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A Gap Theorem For Half-Conformally-Flat 4-Manifolds

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A Gap Theorem For Half-Conformally-Flat 4-Manifolds

Abstract
Given a smooth, compact manifold, an important question to ask is, what are the ``best'' metrics that it admits. A reasonable approach is to consider as ``best'' metrics those that have the least amount of curvature possible. This leads to the study of canonical metrics, that are defined as minimizers of several scale-invariant Riemannian functionals. In this dissertation, we study the minimizers of the Weyl curvature functional in dimension four, which are precisely half-conformally-flat metrics. Extending a result of LeBrun, we show an obstruction to the existence of ``almost'' scalar-flat half-conformally-flat metrics in terms of the positive-definite part of its intersection form. On a related note, we prove a removable singularity result for Hodge-harmonic self-dual 2-forms on compact, anti-self-dual Riemannian orbifolds with non-negative scalar curvature.

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A GAP THEOREM FOR HALF-CONFORMALLY-FLAT 4-MANIFOLDS

Martin Citoler-Saumell

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Supervisor of Dissertation

Brian J. Weber, Asst. Professor of Mathematics

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ABSTRACT

A GAP THEOREM FOR HALF-CONFORMALLY-FLAT 4-MANIFOLDS

Martin Citoler-Saumell

Brian J. Weber

Given a smooth, compact manifold, an important question to ask is, what are the "best" metrics that it admits. A reasonable approach is to consider as "best" metrics those that have the least amount of curvature possible. This leads to the study of canonical metrics, that are defined as minimizers of several scale-invariant Riemannian functionals. In this dissertation, we study the minimizers of the Weyl curvature functional in dimension four, which are precisely half-conformally-flat metrics. Extending a result of LeBrun, we show an obstruction to the existence of "almost" scalar-flat half-conformally-flat metrics in terms of the positive-definite part of its intersection form. On a related note, we prove a removable singularity result for Hodge-harmonic self-dual 2-forms on compact, anti-self-dual Riemannian orbifolds with non-negative scalar curvature.

Contents

Chapter 1

Introduction and background

1.1 Introduction

The classification of smooth manifolds is perhaps one of the most fundamental problems in differential geometry and, closely related, we have the following question.

Question (René Thom)**.** Given a smooth manifold M, what are the "best" Riemannian metrics that it admits? When do they exist?

First we need to decided what do we mean by "best" metrics. Arguably, what Thom had in mind was the uniformization theorem, which says that any closed 2-manifold admits a metric of constant sectional curvature with a specific sign. One can go a little further and consider the resolution of geometrization conjecture, where Einstein manifolds played important role as the building blocks of 3-dimensional manifolds. In any case, it stands to reason that the notion of "best" metric should include constant sectional curvature metrics and Einstein metrics.

The next natural step is to look into 4-manifolds. Unfortunately, in dimension 4 the situation is more complicated and there is no geometrization program in place at the current time. One of the main difficulties is that the geometry is not really controlled. For example, it is well known that any finitely presented group can be realized as the fundamental group of a 4-manifold. As a result, the classification of 4-manifolds seems like a daunting endeavor at best. However, we can still approach Thom's question on 4-manifolds and, hopefully, shed some light onto the problem.

For the rest of this section we assume that M is a smooth, closed, oriented 4-manifold. Recall the decomposition of the Riemannian curvature tensor in dimension 4

$$Rm = W^+ + W^- + \frac{1}{2}\overset{\circ}{Ric} \owedge g + \frac{scal}{24} g \owedge g.$$

As a general rule, flat metrics are considered quite desirable whenever they exist so we should ask "best" metrics to have the least amount of curvature possible in some sense. More specifically, our definition of "best" metrics is those that are the minimizers (or critical points) of scale-invariant Riemannian functionals,

$$\mathcal{F}_C : g \mapsto \int_M |C|^2 \, dV,$$

where C is one of the curvature quantities appearing in the decomposition of Rm. For example, Einstein metrics, whenever they exist, are critical points of all these functionals and they actually minimize $\mathcal{F}_{Rm}$. Indeed, from the Chern-Gauss Bonnet theorem

$$\chi(M) = \frac{1}{32\pi^2}\int_M \left(|W^+|^2 + |W^-|^2 - 2|\overset{\circ}{Ric}|^2 + \frac{scal^2}{6}\right) dV,$$

we can compute

$$\int_M |Rm|^2\, dV = 32\pi^2\chi(M) + 4\int_M |\overset{\circ}{Ric}|^2 dV \geq 32\pi^2\chi(M),$$

with equality only for Einstein metrics. In a similar fashion we can find other minimizers of $\mathcal{F}_{Rm}$. Using Hirzebruch's signature theorem

$$\tau(M) = \frac{1}{48\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) dV,$$

we obtain

$$\begin{aligned}\int_M |Rm|^2\, dV &= -32\pi^2\left(\chi(M) + 3\tau(M)\right) + 4\int_M \left(|W^+| + \frac{scal^2}{12}\right) dV \\ &\geq -32\pi^2\left(\chi(M) + 3\tau(M)\right),\end{aligned} \tag{1.1}$$

with equality only if $W^+ \equiv 0$ and $scal \equiv 0$. This means that anti-self-dual scalar-flat metrics also minimize $\mathcal{F}_{Rm}$. As a result, we have some topological obstructions to the existence of such metrics. It follows that if (M, g) is Einsten, then $\chi(M) \geq 0$ and if (M, g) is anti-self-dual scalar-flat, then $\chi(M) + 3\tau(M) \leq 0$. Along the same lines, we can compute

$$2\chi(M) + 3\tau(M) = \frac{1}{8\pi^2}\int_M \left(|W^+|^2 - |\overset{\circ}{Ric}|^2 + \frac{scal^2}{12}\right) dV,$$

which yields the Hitchin-Thorpe inequality, $2\chi(M) + 3\tau(M) \geq 0$, for Einstein manifolds and its reverse, $2\chi(M) + 3\tau(M) \leq 0$, for anti-self-dual scalar-flat manifolds.

In this dissertation we are mainly concerned with the *Weyl functional*, $\mathcal{F}_W$, which we will usually denote by $\mathcal{W}$. Proceeding as above, we can find minimizers as well. By Hirzebruch's signature theorem,

$$\mathcal{W}(g) = \mp 48\pi^2\tau(M) + 2\int_M |W^{\pm}|^2 dV,$$

which implies that metrics satisfying either $W^+ \equiv 0$ or $W^- \equiv 0$ are minimizers of $\mathcal{W}$. Since changing orientation exchanges W^+ and W^-, we sometimes call these metrics *half-conformally-flat*. For simplicity we will work with anti-self-dual metrics but our results are valid for self-dual metrics as well. As before, we also obtain a topological obstruction to the existence of anti-self-dual metrics. Namely, if (M, g) is anti-self-dual, then $\tau(M) \leq 0$. In fact, not many other topological obstructions are known and, in light of the following result of Taubes, not many are expected.

Theorem 1.1 ([Tau92]). *Let M be a closed 4-manifold. Then $M\#k\overline{\mathbb{CP}}^2$ admits an anti-self-dual metric for all sufficiently large k.*

Nonetheless, if we constrain the scalar curvature, there are some other topological obstructions to the existence of anti-self-dual metrics. Of course, if we actually have $scal \equiv 0$, we are in the previous situation and we have the reversed Hitchin-Thorpe inequality. In addition to this, LeBrun proved in [LeB86] the following result, which we quote as it appears in [LeB04, Proposition 3.5].

