CERN-TH-5598/89
LAPP-TH-270/89

Dynkin-like Diagrams and Representations of the Strange Superalgebra P(n)

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Abstract

Simple root systems (SRS's) are introduced for non-contragredient superalgebras P(n). To each SRS is associated a Dynkin-like diagram. Then the maximal regular sub-(super)algebras are obtained. Moreover, a procedure to construct highest weight irreducible representations of P(n) is presented.

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CERN-TH-5598/89
LAPP-TH-270/89
November 1989

1. Introduction

Simple Lie superalgebras [1a,b,c] (SLA's) are becoming more and more popular these days in different domains of physics in connection with supersymmetry. Classical superalgebras appear quite naturally in extended supergravity as well as in integrable systems (see for example Ref.[2]), while Cartan-type superalgebras show up in two-dimensional superconformal field theories.

Contragredient (i.e. with the same number of positive and negative roots) classical superalgebras, as well as their affine extensions [3] have been studied extensively. However not very much is known about the non-contragredient – or strange – superalgebras P(n) and Q(n).

Therefore it seemed to us of some interest to undertake a study of the superalgebras P(n), the roots of which cannot be divided into two disjoint sets of positive and negative ones in equal number, and to examine how the properties of the usual simple (super)algebras, such as Dynkin diagram, Cartan matrix, representation theory, can be extended to it.

Let us note that the algebra P(n) appears as a maximal* subsuperalgebra of the simple superalgebra SU(n/n)' – obtained from SU(n/n) by deleting the one-dimensional centre – and that P(n) itself contains SU(n–1/1) as a maximal subsuperalgebra. Moreover, one can add that in the same way the quotient $\frac{SU(n) \times SU(n)}{SU(n)_d}$, where $SU(n)_d$ stands for the diagonal subalgebra of SU(n) x SU(n), describes the adjoint representation of SU(n) ; the quotient $\frac{SU(n/n)}{P(n)}$ behaves also as the adjoint representation of the algebra P(n).

The first properties of P(n) algebra are settled down in section 2. We introduce the notion of simple root system and Dynkin diagram for P(n) in section 3. Such an approach allows us in particular to determine the regular sub(super)algebras of P(n) (section 4). Then we turn our attention to highest weight irreducible representations (HWIR's) in section 5, pointing out the pathology resulting from the existence of fermionic roots without opposite counterparts, and computing the decomposition of such a HWIR with respect to the SU(n) bosonic part of P(n). Finally, the example of P(3) is considered in some detail in section 6.

* the (super)algebra B is said to be maximal in the (super)algebra A if and only if no (super)algebra C exists such that $B \subset C \subset A$ (the inclusion being strict).

2. The superalgebra P(n)

The superalgebra P(n) can be represented by the set of 2n x 2n matrices of the form [4]

$$\begin{pmatrix} \lambda & S \\ A & -\lambda^T \end{pmatrix} \tag{2.1}$$

where λ denotes a n x n complex traceless matrix, that is a generator of A_{n-1}. S is a n x n complex symmetric matrix which can be seen as an element of the twofold symmetric representation ([2] in Young tableau notation) of A_{n-1} (of dimension n(n+1)/2), and A is a n x n complex antisymmetric matrix which can be seen as an element of the (n–2)-fold antisymmetric representation ($[1^{n-2}]$ in Young tableau notation) of A_{n-1} (of dimension n(n–1)/2).

The even (bosonic) part has the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\lambda^T \end{pmatrix} \tag{2.2}$$

and forms the Lie algebra A_{n-1} (of dimension n^2-1).

The odd (fermionic) part has the structure

$$\begin{pmatrix} 0 & S \\ A & 0 \end{pmatrix} \tag{2.3}$$

and forms the (reducible) representation (of dimension n^2)

$$[2] \oplus [1^{n-2}] \tag{2.4}$$

of A_{n-1}.

As the adjoint representation $[21^{n-2}]$ cannot be found in the Kronecker product of

$$[2] \otimes [2] \quad \text{or} \quad [1^{n-2}] \otimes [1^{n-2}]$$

we have

$$\{A,A\} = \{S,S\} = 0 \tag{2.5}$$

So only the following anticommutators are non-vanishing

$$\{S,A\} = \lambda \tag{2.6}$$

Now, the roots of the bosonic part, A_{n-1} are given by

$$\Delta_0 = \{\varepsilon_i - \varepsilon_j, \quad 1 \le i \ne j \le n\} \tag{2.7}$$

where $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ constitute an orthonormal basis of the n-dimensional Euclidean space. A simple root system (or SRS) of Δ_0 is given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ (i=1, ..., n−1).

The fermionic roots are given by the weights of the representations [2] and $[1^{n-1}]$ of A_{n-1}, that is [5],

$$\Delta_1 \equiv \left\{ \begin{array}{c} \gamma_i = \delta + \frac{2(n-1)}{n}\varepsilon_i - \frac{2}{n}\sum_{j=1, j\ne i}^{n} \varepsilon_j \\ \pm\beta_\mu = \pm\,[\delta - \frac{2}{n}(\varepsilon_{i_1} + ... + \varepsilon_{i_{n-2}}) + \frac{(n-2)}{n}(\varepsilon_{i_{n-1}} + \varepsilon_{i_n})] \end{array} \right\} \tag{2.8}$$

with i=1, ..., n and μ = 1, ..., n(n−1)/2.
The indices $i_1, i_2, ..., i_n$ are all different with $i_{n-1} < i_n$, and μ specifies the partition $(i_1, ..., i_n)$ and is determined by the following conditions :

$$\begin{array}{c} \mu < \mu' \Rightarrow i_{n-1} + i_n > i_{n-1}' + i_n' \\ \text{or if } i_{n-1} + i_n = i_{n-1}' + i_n' \text{ then } i_n > i_n' \end{array} \tag{2.9}$$

(for example in the case $\mu = 1$, $i_{n-1} = n-1$ and $i_n = n$).

