Definitions: Let $R \subseteq A \times A$ be a relation on a non-empty set A.

- R is **reflexive** if $\forall a \in A, aRa$.
- R is **anti-reflexive** if $\forall a \in A, a \not R a$.
- R is **symmetric** if $\forall a, b \in A, aRb \to bRa$.
- R is **anti-symmetric** if $\forall a, b \in A, \big(aRb \land bRa\big) \to \big(a = b\big)$.
- R is **transitive** if $\forall a, b, c \in A, \big(aRb \land bRc\big) \to aRc$.

1. Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. For each relation below, check all box(es) that apply.

   (a) $R = \Big\{(a, b) \in A \times A \;\mid\; a < b\Big\}$

   ☐ Reflexive ☐ Anti-Reflexive
   ☐ Symmetric ☐ Anti-Symmetric
   ☐ Transitive

   (d) $R = \Big\{(a, b) \in A \times A \;\mid\; a \geq b\Big\}$

   ☐ Reflexive ☐ Anti-Reflexive
   ☐ Symmetric ☐ Anti-Symmetric
   ☐ Transitive

   (b) $R = \Big\{(a, b) \in A \times A \;\mid\; a = b\Big\}$

   ☐ Reflexive ☐ Anti-Reflexive
   ☐ Symmetric ☐ Anti-Symmetric
   ☐ Transitive

   (e) $R = \Big\{(a, b) \in A \times A \;\mid\; a \neq b\Big\}$

   ☐ Reflexive ☐ Anti-Reflexive
   ☐ Symmetric ☐ Anti-Symmetric
   ☐ Transitive

   (c) $R = \Big\{(a, b) \in A \times A \;\mid\; a - b \text{ is even}\Big\}$

   ☐ Reflexive ☐ Anti-Reflexive
   ☐ Symmetric ☐ Anti-Symmetric
   ☐ Transitive

   (f) $R = \Big\{(a, b) \in A \times A \;\mid\; b = 3\Big\}$

   ☐ Reflexive ☐ Anti-Reflexive
   ☐ Symmetric ☐ Anti-Symmetric
   ☐ Transitive

2. For each definition below, write the letter(s) of all the above relations that satisfy the conditions.

   (a) A relation is called a **partial order** if it is reflexive, transitive, and anti-symmetric.

   (b) A relation is called a **strict partial order** if it is anti-reflexive, transitive, and anti-symmetric.

   (c) A relation is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

3. Let $A = \{$a,b,c,d$\}$. For each adjacency matrix below, check all box(es) that apply.

(a) $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(e) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(b) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(f) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(g) $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(d) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

(h) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

☐ Reflexive ☐ Anti-Reflexive
☐ Symmetric ☐ Anti-Symmetric

4. Let U be a universal set. Subset creates a relationship on the power set $\mathcal{P}(U)$, given by

$$R = \Big\{(A, B) \in \mathcal{P}(U) \times \mathcal{P}(U) \ \mid \ A \subseteq B\Big\}$$

   (a) The statement for **anti-reflexive** using set notation is:

$$\forall A \subseteq U, A \not\subseteq A$$

   Is the subset relationship anti-reflexive?

   (b) Write down the statement for **reflexive** using set notation:

   Is the subset relationship reflexive?

   (c) Write down the statement for **symmetric** using set notation:

   Is the subset relationship symmetric?

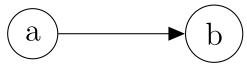
   (d) Write down the statement for **anti-symmetric** using set notation:

   Is the subset relationship anti-symmetric?

   (e) Write down the statement or **transitive** using set notation:

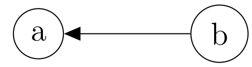
   Is the subset relationship transitive?

5. (a) Consider the following digraph:



- Add any arrow(s) required to make the above digraph symmetric.
- Describe all features of both symmetric and anti-symmetric digraphs.

(b) Consider the following diagraph:

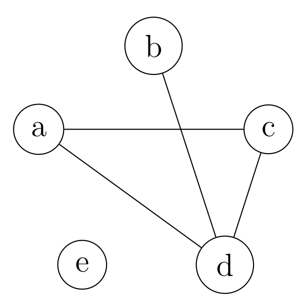


- Add any arrow(s) required to make the above digraph reflexive.
- Describe all features of both reflexive and anti-reflexive digraphs.

6. A (undirected) **Graph** is a collection of **nodes** (points) and **edges** (lines) between two distinct nodes (an edge cannot connect a node to itself). An example of a graph with 5 nodes is shown.

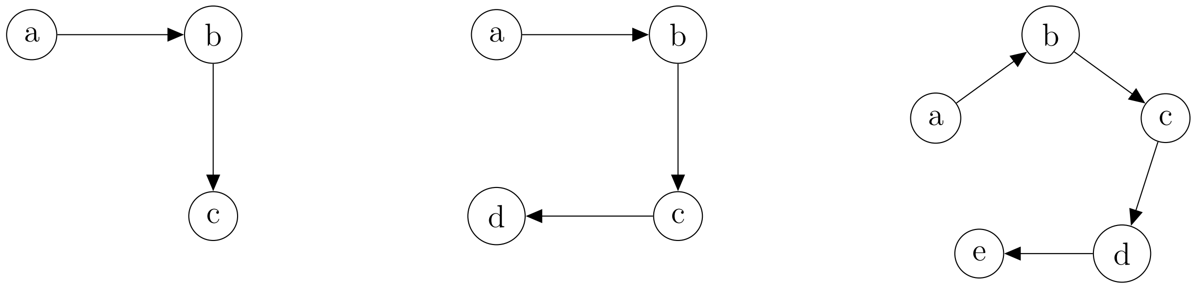
- Is an (undirected) graph symmetric or anti-symmetric?

