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# BEAM MATCHING AND EMITTANCE GROWTH RESULTING FROM SPIRAL INFLECTORS FOR CYCLOTRONS

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A study is made of the emittance growth that results from the mismatch of the beam emittance with respect to the cyclotron eigenellipses, when the beam is injected by means of a spiral inflector. The matching procedure is described. Results are presented for various sizes and tilts of the inflector and for various shapes of the eigenellipses. In all cases, the best match is obtained when a rotation and a skew quadrupole are included in the transport system. Specific results for a 30 MeV $H^-$ cyclotron are also presented. Beam envelopes in the inflector are calculated, including linear space charge effects. An appendix contains an analytical treatment of the matching problem.

KEY WORDS: spiral inflector, inflectors, emittance growth, beam matching, cyclotron

## 1 INTRODUCTION

Spiral inflectors are used to bend an axially injected beam onto the median plane of a cyclotron. In recent years they have gained increased importance. This is due on one hand to progress in external ECR-sources, especially in connection with superconducting cyclotrons, and on the other hand to a strong interest in compact $H^-$ cyclotrons for isotope production.

Since the spiral inflector couples the two transverse phase spaces, an uncorrelated input beam will be correlated when it exits the inflector. For such a coupled transfer system, it is not possible to find an initially uncorrelated beam which matches perfectly to the cyclotron eigenellipses. In cyclotrons, betatron phases are different for different turns at one radial location. Hence, if turns are not separated at extraction, mismatch results in emittance growth. This is especially relevant if extraction is by stripping as in $H^-$ cyclotrons. Even in separated-turn cyclotrons, nonlinear forces arising, for example, from space charge can cause mismatch to contribute to emittance growth.

We used the computer code TRANSOPTR[1] to optimize the input beam for minimum emittance growth. The more commonly used code TRANSPORT is not useful in this application because the equations of motion through an inflector cannot be solved in closed form so a general transfer matrix cannot be specified. Moreover, since in TRANSOPTR the beam transport system is defined by a user supplied FORTRAN subroutine (hereafter this subroutine is referred to as SYSTEM), there are no limitations on the types of optimization or variable parameters which can be specified. This flexibility is convenient when it is required to minimize a mismatch and when it is known *a priori* that perfect matching cannot be obtained. Besides the numerical treatment, an analytical approach to the matching problem is given in Appendix A. A Hamiltonian derivation of the equations of motion inside the spiral inflector is presented in an accompanying paper.[2]

## 2 BEAM MATCHING WITH TRANSOPTR

### 2.1 *The sigma matrix formalism*

The computer code TRANSOPTR is based on the correlation or sigma matrix formalism. This matrix defines the shape of an ellipsoid in phase space. The elements of the sigma matrix are defined as the second order moments of the beam. Assuming that the injected beam is unbunched, it can be described by a 4-dimensional phase space. Denoting the coordinates as $x_1 = x$, $x_3 = y$ and the momenta as $x_2 = p_x$, $x_4 = p_y$, then the elements of the sigma matrix are defined as

$$\sigma_{ij} = \langle x_i x_j \rangle, \tag{1}$$

where the brackets denote averaging over the phase space.*

As usual, we define $p_x$ ($p_y$) as the angle that the particle trajectory makes with the $z$-axis when projected on the $xz$-plane ($yz$-plane). This is not necessarily the same as $x'$ ($y'$) as is shown, for example in our accompanying paper.[2] The formalism presented in that paper for the spiral inflector is fully six-dimensional and includes $x_5 = z$, the longitudinal position of a particle with respect to the reference particle and $x_6 = \Delta p/p_0$. Hence, it is applicable to cases involving bunched beams with energy spread, though here it is applied to unbunched, four-dimensional cases.

Usually the beam from the ion source is uncorrelated. Then the initial sigma matrix can be written as

$$\sigma_0 = \begin{pmatrix} X_0 & O \\ O & Y_0 \end{pmatrix}, \tag{2}$$

where $X_0$ and $Y_0$ are $2 \times 2$ matrices defined by

$$X_0 = \begin{pmatrix} \langle x^2 \rangle_0 & \langle x p_x \rangle_0 \\ \langle x p_x \rangle_0 & \langle p_x^2 \rangle_0 \end{pmatrix}, \tag{3}$$

* It is not essential to define the sigma matrix in this way; it is also possible to define it in terms of the smallest possible ellipsoid which contains all of the beam. The statistical definition given here is the most convenient when space charge forces are to be taken into account.[3]

and a similar expression for $Y_0$. $O$ is the $2 \times 2$ zero matrix.

The rms beam size is defined as $\langle x^2 \rangle^{1/2}$. For an unbunched beam with uniform elliptical cross-section the actual beam size is twice the rms value. The rms emittance in the $x$-$p_x$ phase plane is defined as

$$\epsilon_x = \left( \langle x^2 \rangle \langle p_x^2 \rangle - \langle x p_x \rangle^2 \right)^{1/2} . \tag{4}$$

A similar definition holds for the $y$-$p_y$ phase plane. Note that Eqs. (3) and (4) are consistent with the convention that $\epsilon_x = \det^{1/2}(X_0)$. For a uniform elliptical phase space distribution the actual phase space area is $4\pi$ times the rms emittance. Therefore, the 'ordinary' emittance is defined as four times the rms emittance.

