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S.I.: SUBSTRUCTURAL APPROACHES TO PARADOX

Normalizability, cut eliminability and paradox

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Abstract This is a reply to the considerations advanced by Schroeder-Heister and Tranchini (Ekman's paradox, Unpublished typescript) as *prima facie* problematic for the proof-theoretic criterion of paradoxicality, as originally presented in Tennant (Dialectica 36:265–296, 1982) and subsequently amended in Tennant (Analysis 55:199–207, 1995). Countering these considerations lends new importance to the parallelized forms of elimination rules in natural deduction.

1 Background by way of introduction

Any theory of truth must either *avoid* the logico-semantic paradoxes, by confining itself to semantically *open* languages; or *confront* them, if it makes so bold as to treat of semantically *closed* languages. These paradoxes, loosely understood in broadly *inferentialist* terms, are derivations of absurdity from innocuous-looking first principles, coherent-looking matters of empirical fact, and/or sound-seeming rules governing our *a priori* reasoning. Alternatively, loosely understood in broadly *semantic* or *model-theoretic* terms, paradoxes are sentences or sets of sentences whose evaluations as true or as false appear to be impossible or elusive or unstable—either they are not determined at all, or they are over-determined, or they flip-flop in maddening fashion. By far the greater part of the literature on logical and semantic paradoxes has been occupied with *model-theoretic* means of taming the paradoxes, or at least characterizing them. We have seen a variety of novel extensions of Tarski's paradox-*avoiding* theory of truth for semantically *open* languages, to paradox-*embracing* theories of truth for

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semantically *closed* languages. But inferentialist, or *proof-theoretic*, characterizations of paradoxicality were wanting.

For the benefit of the reader who is not *au fait* with the relevant technicalities, we need to review briefly some of the main ideas in proof theory. The proof-theorist's notion of a *reduction procedure* will be explained in greater detail presently (in Sect. 4). But at this stage we can clarify certain other notions that can be defined in terms of reduction procedures. The result of applying a reduction procedure just once to a proof Π that is eligible for reduction is called a *reduct* of Π. A *reduction sequence* is a sequence of proofs each one of which is succeeded by one of its reducts. A proof in *normal form* is one that is not eligible for the application of any reduction procedure. The *normalization theorem* for any system of natural deduction states that every proof can be transformed (using the reduction procedures but finitely many times) into one in normal form (of the same, or of a possibly strengthened, result). Reduction sequences that eventually produce a proof in normal form *terminate*.

Tennant (1982) proposed a proof-theoretic criterion, or test, for paradoxicality—that of *non-terminating reduction sequences* initated by the 'proofs of $\bot$' associated with the paradoxes in question (p. 271). In that paper, the subsequent focus was on *looping* reduction sequences. These are the proof-theorist's explication of the *vicious circularity* involved in paradoxes. But there are other kinds of non-terminating reduction sequences besides those that enter into loops. One needs to bear this in mind in order to account for the paradox in Yablo (1993). With Yablo's paradox, as was shown in Tennant (1995), the reduction sequence does not so much loop as 'spiral endlessly', ratcheting up a numerical index with each turn. What Yablo's paradox reveals is that we have to contend not only with vicious circles, but also with vicious helices.

Tennant (1982) concentrated on logico-semantic paradoxes, but did also examine Russell's Paradox. Prawitz (1965) had shown how certain naïvely formulated introduction and elimination rules in set theory would—despite the fact that they appeared to admit of a reduction procedure—block the proof that all natural deductions can be brought into normal form. Indeed, the blockage was furnished by an obvious formalization of the reasoning in Russell's Paradox. It is not in normal form, and it cannot be brought into normal form by means of the reduction procedures in question. But Prawitz did not examine the possibility of alternative, non-naïve rules for set-abstraction (in a *free* logic) that might enable one to obtain a normalization theorem. Tennant (1982) examined this prospect in some detail, using the introduction and elimination rules for set-abstraction that were framed and proved complete in Tennant (1978); but found that difficulties still stood in the way of providing a proof in normal form of Russell's Paradox, re-construed now as a proof that the Russell set $\{x|\neg\, x \in x\}$ does not exist.

Schroeder-Heister and Tranchini (Unpublished typescript) have suggested that the conjectural proof-theoretic diagnosis of paradoxicality (which they dub the 'Prawitz–Tennant analysis') mistakenly takes looping reduction sequences (for proofs of $\bot$) as a *sufficient* condition for paradoxicality. They point out that the phenomenon of looping reduction sequences is already manifest in proofs of quite 'ordinary', non-paradoxical cases of inconsistency. They put forward an example taken from Ekman (1998) in order to make this clear. The example in question is the proof of inconsistency of $\{A \rightarrow \neg A, \neg A \rightarrow A\}$. Schroeder-Heister and Tranchini examine the obvious proof of $\bot$ that one would construct in the Gentzen–Prawitz system of natural deduction, using Modus Ponens as the elimination rule for $\to$; and indeed *that* proof is not in normal form. Now, if one tries to normalize the proof, *and* apply, whenever it appears to be called for, the 'compactifying' reduction procedure to be discussed shortly, one finds that the reduction sequence enters a loop.[1] Schroeder-Heister and Tranchini conclude that 'Tennant's test is too coarse, as it induces unmotivated ascriptions of paradoxicality.' They then present a proposal that they argue would solve the problem raised by Ekman's example. This is to insist that admissible reductions should not trivialize identity of proofs.[2] They argue that Ekman's reduction *does* trivialize identity of proofs, and that his would-be counterexample to the proof-theoretic criterion of paradoxicality is therefore ineffectual.

If one does not wish to follow Schroeder-Heister and Tranchini in one's search for a solution to the 'Ekman problem' (on the grounds, say, that the identity of proofs might be less well understood than paradoxes themselves), there is still an intriguing question to address: does the Ekman example provide a good reason to abandon the criterion, for paradoxicality, of non-terminating reduction sequences? Closer reflection will reveal that it does not. Moreover, the resources needed for the solution to be proposed here appear to be more modest than those invoked by Schroeder-Heister and Tranchini.