In the above definitions, we have enlarged the n-dimensional Euclidean space to a Lorentzian one by adding a vector δ with the following properties :

$$(\delta, \varepsilon_i) = 0 \qquad (\delta,\delta) = -\frac{2(n-2)}{n} \tag{2.10}$$

In this way the fermionic roots $\pm\beta_\mu$ will be of vanishing length

$$(\beta_\mu, \beta_\mu) = 0 \tag{2.11a}$$

while

$$(\gamma_i, \gamma_i) = 2 \tag{2.11b}$$

Note that the γ_i's and the β_μ's form the symmetric representation [2] while the $-\beta_\mu$'s span the antisymmetric representation $[1^{n-2}]$ of A_{n-1}. The condition (2.11a) is required by the Jacobi identity involving three fermionic generators (actually two associated to β_μ, and one to $-\beta_\mu$) and the fact that $2\beta_\mu$ is not a root. Let us stress that the γ_i's have no opposite, such a feature resulting from the non-contragredient nature of P(n). From now on, we will call the γ_i's

non-contragredient roots. We remark also that the rank of P(n) is n−1, i.e. the rank of its maximal bosonic subalgebra A_{n-1}.

Denoting by $\alpha_i^{\pm}$, $\beta_\mu^{\pm}$, and γ_i the generators associated to the roots $\pm\alpha_i$, $\pm\beta_\mu$ and γ_i respectively, we can in particular define a Cartan subalgebra of A_{n-1} as

$$h_i = [\,\alpha_i^+, \alpha_i^-\,] \qquad i = 1, ..., n-1 \tag{2.12}$$

We remark that for any μ=1, ..., n(n−1)/2, β_μ^+ and β_μ^- are eigenvectors of the above Cartan generators with opposite eigenvalues. From Eqs (2.6) and (2.8), it follows that for any β_μ^-, at least one γ_k exists such that :

$$\{\,\beta_\mu^-, \gamma_k\} = \alpha_{\mu,k}^{\varepsilon} \tag{2.13}$$

where $\alpha_{\mu,k}^{\varepsilon} \in \Delta_0$, ε being either + or −, and

$$[\alpha_{\mu,k}^{-\varepsilon}, \gamma_k] = \varepsilon\,\beta_\mu^+ \tag{2.14}$$

Now we are ready to prove the following proposition :

Proposition 1 : The anticommutator of a β_μ^+ generator with its opposite β_μ^- belongs to the Cartan subalgebra and satisfies :

$$\{\beta_\mu^+, \beta_\mu^-\} = [\alpha_{\mu,k}^{\varepsilon}, \alpha_{\mu,k}^{-\varepsilon}] \tag{2.15}$$

where $\alpha_{\mu,k}^{\varepsilon} \in \Delta_0$ and is specified by the above Eq. (2.14).

Proof : From the generalized Jacobi identity :

$$[B, \{F, F'\}] + \{F, [F', B]\} + \{F', [F,B]\} = 0 \tag{2.16}$$

it follows that :

$$[h_i, \{\beta_\mu^+, \beta_\mu^-\}] = 0 \quad \forall i = 1,..., n, \text{ and } \forall\mu = 1, ..., n(n-1)/2 \tag{2.17}$$

Therefore $\{\beta_\mu^+, \beta_\mu^-\}$ belongs to the Cartan subalgebra. In order to compute it, we will use again Eq. (2.16) with β_μ^-, γ_k and $\alpha_{\mu,k}^{\varepsilon}$ satisfying Eq. (2.13) and Eq. (2.14), i.e. :

$$\{\beta_\mu^-, [\alpha_{\mu,k}^{-\varepsilon}, \gamma_k]\} = [\alpha_{\mu,k}^{-\varepsilon}, \{\beta_\mu^-, \gamma_k\}] + \{\gamma_k, [\beta_\mu^-, \alpha_{\mu,k}^{-\varepsilon}]\} \tag{2.18}$$

Noticing that the commutator $[\beta_\mu^-, \alpha_{\mu,k}^{\varepsilon}] = 0$, and using Eqs (2.13) and (2.14), we get Eq. (2.15).

3. Simple Root Systems and Dynkin-like diagrams

As can be seen from (2.8), the fermionic roots cannot be divided into two sets (of equal number) of positive and negative roots. Now we introduce the following definition :

Definition 1 - A subset of the root system $\Delta = \Delta_0 \cup \Delta_1$ of P(n) is a simple root system (or SRS), and its elements will be called simple roots if and only if :

i) It is a system of linear independent roots

ii) Any root of Δ can be written as a linear combination of simple roots with integer coefficients of the same sign.

Definition 2 - A root will be called positive (negative) with respect to a SRS if and only if the coefficients of the decomposition in simple roots are positive (negative).

In order to get such a basis, one can easily realize that at least n roots are needed, that is one more than the rank of P(n). This is a peculiar feature because for usual (i.e. contragredient) Lie (super)algebras the dimension of the SRS is equal to the rank of the algebra. We shall choose one of the n roots of a SRS as a γ root, for which no opposite root can be defined and, consequently, to which no Cartan element can be associated in the standard procedure. The remaining roots of the SRS will be of α and/or β type. The simplest system we can think of is made of the usual SRS of A_{n-1} and γ_n, i.e.

$$\Delta_F = \left\{ \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \le i \le n-1),\ \gamma_n = \delta - \sum_{i=1}^{n-1} \frac{2}{n}\varepsilon_i + \frac{2(n-1)}{n}\varepsilon_n \right\} \quad (3.1)$$

To any SRS we can associate a Dynkin diagram (or DD) and a rectangular Cartan matrix. The rules for drawing the DD are the following.

We associate a white ○ (gray ●) dot to each α-type (β-type) root and a box to the γ-type root. So each diagram will be formed by n–1 dots and one box. The dots are joined by a number of lines (always one) given by, ζ denoting an α or β-type root :

$$|(\zeta_i , \zeta_j)| \qquad (3.2)$$

The box is related to an α root if

$$(\gamma,\alpha) \neq 0 \qquad (3.3a)$$

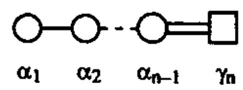
and to a β root if

$$(\gamma, \pm\alpha) \neq 0 \qquad (3.3b)$$

with α being defined by Eq. (2.14).

The number of lines joining γ to a root α (respectively β) (actually two) is given by the absolute value of the expression Eq. (3.3a) (respectively Eq. (3.3b)). Practically, γ will be linked to one and only one root in any SRS.