Theorem 1.2. *Let M be a closed, simply-connected 4-manifold that admits an anti-self-dual scalar-flat metric. Then one of the following holds:*

- *M is homeomorphic to $k\overline{\mathbb{CP}}^2$ for some $k \geq 5$; or*

- *M is diffeomorphic to $\mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2$ for some $k \geq 10$; or else*

- *M is diffeomorphic to $K3$.*

The main ingredient in the proof of this theorem is the following proposition, which gives a restriction on b_+ for anti-self-dual metrics with non-negative scalar curvature.

Proposition 1.3 ([LeB86, Proposition 1])**.** *Let (M, g) be a closed, anti-self-dual 4-manifold with non-negative scalar curvature. Then one of the following holds:*

- *$b_+(M) = 0$; or*

- *$b_+(M) = 1$ and g is scalar-flat Kähler; or else*

- *$b_+(M) = 3$ and g is hyper-Kähler.*

Recall that $b_+(M)$ can be defined as the dimension of the space of Hodge-harmonic self-dual 2-forms on M. The proof of this proposition is an application of the Böchner technique to prove that the 2-forms representing $b_+(M)$ are parallel. As usual, these kind of arguments rely heavily on the non-negativity of some curvature quantity, in this case $scal \geq 0$.

Finally, if one allows the scalar curvature to become negative, not much is known. However, if one considers small negative scalar curvature, there is some hope. Heuristically, the representatives of b_+ are *almost* parallel, which should suffice to obtain a similar result. In this dissertation we prove an obstruction theorem to the existence of "almost" scalar-flat anti-self-dual metrics.

Theorem 1.4. *Fix any $m \geq 2$ and let (M, g) be a closed, unit-volume, anti-self-dual 4-manifold with $\pi_1(M) = 0$. Suppose that there are constants $E_0 < \infty$, $V_0 < \infty$, $S_0 < \infty$, and $C_S > 0$ such that*

$$\left(\int_M |Rm|^2 dV\right)^{\frac{1}{2}} \leq E_0, \tag{1.2}$$

$$\|scal\|_{W^{m+2,4}(M)} \leq S_0, \tag{1.3}$$

$$\mathrm{Vol}\left(B_r(x)\right) \leq V_0 r^4 \text{ for all } x \in M \text{ and } r > 0, \tag{1.4}$$

and

$$C_S\left(M\right) \leq C_S. \tag{1.5}$$

Then there is a constant $\delta_0 = \delta_0(E_0, V_0, S_0, C_S) > 0$ such that if $\|scal\|_{L^1(M)} < \delta_0$, then we have $b_+(M) \leq 3$.

Remark 1.5. *The choice of m in the statement above is rather inconsequential as it only affects the regularity of some convergence results that appear during the course of the proof. One might set $m = 2$ and work with $\|scal\|_{W^{4,4}(M)}$, anyway, the statement includes $m \geq 2$ to highlight the fact that this m is not optimal. In fact, by tweaking some of the arguments, it is possible to only ask for $\|scal\|_{W^{2,4}} \leq S_0$ but we shall not delve into this issue for the sake of a better exposition.*

Now we give a brief description of the methods in the proof. In an argument by contradiction, our result amounts to understanding the convergence theory for anti-self-dual manifolds with small scalar curvature and big b_+. We build on the work of Tian-Viaclovsky [TV05a, TV05b, TV08] (cf. [And05]), where they prove an orbifold compactness theorem for anti-self-dual manifolds with constant scalar curvature. In our case, we obtain limit spaces that are manifolds except for finitely many singular points which might not be orbifold singularities. Nonetheless, this is good enough to obtain parallel Hodge-harmonic self-dual 2-forms on the regular part of the limit space and, similar to the case of smooth manifolds, not many can be supported.

On a related direction, we are able to prove a removable singularity result for Hodge-harmonic self-dual 2-forms on compact Riemannian orbifolds with isolated singularities.

Proposition 1.6. *Let (X, g) be a compact, oriented, smooth Riemannian orbifold with isolated singularities. Also assume that the orbifold metric is anti-self-dual with non-negative scalar curvature. Let X_R denote the regular set of the orbifold. If η is a Hodge-harmonic self-dual 2-form in $L^2(X_R)$, then η must be parallel. In particular, it can be extended across orbifold singularities. Further, if $\eta \not\equiv 0$, then the orbifold is scalar-flat.*

The rest of the dissertation is organized as follows. In the remaining of Chapter 1, we cover some background material that is needed in the later chapters. This includes some basic definitions; important properties of anti-self-dual metrics and 4-manifolds; and the rudiments of the regularity theory for elliptic inequalities. Even though we concentrate our attention to dimension 4, most of the material has analogous formulations in arbitrary dimensions.

In Chapter 2, we prove a geometric ε-regularity theorem for anti-self-dual 4-manifolds. Roughly speaking, it says that in regions where $\|Rm\|_{L^2}$ is sufficiently small, we actually have bounds for $\|Rm\|_{L^\infty}$. Essentially, this comes from the fact that anti-self-dual metrics satisfy a system of elliptic equations. This kind of result is the most important ingredient to understand the convergence theory.

Lastly, Chapter 3 is devoted to give the full details of the proofs of Theorem 1.4 and Proposition 1.6.

1.2 Notation and definitions

Suppose M is a smooth n-manifold and let g be a smooth Riemannian metric on M. We use Rm, Ric and $scal$ to denote, respectively, the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature. Our sign convention is such that the curvature $(0,4)$-tensor is given by

$$\langle R(X,Y)Z, W\rangle = \left\langle \nabla_X \left(\nabla_Y Z\right) - \nabla_Y \left(\nabla_X Z\right) - \nabla_{[X,Y]} Z, W \right\rangle ,$$

where ∇ denotes the Levi-Civita connection. In coordinates, Ric is given by $R_{jk} = g^{il} R_{ijkl}$, and the scalar curvature by, $scal = g^{jk} R_{jk}$. The curvature tensor has nice symmetries that can be expressed by

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}.$$

In fact, given two $(0,2)$-tensors we can use the *Kulkarni-Nomizu product* to produce a $(0,4)$-tensor with curvature symmetries, in coordinates it is given by the following expression,

$$(T \owedge S)_{ijkl} = T_{jk}S_{il} + T_{il}S_{jk} - T_{ik}S_{jl} - T_{jl}S_{ik}.$$

We also use the *comma notation* to denote the covariant derivatives of a tensor. Namely, if T is a (p,q)-tensor, the m-th covariant derivatives has components,

$$(\nabla^m T)_{k_1 \dots k_m i_1 \dots i_p j_1 \dots j_q} = \nabla_{k_1} \cdots \nabla_{k_m} T^{i_1 \dots i_p}_{j_1 \dots j_q} = T^{i_1 \dots i_p}_{j_1 \dots j_q, k_m \dots k_1}.$$