It will emerge, from the considerations given below, that the 'Ekman problem' for the paradoxicality criterion is an artefact of the choice of *serial forms of elimination rules* in natural deduction—in particular, the serial form of ($\to$-E), or *Modus Ponens*:

$$\frac{\varphi \to \psi \qquad \varphi}{\psi}$$

The problem *simply does not arise* if one insists on the use, instead, of the *parallelized form of elimination rule*.[3] The parallelized form of ($\to$-E) is

$$\cfrac{\varphi \to \psi \qquad \begin{array}{c}\Delta\\ \vdots\\ \varphi\end{array} \qquad \begin{array}{c}\overset{\square\text{—}(i)}{\underbrace{\Gamma\ ,\ \psi}}\\ \vdots\\ \theta\end{array}}{\theta}\,(i)$$

[1] Note that the compactifying reduction procedure is not considered by Prawitz. This is why he was able to prove the normalization theorem for intuitionistic logic, despite basing it on the *serial* forms of elimination for $\wedge$, $\to$ and $\forall$. Prawitz was considering as eligible reduction procedures only those provided for the logical operators' introduction-elimination rule-pairs. These reduction procedures are designed to eliminate 'maximal sentence occurrences'—ones standing as conclusions of introductions, and as major premises of the corresponding eliminations.

[2] Space does not permit any further explanation here of the details of Schroeder-Heister's and Tranchini's positive proposal; the interested reader in search of such details is referred to their text. My concern here is to confront the problem they have raised, and to propose a more effective and simpler solution to it.

[3] These are also known as 'generalized' elimination rules. They were first introduced in Schroeder-Heister (1984).

The box next to the discharge stroke over the 'assumption for discharge' in the major subproof indicates that vacuous discharge is not allowed. That is to say, there really must be an assumption of the indicated form in the subordinate proof in question, available to be discharged upon application of the rule.

The parallelized forms of elimination for all the connectives were used in Tennant (1992), in order to make natural deductions in normal form isomorphic to cut-free, thinning-free sequent proofs. The proofs using them were called *hybrid* proofs in that book, and proved themselves to be ideal objects of *automated proof-search*. This was because they combined the economy of macro-structure of proof-trees to be found in the sequent calculus with the economy of sentence-labeling (rather than sequent-labeling) of nodes within the trees, which is the economical feature of natural deduction. I prefer now to call hybrid proofs *core proofs* [See, for example, Tennant (2014)].[4]

When one re-visits the problem of finding a normal-form proof of Russell's Paradox, it turns out that this problem, too, disappears upon the adoption of suitably parallelized forms of the elimination rules for set-abstraction.

2 Serial versus parallelized elimination rules in natural deduction

The Gentzen–Prawitz system allows one to construct natural deductions in which the conclusion of an application of an introduction rule may stand as the major premise of an application of the corresponding elimination rule. Such sentence-occurrences are called *maximal*, and one can get rid of them by applying the well-known reduction procedures. Prawitz's normalization theorem is to the effect that any proof (of, say, the sequent $\Delta : \varphi$) can be turned into one in normal form, establishing some subsequent of $\Delta' : \varphi$ ($\Delta' \subseteq \Delta$) as its result.

Maximal sentence occurrences in proofs represent an undesirable and avoidable prolixity. Another kind of prolixity, which one should be just as anxious to avoid, is that which is evident in the situation that one can call *subproof compactification*. This is when a proof contains a subproof Σ such that Σ proves $\Delta : \varphi$ (say) *and* Σ contains a *proper* subproof (call it Σ') of some subsequent of $\Delta : \varphi$.[5] Surely, one might think, one should be able to make do with just Σ' in place of Σ, and avoid having the 'extra fluff' that Σ has built up around Σ'? To this end, Ekman had a further, 'compactifying' reduction procedure, aiming to do just that—at least, in certain straightforward cases. Ekman's reduction procedure is exactly like the one motivated in Tennant (1982) at p. 270 *infra*, and called 'shrinking' in its applications at pp. 286ff. We shall not need to examine the possible scope of these further reduction procedures here. Suffice it to say that the shrinking reduction was used in the analysis of all the paradoxes covered in

4 The sequent calculus in question is a *single-conclusion* one (succedents of sequents being at most singletons), whose sequent-antecedents are *sets* of sentences, not sequences of sentence-occurrences. Thus the Gentzenian structural rules of Contraction and Interchange (or Permutation) are irrelevant, because unnecessary; and modifications of such rules, in the manner of Zardini (2011), for example, are orthogonal to the discussion undertaken here.

5 A note on notation: we shall use Π, Σ, Ω and Ξ for proofs, and use Δ and Γ for sets of premises.

Tennant (1982), and particularly in the analysis of Russell's Paradox in the non-naïve case discussed above.

Regarding as always eligible for application both the standard Gentzen–Prawitz reduction procedures and the aforementioned additional reduction procedure to get rid of subproof compactification, Schroeder-Heister and Tranchini proceed to examine the reduction sequence that would ensue from the obvious proof in the *Gentzen–Prawitz* system of natural deduction (using the *serial* form of →-Elimination) that establishes

$$A \to \neg A, \neg A \to A : \bot.$$

They demonstrate that the reduction sequence loops. Then, since something like the premise-pair of this example is involved in the proof of $\bot$ in the case of the Liar Paradox, they conclude that the proof-theoretic criterion of paradoxicality is looking for paradoxicality in the wrong place.

In order to call into question the cogency of these considerations, the reader is invited to consider not the proof of $A \to \neg A, \neg A \to A : \bot$ examined by Schroeder-Heister and Tranchini, but rather the obvious proof (to be given presently) of the same result in the natural-deduction system for Core Logic, in which →-Elimination takes the *parallelized* form explained above. Note that parallelized (→-E) is equiform with the sequent rule

$$(\to :) \qquad \frac{\Delta : \varphi \qquad \Gamma, \psi : \theta}{\Delta, \Gamma, \varphi \to \psi : \theta}$$

The reader who might have been unfamiliar with the parallelized form of (→-E) is now in a position to understand proofs that employ it.

$$\text{Let } \begin{matrix}\underbrace{A \to \neg A,\ \neg A \to A}\\ \Pi \\ A\end{matrix} \quad =_{df} \quad \dfrac{\neg A \to A \qquad \dfrac{\dfrac{A \to \neg A \qquad {}^{(2)}\overline{A} \qquad \dfrac{\overline{\neg A}^{(1)} \quad \overline{A}^{(2)}}{\bot}}{\bot}(1)}{\neg A}(2) \qquad \overline{A}^{(3)}}{A}(3)$$

Then the following is the obvious proof foreshadowed earlier, in the parallelized system of natural deduction for Core Logic, of the inconsistency of $\{A \to \neg A, \neg A \to A\}$:[6]