Applying the above rules for the SRS given by Eq. (3.1) we get the following DD



(3.4)

The rectangular Cartan matrix associated to a given SRS – or to its corresponding DD – will be determined as follows :

$$a_{ij} = a_{ji} = (\zeta_i, \zeta_j) \quad 1 \le i,j \le n-1$$
$$a_{in} \text{ will be given by Eq. (3.3a) or (3.3b)} \qquad (3.5)$$

The possible values of a_{in} are therefore either 0 or –2. Thus, to the SRS given by Eq. (3.1) – or to the DD given above – we associate the rectangular (n–1) x n Cartan matrix :

$$\begin{pmatrix} a_{ij} & \begin{matrix} 0 \\ \vdots \\ 0 \\ -2 \end{matrix} \end{pmatrix} \qquad (3.6)$$

where the a_{ij}'s constitute the usual Cartan matrix of A_{n-1} and the elements of the n^{th} column are zero except $a_{n-1,n}$ which is equal to –2. The structure of the P(n) superalgebras, once given a SRS $(\xi_1^+, \ldots, \xi_{n-1}^+, \gamma)$ and the corresponding Cartan matrix, is :

$$[H_i , H_j] = 0$$
$$[H_i , \xi_k^{\pm}] = \pm a_{ik}\, \xi_k^{\pm}$$
$$[H_i , \gamma] = a_{in}\, \gamma$$
$$[\xi_i^+ , \xi_j^-\} = \delta_{ij}\, H_i \qquad (3.7)$$

where [, } is an anticommutator (respectively commutator) depending on wether the two roots ξ_i^+, ξ_j^- are both fermionic or not.

The choice of SRS, Eq. (3.1), has the following characteristics :

a) it immediately makes it evident that the bosonic maximal subalgebra of P(n) is A_{n-1}.

b) it does contain only one fermionic root. It is the analogue of the <u>fundamental</u> (or <u>distinguished</u>) SRS for contragredient Lie superalgebras.

A main difference between a (simple Lie) algebra and superalgebra is the fact that the last one is in general characterized by several inequivalent SRS's. These SRS's can be obtained one from another by means of Weyl reflections with respect to fermionic roots. A Weyl reflection for a contragredient superalgebra is defined as follows :

$$W_{\xi_i}(\xi_j) = \xi_i - a_{ij}\,\xi_j \qquad \text{if } (\xi_i,\xi_i) = 2 \qquad (3.8a)$$

$$W_{\xi_i}(\xi_j) = \xi_j \qquad \text{if } (\xi_i,\xi_i) = (\xi_i,\xi_j) = 0 \;\; i \neq j \qquad (3.8b)$$

$$W_{\xi_i}(\xi_i) = -\xi_i \qquad \text{if } (\xi_i,\xi_i) = 0 \qquad (3.8c)$$

$$W_{\xi_i}(\xi_j) = \xi_i + \xi_j \qquad \text{if } (\xi_i,\xi_i) = 0 \text{ and } (\xi_i,\xi_j) \neq 0 \qquad (3.8d)$$

Let us remark that the above-defined Weyl reflection satisfies $W_\xi^2 = 1$ and that for $(\xi_i,\xi_i) = 2$ (with $\xi_i \neq \gamma$), one recovers the usual action of the Weyl group for A_{n-1}.

Now we would like to generalize such Weyl reflections to the case of the P(n) superalgebras. In order to reach this objective, we have to define a suitable reflection with respect to a γ root W_γ as well as the action of W_ξ on γ with ξ being α or β root. Let us first remark in connection with the SRS Eq. (3.1) that we can still keep the above definition Eq. (3.8a) for a reflection with respect to an α root. In fact, a reflection with respect to a bosonic root not connected with the γ root will give only an equivalent SRS for A_{n-1}. A reflection with respect to α_{n-1} (connected with the γ_n root) will transform the SRS Eq. (3.1) into the following one :

$$\begin{aligned} \alpha_i' &= W_{\alpha_{n-1}}(\alpha_i) = \alpha_i \quad (1 \le i \le n-3) \\ \alpha_{n-2}' &= \alpha_{n-2} + \alpha_{n-1} \\ \alpha_{n-1}' &= -\alpha_{n-1} \\ \gamma_n' &= \gamma_n + 2\alpha_n = \gamma_{n-1} \end{aligned} \qquad (3.9)$$

This new SRS is obviously equivalent to the previous one, as it should be, being obtained by a Weyl reflection with respect to a bosonic root.

Now we have to define a reflection with respect to a γ root. We shall do it in the following way :

$$W_\gamma(\gamma) = \gamma \qquad (3.10a)$$

$$W_\gamma(\xi) = -\xi - \gamma \qquad \text{if } a_{\xi\gamma} \neq 0 \qquad (3.10b)$$

$$W_\gamma(\xi) = -\xi \qquad \text{if } a_{\xi\gamma} = 0 \qquad (3.10c)$$

Let us remark that $W_\gamma^2 = 1$ and that W_γ transforms a bosonic (fermionic) root joined to γ into a fermionic (bosonic) one, as happens with the Weyl reflection with respect to a fermionic root in a contragredient Lie superalgebra. To complete our definition of a Weyl reflection for P(n), we have to define a reflection with respect to a fermionic β-type root linked to γ. We define it as follows :

$$W_\beta(\beta) = \beta \qquad (3.11a)$$

$$W_\beta(\xi) = -\xi \qquad \text{if } (\beta,\xi) = 0 \qquad (3.11b)$$

$$W_\beta(\xi) = -\xi - \beta \qquad \text{if } (\beta,\xi) \neq 0 \qquad (3.11c)$$

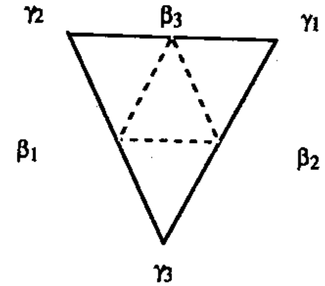
$$W_\beta(\gamma) = -\gamma - 2\beta \qquad (3.11d)$$

Evidently $W_\beta^2 = 1$.

Before showing that the Weyl transformations we have just introduced allow us to construct all the non-equivalent SRS's, let us note :

<u>Proposition 2</u> : Any SRS contains one and only one γ root.

<u>Proof</u> : Let us remark that in the representation [2] of A_{n-1}, the γ roots stand at the vertices of the hyperpolyhedron and each β root is on the middle of a segment joining two γ roots : as an example the 2-fold symmetric SU(3) representation has the form :



Thus, let us suppose that two γ-type roots, γ and γ' belong to the simple root system B. Let β be the β-type root on the segment γ-γ' : then there exists a bosonic root α such that

$\beta+\alpha=\gamma$ and $\beta-\alpha=\gamma'$. One notes that β cannot be a simple root since $\alpha=\beta-\gamma=\gamma'+\beta$. But then the relations $\beta=\alpha+\gamma'=\gamma-\alpha$ contradict the definition of a simple root system. Now, suppose the simple root system B does not contain any γ root. One remarks that it is enough to consider the case of P(3) due to the regular embedding $P(3) \subset P(4) \subset \ldots \subset P(n)$. Then one can easily convince oneself that the three elements of the simple root system must be of the form $\beta, -\beta', \gamma$. Therefore, one could find two roots γ and γ' such that, either $\gamma=\beta+\alpha$ and $\gamma'=\beta-\alpha$, or $\gamma=-(-\beta')+\alpha$ and $\gamma'=-(-\beta')-\alpha$, which is not possible, due to our definition of a simple root system.