Since we can use the metric g to identify TM with T^*M, from now on we often write only subindices. Further, unless otherwise specified, we always use geodesic normal coordinates centered at some point p to write down coordinate expressions and we also adopt the convention of summing over repeated indices. For example, recall the Ricci identities that express the commutator formulas for covariant derivatives. If T is a $(0,2)$-tensor we have

$$T_{ij,kl} - T_{ij,lk} = R_{lkmi}T_{mj} + R_{lkmj}T_{im}. \tag{1.6}$$

Another important fact that is implicitly used is that Riemannian metrics induce inner products on all tensor bundles. In coordinates, given T and S two (p,q)-tensors, we have

$$\langle T, S \rangle = g_{i_1 k_1} \cdots g_{i_p k_p} g^{j_1 l_1} \cdots g^{j_q l_q} T^{i_1 \dots i_p}_{j_1 \dots j_q} S^{k_1 \dots k_p}_{l_1 \dots l_q}.$$

Finally, associated to the metric we can define several operators acting on the differential k-forms on M, which we denote by Λ^k. We have the Hodge-star operator,

$$*: \Lambda^k \to \Lambda^{n-k},$$

which is given by

$$\omega \wedge *\psi = \langle \omega, \psi \rangle dV,$$

where $\omega, \psi \in \Lambda^k$ and dV denotes the volume form; the *Hodge Laplacian*,

$$\Delta_H = \delta d + d\delta,$$

where d denotes the exterior derivative and $\delta = (-1)^{nk+1} * d*$ is the divergence operator, which also is the L^2-adjoint of d; and the *connection Laplacian*,

$$\Delta = \mathrm{tr}(\nabla^2) = g^{ij}\nabla_i \nabla_j,$$

where tr denotes tracing over the first two indices.

1.2.1 Sobolev constants and inequalities

In this dissertation we make repeated use of Sobolev-type inequalities and bounds on the corresponding Sobolev constants. For a compact Riemannian manifold, (M, g), we define the *Sobolev constant* as the smallest positive constant $C_S(M)$ such that

$$\left(\int_M u^{2\gamma} dV\right)^{\frac{1}{\gamma}} \leq C_S(M) \int_M |\nabla u|^2 \, dV + \frac{1}{\mathrm{Vol}\,(M,g)^{\frac{2}{n}}} \int_M u^2 dV, \tag{1.7}$$

for all $u \in \mathcal{C}^1(M)$, where $\gamma = \frac{n}{n-2}$. However, in most cases we are only interested in the local behavior and we can use a local version of the inequality above. For a domain $\Omega \subset M$ we define the *local Sobolev constant* as the biggest positive constant $C_S(\Omega)$ such that

$$C_S(\Omega)\left(\int_\Omega u^{2\gamma}dV\right)^{\frac{1}{\gamma}} \leq \left(\int_\Omega |\nabla u|^2 dV\right)^{\frac{1}{2}}, \tag{1.8}$$

for all $u \in \mathcal{C}_c^1(\Omega)$.

The relation between these two constants is that a bound on the Sobolev constant implies a bound on the local Sobolev constant for domains with sufficiently small volume.

Lemma 1.7. *Let (M, g) be a compact Riemannian manifold and let $\Omega \subset M$ be some domain. Then for any $c \in (0,1)$ we have that* $\mathrm{Vol}(\Omega) \leq c^{\frac{n}{2}} \mathrm{Vol}(M, g)$ *implies* $C_S(\Omega) \geq \frac{c}{C_S(M)}$.

Proof. Given $u \in \mathcal{C}_c^1(\Omega)$, we can extend it to be identically zero on $M \setminus \Omega$ and apply (1.7) to obtain

$$\left(\int_\Omega u^{2\gamma}dV\right)^{\frac{1}{\gamma}} \leq C_S(M)\int_\Omega |\nabla u|^2 dV + \frac{1}{\mathrm{Vol}\,(M,g)^{\frac{2}{n}}}\int_\Omega u^2 dV.$$

By Hölder's inequality

$$\frac{1}{\mathrm{Vol}\,(M,g)^{\frac{2}{n}}}\int_\Omega u^2 dV \leq \frac{\mathrm{Vol}\,(\Omega)^{\frac{2}{n}}}{\mathrm{Vol}\,(M,g)^{\frac{2}{n}}}\left(\int_\Omega u^{2\gamma}dV\right)^{\frac{1}{\gamma}},$$

so the condition on $\mathrm{Vol}(\Omega)$ implies

$$\frac{c}{C_S(M)}\left(\int_\Omega u^{2\gamma}dV\right)^{\frac{1}{\gamma}} \leq \int_\Omega |\nabla u|^2 dV.$$

$\square$

1.3 Anti-self-dual 4-manifolds

In this section we give the definition of anti-self-dual metrics and show how to realize the signature of a 4-manifold using subspaces of Hodge-harmonic 2-forms. We also show how to obtain elliptic equations and inequalities for anti-self-dual metrics.

Suppose (M, g) is a Riemannian n-manifold. By the curvature symmetries, we can think of Rm as an element of the second symmetric power of the bundle of 2-forms, i.e. $Rm \in S^2(\Lambda^2)$. Since Rm satisfies the Bianchi identity, it is in the kernel of the Bianchi map, $b : S^2(\Lambda^2) \to S^2(\Lambda^2)$, which is given by the expression

$$b(R)(X,Y,Z,T) = \frac{1}{3}\left(R(X,Y,Z,T) + R(Y,Z,X,T) + R(Z,X,Y,T)\right).$$

In fact, by studying the representation theory of the set of *algebraic curvature tensors*, $\ker\left(b\big|_{S^2(\Lambda^2)}\right)$, as an $O(n)$-module, we can obtain an orthogonal decomposition for the curvature (0,4)-tensor (see [Bes08, Chapter 1.G] for a more detailed account[1]),

$$Rm = W + \frac{1}{n-2}\mathring{Ric} \owedge g + \frac{scal}{2n(n-1)} g \owedge g, \tag{1.9}$$

where $\mathring{Ric}$ is the *traceless Ricci tensor*, $\mathring{Ric} = Ric - \frac{scal}{n} g$, and W is the *Weyl tensor*. It is important to note that both $\mathring{Ric}$ and W are traceless.

Now we restrict our attention to dimension 4 where the Hodge-star operator acts on 2-forms, $* : \Lambda^2 \to \Lambda^2$. It is straightforward to check that $*$ is self-adjoint and satisfies $*^2 = 1$. This implies that we have an orthogonal decomposition

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^-,$$

[1] Beware that both our curvature tensor and our $\owedge$ have the opposite sign to those in [Bes08].

where $\Lambda^{\pm}$ are the ± 1-eigenspaces of $*$. We call elements in Λ^{+} *self-dual* 2-forms and elements in Λ^{-} *anti-self-dual* 2-forms. It turns out that $W(\Lambda^{\pm}) \subset \Lambda^{\pm}$ (see Lemma A.8) so we can define the restrictions of the Weyl tensor to $\Lambda^{\pm}$, $W^{\pm} = W\big|_{\Lambda^{\pm}}$. Then we say that (M, g) is a *self-dual manifold* if it satisfies $W^{-} = 0$ and we say that g is a *self-dual metric*. Similarly, we use *anti-self-dual* when $W^{+} = 0$. Since reversing the orientation exchanges $W^{\pm}$, we sometimes refer to either one as *half-conformally-flat* manifolds. Roughly speaking, this decomposition of the Weyl tensor arises when one considers the algebraic curvature tensors as an $SO(4)$-module. This is a special phenomenon of dimension 4 and a manifestation of the non-simplicity of $SO(4)$. We refer the reader to [Bes08] and the references therein for further details.