[6] A similar proof, of $\neg((A \to \neg A) \wedge (\neg A \to A))$, using parallelized elimination rules, but using the definition of $\neg\varphi$ as $\varphi \to \bot$, was given by von Plato (2000), at p. 123, by way of solution of Ekman's problem for the serial forms of elimination rules. Note that all applications of (→E) in the proofs that we are giving here are applications of the *parallelized* form of that rule, even if some of those applications involve degenerate (i.e., single-sentence) major subproofs.

$$\dfrac{A\to\neg A \qquad \begin{array}{c}\underbrace{A\to\neg A,\ \neg A\to A}\\ \Pi\\ A\end{array} \qquad \dfrac{{}^{(4)}\overline{\neg A} \qquad \begin{array}{c}\underbrace{A\to\neg A,\ \neg A\to A}\\ \Pi\\ A\end{array}}{\bot}{}^{(4)}}{\bot}$$

This proof, like any Core proof, is in normal form. Moreover, it contains no subproof Σ such that Σ both proves $\Delta : \varphi$ (say) and contains a subproof of some subsequent of $\Delta : \varphi$. That is to say, there is no subproof compactification to worry about. Hence Ekman's so-called 'paradox' is no paradox at all. The inconsistency of $\{A \to \neg A, \neg A \to A\}$ has a perfectly straightforward proof in normal form, calling for no reduction whatsoever in order to get rid of prolixities of any kind at all. With the parallelized form of $\to$-Elimination, as we have just seen, *there is no looping* in the resulting proof of Ekman's example. This is because it is already in normal form, so there is no reduction sequence to be embarked upon.

That it might have been thought otherwise (i.e., that Ekman's example would resist any normal-form proof) is an artefact of the mistaken presumption that a system of natural deduction ought to use the *serial* form of $\to$-Elimination (Modus Ponens) rather than the parallelized form used above.

This issue of serial *versus* parallelized forms of elimination rules is no minor matter. Note that in the original system of natural deduction in Gentzen (1934, 1935), and in the treatment of it in Prawitz (1965), the operators $\wedge$, $\to$ and $\forall$ had their elimination rules stated in *serial* form. (This could not be done, of course, for either $\vee$ or $\exists$; *their* elimination rules, accordingly, were in parallelized form.) Moreover, in Gentzen's original system of sequent-proof for *classical* logic he permitted *multiple succedents*. (The system of Classical Core Logic, by contrast, like that of Curry's system LD, permits at most one sentence in a succedent. See Curry (1952) and Tennant (2015b).)

These fateful decisions on Gentzen's part led to the vexatiously complicated transformations that were needed in order to convert a natural deduction establishing $\Delta : \varphi$ into a sequent-calculus proof of the same result, and conversely. In classical logic, both of the aforementioned features contribute to this complication. In intuitionistic logic, where succedents are at most singletons, only the first feature is at work, but it still causes complications enough. In large measure the troubled relationship between the system of natural deduction and the sequent calculus is owing to the *lack of uniformity* in the formulation of the *elimination* rules in natural deduction. The Gentzen–Prawitz elimination rules for $\neg$, $\vee$ and $\exists$ are in parallelized form; whereas those for $\wedge$, $\to$ and $\forall$ are in *serial* form.

As soon as one insists (i) that *all* elimination rules in natural deduction be *uniformly* stated in *parallelized* form, *and* (ii) that their major premises should *stand proud*, with no proof-work above them, one ensures a gratifying simplification of the formerly troubled relationship between proofs in the system of natural deduction, and corresponding proofs in the sequent calculus. Requirements (i) and (ii) together ensure that any natural deduction establishing $\Delta : \varphi$ is essentially *isomorphic* to the corresponding (cut-free, thinning-free) sequent proof of the same result—provided, of course that in the classical case one insists still on 'single conclusion' sequents and adopts strictly classical rules such as Dilemma or Classical Reductio, as explained in Tennant (2012).

There is yet another advantage to be enjoyed from creating this happy confluence between natural deductions and sequent proofs. The task of *automating deduction* is afforded great savings in efficiency. Normalization theorems can contribute highly exigent constraints on proof-search (without loss of completeness). The proofs for which an automated deducer is searching has the tree-structure *common* to natural deduction and sequent proofs; but, as already pointed out, the *nodes* in that tree can be labeled just by *sentences*, as in natural deduction; they do not need to be labeled by whole *sequents*, as would be the case if one were to work entirely within the sequent calculus. These advantages were essayed upon in Tennant (1992).

Within the system of normal proofs afforded by the combination of requirements (i) and (ii), one can also prove that CUT *can be eschewed*, in the sense of the following metatheorem.

Metatheorem 1 (Cut Elimination for Core Proof) *There is an effective method* [,] *that transforms any two core proofs*

$$\begin{array}{lll} \Delta & \varphi, \Gamma & \\ \Pi & \quad\Sigma & \quad\textit{(where } \varphi \notin \Gamma \textit{ and } \Gamma \textit{ may be empty)} \\ \varphi & \quad\theta & \end{array}$$

into a core proof $[\Pi, \Sigma]$ of θ or of $\bot$ *from (some subset of)* $\Delta \cup \Gamma$.

Proof See Tennant (2012) for the proof of this result for Core Logic, and Tennant (2015b) for its proof for Classical Core Logic.

It is instructive to see the Core sequent proof of Ekman's example, corresponding to the Core natural deduction given above. The following sequent proof will be denoted by $\Pi[A]$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A : A \quad \dfrac{A : A}{\neg A, A :}}{A\to\neg A, A :}}{A\to\neg A : \neg A} \quad A : A}{\neg A\to A, A\to\neg A : A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{A : A \quad \dfrac{A : A}{\neg A, A :}}{A\to\neg A, A :}}{A\to\neg A : \neg A} \quad A : A}{\neg A\to A, A\to\neg A : A}}{\neg A\to A, A\to\neg A, \neg A :}}{\neg A\to A, A\to\neg A :}$$

Note that $\Pi[A]$ contains no cuts or thinnings; it uses only the right- and left-rules for $\neg$ and the left-rule for $\to$; and no sequent σ within it is a subsequent of any sequent below σ, on a branch containing σ.

Ekman's example can now be seen to qualify no longer as a 'false positive' embarrassing the proof-theoretic criterion for paradoxicality. Although its proof in a system of natural deduction with *serial* $\to$-Elimination initiates a looping reduction sequence, the same is not true of its proof in the system with *parallelized* $\to$-Elimination. The form of elimination rule matters; and we must be careful from now on to frame the criterion for paradoxicality with reference only to the system of parallelized natural deduction.