Now we can prove the following statement :

<u>Proposition 3</u> : A Weyl reflection with respect to a fermionic root of a SRS B_1 gives a SRS B_2 not equivalent to B_1.

<u>Proof</u> : If the system B_2 obtained from B_1 by a reflection with respect to a fermionic root is still a SRS, it will not be equivalent to B_1, since it will differ from it by the number of α or β roots. So we have just to show that the B_2 is still a SRS according to Definition 1. We will consider separately all the different cases.

i) Reflection with respect to a β-root (W_β) not connected to the γ-root ($a_{\beta\gamma}=0$).

With respect to the SRS $(\xi_1, \ldots, \xi_{n-1}, \gamma_n)$ any positive root $\zeta_+ \in \Delta_+$, $\zeta_+ \neq \beta$ can be written :

$$\zeta_+ = \sum_{i=1}^{n-1} c_i \xi_i + c_\gamma \gamma = \Sigma_j c_j \xi_j + \Sigma_\mu c_\mu \xi_\mu + c_\beta \beta + c_\gamma \gamma \tag{3.12}$$

where the ξ_j, respectively the ξ_μ, are such that

$$(\beta,\xi_j)=0 \quad \text{and} \quad (\beta,\xi_\mu)\neq 0 \tag{3.13}$$

Then

$$\zeta_+ = \Sigma_j c_j \xi_j + \Sigma_\mu c_\mu(\xi_\mu+\beta) + c_\gamma\gamma + (\Sigma_\mu c_\mu - c_\beta)(-\beta)$$

$$= \Sigma_j c_j \xi_j + c_\gamma\gamma + \Sigma_\mu c_\mu W_\beta(\xi_\mu) + n'_\beta W_\beta(\beta) \tag{3.14}$$

where $n'_\beta = \Sigma_\mu c_\mu - c_\beta$ and we have used Eq. (3.8).

Now, to prove that $W_\beta(B_1)$ is still a SRS, we have to show that $n'_\beta \geq 0$. It is enough to show that there does not exist any root of the form $(2\beta+\xi)$. Let us remark that the β's are states of the symmetric representation which are eigenvectors of the Cartan subalgebra of A_{n-1} with the same eigenvalues as the states of the two-fold antisymmetric representation $[1^2]$.

Therefore 2β is an eigenvector of the Cartan subalgebra with the same eigenvalue as the states of $[2^2]$, and

$$[2^2] \otimes \begin{cases} [21^{n-2}] \\ [1^{n-2}] \end{cases} \not\ni [21^{n-2}] \text{ or } [1^{n-2}] \text{ or } [2] \tag{3.15}$$

So our SRS B_2 will be formed by $(\xi_j, W_\beta(\xi_\mu), -\beta, \gamma) = W_\beta(B_1)$.

ii) Reflection with respect to a fermionic root β linked with γ ($a_{\beta\gamma}\neq 0$ and $\beta \in [1^{n-2}]$).

By definition, any negative root $\zeta_- \in \Delta_-$, $\zeta_- \notin -\beta$, can be written as

$$\zeta_- = \sum_{i=1}^{n-1} (-c_i)\,\xi_i + (-c_\gamma)\gamma$$

$$= \Sigma_j c_j(-\xi_j) + \Sigma_\mu c_\mu(-\xi_\mu) + (-c_\beta)\beta + c_\gamma(-\gamma)$$

$$= \Sigma_j c_j W_\beta(\xi_i) + \Sigma_\mu c_\mu W_\beta(\xi_\mu) + c_\gamma W_\beta(\gamma) + c''_\beta \beta \tag{3.16}$$

where the subsets Σ_j and Σ_μ are defined by Eq.(3.13), $c''_\beta = \Sigma_\mu c_\mu + 2c_\gamma - c_\beta$ and we have used Eq. (3.11). By an argument completely analogous to that of Eq. (3.15), one can see that $c''_\beta \geq 0$.

Thus, we have obtained a new simple root system formed by $(W_\beta(\xi_j)\ (\xi_j \neq -\beta), W_\beta(\xi_\mu), W_\beta(\gamma), \beta) = W_\beta(B_1)$.

iii) Reflection with respect to the fermionic root γ.

Any negative root ζ_- can be written as (the ξ_μ being linked to γ) :

$$\zeta_- = \sum_{i=1}^{n-1} (-c_i)\,\xi_i + (-c_\gamma)\gamma = \Sigma_j c_j(-\xi_j) + \Sigma_\mu c_\mu(-\xi_\mu-\gamma) + c_\gamma'\gamma \tag{3.17}$$

where $c_\gamma' = \Sigma_\mu c_\mu - c_\gamma$ and we have used Eq. (3.11).

So we have to show that c_γ' is not negative. It is enough to prove that we cannot have a root of the type $(2\gamma+\xi_\mu)$. One notes that the states 2γ have the same weights as the external points of the hyperpolyhedron in the (n−1) Euclidean space describing the 4-symmetric representation of A_{n-1}, and we have

$$[4] \otimes \begin{cases} [1^{n-2}] \\ [21^{n-2}] \end{cases} \not\in [21^{n-2}] \text{ or } [1^{n-2}] \text{ or } [2] \quad (3.18)$$

which concludes the proof of Proposition 3.

It remains to be shown that the Weyl reflections associated with fermionic roots allow us to construct all the non-equivalent simple root systems. Owing to the above Proposition 3, we can use the same argument as the one given by Leites et al. (see theorem in Appendix of Ref.[6]) in the case of contragredient superalgebras, to prove this property. Therefore, starting from one simple root system, one can describe all the set of non-equivalent simple root systems by a sequence of generalized Weyl transformations.

Note a slight difference with respect to the case of contragredient superalgebras. According to our definition (3.10a), the Weyl reflection with respect to γ does not change itself, and by respected application of reflections, we could get two simple root systems which are equivalent, i.e. which can be obtained one from another by reflections with respect to bosonic roots.

