturalness and simplicity. We are developing explicitly Π^0_1 sentences of mathematical naturalness and simplicity that are related to such equations. We will present this research elsewhere.

THEOREM 5.3. There is an algorithm which, given order theoretic $R \subseteq Q[0,1]^{2k}$, produces a sentence $\varphi(R)$ in first order predicate calculus with equality, a binary relation symbol $<$, a k-ary relation symbol, and finitely many constant symbols, such that $R[A] = A^c$ has a solution if and only if $\varphi(R)$ has a countable model.

COROLLARY 5.4. There is an algorithm which, given order theoretic R, produces a Π^0_1 sentence $\varphi(R)$ such that $R[A] = A^c$ has a solution if and only if $\varphi(R)$.

Proof: This follows immediately from Theorem 5.3 via Gödel's completeness theorem. The latter tells us that "having a countable model" is the same as "being consistent with the usual axioms and rules of first order predicate calculus with equality".

In fact, there is no difficulty in generalizing Theorem 5.3 and Corollary 5.4 to common solutions for finite sets of our equations.

THEOREM 5.5. There is an algorithm which, given order theoretic $R_1,...,R_n,S_1,...,S_m \subseteq Q[0,1]^{2k}$, produces a sentence $\varphi(R_1,...,R_n,S_1,...,S_m)$ in first order predicate calculus with equality, a binary relation symbol <, n+m k-ary relation symbols, and finitely many constant symbols, such that $R_1[A]= A,...,R_n[A] = A,S_1[A] = A^c,...,S_m[A] = A^c$ have a common solution if and only if $\varphi(R_1,...,R_n,S_1,...,S_m)$ has a countable model.

COROLLARY 5.6. There is an algorithm which, given order theoretic $R_1,...,R_n,S_1,...,S_m$, produces a Π^0_1 sentence $\varphi(R_1,...,R_n,S_1,...,S_m)$ such that $R_1[A] = A,...,R_n[A] = A,S_1[A] = A^c,...,S_m[A] = A^c$ have a common solution if and only if $\varphi(R_1,...,R_n,S_1,...,S_m)$.

Once again, computational complexity issues arise that may be of some interest.

6. ORDER THEOREITC CONJECTURES.

In this section, we formulate a number of conjectures that do not refer to systems of set theory. In the next section, we refine the conjectures using systems based on ZFC and ZFC extended with large cardinal hypotheses.

We start with the least ambitious of our conjectures.

CONJECTURE 1. There is an algorithm for determining whether there is a solution to $R[A] = A^c$, for any given order theoretic R.

We now move to yet more difficult challenges.

CONJECTURE 2. There is an algorithm for determining whether there is a common solution to $R_1[A] = A,...,R_n[A] = A,S_1[A] = A^c,...,S_m[A] = A^c$, for any given order theoretic $R_1,...,R_n,S_1,...,S_m$.

We have seen that the most natural R for R[A] = A are arguably the equivalence relations, and the most natural S for $S[A] = A^c$ are arguably the graphs. In both cases, we have a good understanding of the solutions. In the former case, the unions of cosets of R. In the latter case, the independent sets in S.

So a particularly natural weakened form of Conjecture 2 is

CONJECTURE 3. There is an algorithm for determining whether there is a common solution to $R_1[A] = A,...,R_n[A] = A,S_1[A] = A^c,...,S_m[A] = A^c$, for any given order theoretic equivalence relations $R_1,...,R_n$ and order theoretic graphs $S_1,...,S_m$.

There are a number of additional weakened forms of Conjectures 2,3. Obviously, these will assume special importance if Conjectures 2 or 3 are refuted.

1. Fix the dimension of the R's and S's. We expect the case k = 1 (R's and S's of dimension 2) to be reasonably straightforward. The case k = 2 should be seriously challenging but within reach. Already k = 3 may pose monumental difficulties. These are just guesses, as we have not seriously investigated these cases.

2. Fix n,m. Already fixing n = m = 1 even in Conjecture 3, runs into issues of a totally unexpected nature - see section 7. The case n = 0 and m = 2, even for Conjecture 3, is probably much more difficult than Conjecture 1.

3. Restrict the number of constants of (some of) the relations. Of particular interest is the case where some or all of $R_1,...,R_n,S_1,...,S_m$ are required to have no constants (purely order theoretic). For a more refined study, specify the set of constants for each of the relations.

These restrictions 1-3 are independent of each other. They can and should be combined. In particular, the very preliminary work that we have done has focused on combining restrictions 2,3 on Conjecture 3, with some initial consideration of using all three restrictions 1,2,3 on Conjecture 3. In particular, we have looked at

CONJECTURE 4. There is an algorithm for determining whether there is a common solution to $R[A] = A$ and $S[A] = A^c$, for any given order theoretic equivalence relation R and purely order theoretic graph S.

We have some simple examples of relevant R that work for ALL relevant S (provided, of course, that R,S are of the same dimension). This suggests the following conjecture.