On reflection, it is only right that there should be a straightforward proof of inconsistency of $\{A \to \neg A, \neg A \to A\}$, for otherwise one would be condemned to regarding as paradoxical the straightforward logical truth that no one shaves all and only those who do not shave themselves. Ekman's sequent has the parametric instance $\{Saa \to \neg Saa, \neg Saa \to Saa\}$, from which the desired result

$$\emptyset : \neg\exists x \forall y(Syx \leftrightarrow \neg Syy)$$

would be obtained as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi[Saa]}{Saa \to \neg Saa, \neg Saa \to Saa : \bot}}{Saa \leftrightarrow \neg Saa : \bot}}{\forall y(Sya \leftrightarrow \neg Syy) : \bot}}{\exists x \forall y(Syx \leftrightarrow \neg Syy) : \bot}}{: \neg\exists x \forall y(Syx \leftrightarrow \neg Syy)}$$

If no Core proof like $\Pi[A]$ were available, one would be in a terrible fix. But of course one is not, as has been shown above. □

3 Re-visiting Russell's Paradox

The Core proof just given, showing that no one shaves all and only those who do not shave themselves, can be adapted to show that no set has as members all and only those sets that do not have themselves as members:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi[a \in a]}{a \in a \to \neg a \in a, \neg a \in a \to a \in a : \bot}}{a \in a \leftrightarrow \neg a \in a : \bot}}{\forall y(y \in a \leftrightarrow \neg y \in y) : \bot}}{\exists x \forall y(y \in x \leftrightarrow \neg y \in y) : \bot}}{: \neg\exists x \forall y(y \in x \leftrightarrow \neg y \in y)}$$

The conclusion of this proof exactly captures the thought that there is no set of all and only those sets that do not contain themselves. This is Russell's Paradox made unparadoxical (in the sense that occupies us here). The would-be paradox has been converted into a straightforward proof of the negative existential theorem $\neg\exists x \forall y(y \in x \leftrightarrow \neg y \in y)$.[7]

[7] With an eye to the Paradox's historical (i.e., Fregean) origins, we see also (at second order) that Naïve Abstraction is refutable:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Pi[a \in a]}{a \in a \to \neg a \in a, \neg a \in a \to a \in a : \bot}}{a \in a \leftrightarrow \neg a \in a : \bot}}{\forall y(y \in a \leftrightarrow \neg y \in y) : \bot}}{\exists x \forall y(y \in x \leftrightarrow \neg y \in y) : \bot}}{\forall \Phi \exists x \forall y(y \in x \leftrightarrow \Phi y) : \bot}$$

Understandably, one would expect the same to hold when the proof is of the different, but theoretically equivalent, conclusion $\neg\exists y\, y = \{x|x \notin x\}$, where the logically primitive *term-forming operator of set abstraction* is applied to the open sentence $\neg\, x \in x$ in order to form the singular term $\{x|\neg\, x \in x\}$. The expectation in question would be that in the free logic of set-terms, with either

(i) suitable natural-deduction rules ({ }-I) and ({ }-E) for the introduction and elimination of the set-abstraction operator, or
(ii) suitable sequent rules ({ }:) and (:{ }) for the introduction of the set-abstraction operator on the left or on the right of the colon,

there should be a Core disproof of the *reductio* assumption $\exists y\, y = \{x|x \notin x\}$.[8]

In Tennant (1982) the alternative (i) was investigated. Use was made of the introduction and elimination rules for the set-abstraction operator that had been formulated, and supplied with a completeness proof, in Tennant (1978). The rules in question were:

$$(\{\,\})\text{-I}\qquad \dfrac{\begin{matrix}\underbrace{\overline{\varphi^x_a}^{(i)}\ ,\ \overline{\exists! a}^{(i)}} \\ \vdots \\ a \in t\end{matrix}\qquad \exists! t \qquad \begin{matrix}\overline{a \in t}^{(i)} \\ \vdots \\ \varphi^x_a\end{matrix}}{t = \{x|\varphi\}}{\scriptstyle (i)}$$

and, corresponding to the three subordinate proofs for ({ })-I, the three respective elimination rules

$$(\{\,\})\text{-E}\qquad \frac{t = \{x|\varphi\} \quad \varphi^x_u \quad \exists! u}{u \in t} \qquad \frac{t = \{x|\varphi\}}{\exists! t} \qquad \frac{t = \{x|\varphi\} \quad u \in t}{\varphi^x_u},$$

of which the middle rule is already covered as a special instance of the Rule of Denotation for free logic, namely

$$\frac{A(\ldots, t, \ldots)}{\exists! t}, \text{ where } A \text{ is a primitive predicate.}$$

The corresponding sequent form of the Rule of Denotation is the additional rule of initial sequents

$$\overline{A(\ldots, t, \ldots) : \exists! t}$$

The investigation in Tennant (1982) of Russell's Paradox on the basis of these rules returned the untoward result that the *reductio* proof for the assumption

[8] Recall that a *free* logic is one that is not based on the standard presupposition that all singular terms denote. Put another way: free logic allows for non-denoting singular terms. If in addition the logic is not based on the standard presupposition that the universe is non-empty, then it is called a *universally* free logic. Such a logic was presented in Tennant (1978), Ch. 7. A completeness proof was also given there for the universally free logic of a first-order language with identity and the familiar primitive variable-binding operator $\{x|\ldots x\ldots\}$ for forming (singular) *set-abstraction terms*.

$\exists y\, y = \{x|x \notin x\}$ initiated a looping reduction sequence—in generating which, the reduction called the *shrinking* reduction was always applicable. The conclusion that I drew at that stage—a conclusion to which this study *demurs*—was the overly pessimistic, because over-hasty, claim that Russell's Paradox had somehow earned its label as a paradox, on the proof-theoretic construal of paradox that was formulated in that paper.