Often, the initial emittances in the two transverse phase planes are equal. In some cases, the beam from the ion source has $x$-$y$ correlation. This occurs whenever the charged particles are created inside and expelled from a longitudinal magnetic field, as in an ECR source. In those cases the $\sigma$-matrix contains elements $\langle x p_y \rangle$ and $\langle y p_x \rangle$. If these correlations are not included in the transport calculation, the predicted beam envelopes are in general too large.[4]

In a beam transport system with a transfer matrix $M$, the sigma matrix transforms according to the following relation:

$$\sigma = M \sigma_0 M^T , \tag{5}$$

where $M^T$ is the transpose of $M$. However, when space charge effects must be included, this simple matrix multiplication is not appropriate since the space charge forces depend upon the properties of the beam itself, i.e. on $\sigma$. For dealing with space charge, TRANSOPTR employs the infinitesimal transfer matrix approach.[3,5] The infinitesimal transfer matrix $F(s)$ is defined as $(T - I)/ds$ where $T$ is the transfer matrix from $s$ to $s + ds$, $I$ is the identity matrix and $s$ denotes longitudinal position along the transport system. Then for individual particles we have

$$\frac{d\mathbf{x}}{ds} = F\mathbf{x}, \qquad \text{where} \qquad \mathbf{x} \equiv \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} . \tag{6}$$

For beams of particles the sigma matrix and the transfer matrix $M$ are calculated by numerical integration of the equations

$$\frac{d\sigma}{ds} = F\sigma - \sigma F^T , \tag{7}$$

$$\frac{dM}{ds} = FM . \tag{8}$$

With space charge, $F$ depends on $\sigma$ so that Eq. (7) is non-linear.

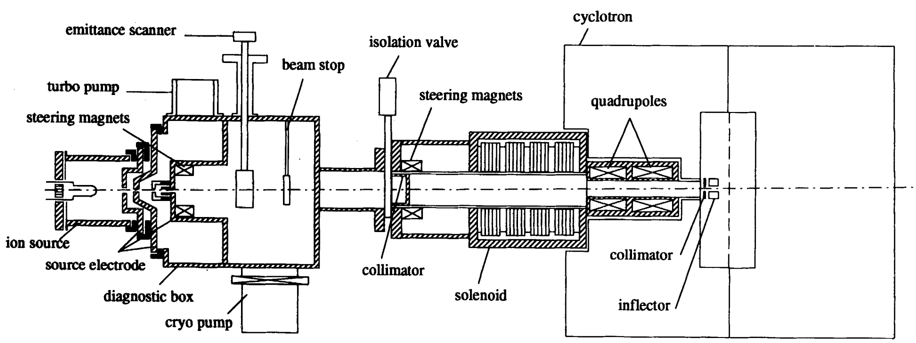


FIGURE 1: Lay-out of the experimental arrangement used for the Centre Region Model Cyclotron which was built as a full scale model of the TR30 centre region. Note that the horizontal orientation of the injection line pertains to the model only. For TR30 the injection line is vertical and the median plane horizontal.

### *2.2 Optimization parameters of the transport system*

The transport system assumed for this study consisted of ordinary (uncoupled) optics, a rotation, a skew quadrupole, the cyclotron axial magnetic field, and the inflector. Therefore, the overall transfer matrix from ion source to the first turn in the cyclotron has the following form,

$$M = [\mathtt{INFL}][\mathtt{MAGN}][\mathtt{SKEW}][\mathtt{TWIST}][\mathtt{ENV}]. \tag{9}$$

It is not essential that the transfer matrix explicitly be of this form: for example, the skew quad can occur elsewhere or the ordinary optics can be partly immersed in the axial magnetic field of the magnet bore. However, in terms of emittance growth, there is no loss of generality in assuming this form.

The matrix [ENV] contains four independent parameters. These can be thought of as the sizes and divergences in $x$ and $y$ of the uncoupled beam before the coupling optics. Alternatively, they can be thought of as the settings of the ordinary optical elements between the ion source and the coupling optics. For example, in the 30 MeV $H^-$ cyclotron TR30,[6] the injection line[7] consists of a solenoid and two quadrupoles. The four optimization parameters are the source-to-solenoid drift length and the strengths of the solenoid and the quads. For TR30 a 1 MeV Centre Region Cyclotron Model[8] was built to test the design of the critical components and to study the beam properties out to the 5th turn. Figure 1 shows a lay-out of the ion source and injection line which was used in this experimental set-up.

The matrix [TWIST] defined by

$$[\mathtt{TWIST}] = \begin{pmatrix} I\cos\theta & I\sin\theta \\ -I\sin\theta & I\cos\theta \end{pmatrix}, \tag{10}$$

where $I$ is the $2 \times 2$ identity matrix, is a rotation of the transverse axes by an angle $\theta$. For example, in the TR30 cyclotron, $\theta$ is varied by mechanically rotating the two quadrupoles, since the transport system up to this point is axially symmetric. Emittance growth was calculated both for cases where the rotation is not available ($\theta = 0$), and for cases where the quadrupoles are rotatable ($\theta$ optimized).

The matrix [SKEW] represents a thin skew (45° rotated) quadrupole. This matrix couples the transverse phase planes of the initially uncorrelated input beam and may therefore partly compensate the correlation introduced by the inflector. Calculations done with and without the skew quad show that it can indeed provide a considerably better match to the cyclotron. The matrix is given by

$$[\mathtt{SKEW}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k_s & 0 \\ 0 & 0 & 1 & 0 \\ k_s & 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

where $k_s$ is the strength (reciprocal focal length) of the quadrupole.