The number of non-equivalent simple root systems for P(n) is 2^{n-1}. Starting from the fundamental simple root system, given in Eq. (3.1), all of them can be obtained by Weyl reflections with respect to β or γ roots, stopping the procedure after the reflection relative to $\gamma_{[n/2]}$.

The case of P(3) will be treated in great detail in Sect. 6.

4. Regular subalgebras of P(n)

In this section we would like to show how one can determine the maximal sub(super)algebras of P(n) using the same kind of argument that is used in the usual contragredient (super)algebra case.

Let us recall the method, due to Dynkin, in the case of simple Lie algebras.

Let Δ be the root system of a simple Lie algebra $\mathcal{G}$. One defines a regular subsystem Γ of Δ by the two conditions :

$$\Gamma = \{\alpha \in \Delta \quad \text{such that} \quad \text{(i) the } \alpha\text{'s are linear independent} \quad \text{(ii) } \forall\, \alpha,\beta \in \Gamma,\ \alpha-\beta \notin \Delta\} \quad (4.1)$$

The main point is that any regular subsystem of Δ is a simple root system of a regular subalgebra of $\mathcal{G}$. Therefore, in order to find the (maximal) regular subalgebras of $\mathcal{G}$, it is sufficient to determine the (maximal) regular subsystems of the root system Δ of $\mathcal{G}$.

The method is the following : let Π be a SRS of $\mathcal{G}$. One extends Π by a root $-h \in \Delta$ such that $\Pi^e = \Pi \cup \{-h\}$ (which is necessarily a system of dependent roots) still satisfies the property (ii). It is clear that deleting any root of Π^e will lead us to a subsystem of Δ which satisfies now both properties (i) and (ii), i.e. which is a regular subsystem. Repeating the procedure of the so-obtained regular subsystem will give all the regular subalgebras of $\mathcal{G}$. Of course, only the first step has to be achieved in order to find the maximal regular subalgebras of $\mathcal{G}$.

At this point, two remarks are in order.

First [7], it is worthwhile to emphasize that the *method also holds for contragredient simple Lie superalgebras* (or basic superalgebras), the point being that one has to apply the procedure to *all the non-equivalent DD's* of the superalgebra.

Second, the way to extend Π into Π^e by the root $-h \in \Delta$ is unique for the property (ii) to remain valid. The root h is simply the highest root with respect to the SRS Π.

Now we turn to the case of P(n). The main difficulty is that the root system Δ of P(n) is a non-contragredient one. Consider the distinguished DD of P(n) :

$\alpha_1 \quad \alpha_2 \quad \alpha_{n-1} \quad \gamma_n$

associated to the distinguished SRS

$$\Pi = \{ \alpha_1, \alpha_2, ..., \alpha_{n-1}, \gamma_n \} \quad (4.2)$$

The highest root with respect to Π is γ_1 :

$$\gamma_1 = 2\alpha_1 + 2\alpha_2 + ... + 2\alpha_{n-1} + \gamma_n \quad (4.3)$$

However such a root cannot be used to extend the SRS Π because it is not a contragredient root and the arguments developed previously would not apply. Therefore, the most natural way to extend Π is to find the "highest contragredient root" (or h.c.r.) which here turns out to be

$$\beta_A = \alpha_1 + 2\alpha_2 + ...+ 2\alpha_{n-1} + \gamma_n = \gamma_1 - \alpha_1 \quad (4.4)$$

If we extend Π into Π^e by adding the root $-\beta_A$ to Π, one obtains

$$\Pi^e = \{\alpha_1, \alpha_2, ..., \alpha_{n-1}, \gamma_n, -\beta_A\} \tag{4.5}$$

which satisfies neither property (i) nor (ii)!

Notice that property (ii) is violated only because

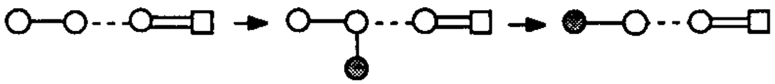
$$\alpha_1 \text{ and } -\beta_A \in \Pi^e \quad \text{and} \quad \alpha_1 - (-\beta_A) = \alpha_1 + \beta_A = \gamma_1 \in \Delta \tag{4.6}$$

Thus if we extend Π into $\tilde{\Pi}^e$ by adding $-\beta_A$ (i.e. the opposite of the h.c.r.) and deleting the simple root α_1 at the same time, one obtains

$$\tilde{\Pi}^e = \{\alpha_2, ..., \alpha_{n-1}, \gamma_1, -\beta_A\} \tag{4.7}$$

which now satisfies the two conditions of a regular subsystem of Δ.

It is therefore a simple root system of a regular subalgebra P(n). Actually, this SRS is one of the non-equivalent SRS's of P(n). The procedure is resumed in the following drawing :



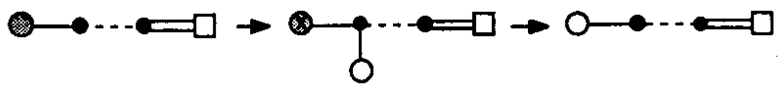
Consider now the case of an arbitrary DD of P(n). The diagram can be of the two following forms :

case 1 case 2

where the little black dot is either a white dot or a grey dot.

In the first case, the h.r. is of γ-type and one extends therefore the SRS by minus the h.c.r. which is of β-type. In order to satisfy the property (ii), one deletes the marked root and obtains a DD of P(n) :

In the second case, the h.r. is of α-type. One extends the SRS by minus the h.r. and one deletes the marked root in order to satisfy the property (ii). One obtains again a DD of P(n) :



Finally, the "extended" Dynkin diagrams of P(n) are equivalent to the DD of P(n). The maximal regular sub(super)algebras of P(n) are determined by deleting an arbitrary dot in every

non-equivalent DD of P(n). The reader can notice that this is also the case of the unitary simple Lie algebras SU(n) and unitary simple Lie superalgebras SU(m/n) where the maximal regular sub(super)algebras are determined by deleting one dot in the DD of the (super)algebra.

Let us remark that we can use the extended DD to obtain all the non-equivalent DD's of P(n). Indeed, starting from the distinguished one by adding minus the h.c.r. and deleting the root α_1, one obtains a DD with a fermionic root β; then one deduces all the non-equivalent SRS's by means of the Weyl reflections with respect to fermionic root β, without using W_γ, Eq. (3.10).