1.3.1 2-forms and topology

As illustrated in the introduction, not many obstructions to the existence of half-conformally-flat metrics are known. Nonetheless, we can use the decomposition of the bundle of 2-forms seen above to gain some insight into some topological invariants of M. Namely, we can encode the signature of M using Hodge-harmonic 2-forms in $\Lambda^{\pm}$. We follow the exposition in [LeB04].

Recall that for any closed 4-manifold we have the intersection form

$$\mathcal{Q} : H^2_{dR}(M) \times H^2_{dR}(M) \to \mathbb{R}, \tag{1.10}$$

given by $\mathcal{Q}\left([\alpha], [\beta]\right) = \int_M \alpha \wedge \beta$. Note that the wedge product is commutative on 2-forms and Poincaré duality says that $\mathcal{Q}$ is non-degenerate. Thus, $\mathcal{Q}$ is a symmetric bilinear form and we can choose a basis for $H^2_{dR}(M)$ so that $\mathcal{Q}$ takes the form of a diagonal matrix with only ± 1 entries and zeros. Since this only depends on the topology of M, we can define

$$b_{\pm}(M) = \text{“dimension of } \pm\text{-definite subspace of } H^2_{dR}(M) \text{ with respect to } \mathcal{Q}\text{”},$$

and the *signature of* M is the signature of $\mathcal{Q}$

$$\tau(M) = b_+(M) - b_-(M).$$

By the Hodge decomposition theorem, we can restrict $\mathcal{Q}$ to Hodge-harmonic 2-forms

$$\mathcal{H}_g(M) = \left\{ \eta \in \Lambda^2 : \Delta_H \eta = 0 \right\}.$$

Moreover, since $*\Delta_H = \Delta_H *$, we have that $\Delta_H : \Lambda^{\pm} \to \Lambda^{\pm}$. This implies that the orthogonal decomposition, $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, restricts to Hodge-harmonic 2-forms

$$\mathcal{H}_g(M) = \mathcal{H}^+_g(M) \oplus \mathcal{H}^-_g(M),$$

where $\mathcal{H}^{\pm}_g(M) = \{\eta \in \Lambda^{\pm} : \Delta_H \eta = 0\}$. Finally, given $\eta \in \mathcal{H}^{\pm}_g$ we have

$$\mathcal{Q}(\eta, \eta) = \int_M \eta \wedge \eta = \pm \int_M \eta \wedge *\eta = \pm \int_M |\eta|^2 dV,$$

so $\mathcal{Q}$ restricted to $\mathcal{H}^{\pm}_g$ is $\pm$-definite and we have

$$b_{\pm}(M) = \dim \mathcal{H}^{\pm}_g(M).$$

1.3.2 Geometric elliptic equations and inequalities

Recall that any Riemannian manifold satisfies an elliptic equation for the Riemannian curvature tensor (see Lemma A.5)

$$\Delta Rm = Rm * Rm + L(\nabla^2 Ric), \tag{1.11}$$

where L denotes a linear combination of the components of the tensor and $A * B$ denotes a generic linear combination of contractions of the tensors A and B. One of the key features of half-conformally-flat 4-manifolds is that they satisfy an additional elliptic equation for the Ricci tensor

$$\Delta Ric = Rm * Ric + \frac{1}{6}(\Delta scal)g + \frac{1}{3}\nabla^2 scal. \tag{1.12}$$

These two equations are at the core of the geometric regularity theorem from the next chapter, which is the main tool that allows us to control the geometry of anti-self-dual 4-manifolds.

Derivation of equation for Ric (1.12)

The following derivation can be found in [TV05a] for the case of constant scalar metrics and in [CW11] for the case of extremal Kähler metrics. Our computations are essentially the same but keeping track of the scalar curvature. See [Der83, Section 2].

As observed in [Der83, Lemma 1] (cf. [Bac21]), the Euler-Lagrange equations of the Weyl curvature functional

$$\mathcal{W} : g \mapsto \int_M |W|^2 dV,$$

are given in local coordinates by

$$W_{ikjl,lk} + \frac{1}{2} R_{kl} W_{ikjl} = 0. \tag{1.13}$$

This expression can be use to define the *Bach tensor*, $B_{ij} = W_{ikjl,lk} + \frac{1}{2} R_{kl} W_{ikjl}$. Metrics satisfying $B_{ij} = 0$ are called *Bach-flat*. Furthermore, by [ACG03, Section 1.5] we actually have

$$B_{ij} = W^{\pm}_{ikjl,lk} + \frac{1}{2} R_{kl} W^{\pm}_{ikjl},$$

so, in particular, half-conformally-flat metrics are examples of Bach-flat metrics. Alternatively, we can also see this using Hirzebruch's signature theorem. Indeed, recall that

$$\tau(M) = \frac{1}{48\pi^2} \int_M \left(|W^+|^2 - |W^-|^2 \right) dV,$$

so we obtain

$$\mathcal{W}(g) \pm 48\pi^2 \tau(M) = 2 \int_M |W^{\pm}|^2 dV \geq 0,$$

which implies $\mathcal{W}(g) \geq \mp 48\pi^2 \tau(M)$ and equality is attained only when $W^{\pm} = 0$. Therefore, half-conformally-flat metrics are critical points of the Weyl functional and they must satisfy the corresponding Euler-Lagrange equations.

Now we go back to the Bach-flat equation (1.13). Using the divergence formula for the Weyl tensor, $W_{ijkl,i} = S_{jk,l} - S_{jl,k}$ (A.3), we obtain

$$S_{ji,kk} = S_{jk,ik} + \frac{1}{2} R_{kl} W_{ikjl},$$

where $S = \frac{1}{2}\left(Ric - \frac{scal}{6} g \right)$ is the Schouten tensor. Thus, we have

$$\frac{1}{2}\left(R_{ij,kk} - \frac{\Delta scal}{6} g_{ij} \right) = \frac{1}{2}\left(R_{jk,ik} - \frac{1}{6} scal_{,ij} \right) + \frac{1}{2} R_{kl} W_{ikjl}.$$

Then we use the Ricci identities (1.6) followed by the divergence formula, $\operatorname{div}(Ric) = \frac{1}{2}\,\mathrm{d}\,scal$ (A.2), to obtain

$$\frac{1}{2}\left(R_{ij,kk} - \frac{\Delta scal}{6} g_{ij}\right) = \frac{1}{2}\left(\frac{1}{3} scal_{,ij} + R_{ikmj}R_{mk} - R_{im}R_{jm}\right) + \frac{1}{2} R_{kl}W_{ikjl},$$

which yields the coordinate expression of (1.12)

$$R_{ij,kk} = R_{ikmj}R_{mk} - R_{im}R_{jm} + R_{kl}W_{ikjl} + \frac{1}{6}(\Delta scal)g_{ij} + \frac{1}{3} scal_{,ij}.$$

Remark 1.8. *In particular, Bach-flat metrics also satisfy* (1.12).