That over-hasty conclusion was in error. This has become evident only thanks to the reflections prompted by the interesting challenge posed by Schroeder-Heister and Tranchini (*loc. cit.*). It turns out that the looping reduction sequence is once again an artefact of the *serial form of the elimination rules for set-abstraction*, and in particular the rules that from now on we shall call { }-E$_1$ and { }-E$_2$:

$$\{\}\text{-E}_1\quad \frac{t = \{x|\varphi\}\quad \varphi^x_u\quad \exists! u}{u \in t} \qquad\qquad \{\}\text{-E}_2\quad \frac{t = \{x|\varphi\}\quad u \in t}{\varphi^x_u}$$

These two rules need to be parallelized, in order to enable the construction of a *normal* disproof for Russell's 'Paradox', thereby depriving it of the status of a genuine paradox. We propose the following parallelized versions, which we shall simply call E$_1$ and E$_2$:

$$\text{E}_1\quad \frac{t = \{x|\varphi\}\quad \varphi^x_u\quad \exists! u\quad \begin{array}{c}\overline{u \in t}^{\,\square\,(i)}\\ \vdots\\ \psi\end{array}}{\psi}{\scriptstyle (i)} \qquad\qquad \text{E}_2\quad \frac{t = \{x|\varphi\}\quad u \in t\quad \begin{array}{c}\overline{\varphi^x_u}^{\,\square\,(i)}\\ \vdots\\ \psi\end{array}}{\psi}{\scriptstyle (i)}$$

For a Core *reductio* of the assumption $\exists y\, y = \{x|\neg x \in x\}$ we shall avail ourselves of the instances of E$_1$ and of E$_2$ where for φ we take $\neg x \in x$; for ψ we take $\bot$; and for both terms t and u we take the parameter a:

$$\text{E}_1\quad \frac{a = \{x|\neg x \in x\}\quad \neg a \in a\quad \exists! a\quad \begin{array}{c}\overline{a \in a}^{\,\square\,(i)}\\ \vdots\\ \bot\end{array}}{\bot}{\scriptstyle (i)} \qquad\qquad \text{E}_2\quad \frac{a = \{x|\neg x \in x\}\quad a \in a\quad \begin{array}{c}\overline{\neg a \in a}^{\,\square\,(i)}\\ \vdots\\ \bot\end{array}}{\bot}{\scriptstyle (i)}$$

Now let $\begin{array}{c}\underbrace{a = \{x|\neg x \in x\}\ ,\ a \in a}\\ \Sigma\\ \bot\end{array}$ be the Core natural deduction

$$(\text{E}_2)\frac{a = \{x|\neg x \in x\}\quad a \in a\quad (\text{E}_1)\dfrac{a = \{x|\neg x \in x\}\quad \overline{\neg a \in a}^{(2)}\quad \dfrac{a = \{x|\neg x \in x\}}{\exists! a}\quad \dfrac{\overline{\neg a \in a}^{(2)}\quad \overline{a \in a}^{(1)}}{\bot}}{\bot}{\scriptstyle (1)}}{\bot}{\scriptstyle (2)}$$

We can now provide the following Core *reductio* of the assumption $a = \{x|\neg x \in x\}$:

$$
(\mathrm{E}_1)\dfrac{a=\{x|\neg x\in x\} \qquad \dfrac{\overset{\displaystyle\underbrace{a=\{x|\neg x\in x\}\ ,\ \overline{a\in a}^{(3)}}}{\overset{\Sigma}{\bot}}}{\neg a\in a}(3) \qquad \dfrac{a=\{x|\neg x\in x\}}{\exists!a} \qquad \overset{\displaystyle\underbrace{a=\{x|\neg x\in x\}\ ,\ \overline{a\in a}^{(4)}}}{\overset{\Sigma}{\bot}}}{\bot}(4)
$$

A final step of (∃-E) then yields the sought Core *reductio* of the existential assumption $\exists y\, y = \{x|\neg x \in x\}$. Inspection will reveal that no shrinking reduction is ever called for.

It is instructive to prove the same result in the sequent calculus for Core Logic, because with sequent proofs it is easier to check that there is no subproof compactification. The two rules E_1 and E_2 become the following left sequent rules:

$$
(\{\,\}:)_1\dfrac{\Delta : \varphi^x_u \qquad \Gamma : \exists!u \qquad \Theta, u\in t : \psi}{t=\{x|\varphi\}, \Delta, \Gamma, \Theta : \psi} \qquad (\{\,\}:)_2\dfrac{\Delta : u\in t \qquad \Gamma, \varphi^x_u : \psi}{t=\{x|\varphi\}, \Delta, \Gamma : \psi}
$$

The foregoing Core *reductio* is rendered in the Core sequent calculus as follows. Let Σ be the following sequent proof of $a = \{x|\neg x \in x\}, a \in a : \bot$.

$$
\Sigma : \quad (\{\,\}:)_2\dfrac{a\in a : a\in a \qquad (\{\,\}:)_1\dfrac{\neg a\in a : \neg a\in a \qquad \overline{a=\{x|\neg x\in x\} : \exists!a} \qquad \dfrac{a\in a : a\in a}{\neg a\in a, a\in a : \bot}}{a=\{x|\neg x\in x\}, \neg a\in a : \bot}}{a=\{x|\neg x\in x\}, a\in a : \bot}
$$

Then use Σ to complete a Core *reductio* (sequent) proof for Russell's Paradox:

$$
\dfrac{\dfrac{(\{\,\}:)_1\dfrac{\dfrac{\overset{\Sigma}{a=\{x|\neg x\in x\}, a\in a : \bot}}{a=\{x|\neg x\in x\} : \neg a\in a} \qquad \overline{a=\{x|\neg x\in x\} : \exists!a} \qquad \overset{\Sigma}{a=\{x|\neg x\in x\}, a\in a : \bot}}{a=\{x|\neg x\in x\} : \bot}}{\exists y\, y=\{x|\neg x\in x\} : \bot}}{: \neg\exists y\, y=\{x|\neg x\in x\}}
$$

That completes our discussion here of Russell's Paradox. We have seen that there are perfectly good proofs, in normal form, of both versions:

$$
\neg\exists x\forall y(y\in x \leftrightarrow \neg y\in y), \text{ and } \neg\exists y\, y=\{x|\neg x\in x\}.
$$

The proof of the former is in normal form, so there are no Ekmanesque problems with it. The proof of the latter, too, is in normal form, thereby revealing that Russell's Paradox is not a genuine paradox at all, in the sense being explicated by Tennant (1982). The explicans on offer there was, rather, for genuinely *logico-semantical* paradoxes, of the kind that arise within semantically closed languages, rather than of any of the so-called 'paradoxes' in the foundations of mathematics. The latter paradoxes, once they are resolved by suitable reformulations of first principles, are thereby 'tamed' as straightforward proofs of negative existentials—proofs that *can* be brought into normal form, if they are not already in normal form.