CONJECTURE 5. There is an algorithm for determining, for any given order theoretic equivalence relation R, whether for every purely order theoretic graph S of the same dimension, R[A] = A and S[A] = A^c have a common solution.

Conjecture 5 is also subject to natural restrictions like 1,3 above.

There is an interesting graph theoretic formulation of Conjecture 5.

CONJECTURE 6. There is an algorithm for determining, for any given order theoretic equivalence relation, whether in every purely ordered theoretic graph with the same ambient space, some union of cosets of R is maximally independent.

7. ORDER THEORETIC CONJECTURES AND SET THEORY.

It is natural to expect that the following ZFC form of Conjecture 1 holds.

CONJECTURE 1/ZFC. For any given order theoretic R, it is provable or refutable in ZFC that there is a solution to R[A] = A^c.

Note that if Conjecture 1/ZFC holds then we obtain the algorithm called for by Conjecture 1 by declaring the existence of a solution if we find a proof, and declaring the nonexistence of a solution if we find a refutation. There is a subtlety in this argument, as it requires that ZFC be "sound" in the following sense.

DEFINITION 7.1. A formal system is 1-consistent if and only if for any given TM, if it proves that TM halts at the empty input tape, then TM actually halts at the empty input tape.

THEOREM 7.1. If ZFC is 1-consistent and Conjecture 1/ZFC holds, then Conjecture 1 holds.

Now consider these ZFC forms of other Conjectures from section 6:

A. For any given order theoretic $R_1,...,R_n,S_1,...,S_m$, it is provable or refutable in ZFC that there is a common solution to $R_1[A] = A,...,R_n[A] = A,S_1[A] = A^c,...,S_m[A] = A^c$.

B. For any given order theoretic equivalence relation R and graph S, it is provable or refutable in ZFC that there is a common solution to $R[A] = A$ and $S[A] = A^c$.

C. For any given order theoretic equivalence relation R, it is provable or refutable in ZFC that for all purely order theoretic graphs S of the same dimension, $R[A] = A$ and $S[A] = A^c$ have a common solution.

We have refuted A,B,C, assuming the 1-consistency of SRP (see below). For unprovability in ZFC, consistency of ZFC suffices. For unrefutability in ZFC, the 1-consistency of SRP suffices, and almost certainly, the 1-consistency of ZFC suffices.

We now reinstate these conjectures using the system SRP.

DEFINITION 7.2. Let λ be a limit ordinal. $E \subseteq \lambda$ is stationary if and only if E meets every closed unbounded subset of λ. For $k \geq 1$, λ has the k-SRP if and only if every partition of the unordered k tuples from λ into two pieces has a homogenous set which is stationary in λ.

Here SRP abbreviates "stationary Ramsey property".

DEFINITION 7.3. SRP is the formal system ZFC + $\{(\exists\lambda)(\lambda$ is k-SRP$)\}_k$. SRP^+ is ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-SRP$)$. SRP_k is ZFC + $(\exists\lambda)(\lambda$ is k-SRP$)$. WKL_0 is the second system of Reverse Mathematics. See [WIKI].

CONJECTURE 2/SRP. For any given order theoretic $R_1,...,R_n,S_1,...,S_m$, it is provable or refutable in SRP that there is a common solution to $R_1[A] = A,...,R_n[A] = A,S_1[A] = A^c,...,S_m[A] = A^c$.

CONJECTURE 3/SRP. For any given order theoretic equivalence relation R and graph S, it is provable or refutable in SRP that there is a common solution to $R[A] = A$ and $S[A] = A^c$.

CONJECTURE 4/SRP. For any given order theoretic equivalence relation R, it is provable or refutable in SRP that for all purely order theoretic graphs S of the same dimension, $R[A] = A$ and $S[A] = A^c$ have a common solution.

THEOREM 7.2. Conjectures 2-4/SRP are false if we replace SRP by any SRP_k (assuming SRP is 1-consistent).

We now provide more specifics about our refutations of A,B,C. Note that B is a specialization of A.

THEOREM 7.3. There is an order theoretic equivalence relation R and a purely order theoretic graph S such that SRP proves that there is a common solution to $R[A] = A$ and $S[A] = A^c$, but ZFC does not (assuming ZFC is consistent). The dimension of R,S and the number of constants for R can be taken to be small.

How small? This requires very careful study. We have set the following target: $R,S \subseteq Q[0,1]^8$, R with 4 constants. Thus solutions A are 4 dimensional.

We can be specific about the R's that we use for Theorem 7.2 and the refutation of C.

DEFINITION 7.3. $EQR_k \subseteq Q[0,1]^{2k}$ is the equivalence relation that relates $x,y \in Q[0,1]^k$ if and only if $x = y \vee$ (x,y are strictly increasing and after their initial common part, stay within $\{1,1/2,...,1/k\}$).

THEROEM 7.4. There exists k and a purely order theoretic graph $S \subseteq Q[0,1]^{2k}$ such that SRP proves that there is a common solution to $EQR_k[A] = A$ and $S[A] = A^c$, but ZFC does not (assuming ZFC is consistent). There exists k such that "for all purely order theoretic graphs $S \subseteq Q[0,1]^k$, there is a common solution to $EQR_k[A] = A$ and $S[A] = A^c$" is provable in SRP but not in ZFC.