Before entering the inflector, the beam must pass into the axial magnetic field of the cyclotron. [MAGN] is the transfer matrix through the rising axial magnetic field to the inflector entrance. As shown in the accompanying paper,[2] it can be found by integrating Eq. (8) using an $F$-matrix which depends upon the choice of momenta. In the idealized hard-edge case with transverse velocity components as the momenta, [MAGN] is essentially the product of a solenoid entrance and a solenoid body (of appropriate length), with no exit. (See Banford[9] for entrance, exit, and body matrices.) If canonical momenta are used in the hard-edge case, [MAGN] is simply a solenoid, including both entrance and exit. This is because the matrix for hard-edge exit is equal to the matrix to transform transverse velocity components to canonical momenta.

Lastly, [INFL] is the transfer matrix of the device used to bend the beam onto the median plane of the cyclotron. There are many types of such devices, but we deal here with one called a spiral inflector. The simplest (i.e. non-tilted) spiral inflector is basically an electrostatic 90° bend with its electrodes deformed into a spiral so that the electric field remains orthogonal to the beam as it is also bent by the cyclotron's magnetic field. In the tilted case, the electrodes gradually tilt in such a way that the bending imparted by the magnetic field is supplemented by electric bending. This provides an additional parameter $k'$ (besides the electric bend radius, $A$, or height), which allows the designer to place the injected beam at the correct radial location in the cyclotron. The tilt parameter $k'$ is defined as the tangent of the angle between the magnetic field and the electric field at the inflector exit. [INFL] is determined by numerical integration of Eq. (8), with the $F$-matrix derived in the accompanying paper.[2] The $F$-matrix approach is used because in a spiral inflector there is a complicated dependence of the focusing forces on $s$, and so an analytical transfer matrix is not available. Note that for a fixed inflector design, if there is no space charge, the integration through the magnet bore and the inflector needs to be done only the first time that SYSTEM is called. Besides non-negligible space charge, the other type of TRANSOPTR run where a fixed transfer matrix cannot be specified for the inflector is where the inflector parameters themselves are used to optimize beam matching to the cyclotron. For example, if there is no size constraint for the inflector, $A$ can be left as a variable parameter with $k'$ specified in SYSTEM as a function of $A$ to give correct beam centering.[10]

*2.3 The mismatch parameter*

In the cyclotron, the circulating emittance in each plane is determined by the shape of the projected beam ellipse at the inflector exit and the shape of the cyclotron eigenellipse. Consider as an example the $x$-$p_x$ phase plane. The projected beam ellipse in this plane is described by the matrix

$$X = \begin{pmatrix} \langle x^2 \rangle & \langle xp_x \rangle \\ \langle xp_x \rangle & \langle p_x^2 \rangle \end{pmatrix}. \tag{12}$$

The phase space ellipse associated with this $2 \times 2$ sigma matrix is given by

$$\gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2 = \epsilon_x, \tag{13}$$

where the 'Twiss-parameters' $\gamma_x$, $\alpha_x$, and $\beta_x$ are defined as

$$\begin{aligned} \gamma_x &= \langle p_x^2 \rangle / \epsilon_x, \\ -\alpha_x &= \langle xp_x \rangle / \epsilon_x, \\ \beta_x &= \langle x^2 \rangle / \epsilon_x, \end{aligned} \tag{14}$$

with $\epsilon_x$ the emittance given in Eq. (4). Let the cyclotron eigenellipse in the $x$-$p_x$ phase plane be defined as

$$\gamma_{ex} x^2 + 2\alpha_{ex} x p_x + \beta_{ex} p_x^2 = \text{constant}, \tag{15}$$

with

$$\beta_{ex}\gamma_{ex} - \alpha_{ex}^2 = 1. \tag{16}$$

In cyclotrons which do not have separated turns at extraction, the effective circulating emittance is given by the area of the ellipse that encloses the beam ellipse and is similar to the eigenellipse (see Fig. 2). This emittance is given by[11]

$$\epsilon_{cx} = \left( \mathcal{D} + (\mathcal{D}^2 - 1)^{1/2} \right) \epsilon_x, \tag{17}$$

where $\mathcal{D}$ is defined as

$$\begin{aligned} \mathcal{D} &= \frac{1}{2}(\gamma_{ex}\beta_x - 2\alpha_{ex}\alpha_x + \beta_{ex}\gamma_x) \\ &= \frac{1}{2\epsilon_x} \left( \gamma_{ex}\langle x^2 \rangle + 2\alpha_{ex}\langle xp_x \rangle + \beta_{ex}\langle p_x^2 \rangle \right). \end{aligned} \tag{18}$$

Minimization of the circulating emittance is achieved by minimizing the mismatch parameter $\mathcal{D}$. The actual quantity that was minimized in the TRANSOPTR calculations is the sum of the transverse mismatch parameters, namely

$$\begin{aligned} \bar{\mathcal{D}} &= \bar{\mathcal{D}}_x + \bar{\mathcal{D}}_y \\ &= \gamma_{ex}\langle x^2 \rangle + 2\alpha_{ex}\langle xp_x \rangle + \beta_{ex}\langle p_x^2 \rangle + \gamma_{ey}\langle y^2 \rangle + 2\alpha_{ey}\langle yp_y \rangle + \beta_{ey}\langle p_y^2 \rangle. \end{aligned} \tag{19}$$

Note that minimizing $\bar{\mathcal{D}}_x + \bar{\mathcal{D}}_y$ is not exactly equivalent to minimizing $\epsilon_{cx} + \epsilon_{cy}$ because of the square root in Eq. (17). However, for all the cases studied, the emittance growth