The table of the maximal regular sub(super)algebras of P(n) is therefore the following :

P(n)	
SU(n)	
SU(n−k/k)	$1 \le k \le [n/2]$
P(k) ⊕ SU(n−k)	$3 \le k \le n-2$
P(k) ⊕ SU(n−m/m−k)	$3 \le k \le n-2$ and $1 \le m \le k$

5. Representations of P(n)

Hereafter, we want to present a procedure to build up a class of representations and to discuss their decomposition in terms of irreducible representations (IR's) of some sub(super)-algebras.

In the following we shall consider only the fundamental SRS defined by Eq.(3.1). We will restrict our study to a class of representations satisfying the following properties:

a) The representation has a highest weight (h.w.), Λ, defined as follows : Λ is annihilated by the positive roots Δ_0^+ of A_{n-1} (i.e. acts as a h.w. of an IR of A_{n-1}) and by the fermionic roots belonging to the (n−2)-antisymmetric representation of A_{n-1} (that is, the negative fermionic roots).

b) It decomposes into a sum of IR's of A_{n-1}.

c) Any two vectors of the representation space connected by a positive root of P(n) (raising operator) will also be connected by a negative root (lowering operator).

We call such a representation a highest weight irreducible representation (or H.W.I.R.).

The different A_{n-1} IR's are obtained one from the other by the action of the fermionic roots of P(n). Assuming that the IR (of A_{n-1}) to which belongs the h.w. is "fermionic", any IR of A_{n-1} appearing in the P(n) representation will be fermionic (bosonic) if obtained from the h.w. by application of an even (odd) number of fermionic roots. According to the point c) we have to factor out the states resulting from the action of the (raising) operator associated to a γ-root, since this operator has no inverse (lowering) operator. The existence of the γ roots leads to an ever-present peculiar feature of the representations which reminds us of the atypicality in the contragredient classical SLA's (except OSp(1/n)) [1b]. Let us remark that for P(n), due to the fact that each of the \f(n(n–1);2) anticommutators, $\{\beta_\mu^+,\beta_\mu^-\}$ is proportional to one of the $\frac{n(n-1)}{2}$ commutators $[\alpha^+,\alpha^-]$, we have no atypicality conditions of the usual type. This can also be inferred by the existence of a SRS, the fundamental one, which does not contain any β root.

We can limit ourselves to considering the fermionic roots and their action on the h.w. Λ since the bosonic roots span all the states of the IR of A_{n-1}.

At first level, i.e. by the first action of the positive fermionic roots, we will necessarily find IR's which are contained in the Kronecker product

$$[\Lambda] \otimes [2] \qquad (5.1)$$

as, by definition, the action of $[1^{n-2}]$ on Λ will give zero. According to our definition of the HWIR of P(n) we have to find out the representation obtained by the γ roots. In order to do this on the Kronecker product Eq. (5.1), we must not take into account all Young tableaux obtained from $[\Lambda]$ by adding two boxes in the same row.

At second level, we have to perform the Kronecker product of $[\Lambda]$ with the antisymmetric product of two symmetric representations, due to the vanishing of the anti-commutator $\{S,S\}$, i.e.

$$([2] \otimes [2])_A = [31] \qquad (5.2)$$

However, the Kronecker product $[\Lambda] \otimes [31]$ contains some Young tableaux which appear in the Kronecker product of the omitted representations of the first level by the symmetric representation [2]. These Young tableaux have also to be drawn away, and so on. We shall illustrate this procedure more in detail in the case in which $[\Lambda]$ is a completely symmetric $[\lambda]$ or k-fold antisymmetric $[1^k]$ representation.
We have

$$[\lambda] \otimes [2] = [\lambda+2] + [\lambda+1,1] + [\lambda,2] \qquad (5.3a)$$

$$[1^k] \otimes [2] = [31^{k-1}] + [21^k] \qquad (5.3b)$$

As commented before, the representations $[\lambda+2]$, $[\lambda,2]$ and $[31^{k-1}]$ do not have to be considered. At second level, we have to study the Kronecker product of λ with the anti-symmetric product of $[2] \otimes [2]$, i.e.

$$[\lambda] \otimes [31] \quad \text{or} \quad [1^k] \otimes [31] \qquad (5.4)$$

From the IR's which appear, we have to take away those appearing in the Kronecker product of the neglected representation of the first level, i.e. $[\lambda+2]$, $[\lambda,2]$ or $[31^{k-1}]$, with the [2]. Only the representations $[\lambda+2,1,1]$ in the first case and $[31^{k+1}]$ in the second case will survive.

We go on performing the Kronecker product of $[\lambda]$ or $[1^k]$ with the antisymmetric product of three representations [2], i.e. $[3^2] \oplus [41^2]$, and taking away the representations which appear in the product of $[\lambda+2] \oplus [\lambda,2]$ or $[31^{k-1}]$ with the antisymmetric product of the [2]. In the first case it will survive only the representation $[\lambda+3,1^3]$ and in the second case $[4,1^{k+2}]$. At a further level, we have to multiply $[\lambda]$ or $[1^k]$ with the antisymmetric product of four representations [2], i.e. $[431] \oplus [5111]$, and to take away the IR's which appear in the Kronecker product of $[\lambda+2] \oplus [\lambda,2]$ or $[31^{k-1}]$ with $[3^2] \oplus [41^2]$, and so on.

In particular for k=n–2 we get the adjoint representation of P(n) and the procedure stops at the second level.

Given a HWIR, we can get a complex conjugate one which has as highest weight the complex conjugate of the lowest weight. For example, the complex conjugate of the adjoint of P(n), and which is not real, will be

$$[2^{n-1}] \oplus [21^{n-2}] \oplus [1^2] \qquad (5.5)$$

Taking as highest weight $[\Lambda]$ the representation $[1^{n-1}]$, we find a representation that we can consider as the fundamental one of P(n), which decomposes as

$$[1^{n-1}]_F \oplus [1]_B \qquad (5.6)$$

This representation is real. In this case the procedure stops at the first level.

Let us close this section by giving the decomposition of the adjoint representation of P(n) in terms of the representations of the maximal sub(super)algebras SU(n–1/1) and SU(n–2/2).

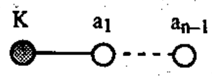
The adjoint of P(n) decomposes under SU(n–1/1) as:

n = 3	$(4;0)_F \oplus$	$(1;1) \oplus$	$(-2;2)$	(5.7)
dimension	4	8	5	

$n > 3$ $\quad (-\frac{2}{n-2}\,;\,0^{n-3},1,0) \oplus (1\,;\,0^{n-2},1) \oplus (\frac{n-2}{n}\,;\,1,0^{n-2})$ (5.8)

dimension $\quad \frac{n(n-1)}{2}+1 \qquad n^2-1 \qquad \frac{n(n+1)}{2}+1$

where the first label is the value of <u>the K number</u> and the (n–1) following labels identify the weight of the representation of A_{n-1} in the standard Dynkin diagram (see Refs [1b] and [8]) :



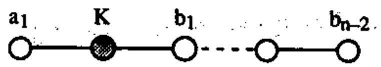
Let us remark that the second representation of Eqs (5.7) and (5.8) corresponds to the adjoint of SU(n–1/1).