Elliptic inequalities for curvature quantities

In applications, it is sometimes more convenient to work with elliptic inequalities instead of the full equations. To make this transition we use the following well-known identity (A.7)

$$|T|\,\Delta\,|T| + |\nabla\,|T||^2 = \langle T, \Delta T\rangle + |\nabla T|^2,$$

where T is any tensor. Using Kato's inequality (A.6) and Cauchy-Schwarz, we can also obtain

$$\Delta\,|T| \geq -|\Delta T|, \tag{1.14}$$

which is valid when $|T| \neq 0$ and it holds in the sense of distributions otherwise. Applying this inequality to the equations for Rm and Ric, (1.11) and (1.12), we obtain

$$\Delta|Ric| \geq -|Rm||Ric| - |\nabla^2 scal|, \tag{1.15}$$

$$\Delta|Rm| \geq -|Rm|^2 - |\nabla^2 Ric|, \tag{1.16}$$

where we used $|\Delta scal| \leq 2|\nabla^2 scal|$ for the first inequality and we also ignored dimensional constants.

1.4 Regularity theory for elliptic inequalities

Suppose that (M, g) is a complete smooth Riemannian manifold and that we have chosen a point x in M and some geodesic ball centered around it, $B_r(x)$. Also suppose that we have non-negative functions, $u, f, s : B_r(x) \to \mathbb{R}$ in $L^p(B_r(x))$. In this section we review some of the basic regularity theory for elliptic inequalities of the type

$$\Delta u \geq -fu - s. \tag{1.17}$$

In this section we are only concerned about the local behavior of u, so one expects the same kind of behavior as with elliptic equations in Euclidean space. Generally speaking, any function satisfying an inequality of this type will enjoy better regularity properties than a priori assumed. The main difference arises when one tries to obtain estimates for higher order derivatives because curvature terms start to appear in the equations. This phenomenon will manifest itself later on in this dissertation but for now we are only concerned with two basic results. First, as long as the $L^{\frac{n}{2}}$-norm of f is small enough and we have suitable a priori control of s, we can improve the initial regularity of u to any L^q-space with $p \leq q < \infty$. Second, we can actually obtain the limit case $q = \infty$ as long as we have better than $L^{\frac{n}{2}}$-control on f. All these results are really well-known and have been extensively used in the literature. We follow the presentations from [BKN89, Section 4] and [TV05a, Section 3].

In an essential way, most of this section stems from the local Sobolev inequality (1.8). For clarity in the exposition, we restrict to dimension 4 and we adopt a few conventions to make the notation leaner: we write $C_S = C_S(\Omega)$, note that we only use the local Sobolev inequality in this chapter so there is no ambiguity; we omit the volume form in the integrals; and, when clear from the context, we omit the domains of integration as well.

It is also important to reinforce the notion that these kind of estimates will play a significant role in the rest of the dissertation, where we will work with inequalities like (1.17) that are satisfied by curvature quantities and Hodge-harmonic self-dual 2-forms. See for example, (1.15), (1.16) and (3.9).

Lemma 1.9 ([BKN89])**.** *Fix some $p \geq p_0 > 1$ and suppose that $u \in L^p\left(B_r(x)\right)$ and $f \in L^2\left(B_r(x)\right)$ satisfy $\Delta u \geq -fu$. Then there are constants $\varepsilon_0 = \varepsilon_0(p, C_S) > 0$ and $C = C(p_0, C_S) < \infty$ such that if $\left(\int_{B_r(x)} f^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\left(\int_{B_{\frac{r}{2}}(x)} u^{2p}\right)^{\frac{1}{2}} \leq Cr^{-2}\int_{B_r(x)} u^p.$$

Proof. Let ϕ be any smooth cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. By the Sobolev inequality (1.8) applied to $\phi u^{\frac{p}{2}}$ and Cauchy's inequality we have

$$C_S\left(\int \phi^4 u^{2p}\right)^{\frac{1}{2}} \leq 2\int |\nabla\phi|^2 u^p + \frac{p^2}{2}\int \phi^2 u^{p-2}|\nabla u|^2.$$

Note that we can rewrite the last term as

$$p^2\int \phi^2 u^{p-2}|\nabla u|^2 = \frac{p^2}{p-1}\int \phi^2\left\langle \nabla u^{p-1}\nabla u\right\rangle,$$

so integrating by parts we obtain

$$
\begin{aligned}
p^2 \int \phi^2 u^{p-2} |\nabla u|^2 &= -\frac{2p^2}{p-1} \int \phi u^{p-1} \langle \nabla \phi, \nabla u \rangle - \frac{p^2}{p-1} \int \phi^2 u^{p-1} \Delta u \\
&\leq \frac{p^2}{2} \int \phi^2 u^{p-1} |\nabla u|^2 + \frac{2p^2}{(p-1)^2} \int |\nabla \phi|^2 u^p + \frac{p^2}{p-1} \int \phi^2 u^p f,
\end{aligned}
$$

where we used Cauchy's inequality and $\Delta u \geq -fu$ on the second line. This yields

$$
C_S \left(\int \phi^4 u^{2p} \right)^{\frac{1}{2}} \leq 2 \left(1 + \frac{p^2}{(p-1)^2} \right) \int |\nabla \phi|^2 u^p + \frac{p^2}{p-1} \int \phi^2 u^p f.
$$

By Hölder's inequality,

$$
\int \phi^2 u^p f \leq \left(\int \phi^4 u^{2p} \right)^{\frac{1}{2}} \left(\int_{B_r(x)} f^2 \right)^{\frac{1}{2}},
$$

so choosing $\varepsilon_0 = \frac{C_S(p-1)}{2p^2}$ implies

$$
\frac{C_S}{2} \left(\int \phi^4 u^{2p} \right)^{\frac{1}{2}} \leq 2 \left(1 + \frac{p^2}{(p-1)^2} \right) \int |\nabla \phi|^2 u^p. \tag{1.18}
$$

Now observe that $\frac{p^2}{(p-1)} \leq \frac{p_0^2}{(p_0-1)^2}$, so we can obtain the desired inequality by choosing ϕ such that $\phi|_{B_{\frac{r}{2}(x)}} \equiv 1$ and $|\nabla \phi| \leq 2r^{-1}$. $\square$

If we iterate the inequality above, we can prove that u is in L^q for any $q > p$ provided $||f||_{L^2}$ is small enough. Indeed, let k be the first positive integer such that $2^k p > q$. Then, using Hölder's inequality, we have

$$
\begin{aligned}
\left(\int_{B_{2^{-k}r}(x)} u^q \right)^{\frac{1}{q}} &\leq (\operatorname{Vol} B_{2^{-k}r}(x))^{\frac{1}{q} - \frac{1}{2^k p}} \left(\int_{B_{2^{-k+1}r}(x)} u^{2^k p} \right)^{\frac{1}{2^k p}} \\
&\leq (\operatorname{Vol} B_{2^{-k}r}(x))^{\frac{1}{q} - \frac{1}{2^k p}} \prod_{l=1}^{k} (Cr^{-2})^{\frac{1}{2^{l-1}p}} \left(\int_{B_r(x)} u^p \right)^{\frac{1}{p}}.
\end{aligned}
$$

Further, since $\|u\|_{L^q} \to \|u\|_{L^\infty}$ as $q \to \infty$, it stands to reason that we can extend this result to the case $q = \infty$. Of course, the caveat is that the bigger q is, the smaller the

L^2-bound on f must be. As we mentioned above, this can be circumvented whenever we have better control on f. We require L^4-bounds, which is enough for our purposes, but it is possible to achieve the same result with only $L^{2+\varepsilon}$-bounds where $\varepsilon > 0$.