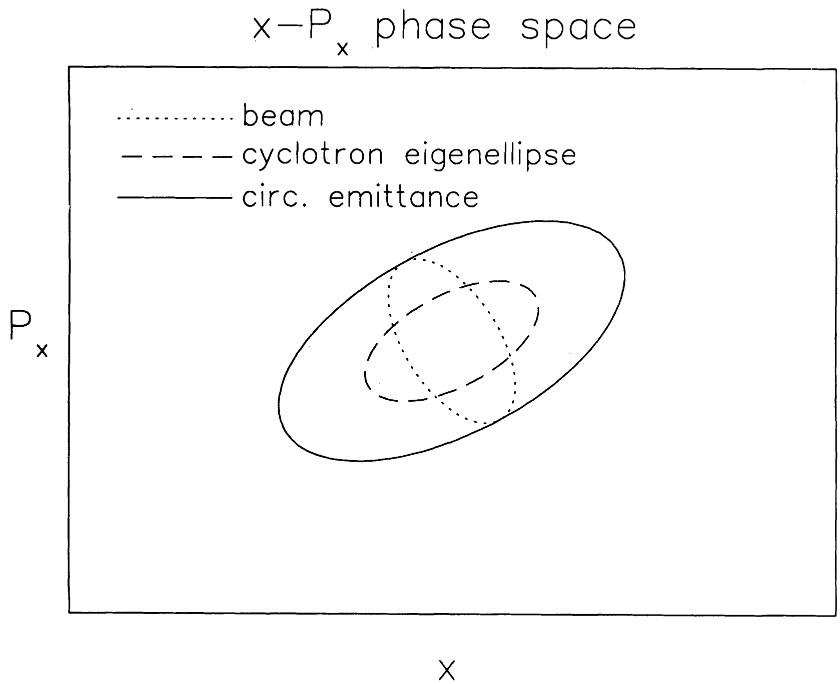


FIGURE 2: Emittance growth due to mismatch: the circulating emittance is the area/$\pi$ of the ellipse that is similar to the eigenellipse and that encloses the beam ellipse.

factors in both planes were found to be approximately equal so that Eq. (19) is a good approximation. For matching problems where there is a large difference between these two emittance growth factors, one must decide whether it is the sum of the emittances or the sum of the mismatch parameters which should be minimized. It should also be noted that any linear combination of $\bar{\mathcal{D}}_x$ and $\bar{\mathcal{D}}_y$ can be specified in TRANSOPTR as a parameter to be minimized. This is useful if different emittance growth factors in the two phase planes is tolerated.

## 3 RESULTS

The emittance growth depends both on the spiral inflector used and on the shape of the eigenellipses. The spiral inflector is characterized by two parameters, namely, $A/\rho$ and $k'$; whereas the cyclotron eigenellipses can be described by four parameters, namely, the ratio of the semi-axes of each ellipse and the orientation of each ellipse. The total parameter space is large and the calculation has therefore been restricted to a few examples. An upright radial eigenellipse with an aspect ratio determined by a radial oscillation frequency $\nu_r = 1$ has been used in all cases except the TR30 case. The dependence on vertical eigenellipse is studied in section 3.1. It is shown in Appendix A that for a given inflector design, fixing the radial eigenellipse and studying mismatch as a function

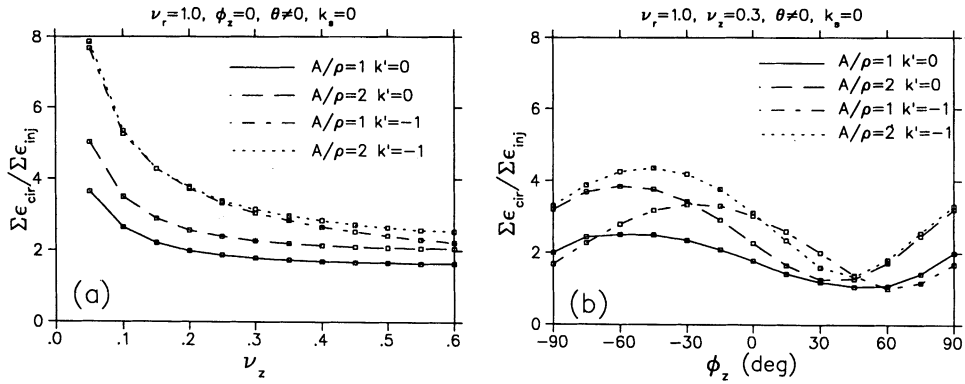


FIGURE 3: (a) Total emittance growth due to mismatch as a function of the vertical tune $\nu_z$ for a beam injected into a cyclotron with upright eigenellipses and $\nu_r = 1$. (b)Total emittance growth as a function of the orientation angle $\phi_z$ of the vertical eigenellipse. In both (a) and (b), the input beam was assumed to be uncorrelated ($k_s = 0$) but allowed to be rotated around its axis ($\theta \neq 0$).

of vertical eigenellipse involves no loss of generality since the radial eigenellipse can always be transformed into the unit circle. In section 3.2 the dependence of emittance growth on inflector parameters is considered. Calculations for the emittance growth for the TR30 cyclotron are presented in section 3.3.

In the following, the transverse divergences $p_x$ and $p_y$ have length units because they have been multiplied by $\rho$. This is a convenient normalization not only because then the orientation angle $\phi$ of an ellipse in phase space is well-defined, but also because then the eigenellipse for a tune of 1 in a flat magnetic field is a circle.