It remains now to ascertain that these insights, and this re-classification of Russell's Paradox, do not impugn the original proof-theoretic test for paradoxicality in terms of non-terminating reduction sequences. To this end we have space only to examine the Liar Paradox, as the best representative of the class of logico-semantical paradoxes. We shall show that the Liar remains genuinely paradoxical on the amended account on offer, even though that account lays preferential stress on parallelized elimination rules.

4 Rules of truth, and *id est* rules: the Liar is still paradoxical

We state below the rules (T-I) and (T-E) for introducing and eliminating the truth predicate, followed by the reduction procedure for T; and the rules (λ-I) and (λ-E) for introducing and eliminating the Liar sentence λ, followed by the reduction procedure for λ. (λ-I) and (λ-E) are the 'id est' rules for the Liar (so-called because of the familiar transitions 'λ, i.e. $\neg T\lambda$' and '$\neg T\lambda$, i.e. λ'). The rules (λ-I) and (λ-E) ensure that the sentence called λ is *interdeducible with* $\neg T\lambda$—certainly a necessary (even if not sufficient) condition for the former to *be* the latter.[9]

In framing (T-I) and (T-E) we suppress use of corner quotes, and write '$T\varphi$' in place of '$T\ulcorner\varphi\urcorner$'. Note, however, that since 'λ' is intended to be understood as the *name* of the sentence '$\neg T\lambda$', there would never be any occasion to write '$T\ulcorner\lambda\urcorner$'. Indeed, the latter expression is sortally incorrect, for the simple reason that the truth predicate can be satisfied only by sentences, not by their names. Thus with the special instance '$T\lambda$' there are actually no 'implicit' corner quotes that are being suppressed.

After stating each pair of Introduction and Elimination rules, we give the *reduction procedure* that shows that the two rules are in harmony with one another. Here by 'reduction' we mean, in effect, what Prawitz meant. Prawitz intended a reduction to get rid of any unnecessarily complex sentence-occurrence standing both as the conclusion of an application of the Introduction rule for its dominant logical expression, and as the major premise in an application of the corresponding Elimination rule. Natural deductions in the Gentzen–Prawitz form can contain such unnecessarily complex sentence-occurrences because MPEs are not required by Gentzen and Prawitz to stand proud. In Core Logic, by contrast, there is such a requirement. Accordingly, any reduction procedure in Core Logic is aimed at getting rid of a sentence φ that enjoys both an introductory conclusion-occurrence in one proof (call it Π) and an eliminative major-premise occurrence in another proof (call it Σ), in a circumstance where one wishes to know what core proof might result by 'cutting' on φ as an interpolant (a deductive 'halfway house'). Thus φ features as in the statement of Metatheorem 1 above [See Tennant (2012) and Tennant (2015b) for further details].

[9] Here I am indebted to Elia Zardini.

Here, now, are the promised rules for T and λ.

$$(\text{T-I})\quad \dfrac{\begin{array}{c}\Delta\\ \vdots\\ \varphi\end{array}}{T\varphi} \qquad\qquad (\text{T-E})\quad \dfrac{T\varphi \qquad \begin{array}{c}\underbrace{\Delta\ ,\ \overline{\varphi}^{\,(i)}}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle (i)}$$

Reduction procedure for T:

$$\left[\begin{array}{c}\Delta\\ \Pi\\ \dfrac{\varphi}{T\varphi}\end{array}\ ,\quad \dfrac{T\varphi \qquad \begin{array}{c}\underbrace{\Gamma\ ,\ \overline{\varphi}^{\,(i)}}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (i)}\right] =_{df} \left[\Pi\ ,\ \Sigma\right]$$

$$(\lambda\text{-I})\quad \dfrac{\begin{array}{c}\underbrace{\Delta\ ,\ \overline{T\lambda}^{\,(i)}}\\ \vdots\\ \bot\end{array}}{\lambda}{\scriptstyle (i)} \qquad\qquad (\lambda\text{-E})\quad \dfrac{\lambda \qquad \begin{array}{c}\underbrace{\Delta\ ,\ \overline{\neg T\lambda}^{\,(i)}}\\ \vdots\\ \theta\end{array}}{\theta}{\scriptstyle (i)}$$

Reduction procedure for λ:

$$\left[\dfrac{\begin{array}{c}\underbrace{\Delta\ ,\ \overline{T\lambda}^{\,(i)}}\\ \Pi\\ \bot\end{array}}{\lambda}{\scriptstyle (i)}\ ,\quad \dfrac{\lambda \qquad \begin{array}{c}\underbrace{\Gamma\ ,\ \overline{\neg T\lambda}^{\,(j)}}\\ \Sigma\\ \theta\end{array}}{\theta}{\scriptstyle (j)}\right] =_{df} \left[\dfrac{\begin{array}{c}\underbrace{\Delta\ ,\ \overline{T\lambda}^{\,(i)}}\\ \Pi\\ \bot\end{array}}{\neg T\lambda}{\scriptstyle (i)}\ ,\quad \begin{array}{c}\underbrace{\Gamma, \neg T\lambda}\\ \Sigma\\ \theta\end{array}\right]$$

Using the foregoing natural-deduction rules, one can choose to

1. both refute λ and prove λ; or
2. both refute $T\lambda$ and prove $T\lambda$.

(1) offers the shortest way to achieve either of these goals, by constructing the following shortest possible proof of λ and shortest possible refutation of λ.

$$\Omega:\quad \dfrac{(\text{T-E})\dfrac{\overline{T\lambda}^{\,(3)} \qquad (\lambda\text{-E})\dfrac{\overline{\lambda}^{\,(2)} \qquad \dfrac{\overline{\neg T\lambda}^{\,(1)} \quad \overline{T\lambda}^{\,(3)}}{\bot}(\neg\text{-E})}{\bot}(1)}{\bot}(2)}{\lambda}(\lambda\text{-I})\,(3)$$

$$\Xi:\quad (\lambda\text{-E})\dfrac{\lambda \qquad \dfrac{\overline{\neg T\lambda}^{\,(1)} \quad \dfrac{\lambda}{T\lambda}(\text{T-I})}{\bot}(\neg\text{-E})}{\bot}(1)$$

The sequent rules corresponding to the parallelized natural deduction rules for T and for λ are as follows.