- Is an (undirected) graph reflexive or anti-reflexive?



Discussion: Describe the two main differences between a **graph** and a **digraph**.

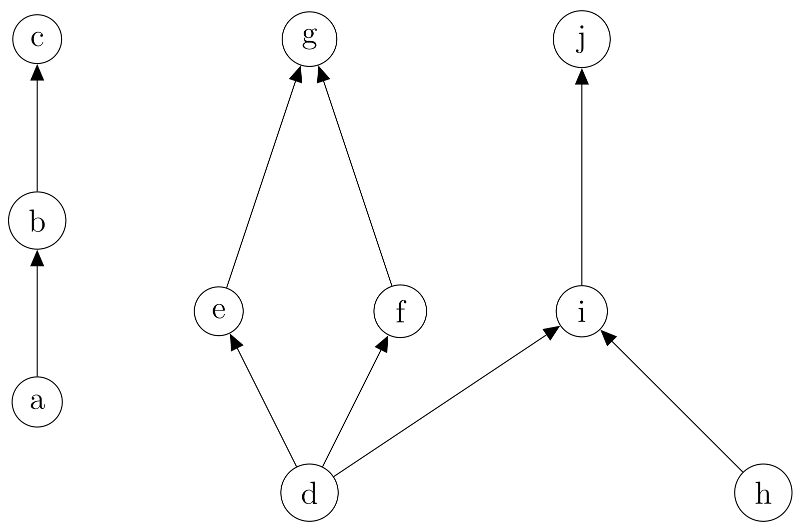
7. Consider the following three digraphs:



   (a) Add the minimal number of arrows required to make each digraph above transitive.

   (b) **Discussion:** If it was known a relationship was transitive, does adding the additional arrows provide any information? Do the additional arrows make the graph easier or harder to read?

8. Add the minimal number of arrows required to make the following digraph transitive and reflexive.



Discussion Points: Reflexive, transitive, and anti-symmetric relation orders the elements in A:

- Every element is related to itself (reflexive).
- Elements on the bottom are smaller than the elements on the top.
- No element is related to an element smaller than it (arrows point up or anti-symmetric).
- Not every pair of elements are related or are comparable.
- Two main examples of orders are $\leq$ (less than or equal to) and $\subseteq$ (subset).

9. Statements to prove (optional):

   (a) Prove that the following relations on $\mathbb{Z}$ are reflexive, symmetric, and transitive:

      i. $a \,\$\, b$ if $a - b$ is divisible by 2

      ii. $a \,\&\, b$ if $a - b$ is divisible by 7

      iii. $a \,@\, b$ if $5a + 13b$ is divisible by 9

   (b) Let I, S, B be the relations on sets defined by: XIY if there is an injective function from X to Y, XSY if there is a surjective function from X to Y, XBY if there is a bijective function from X to Y. Prove that each of these three relations is reflexive and transitive, but not symmetric or anti-symmetric.

   (c) Let C be the set of continuous functions $f : [0, 1] \to \mathbb{R}$. Define $f \leq g$ if $f(x) \leq g(x)$ for all x. Show that this relation is reflexive, anti-symmetric, and transitive. Give an example of functions f, g that are incomparable (neither $f \leq g$, nor $g \leq f$).

   (d) The *Hasse diagram* of a partial order shows the elements of the set, with smaller elements below larger ones, and lines connecting elements x, y whenever $x \leq y$. Draw the Hasse diagrams of the following partially ordered sets:

      - The subsets of $\{1, 2, 3\}$, ordered by subset inclusion (try $\{1, 2, 3, 4\}$)
      - The integers $\{1, \ldots, 10\}$, ordered by divisibility
      - The positive divisors of 72, ordered by divisibility (try 60)
      - The bitstrings of length less than or equal to 3, ordered by prefix inclusion: $x \leq y$ if the first bits of y consist of x. For example $11 \leq 110$, but $10 \not\leq 110$ because 110 doesn't start with 10.

   (e) In a partially ordered set S, a *chain* is a subset $C \subseteq S$ such that every two elements $x, y \in C$ are comparable: $x \leq y$ or $y \leq x$. An *antichain* is a subset $A \subseteq S$ such that for two elements $x, y \in A$ are incomparable: neither $x \leq y$, nor $y \leq x$. Find chains and antichains in the partially ordered sets: the subsets of $\{1, 2, 3\}$, and the positive divisors of 72.

   (f) Suppose that C is any chain and A is any antichain. Prove:

      i. Any subset $C' \subseteq C$ is a chain

      ii. Any subset $A' \subseteq A$ is an antichain

      iii. The intersection $A \cap C$ has at most one element