Lemma 1.10 ([BKN89]). *Fix some $q \geq 2$ and suppose $u \in L^q(B_r(x))$ and $f \in L^4(B_r(x))$ satisfy $\Delta u \geq -fu$. Then there is a constant $C = C(q, C_S) < \infty$ such that if $\left(\int_{B_r(x)} f^4\right)^{\frac{1}{2}} \leq A < \infty$, then we have*

$$\sup_{B_{\frac{r}{2}}(x)} u \leq C\left(A + r^{-2}\right)^{\frac{2}{q}} \left(\int_{B_r(x)} u^q\right)^{\frac{1}{q}}. \tag{1.19}$$

Proof. Let ϕ be any smooth cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. We can proceed exactly as in the proof of Lemma 1.11 to obtain

$$C_S\left(\int \phi^4 u^{2p}\right)^{\frac{1}{2}} \leq 10 \int |\nabla\phi|^2 u^p + 2p\int \phi^2 u^p f,$$

where now we are assuming $p \geq 2$. The key point is that we can improve the estimate of the last term. By Hölder's inequality with exponents $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, we obtain

$$\begin{aligned} 2p\int \phi^2 u^p f &\leq 2p\left(\int_{B_r(x)} f^4\right)^{\frac{1}{4}} \left(\int \phi^2 u^p\right)^{\frac{1}{2}} \left(\int \phi^4 u^{2p}\right)^{\frac{1}{4}} \\ &\leq \frac{p^2}{\delta}\left(\int_{B_r(x)} f^4\right)^{\frac{1}{2}} \int \phi^2 u^p + \delta\left(\int \phi^4 u^{2p}\right)^{\frac{1}{2}}, \end{aligned}$$

where we used the δ-Cauchy inequality on the second line. Thus, setting $\delta = \frac{C_S}{2}$ yields

$$\left(\int \phi^4 u^{2p}\right)^{\frac{1}{2}} \leq \frac{20}{C_S}\int |\nabla\phi|^2 u^p + \frac{4p^2}{C_S^2}\left(\int_{B_r(x)} f^4\right)^{\frac{1}{2}} \int \phi^2 u^p.$$

Now we pick appropriate cutoff functions. For each $i \in \mathbb{N}$ chose ϕ_i such that it also satisfies

$$supp(\phi_i) \subset B_{r(2^{-1}+2^{-i})}(x), \quad \phi_i\big|_{B_{r\left(2^{-1}+2^{-(i+1)}\right)}(x)} \equiv 1 \quad , \quad \sup|\nabla\phi_i| \leq 2^{i+1} r^{-1}.$$

Then we can replace ϕ with ϕ_i and p with $2^{i-1}q$ in the inequality above to obtain

$$\left(\int_{B_{r\left(2^{-1}+2^{-(i+1)}\right)}(x)} u^{2^i q}\right)^{\frac{1}{2^i}} \leq C_i(q)^{\frac{1}{2^{i-1}}} \left(\int_{B_{r\left(2^{-1}+2^{-i}\right)}(x)} u^{2^{i-1}q}\right)^{\frac{1}{2^{i-1}}},$$

where $C_i(q) = \frac{20}{C_S} 2^{2i-2}\left(\frac{q^2 A}{5C_S} + 2^4 r^{-2}\right)$. We can reiterate this inequality to achieve

$$\|u\|^q_{L^{2^k q}\left(B_{\frac{r}{2}}(x)\right)} \leq \|u\|^q_{L^{2^k q}\left(B_{r\left(2^{-1}+2^{-(k+1)}\right)}(x)\right)} \leq \left(\prod_{i=1}^{k} C_i(q)^{\frac{1}{2^{i-1}q}}\right) \int_{B_r(x)} u^q,$$

for any positive integer k. Since $\|u\|_{L^a} \to \|u\|_{L^\infty}$ as $a \to \infty$, we only need to bound the constant appearing above. A simple computation yields

$$\begin{aligned}\prod_{i=1}^{k} C_i(q)^{\frac{1}{2^{i-1}}} &= \left(\frac{20}{C_S}\right)^{2(1-2^{-k})} 2^{4\left(3-(k+3)2^{-k}\right)} \left(\frac{q^2 A}{2^4 \cdot 5C_S} + r^{-2}\right)^{2(1-2^{-k})} \\ &\leq C\left(A + r^{-2}\right)^2 \left(\frac{20}{C_S}\right)^{-2^{-k+1}} \left(\frac{q^2 A}{2^4 \cdot 5C_S} + r^{-2}\right)^{-2^{-k+1}},\end{aligned}$$

which is uniformly bounded for large enough k. □

Now we can prove the corresponding statements for non-zero s in (1.17). First we show the L^p estimates and then the L^∞ estimate.

Lemma 1.11 ([BKN89]). *Fix some $p \geq p_0 > 1$ and suppose that $u, s \in L^p\left(B_r(x)\right)$ and $f \in L^2\left(B_r(x)\right)$ satisfy $\Delta u \geq -fu - s$. Then there are constants $\varepsilon_0 = \varepsilon_0(p, C_S) > 0$ and $C = C\left(p_0, C_S\right) < \infty$ such that if $\int_{B_r(x)} f^2 \leq \varepsilon_0$, then we have*

$$\left(\int_{B_{\frac{r}{2}}(x)} u^{2p}\right)^{\frac{1}{2}} \leq C\left(r^{-2}\int_{B_r(x)} u^p + p\left(\int_{B_r(x)} u^p\right)^{\frac{p-1}{p}} \left(\int_{B_r(x)} s^p\right)^{\frac{1}{p}}\right).$$

Proof. Let ϕ be any smooth cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. We can follow the argument in the proof of Lemma 1.9 word by word but carrying the extra term coming from s. In the end we obtain

$$C\left(\int \phi^4 u^{2p}\right)^{\frac{1}{2}} \leq \int \phi^2 s u^{p-1} + \int u^p |\nabla\phi|^2, \tag{1.20}$$

instead of (1.18). By Hölder's inequality

$$\int \phi^2 s u^{p-1} \leq \left(\int \phi^{\frac{p}{p-1}} u^p\right)^{\frac{p-1}{p}} \left(\int \phi^p s^p\right)^{\frac{1}{p}},$$

and the result follows choosing ϕ with $\phi|_{B_{\frac{r}{2}}(x)} \equiv 1$ and $|\nabla\phi| \leq 2r^{-1}$. □

Lemma 1.12 ([TV05a, Lemma 3.9])**.** *Fix $q \geq 2$ and suppose that $u \in L^q(B_r(x))$, $f \in L^4(B_r(x))$ and $s \in L^4(B_r(x))$ satisfy $\Delta u \geq -fu - s$. Then there is a constant $C = C(q, C_S) < \infty$ such that if $\left(\int_{B_r(x)} f^4\right)^{\frac{1}{2}} \leq A < \infty$, then we have*

$$\sup_{B_{\frac{r}{2}}(x)} u \leq C\left(A + r^{-2}\right)^{\frac{2}{q}} \left(\|u\|_{L^q(B_r(x))} + r\|s\|_{L^4(B_r(x))} (\mathrm{Vol}\, B_r(x))^{\frac{1}{q}}\right).$$