The adjoint of P(n) decomposes into the two following sums of three IR's of SU(n – 2/2):

$n > 5$ $\quad (0\,;\,-4\,;\,0^{n-5},1,0) \oplus (1\,;\,1\,;\,0^{n-4},1) \oplus (0\,;\,2(n-2)\,;\,0^{n-3})$ (5.9)

dimension $\quad \frac{n(n-1)}{2}+2 \qquad n^2-1 \qquad \frac{n(n+1)}{2}-2$

where the labels are specified by [1b]



6. An explicit example : P(3)

In this section we present a detailed discussion of the superalgebra P(3).

The maximal bosonic subalgebra A_2 is spanned by the eight (6 x 6) matrices

$$B_i = \frac{1}{2}\begin{pmatrix}\lambda_i & 0\\ 0 & -\lambda_i^T\end{pmatrix} \qquad (i=1,\dots,8) \tag{6.1}$$

where λ_i are the standard (3 x 3) Gell-Mann matrices.

The fermionic part is spanned by the three matrices

$$F_a = \begin{pmatrix}0 & 0\\ A_a & 0\end{pmatrix} \qquad (a=1,2,3) \tag{6.2}$$

where $\quad \{A\} = \frac{1}{\sqrt{2}}\,\{-\lambda_2, -\lambda_5, \lambda_7\} \equiv$ (s,d,u) quarks (6.3)

and by the six matrices

$$F_\mu' = \begin{pmatrix}0 & S_\mu\\ 0 & 0\end{pmatrix} \qquad (\mu = 1,\dots,6) \tag{6.4}$$

where $\quad \{S\} = \frac{i}{\sqrt{2}}\,\{\lambda_1, \lambda_3, \lambda_4, \lambda_6, \lambda_8, \lambda_0 = \sqrt{\tfrac{2}{3}}\,\mathbb{1}\}$ (6.5)

$\mathbb{1}$ being the 3 x 3 identity matrix.

With the above choice of the coefficients, the F_a's and F_μ's form respectively a set of orthonormal vectors for the A_2 representations $\underline{3} \equiv [1]$ and $\underline{6} \equiv [2]$ with the scalar product

$$(\lambda_m, \lambda_n) = \mathrm{Tr}\,(\lambda_m^*, \lambda_n) = \delta_{mn} \qquad (m,n = 0,\dots,8) \tag{6.6}$$

λ_m^* being the complex conjugate of λ_m.

Of course we have

$$[B_i, B_j] = i f_{ijk}\, B_k \tag{6.7}$$

It is easy to verify that

$$\{F_a, F_b\} = \{F_\mu', F_\nu'\} = 0 \tag{6.8}$$

and that

$$\left\{\begin{pmatrix}0 & S_\mu\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ A_a & 0\end{pmatrix}\right\} = \begin{pmatrix}S_\mu A_a & 0\\ 0 & -(S_\mu A_a)^T\end{pmatrix} \tag{6.9}$$

where $\quad S_\mu,\ A_a \in \{\lambda_i\} \qquad i=1,\dots,8$

The explicit form of the r.h.s. of Eq. (6.9) can be computed by (i,j,k = 1,...,8):

$$\lambda_i\lambda_j = \frac{2}{3}\,\delta_{ij}\,\mathbb{1} + i f_{ijk}\,\lambda_k + d_{ijk}\,\lambda_k \tag{6.10}$$

$$\lambda_i\lambda_0 = i\sqrt{\tfrac{2}{3}}\,\lambda_i \tag{6.11}$$

where f_{ijk} (d_{ijk}) are completely antisymmetric (symmetric) in the indices.

The connection with the Cartan notation is given by

$$\begin{aligned} T_\pm &= E_{\pm\alpha_1} & h_1 &= 2T_3 \\ U_\pm &= E_{\pm\alpha_2} & h_2 &= 2U_3 = \tfrac{3}{2}Y + T_3 \qquad \left(Y = \frac{\lambda_8}{\sqrt{3}}\right) \\ V_\pm &= E_{\pm\alpha_3} & h_3 &= h_1 + h_2 = 2V_3 = \tfrac{3}{2}Y_1 + T_3 \end{aligned} \tag{6.12}$$

The weights of the fundamental representation are

$$-\beta_1' = \tfrac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \equiv u \equiv \frac{1}{\sqrt{2}}\lambda_7$$

$$-\beta_2' = \tfrac{1}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) \equiv d \equiv -\frac{1}{\sqrt{2}}\lambda_5 \qquad (6.13)$$

$$-\beta_3' = \tfrac{1}{3}(-\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3) \equiv s \equiv -\frac{1}{\sqrt{2}}\lambda_2$$

$$-\beta_1' \xrightarrow{-\alpha_1} -\beta_2' \xrightarrow{-\alpha_2} -\beta_3' \qquad (6.14)$$

with the normalization

$$(-\beta_a', -\beta_b') = \delta_{ab} - \tfrac{1}{3} \qquad (6.15)$$

We shall add to the above weights the "fermionic" vector δ, with the minus sign

$$(\delta,\delta) = -\tfrac{2}{3} \qquad (6.16)$$

getting

$$(-\beta_a, -\beta_b) = (-\beta_a' - \delta, -\beta_b' - \delta) = \delta_{ab} - 1 \qquad (6.17)$$

The weights of the symmetric representation are

$$\gamma_1' = \tfrac{2}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \equiv uu \equiv \frac{i}{\sqrt{2}}\left(\frac{\sqrt{2}}{3}\mathbb{1} + \frac{1}{\sqrt{2}}\lambda_3 + \frac{1}{\sqrt{6}}\lambda_2\right)$$