### *3.1 Dependence of emittance growth on eigenellipse*

Figure 3(a) gives the total emittance growth factor (i.e. summed over the two transverse directions) for an upright vertical eigenellipse as a function of the vertical oscillation frequency $\nu_z$, for four different inflectors. In all four cases the emittance growth becomes less with increasing $\nu_z$, indicating that strong vertical focusing is profitable. However, the curves clearly saturate with increasing $\nu_z$.

Figure 3(b) gives the total emittance growth as a function of the orientation of the vertical eigenellipse, $\phi_z$. As can be seen, there is strong variation with $\phi_z$ and for suitably chosen values, the emittance growth factor becomes close to 1. In practice however, it will be difficult to influence the shape and angle of the eigenellipse in order to match it to a given spiral inflector.

### *3.2 Dependence of emittance growth on inflector parameters*

Figure 4 gives the calculated total emittance growth factor as a function of the tilt parameter $k'$. Calculations were done for four values of $A/\rho$, namely 0.5, 1.0, 1.5 and 2.0. In practice, most inflectors have an $A/\rho$ value within this range. Matching was done into a cyclotron with a radial tune $\nu_r = 1$ and a vertical tune $\nu_z = 0.3$. For each inflector four cases were studied:

- no rotation or skew quadrupole ($\theta = 0, k_s = 0$),
- rotation optimized but no skew quadrupole ($\theta \neq 0, k_s = 0$),
- no rotation but skew quadrupole optimized ($\theta = 0, k_s \neq 0$),
- both rotation and skew quadrupole optimized ($\theta \neq 0, k_s \neq 0$).

The results show that for the first case the emittance growth factor may become unacceptably high. A significant improvement is obtained in the second case, where the matching quadrupoles are rotatable around their axis. The best result by far is obtained in the fourth case where the input beam is also allowed to be correlated by a skew quadrupole. In that case the remaining emittance growth is almost insignificant.

### *3.3 Emittance growth for the TR30 cyclotron*

TR30[6] is a four-sector, 30 MeV $H^-$ cyclotron. Near the centre the average magnetic field is 1.2 T, giving for the 25 keV injected beam a bend radius of $\rho$=1.9 cm. The source emittance is 50 $\mu$m (=phase space area/$\pi$). The eigenellipses were obtained by retracing eigenellipses at an outer radius backward to the exit of the inflector.[12] The parameters describing these ellipses are given in Fig. 5. The divergences in this figure have been multiplied by $\rho$. The inflector parameters $k'$ and $A/\rho$ could not be used to optimize the match, as they were fixed by space and centering constraints.[13] The tilt parameter of the TR30 inflector is $k' = -0.83$ and the height $A$=2.5 cm. The parameter $A/\rho$ was therefore assumed to be 1.32, although the magnetic field in the inflector is not completely homogeneous. Space charge was taken into account by adding the linear parts of the space charge forces to $F_{21}$, $F_{23}$, $F_{41}$ and $F_{43}$ in the inflector $F$-matrix, as described by de Jong.[5] Image charges in the inflector electrodes were not included in the calculation. With a beam size of about half the electrode spacing these charges constitute only about 10% of the total space charge force.

Figure 6 shows the inflector-induced total emittance growth as a function of the injected dc beam current (note that only $\approx$ 10% of this current is longitudinally accepted by the cyclotron). Clearly, at the few mA level, space charge only slightly modifies the $x$-$y$ coupling in the inflector. (Note that the modification of the cyclotron eigenellipses due to space charge has not been taken into account. One can infer from Fig. 3(a), however, that the effect will be small). For TR30, the quadrupoles are rotatable so that the curve labeled with $\theta \neq 0$ and $k_s = 0$ applies. At 5 mA an emittance growth factor of 5 is predicted from the calculations. This is in agreement with the measurement of $5 \pm 1$ obtained from beam profile measurements in the TR30 cyclotron Centre Region Model.[8]

Given in Fig. 7 are the beam envelopes (i.e. twice the rms size) in the inflector for zero current (no space charge) and for a 10 mA dc input beam, both with an input ($4\pi$ rms) phase space area of $50\,\pi\,\mu$m. The envelopes were plotted in the direction defined by the local orientation of the inflector electrodes: $\alpha_r$ is parallel to the electric field and $\beta_r$ is perpendicular to it (and to the central trajectory). This is convenient because the electrode can be plotted on the $\alpha_r$-envelope graph. This coordinate system is obtained