Once again, k can be taken to be small, with a target of k = 4.

Here is a single statement concerning common solutions to the two equations.

PROPOSITION 7.5. For all purely order theoretic graphs $S \subseteq Q[0,1]^{2k}$, there is a common solution to $EQR_k[A] = A$ and $S[A] = A^c$.

THEOREM 7.6. Proposition 7.3 is provable in SRP^+ but not in SRP. For each k, Proposition 7.5 is provable in SRP. For any k greater than some small number, Proposition 7.5 is not provable in ZFC (assuming ZFC is consistent).

Proposition 7.6 is provably equivalent, over WKL_0, to the consistency of SRP.

8. MAXIMAL CLIQUES AND MAXIMAL SQUARES.

Theorem 7.5 has the following clear graph theoretic formulation (dual forms).

PROPOSITION 8.1. In every purely order theoretic graph on $Q[0,1]^k$, some union of cosets of EQR_k is maximally independent.

PROPOSITION 8.2. In every purely order theoretic graph on $Q[0,1]^k$, some union of cosets of EQR_k is a maximal clique.

We also give the following version without graphs.

DEFINITION 8.1. In $R \subseteq X^2$, a square is a set $E^2 \subseteq R$. A maximal square is a square which is not a proper subset of a square.

PROPOSITION 8.3. In every purely order theoretic subset of $Q[0,1]^{2k}$, some union of cosets of EQR_{2k} is a maximal square.

THEOREM 8.4. Propositions 8.1 - 8.3 are provable in SRP^+ but not in SRP. For each k, Propositions 8.1 - 8.3 are provable in SRP. For any k greater than some small number, none of Propositions 8.1 - 8.3 are provable in ZFC (assuming ZFC is consistent). Propositions 8.1 - 8.3 are provably equivalent, over WKL_0, to the consistency of SRP.

CONJECTURE 5/SRP. For any given order theoretic equivalence relation $R \subseteq Q[0,1]^{2k}$, it is provable or refutable in SRP that for all purely order theoretic graphs S on $Q[0,1]^k$, some union of cosets of R is maximally independent.

CONJECTURE 6/SRP. For any given order theoretic equivalence relation $R \subseteq Q[0,1]^{2k}$, it is provable or refutable in SRP that for all purely order theoretic graphs S on $Q[0,1]^k$, some union of cosets of R is a maximal clique.

CONJECTURE 7/SRP. For any given order theoretic equivalence relation $R \subseteq Q[0,1]^{4k}$, it is provable or refutable in SRP that for all purely order theoretic $S \subseteq Q[0,1]^{2k}$, some union of cosets of R is a maximal square.

Obviously Conjectures 5/SRP and 6/SRP are equivalent by duality.

THEOREM 8.5. Conjectures 5-7/SRP are false if we replace SRP by any SRP_k (assuming SRP is 1-consistent).

These conjectures are within reach.

9. J REPLACING Q[0,1].

DEFINITION 9.1. A rational interval is a $J \subseteq Q$ such that $a < b < c \wedge a,c \in J \rightarrow b \in J$. The endpoints of nonempty rational intervals J are inf(J) and sup(J), which are allowed to be $-\infty,\infty$.

Sections 3-7 are based on the ambient spaces Q[0,1]k. What if we use the ambient spaces Jk, where J is a rational interval? Naturally, constants for order theoretic subsets of Jk are elements of J.

Theorems 3.1 - 3.5, 4.1 - 4.4, 5.1 - 5.6 still hold, and Conjectures 1-6 still stand. Conjecture 1/ZFC still stands.

Conjectures 2-4/ZFC have still been refuted using the J^k, provided J contains at least one of its two distinct endpoints. Conjectures 2-4/SRP still stand. Theorems 7.2, 7.3, 7.4, 7.6, 8.4, 8.5 still hold if J contains at least one of its two distinct endpoints (and EQR_k is adjusted in an obvious way). Conjectures 5-7/SRP still stand.

Thus the only open issue concerns the effect of replacing Q[0,1] by Q. None of our unprovability results work, as they all rely on J containing at least one of its two endpoints.

10. PROOFS.

All missing proofs are reasonably straightforward except the refutations of Conjectures 2-4/ZFC and Theorems 7.2, 7.3, 8.4. The provability in SRP is done almost exactly as in section 9 of [Fr14] (and earlier in section 4 of [Fr11]). The unprovability is essentially done in a rather ponderous way in section 5 of [Fr11]. The section 5 unprovability from [Fr11] has to be substantially upgraded in order to get reasonably sized k for Theorems 7.2, 7.3, 8.4, let alone the target of k = 4 which we are presently attempting.

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[Fr11] H. Friedman, Invariant Maximal Cliques and Incompleteness, Downloadable Manuscripts, #71, October 7, 2011, 132 pages.

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