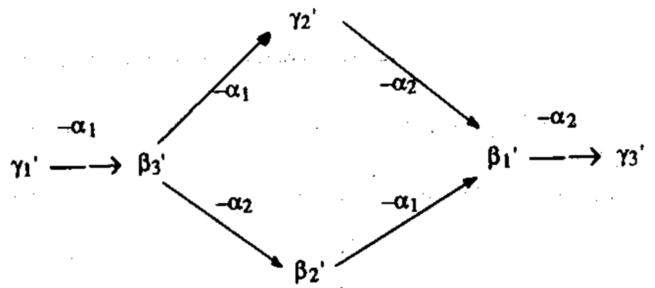
$$\beta_3' = \tfrac{1}{3}(\varepsilon_1 + \varepsilon_2 - 2\varepsilon_3) \equiv \frac{1}{\sqrt{2}}(du + ud) \equiv \frac{i}{\sqrt{2}}\lambda_1$$

$$\gamma_2' = \tfrac{2}{3}(-\varepsilon_1 + 2\varepsilon_2 - \varepsilon_3) \equiv dd \equiv \frac{i}{\sqrt{2}}\left(\frac{\sqrt{2}}{3}\mathbb{1} - \frac{1}{\sqrt{3}}\lambda_3 + \frac{1}{\lambda_6}\lambda_8\right)$$

$$\beta_2' = \tfrac{1}{3}(\varepsilon_1 - 2\varepsilon_2 + \varepsilon_3) \equiv \frac{1}{\sqrt{2}}(su + us) \equiv \frac{i}{\sqrt{2}}\lambda_4 \qquad (6.18)$$

$$\beta_1' = \tfrac{1}{3}(-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \equiv \frac{1}{\sqrt{2}}(sd + ds) \equiv \frac{i}{\sqrt{2}}\lambda_6$$

$$\gamma_3' = \tfrac{2}{3}(-\varepsilon_1 - \varepsilon_2 + 2\varepsilon_3) \equiv ss \equiv \left(\frac{i}{\sqrt{3}}\mathbb{1} - \lambda_8\right)$$



(6.19)

with the normalization

$$(\gamma_a', \gamma_b') = 4\delta_{ab} - \tfrac{4}{3} \quad \text{and} \quad (\beta_a', \beta_b') = \delta_{ab} - \tfrac{1}{3} \qquad (6.20)$$

Adding to the above weights the fermionic vector δ we get

$$(\gamma_a, \gamma_b) = (\delta + \gamma_a', \delta + \gamma_b') = 2(\delta_{ab} - 2) \qquad (6.21)$$

$$(-\beta_a, \beta_b) = (-\delta - \beta_a', \delta + \beta_b') = 1 - \delta_{ab} \qquad (6.22)$$

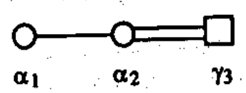
$$(-\beta_a, \gamma_b) = (-\delta - \beta_a', \delta + \gamma_b') = \delta_{ab} \qquad (6.23)$$

The fundamental SRS of P(3) is formed by $(\alpha_1, \alpha_2, \gamma_3)$. The positive roots will be

$$\begin{array}{ll} \alpha_3 = \alpha_1 + \alpha_2 & \beta_1 = \alpha_2 + \gamma_3 \\ \gamma_1 = 2\alpha_2 + \gamma_3 & \beta_2 = \alpha_2 + \alpha_2 + \gamma_3 \\ \gamma_2 = 2\alpha_1 + 2\alpha_2 + \gamma_3 & \beta_3 = \alpha_1 + 2\alpha_2 + \gamma_3 \end{array} \qquad (6.24)$$

while the negative roots will be the negative roots of A_2 and the states of the fundamental representation $(-\beta_a)$.

The associated DD is



(6.25)

To compute the links of γ_3 to α_1 and α_2 we have used our definition Eq. (3.27), i.e.

$$(\gamma_3, \alpha_1) = 0 \qquad (\gamma_3, \alpha_2) = -2 \qquad (6.26)$$

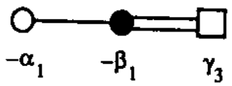
The corresponding Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \end{pmatrix} \qquad (6.27)$$

Making a Weyl reflection with respect to the fermionic root γ_3, we have a new SRS given by, according to Eq. (3.10)

$$\begin{aligned} W_{\gamma_3}(\gamma_3) &= \gamma_3 \\ W_{\gamma_3}(\alpha_2) &= -\alpha_2 - \gamma_3 = -\beta_1 \\ W_{\gamma_3}(\alpha_1) &= -\alpha_1 \end{aligned} \qquad (6.28)$$

The associated DD is



(6.29)

with Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 0 & -2 \end{pmatrix} \qquad (6.30)$$

The links (and the entries of the matrix Eq. (6.30)) have been computed by

$$(-\alpha_1, -\beta_1) = -1 \qquad (-\alpha_1, \gamma_3) = 0$$

and as

$$\{\beta_1^-, \beta_1^+\} = H_2 = -h_2$$

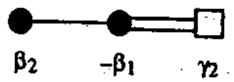
one obtains

$$(\gamma_3, \alpha_2) = -2 \qquad (6.31)$$

A reflection with respect to the fermionic root $-\beta_1$ gives, according to Eq. (3.11)

$$\begin{aligned} W_{-\beta_1}(-\beta_1) &= -\beta_1 \\ W_{-\beta_1}(-\alpha_1) &= \alpha_1 + \beta_1 = \beta_2 \\ W_{-\beta_1}(\gamma_3) &= -\gamma_3 + 2\beta_1 = \gamma_2 \end{aligned} \qquad (6.32)$$

The associated DD is



(6.33)

with Cartan matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \qquad (6.34)$$

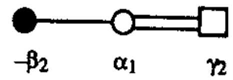
As

$$\{\beta_2^+, \beta_2^-\} = H_1' = h_1 + h_2$$

the links have been computed as

$$(\beta_2, -\beta_1) = 1 \qquad (\gamma_2, \alpha_1 + \alpha_2) = 0 \qquad (\gamma_2, -\alpha_2) = -2 \qquad (6.35)$$

A reflection with respect to the root β_2 gives the (last) non-equivalent SRS. From Eq. (3.11) we get



(6.36)

and Cartan matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -2 \end{pmatrix} \qquad (6.37)$$

the links being computed by

$$(-\beta_2, \alpha_1) = -1 \qquad (\alpha_1, \gamma_2) = -2 \qquad (6.38)$$

One can check that any further reflection with respect to fermionic roots will give simple root systems equivalent to the ones already obtained.

Drawing the box from DD Eq. (6.25) we get the subalgebra A_2 while drawing away the box from the DD's Eqs (6.29), (6.33) and (6.36), we get the two non-equivalent DD's of SU(2/1) (as the diagram obtained by Eq. (6.29) is equivalent to the one obtained by Eq.(6.36)).

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