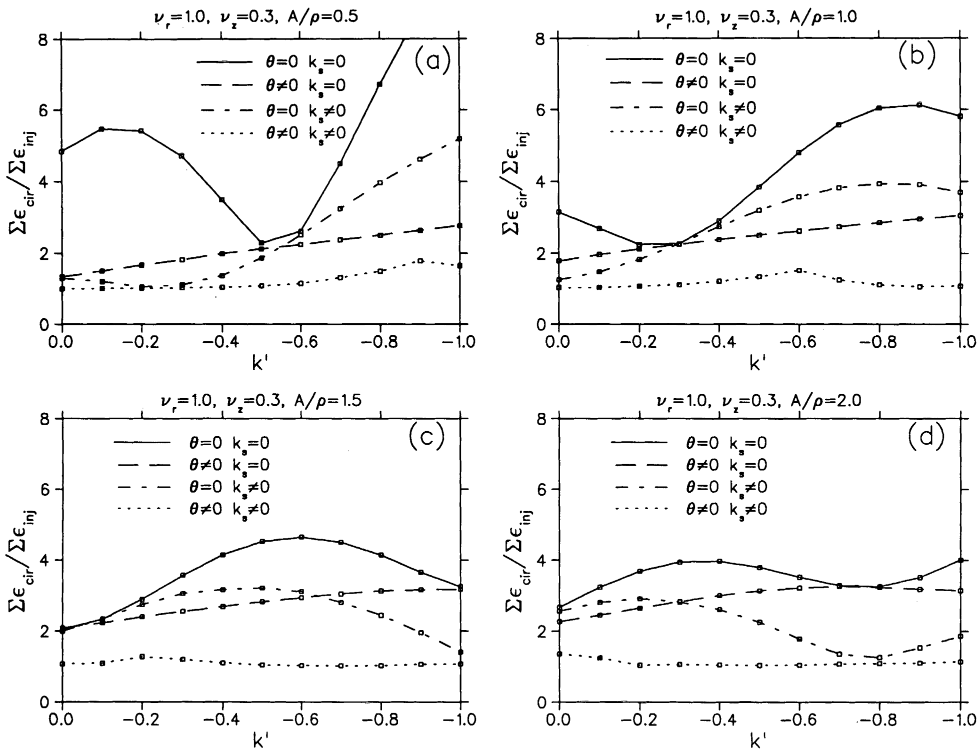


FIGURE 4: Total emittance growth as a function of the tilt parameter $k'$ and for four $A/\rho$ values: (a)$A/\rho = 0.5$, (b)$A/\rho = 1.0$, (c)$A/\rho = 1.5$, and (d)$A/\rho = 2.0$. For each inflector four different matching procedures were studied as explained in the text.

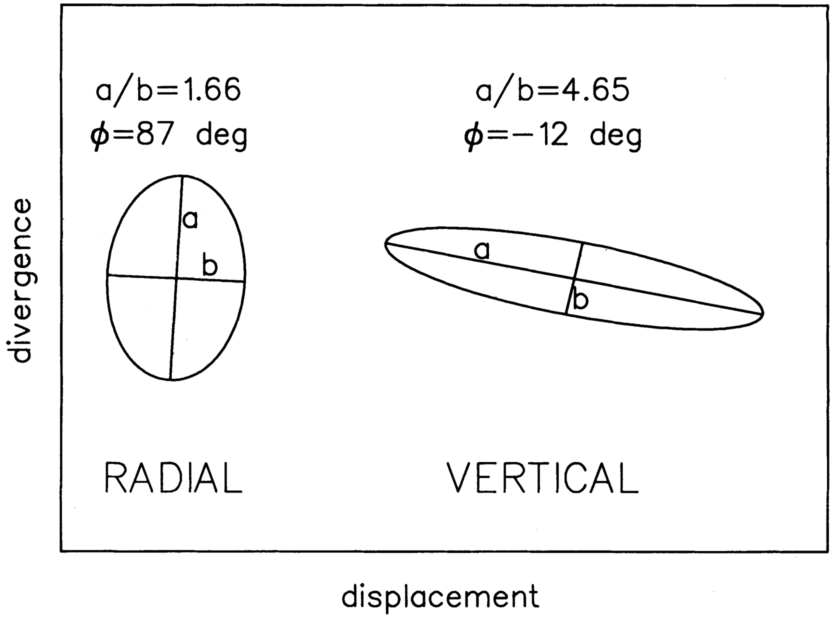


FIGURE 5: Radial and vertical TR30 eigenellipses at the inflector exit, as obtained from backward tracing of the eigenellipses at an outer radius. The divergences have been normalized with respect to the magnetic radius of the injected beam ($\rho$=1.9 cm) and the displacement and divergence length scales have been made commensurate.

from the optical coordinate system $(\alpha, \beta, \gamma)$ (as defined in the accompanying paper[2]) by a rotation by the tilt angle $\theta_t$, using the $\gamma$-axis as the axis of rotation.

Results are given for cases with and without skew quadrupole correlation. Note that the correlated beam gives the best match to the cyclotron but, on the other hand, it may result in a too large beam at the inflector entrance.

## 4 CONCLUSIONS

In a beam optical system with no coupling between the transverse directions, the emittances in the two transverse subspaces are conserved and only four parameters are needed to match the given transverse phase space ellipses to the corresponding eigenellipses. In a spiral inflector, there is strong coupling between the two transverse directions and even though the beam's four-dimensional phase space is conserved, the projected areas onto the transverse subspaces cannot be matched by using just the four parameters of an uncorrelated input beam. For optimized matching into a given cyclotron, the emittance growth factor is found to be sensitive to the inflector bend radius and tilt and to the orientation and aspect ratios of the cyclotron eigenellipses.

In the particular case of the TR30 cyclotron, emittance growth is typically a factor of 6 when using only the four usual matching parameters. If both a rotation and a skew