Proof. We can assume that $\|s\|_{L^4} > 0$, otherwise we can just use (1.19). Consider the auxiliary function $v = u + r\|s\|_{L^4}$. Since $\Delta v = \Delta u$ and $u \leq v$, we have

$$\Delta v \geq -fu - s \geq -(fv + s) \geq -\left(f + \frac{s}{v}\right) v \geq -\left(f + \frac{s}{r\|s\|_{L^4}}\right) v,$$

which is the same type of inequality as in Lemma 1.10. Note that

$$\left\| f + \frac{s}{r\|s\|_{L^4}} \right\|_{L^4} \leq \|f\|_{L^4} + \left\| \frac{s}{r\|s\|_{L^4}} \right\|_{L^4} \leq A^{\frac{1}{2}} + r^{-1},$$

and

$$\|v\|_{L^q} \leq \left(\|u\|_{L^q} + r^{\frac{a}{2}} \|s\|_{L^4} (\mathrm{Vol}\, B_r(x))^{\frac{1}{q}}\right),$$

so we can use (1.19) to obtain

$$\begin{aligned}\sup_{B_{\frac{r}{2}}(x)} v &\leq C\left[\left(A^{\frac{1}{2}} + r^{-1}\right)^2 + r^{-2}\right]^{\frac{2}{q}} \left(\int_{B_r(x)} v^q\right)^{\frac{1}{q}} \\ &\leq C\left[A + r^{-2}\right]^{\frac{2}{q}} \left(\int_{B_r(x)} v^q\right)^{\frac{1}{q}},\end{aligned}$$

and the result follows because $u \leq v$. $\square$

Chapter 2

ε-regularity theorem for anti-self-dual 4-manifolds

In this chapter we prove an ε-regularity theorem for anti-self-dual 4-manifolds with $W^{m,4}$-bounds on the scalar curvature. This is the basic ingredient that allows us to control the geometry and obtain convergence results for the manifolds under consideration. The proof closely follows the analogous result of Tian-Viaclovsky [TV05a, Theorem 3.1], where constant scalar curvature is assumed. Of course, the main difference is the extra scalar curvature terms which makes some of the analysis a more convoluted. See also [CW11] for the corresponding result in the context of extremal Kähler metrics.

Theorem 2.1. *Fix some integer $m \geq 0$ and let (M, g) be a closed 4-manifold satisfying the elliptic equation for Ric* (1.12). *Suppose that there are some constants*

$C_S > 0$, $V_0 < \infty$ and $S_0 < \infty$ such that $\mathrm{Vol}(B_r(x)) \leq V_0 r^4$, $C_S\,(B_r(x)) \geq C_S$ *and* $||\nabla^2 scal||_{W^{m,4}(B_r(x))} \leq S_0$. *Then there are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C_m = C_m(r, C_S, S_0, V_0) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\sup_{B_{\frac{r}{2}}(x)} |\nabla^m Rm| \leq C_m \varepsilon_0^{\frac{1}{2}}.$$

If in addition we assume that the scalar curvature is constant, we recover Tian-Viaclovsky's result. There are constants $\varepsilon_1 = \varepsilon_1(C_S)$ and $C_1 = C_1(m, C_S)$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_1$, then we have

$$\sup_{B_{\frac{r}{2}}(x)} |\nabla^m Rm| \leq \frac{C_1}{r^{m+2}} \varepsilon_1.$$

Note that no assumptions on the upper volume growth are needed for this case.

The overall goal of the proof is to be able to apply Lemma 1.12 to the elliptic inequality for $|Rm|$ (1.16). In order to accomplish this, we need to prove estimates for several curvature quantities. The strategy is to use equation (1.12)

$$\Delta Ric = Rm * Ric + \nabla^2 scal,$$

to obtain estimates for Ric which in turn can be used together with (1.11)

$$\Delta Rm = Rm * Rm + \nabla^2 Ric,$$

to prove estimates for Rm. The improved regularity on Rm implies that we can extract better estimates for Ric from (1.12) which in turn enables improved estimates for Rm. We can repeat this process as long as we have control on the scalar curvature.

Remark 2.2. *To simplify some computations we always assume that $\varepsilon_0 \leq 1$ without further notice.*

2.1 L^2 and L^4 bounds for curvature quantities

The goal of this section is to prove the following L^2 and L^4 estimates.

Proposition 2.3. *There are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(m, C_S) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\left(\int_{B_{\frac{r}{2}}(x)} |\nabla^m Ric|^4\right)^{\frac{1}{2}} \leq C\left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Ric|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

$$\int_{B_{\frac{r}{2}}(x)} |\nabla^{m+1} Ric|^2 \leq C\left(r^{-2} + P_{m+1}\right)^{m+1} \left(\int_{B_r(x)} |Ric|^2\right)^{\frac{1}{2}}, \tag{2.2}$$

$$\int_{B_{\frac{r}{2}}(x)} |\nabla^m Rm|^2 \leq C\left(r^{-2} + P_m\right)^{m} \left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}}, \tag{2.3}$$

$$\left(\int_{B_{\frac{r}{2}}(x)} |\nabla^m Rm|^4\right)^{\frac{1}{2}} \leq C\left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}}, \tag{2.4}$$

where $P_m = \sum_{k=0}^{m} \left\|\nabla^{k+2} scal\right\|_{L^2(B_r(x))}^{\frac{2}{k+2}}$.

Remark 2.4. *Note that if we scale the metric $\widetilde{g} = c^2 g$, P_m scales like c^2 and the inequalities are scale invariant.*

In order to set up an induction argument, we first prove a series of lemmas that establish the cases with $m = 0$.

Lemma 2.5. *There are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(C_S) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\left(\int_{B_{\frac{r}{2}}(x)} |Ric|^4\right)^{\frac{1}{2}} \leq C\left(r^{-2} + \left\|\nabla^2 scal\right\|_{L^2(B_r(x))}\right) \left(\int_{B_r(x)} |Ric|^2\right)^{\frac{1}{2}}. \tag{2.5}$$

Proof. Since we have the inequality for Ric (1.15), we can directly apply Lemma 1.11 with $s = \nabla^2 scal$ and $p = 2$. More specifically, we have the inequality (cf. (1.20))

$$\frac{C_S}{2}\left(\int \phi^4 |Ric|^4\right)^{\frac{1}{2}} \leq \left(\int_{B_r(x)} |\nabla^2 scal|^2\right)^{\frac{1}{2}} \left(\int_{B_r(x)} |Ric|^2\right)^{\frac{1}{2}} + \int |Ric|^2 |\nabla\phi|^2, \quad (2.6)$$

where ϕ is a cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. The result can be obtained by choosing ϕ that also satisfies $\phi|_{B_{\frac{r}{2}}(x)} \equiv 1$ and $|\nabla\phi| \leq 2r^{-1}$. $\square$

Lemma 2.6. *There are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(C_S) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\int_{B_{\frac{r}{2}}(x)} |\nabla Ric|^2 \leq C\left(r^{-2} + \left\|\nabla^2 scal\right\|_{L^2(B_r(x))}\right)\left(\int_{B_r(x)} |Ric|^2\right)^{\frac{1}{2}}. \quad (2.7)$$