$$(:\text{T})\quad \dfrac{\Delta : \varphi}{\Delta : T\varphi} \qquad\qquad (\text{T}:)\quad \dfrac{\Delta, \varphi : \theta}{\Delta, T\varphi : \theta}$$

$$(:\lambda)\quad \dfrac{\Delta, T\lambda : \bot}{\Delta : \lambda} \qquad\qquad (\lambda:)\quad \dfrac{\Delta, \neg T\lambda : \theta}{\Delta, \lambda : \theta}$$

Using these sequent rules, one can give the following sequent-proofs corresponding to Ω and Ξ:

$$\Omega' :\quad \dfrac{(T:)\dfrac{(\lambda:)\dfrac{(\neg:)\dfrac{T\lambda : T\lambda}{\neg T\lambda, T\lambda : \bot}}{\lambda, T\lambda : \bot}}{T\lambda : \bot}}{: \lambda}(:\lambda)$$

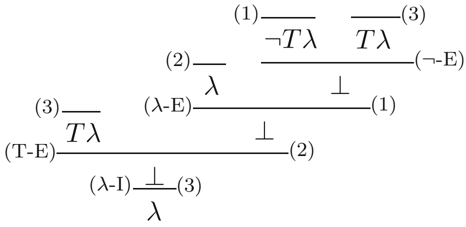
$$\Xi' :\quad (\lambda:)\dfrac{(\neg:)\dfrac{\dfrac{\lambda : \lambda}{\lambda : T\lambda}(:T)}{\neg T\lambda, \lambda : \bot}}{\lambda : \bot}$$

It would be the orthodox expectation that one could now put these two proofs together, ‘cutting on λ’, so as to produce an ‘outright’ sequent proof of $: \bot$ (i.e., the empty sequent), thereby completing the embarrassment that is the Liar Paradox. It would be a further expectation on the part of some that the resulting overall ‘proof of ⊥’ could be normalized, or made cut-free.

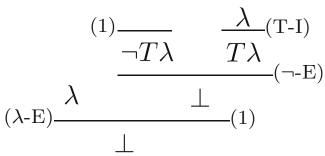
But inspection reveals that the proofs Σ and Ω, despite the reduction procedures just stated for T and λ, cannot be put together to produce a proof, in normal form, of

$\bot$. Let us consider the matter from the perspective of the ordinary natural deduction theorist, who, unlike the Core logician, allows major premises for eliminations to be conclusions of non-trivial proof-work. Copies of the proof Ω of conclusion λ would have to be grafted onto the undischarged assumption-occurrences of λ within the proof Ξ.

Ω :



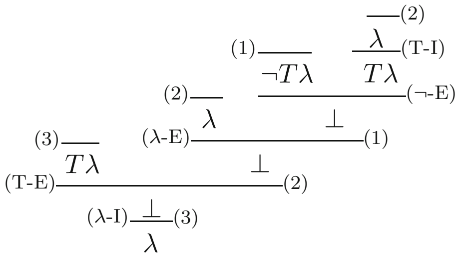
Ξ :



Such grafting would make the leftmost occurrence of λ in Ξ, which stands as a major premise for λ-Elimination, stand also as the conclusion (which it is within Ω) of λ-Introduction. So the reduction procedure for λ would be applicable. The reader can check that the ‘accumulated’ proof

$$\begin{array}{c}\Omega\\(\lambda)\\\Xi\\\bot\end{array}$$

becomes, upon λ-reduction, a proof that calls for an application of $\neg$-reduction ; and that the result of the latter reduction calls for an application of T-reduction … whereupon we are back with the input for the initial λ-reduction. It is easy to verify also that the same pathology is evident when for Ω we use the slightly longer proof



In pursuit of choice (1) above (so I contend—conjecturally, to be sure, but with a high degree of moral certainty) we find that putting together any proof of λ with any refutation of λ results in a looping reduction sequence. The same holds (so I contend) with any attempted pursuit of choice (2): putting together any proof of $T\lambda$ with any refutation of $T\lambda$ results in a looping reduction sequence.

The reduction sequences loop, despite the fact that there is no call for subproof compactification. Even with parallelized elimination rules, the Liar Paradox remains genuinely paradoxical according to our current modification of my earlier proof-theoretic test—unlike Russell's Paradox.

4.1 Digression on cut and transitivity of deduction

If one has two proofs (natural deductions) of the respective forms

$$\begin{array}{c} \Delta \\ \Pi \\ \varphi \end{array} \quad \text{and} \quad \begin{array}{c} \underbrace{\Gamma, \varphi} \\ \Sigma \\ \psi \end{array}$$

how is one to obtain from them the (usually expected) 'target result'

$$\Delta, \Gamma : \psi \quad ?$$

That is to say, how is one to ensure that one may 'perform the cut' that is invited?

The usual answer from the natural-deduction theorist is that one can simply 'accumulate' the proofs Π and Σ, by placing a copy of Π over every undischarged assumption-occurrence of φ within Σ:

$$\begin{array}{c} \Delta \\ \Pi \\ \underbrace{(\varphi)\,, \Gamma} \\ \Sigma \\ \psi \end{array}$$

The usual definition of the notion of proof for the system of natural deduction allows the result of such grafting to count as a proof. In brief: such *accumulations of proofs are always proofs* (even if they happen to fail to be in normal form). This is a signal feature of the Gentzen–Prawitz systems of natural deduction.

The corresponding usual answer from the sequent-calculus theorist is that one can apply the (unrestricted) rule of CUT as a rule *of the system*, thereby obtaining a new proof, in the system, from the two proofs Π and Σ. In the schematic notation of sequent-proofs, this would amount to constructing a sequent proof of the following form:

$$\frac{\begin{matrix}\Pi \\ \Delta : \varphi\end{matrix} \qquad \begin{matrix}\Sigma \\ \Gamma, \varphi : \psi\end{matrix}}{\Delta, \Gamma : \psi}\,(\text{CUT})$$

The usual definition of the notion of proof for the sequent calculus allows the result of such an application of (unrestricted) CUT to count as a proof. In brief: such *unrestricted cuts on proofs always produce proofs* (even if they happen to produce results that fail certain obvious tests for relevance of premises to conclusion).

One of the Holy Grails of proof theory has always been to prove (for whichever system is under consideration) a Gentzenian *Hauptsatz* to the effect that if a given sequent can be proved using CUT, then it can be proved *without* using CUT. That is, all applications of CUT are *eliminable* from sequent proofs. It is strange, indeed, that it should be a central concern to establish the dispensability of a rule on which almost every logician insists. The peculiarity of this predicament is discussed at greater length in Tennant (2016).