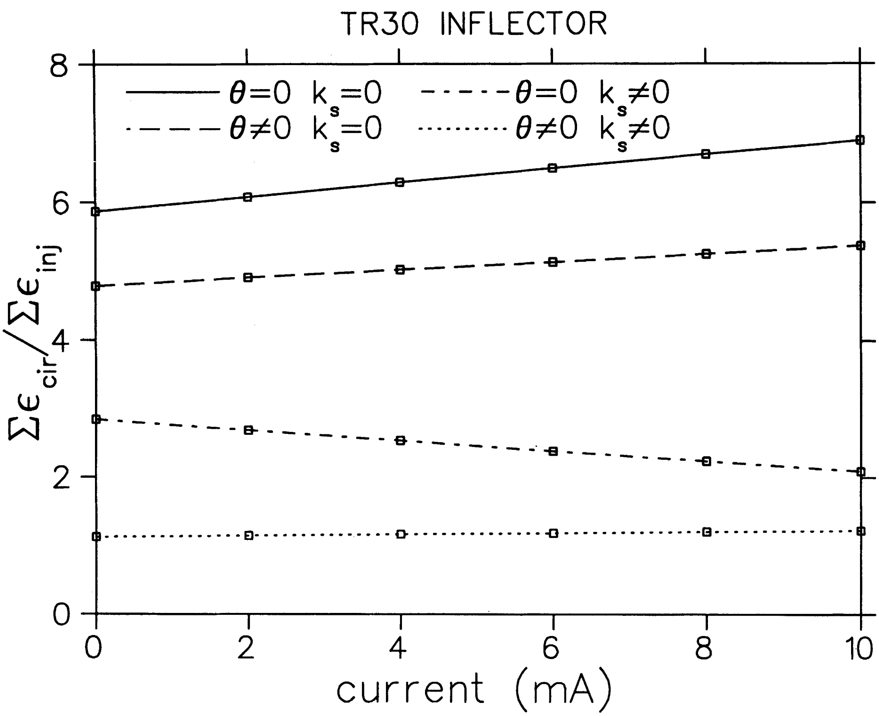


FIGURE 6: Total emittance growth as a function of the injected dc beam current. The curve labeled $\theta \neq 0$ and $k_s = 0$ applies to the situation actually realized in the TR30 cyclotron. Unlike the no space charge case, these growth factors are not independent of emittance, since space charge forces depend on beam size. The emittance used here is 50 $\mu$m (4 times rms).

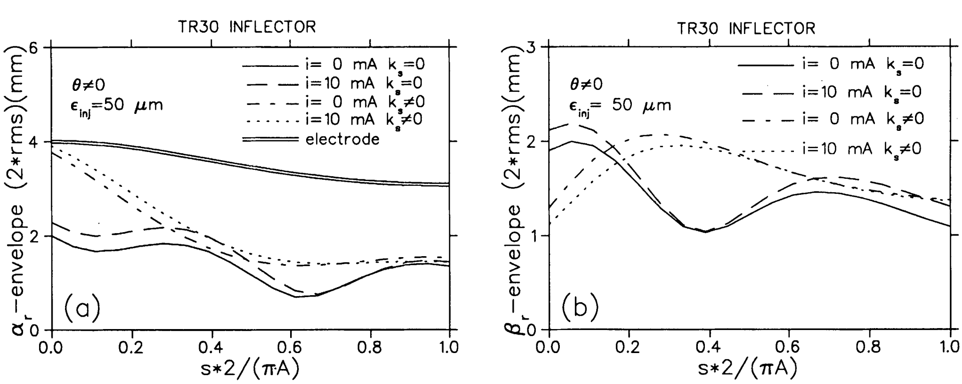


FIGURE 7: Beam envelopes (twice the rms-values) in the TR30 inflector for an input emittance of 50 $\mu$m (4 times rms). The $\alpha_r$-envelope (a) is measured normal to the electrode surfaces and the $\beta_r$-envelope (b) parallel to the electrode surfaces. The independent variable $s \div (\pi/2)A$ is the distance $s$ measured along the central trajectory normalized with respect to the total length $A(\pi/2)$ of the inflector.

quadrupole are optimized in addition to the four other parameters, the emittance growth drops to the 10-20% level, i.e. almost negligible.

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## APPENDIX: An Analytical Approach to Matching

### *A.1 Input beam optimization*

The mismatch function $\bar{\mathcal{D}}$ can be written as

$$\bar{\mathcal{D}} = \bar{\mathcal{D}}_x + \bar{\mathcal{D}}_y = \mathrm{Tr}(E_x X + E_y Y), \tag{20}$$

where Tr is the trace operator and where $E_x$ and $X$ are defined as

$$E_x = \begin{pmatrix} \gamma_{ex} & \alpha_{ex} \\ \alpha_{ex} & \beta_{ex} \end{pmatrix}, \quad X = \begin{pmatrix} \langle x^2 \rangle & \langle xp_x \rangle \\ \langle xp_x \rangle & \langle p_x^2 \rangle \end{pmatrix}, \tag{21}$$

and with similar definitions for $E_y$ and $Y$.

Let us now write the transfer matrix $M$ as follows,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{22}$$

where $A$, $B$, $C$ and $D$ are $2 \times 2$ matrices. With this definition of $M$ the submatrices $X$ and $Y$ become

$$\begin{aligned} X &= AX_0A^T + BY_0B^T, & (23)\\ Y &= CX_0C^T + DY_0D^T, & (24)\end{aligned}$$

where $X_0$ and $Y_0$ represent the uncorrelated input beam.

We substitute these expressions for $X$ and $Y$ into Eq. (20) and note that the trace of the product of two matrices does not depend on the multiplication order. The function $\bar{\mathcal{D}}$ can therefore be written as:

$$\bar{\mathcal{D}} = \mathrm{Tr}(PX_0 + QY_0), \tag{25}$$

with $P$ and $Q$ defined by

$$\begin{aligned} P &= A^TE_xA + C^TE_yC, & (26)\\ Q &= B^TE_xB + D^TE_yD. & (27)\end{aligned}$$

Expanding $\bar{\mathcal{D}}$ in Eq. (25) gives

$$\begin{aligned} \bar{\mathcal{D}} &= P_{11}\langle x^2\rangle_0 + 2P_{12}\langle xp_x\rangle_0 + P_{22}\langle p_x^2\rangle_0 \\ &+ Q_{11}\langle y^2\rangle_0 + 2Q_{12}\langle yp_y\rangle_0 + Q_{22}\langle p_y^2\rangle_0. \end{aligned} \tag{28}$$

This expression for $\bar{\mathcal{D}}$ is the sum of two terms, the first one depending only on $\langle x^2\rangle_0$, $\langle xp_x\rangle_0$ and $\langle p_x^2\rangle_0$, and the second one depending only on $\langle y^2\rangle_0$, $\langle yp_y\rangle_0$ and $\langle p_y^2\rangle_0$. These two terms can therefore be minimized independently. This minimization must, however, be done such that the initial emittances remain constant, i.e.

$$\epsilon_{x0}^2 = \langle x^2\rangle_0\langle p_x^2\rangle_0 - \langle xp_x\rangle_0^2 = \text{constant} \tag{29}$$

and a similar expression for $\epsilon_{y0}^2$. The easiest way to do this is with the technique of Lagrange multipliers where we minimize the function

$$\bar{\mathcal{D}}_{x0} = P_{11}\langle x^2\rangle_0 + 2P_{12}\langle xp_x\rangle_0 + P_{22}\langle p_x^2\rangle_0 + \lambda(\langle x^2\rangle_0\langle p_x^2\rangle_0 - \langle xp_x\rangle_0^2 - \epsilon_{x0}^2) \tag{30}$$

with respect to the initial moments. This leads to three equations for $\langle x^2\rangle_0$, $\langle p_x^2\rangle_0$ and $\langle xp_x\rangle_0$, with the parameter $\lambda$ as an unknown variable. This parameter can be eliminated however, by substituting the solution into Eq. (29). The procedure is repeated for the other transverse direction. The solution is

$$\begin{aligned} X_0 &= \epsilon_{x0}\det{}^{1/2}(P)P^{-1}, & (31)\\ Y_0 &= \epsilon_{y0}\det{}^{1/2}(Q)Q^{-1}. & (32)\end{aligned}$$

With the input beam known, the beam circulating in the cyclotron follows from Eq. (5) and the circulating emittances follow from Eqs. (17) and (18).

The minimum value of $\bar{\mathcal{D}}$ is obtained by substituting Eqs. (31) and (32) into (25). This gives:

$$\bar{\mathcal{D}}_{\min} = 2\epsilon_{x0}\det^{1/2}(P) + 2\epsilon_{y0}\det^{1/2}(Q). \tag{33}$$

By Brown and Servranckx[14] it has been shown that when $M$ is symplectic, the determinants of $P$ and $Q$ are equal. Therefore, Eq. (33) becomes

$$\bar{\mathcal{D}}_{\min} = 2(\epsilon_{x0} + \epsilon_{y0})\det^{1/2}(A^T E_x A + C^T E_y C). \tag{34}$$

Thus far it has been assumed that the input beam is uncorrelated. Let us now assume that the input beam is allowed to be rotated over an angle $\theta$ so that this angle is an additional optimization parameter. This can be taken into account by multiplying $M$ in Eq. (22) by the rotation matrix given in Eq. (10). The matrices $P$ and $Q$ in Eq. (28) now depend upon $\theta$. Since $\theta$ is a new independent variable and it does not matter in which order the variables are minimized, we can optimize $\theta$ by solving

$$\frac{d\bar{\mathcal{D}}_{\min}}{d\theta} = 0, \tag{35}$$

with $\bar{\mathcal{D}}_{\min}$ given by Eq. (34).

A similar equation applies if we wish to optimize a skew quadrupole instead of a rotation. In that case the differentiation is to be done with respect to $k_s$ and the submatrices $A$ and $C$ depend upon $k_s$.

### *A.2 Invariance of the matching problem to eigenellipse transformations*

Another interesting point to be mentioned is the invariance of the emittance growth problem for certain transformations applied to the eigenellipses. Let us consider an uncoupled symplectic transformation $U$ defined by

$$U = \begin{pmatrix} U_x & O \\ O & U_y \end{pmatrix}, \tag{36}$$

where $U_x$ and $U_y$ are symplectic $2 \times 2$ matrices. It can be shown that if $U_x$ and $U_y$ obey the condition

$$A^{-1} U_x A = C^{-1} U_y C, \tag{37}$$

then $U^{-1}$ 'commutes' with the transfer matrix $M$ defined in Eq. (22), in the sense that

$$U^{-1} M = MT, \tag{38}$$

where $T$ is the uncoupled symplectic transformation

$$T = \begin{pmatrix} A^{-1} U_x^{-1} A & O \\ O & D^{-1} U_y^{-1} D \end{pmatrix}. \tag{39}$$

This means that we can write for the transfer matrix

$$M = UU^{-1}M = UMT. \tag{40}$$

Since $T$ is an uncoupled transformation, it will not change the result of the minimization process for uncoupled initial beams. This proves that the problem is invariant for transformations $U$ applied to the cyclotron eigenellipses, when $U$ satisfies Eq. (37). The matrix $U_x$ can now be chosen such that it transforms $E_x$ into the unit circle. The new $y$-eigenellipse is fully specified by two parameters, namely, the ratio of the two semi-axes and the orientation of the ellipse. This means that for a given transfer matrix $M$ (i.e. for a given inflector), two parameters are sufficient to describe the cyclotron eigenellipses, as far as the emittance growth is concerned.