Let us re-visit the question posed above, but express it now wholly in terms of sequents:

> if one has two sequent proofs $\begin{matrix}\Pi \\ \Delta : \varphi\end{matrix}$ and $\begin{matrix}\Sigma \\ \Gamma, \varphi : \psi\end{matrix}$, how is one to obtain from them the (usually expected) 'target result'
>
> $$\Delta, \Gamma : \psi \quad ?$$
>
> That is to say, how is one ensure that one may 'perform the cut' that is invited?

In Core Logic, neither CUT nor THINNING are applicable rules of the system of sequent proof. So how is the Core Logician to respond to this very understandable question, in a way that can allay the concern behind it? The concern, of course, is that even with *cut-free* proofs Π and Σ, one might fail to find a *cut-free* proof of the target result (let alone: a cut-free, *thinning-free* proof of it). The Core Logician takes this concern very seriously, and has a perfectly adequate answer to the question posed. There is an effective binary operation on Core proofs, denoted by $\left[\Pi_1, \Pi_2\right]$, such that

$$\left[\begin{matrix}\Pi \\ \Delta : \varphi\end{matrix}, \begin{matrix}\Sigma \\ \Gamma, \varphi : \psi\end{matrix}\right]$$

is a Core proof of *some subsequent of* the target sequent $\Delta, \Gamma : \psi$. This provides all the transitivity of deduction that one could possibly need.

4.2 The inadmissibility of cut in languages containing paradoxes

The orthodox proof-theorist who works with a sequent calculus in which CUT is an applicable rule will not be able to prove cut-elimination for that calculus if the language in question is a semantically closed one in which the Liar can be formulated. Likewise, the Core logician, who does *not* have CUT as a structural rule of his sequent calculus, will not be able to prove that cut is nevertheless admissible for any such language. To see this, we need only note that, *with* CUT, we can prove the 'empty sequent' $\emptyset : \bot$, by using the sequent proofs Ω' and Ξ' given above:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{T\lambda : T\lambda}{\neg T\lambda, T\lambda : \bot}}{\lambda, T\lambda : \bot}}{T\lambda : \bot}}{\emptyset : \lambda} \qquad \dfrac{\dfrac{\dfrac{\lambda : \lambda}{\lambda : T\lambda}}{\neg T\lambda, \lambda : \bot}}{\lambda : \bot}}{\emptyset : \bot}\,(\textsc{cut})$$

But the sequent rules for ¬, T, and λ all have at least one sentence on the left or on the right of their conclusion-sequents. So no arrangement of steps in accordance with those rules could possibly be a proof of the empty sequent. Hence CUT-elimination fails for the sequent calculus based on those rules.

This is the fundamental proof-theoretic lesson to be drawn from the logico-semantical paradoxes. It does not mean, however, that one should simply throw in the towel and avoid semantically closed languages altogether (as Tarski did). Rather, one should have one's eyes wide open, as it were, for localized failures of cut, *because they are symptomatic of paradoxicality*. All would-be *fully formalized* proofs 'by transitivity' (i.e. calling for CUTs) in the semantically open fragment of a semantically closed language, along with many innocuous such proofs in the semantically closed part as well, will turn out to be normalizable (equivalently, in the terminology of sequents: they will turn out to have sequent proofs using just the left and right rules). The normal-form proofs thus obtained will guarantee transmission of warrants-for-assertion from their premises to their conclusions. If the proof's conclusion is ⊥, then the proof's premises will have been revealed as jointly incoherent. If, however, a would-be proof of ⊥ 'by transitivity' of normal proofs *is not itself normalizable*, then no such incoherence will have been established. On the present account, the reasoning associated with paradoxes *does not reveal inconsistency* of the usual suspects. It provides no warrant at all for joint denial. This is because the regimented reasoning cannot be brought into normal form. (This view has been developed further, with replies to anticipated objections, in Tennant (2015a).)

5 Summary and conclusion

We have seen that the proof-theoretic criterion of paradoxicality has not been found to rule as paradoxical proofs of absurdity that are not genuinely paradoxical; in particular, Ekman's 'paradox' is no paradox at all. The initial impression to the contrary that Schroeder-Heister and Tranchini (*loc. cit.*) articulated in some detail can be dispelled by observing that there is a crucial difference between serial and parallelized elimination rules. Ekman's proof of ⊥ from $\{A \rightarrow \neg A, \neg A \rightarrow A\}$ appears to initiate a non-terminating (because looping) reduction sequence only because it uses the serial form of elimination rule for →. We have seen that if the parallelized form is used instead, then the proof can be given in normal form. Moreover, the proof of absurdity using the parallelized elimination rules in the case of the Liar Paradox turns out still to be a starting point for non-terminating reduction procedures, no matter how the reductions are applied. Thus the Liar's intuitive paradoxicality is borne out by the formal proof-theoretic criterion that I originally proposed, but incorporating now the necessary qualification that all elimination rules should be in parallelized form.

Essentially, this means that I am making my original point about paradoxicality by framing it in terms of cut-free, thinning-free sequent proofs (equivalently: core natural deductions), rather than in terms of natural deductions of Gentzen–Prawitz form. We find also that Russell's Paradox enjoys a proof in normal form, so that it is not genuinely paradoxical. This brings the proof-theoretic criterion of paradoxicality more closely into line with Ramsey's famous (and now, we see, perhaps more deeply principled) distinction between the 'Group A contradictions'—that is, the mathematical paradoxes such as Russell's Paradox—and the 'Group B contradictions'—that is, the logico-semantical paradoxes such as the Liar.[10] The latter paradoxes call for a proof-theoretic clarification of the vicious circles and helices within them—which is what I am seeking to provide. The former 'paradoxes', however, like Russell's, call for a re-thinking of both our abstract ontology and the first principles governing our thinking about it. In the latter case (for example, the Liar Paradox) we cannot have normality of (dis)proof. In the former case (for example, Russell's Paradox) we can; and we thereby obtain important negative existentials.

We cannot, and should not, hope for or expect anything remotely similar to happen in the case of the genuine logico-semantical paradoxes. One simply has to find a way to live with them—such as the way (perhaps) of the anti-realist that is described in Tennant (2015a). For lack of space, however, we cannot expound on that account in this study.

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[10] See Ramsey (1926), at pp. 352–3.

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