Proof. Let ϕ be any smooth cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. Integrating by parts we obtain

$$\int \phi^2 |\nabla Ric|^2 = -\int \phi^2 \langle \Delta Ric, Ric\rangle - 2\int \phi \langle \mathrm{tr}(\nabla\phi \otimes \nabla Ric), Ric\rangle,$$

so by Cauchy-Schwarz's inequality and the elliptic inequality for Ric (1.15),

$$\int \phi^2 |\nabla Ric|^2 \leq \int \phi^2 |Rm||Ric|^2 + \int \phi^2 |\nabla^2 scal||Ric| + 2\int \phi |\nabla\phi||\nabla Ric||Ric|.$$

We use Hölder's inequality to bound the first and second terms

$$\int \phi^2 |Rm||Ric|^2 \leq \left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \left(\int \phi^4 |Ric|^4\right)^{\frac{1}{2}},$$

and

$$\int \phi^2 |\nabla^2 scal||Ric| \leq \left(\int \phi^2 |\nabla^2 scal|^2\right)^{\frac{1}{2}} \left(\int \phi^2 |Ric|^2\right)^{\frac{1}{2}},$$

and we bound the third term with the δ-Cauchy inequality

$$2\int \phi|\nabla\phi||\nabla Ric||Ric| \leq \delta\int \phi^2|\nabla Ric|^2 + \frac{1}{\delta}\int |\nabla\phi|^2|Ric|^2.$$

Therefore, using (2.6) to bound $\int \phi^2|Ric|^4$ and setting $\delta = \frac{1}{2}$, we obtain

$$\int \phi^2|\nabla Ric|^2 \leq C\left(\int |\nabla\phi|^2|Ric|^2 + \left(\int_{B_r(x)} |\nabla^2 scal|^2\right)^{\frac{1}{2}}\left(\int_{B_r(x)} |Ric|^2\right)^{\frac{1}{2}}\right), \quad (2.8)$$

and the result follows choosing ϕ such that $\phi|_{B_{\frac{r}{2}}(x)} \equiv 1$ and $|\nabla\phi| \leq 2r^{-1}$. □

Lemma 2.7. *There are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(C_S) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\left(\int_{B_{\frac{r}{2}}(x)} |Rm|^4\right)^{\frac{1}{2}} \leq C\left(r^{-2} + \left\|\nabla^2 scal\right\|_{L^2(B_r(x))}\right)\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}}. \quad (2.9)$$

Proof. Let ϕ be any smooth cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. Using the Sobolev inequality followed by integration by parts we obtain

$$C_S\left(\int \phi^4|Rm|^4\right)^{\frac{1}{2}} \leq \int |\nabla\phi|Rm||^2 = -\int \phi^2|Rm|\Delta|Rm| + \int |\nabla\phi|^2|Rm|^2.$$

Since we have $|Rm|\Delta|Rm| \geq \langle Rm, \Delta Rm\rangle$ by (A.7), using the equation for Rm (1.11) yields

$$-\int \phi^2|Rm|\Delta|Rm| \leq -\int \phi^2\langle Rm, Rm * Rm\rangle - \int \phi^2\langle Rm, \nabla^2 Ric\rangle.$$

Then, using Cauchy-Schwarz's inequality on the first summand and integrating by parts the second one, we obtain

$$C_S\left(\int \phi^4|Rm|^4\right)^{\frac{1}{2}} \leq \int \phi^2|Rm|^3 + \underbrace{\int \langle \operatorname{tr}\nabla(\phi^2 Rm), \nabla Ric\rangle}_{II} + \int |\nabla\phi|^2|Rm|^2. \quad (2.10)$$

Now the second term is just

$$II = \int \phi^2 \langle \mathrm{tr}\, \nabla Rm, \nabla Ric\rangle + 2\int \phi \langle \mathrm{tr}\left(\nabla\phi \otimes Rm\right), \nabla Ric\rangle,$$

which can be bounded using Cauchy-Schwarz's and Cauchy's inequalities

$$\begin{aligned} II &\leq \int \phi^2 |\nabla Rm||\nabla Ric| + 2\int \phi |\nabla\phi||Rm||\nabla Ric| \\ &\leq \frac{1}{2}\int \phi^2|\nabla Rm|^2 + \frac{3}{2}\int \phi^2 |\nabla Ric|^2 + \int |\nabla\phi|^2|Rm|^2. \end{aligned}$$

Next we want to bound the term $\int \phi^2|\nabla Rm|^2$. Integrating by parts and plugging in the equation for Rm (1.16),

$$\begin{aligned} \int \phi^2|\nabla Rm|^2 = -\int \phi^2 \langle Rm * Rm, Rm\rangle - \int \phi^2\langle \nabla^2 Ric, Rm\rangle \\ - 2\int \phi \langle \mathrm{tr}(\nabla\phi\otimes\nabla Rm), Rm\rangle. \end{aligned}$$

We also integrate by parts the second term

$$-\int \phi^2 \langle \nabla^2 Ric, Rm\rangle = \int \langle \nabla Ric, \phi^2 \nabla Rm + 2\phi\, \mathrm{tr}(\nabla\phi\otimes Rm)\rangle,$$

thus, Cauchy-Schwarz's inequality yields

$$\begin{aligned} \int \phi^2|\nabla Rm|^2 \leq \int \phi^2 |Rm|^3 + \int \phi^2 |\nabla Ric||\nabla Rm| \\ + 2\int |\nabla Ric|\phi|\nabla\phi||Rm| + 2\int \phi|\nabla\phi||Rm||\nabla Rm|. \end{aligned}$$

Then we bound each of the last three terms using the δ-Cauchy inequality

$$\begin{aligned} \int \phi^2|\nabla Rm|^2 \leq \int \phi^2|Rm|^3 + \left(\frac{\delta_1}{2}+\delta_2\right)\int \phi^2|\nabla Ric|^2 \\ + \left(\frac{1}{2\delta_1}+\delta_3\right)\int \phi^2|\nabla Rm|^2 + \left(\frac{1}{\delta_2}+\frac{1}{\delta_3}\right)\int |\nabla\phi|^2|Rm|^2. \end{aligned}$$

Regrouping the term $\int \phi^2 |\nabla Rm|^2$ and choosing δ_i's small enough we have

$$C \int \phi^2 |\nabla Rm|^2 \leq \int \phi^2 |Rm|^3 + \int \phi^2 |\nabla Ric|^2 + \int |\nabla \phi|^2 |Rm|^2. \tag{2.11}$$

Putting it all back together with (2.10) yields

$$C \left(\int \phi^4 |Rm|^4 \right)^{\frac{1}{2}} \leq \int \phi^2 |Rm|^3 + \int \phi^2 |\nabla Ric|^2 + \int |\nabla \phi|^2 |Rm|^2.$$

By Hölder's inequality $\int \phi^2 |Rm|^3 \leq \left(\int \phi^4 |Rm|^4 \right)^{\frac{1}{2}} \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}}$. Thus, we can choose ε_0 small enough so that

$$C \left(\int \phi^4 |Rm|^4 \right)^{\frac{1}{2}} \leq \int \phi^2 |\nabla Ric|^2 + \int |\nabla \phi|^2 |Rm|^2,$$

which combined with (2.8) yields

$$C \left(\int \phi^4 |Rm|^4 \right)^{\frac{1}{2}} \leq \int |\nabla \phi|^2 |Rm|^2 + \left(\int_{B_r(x)} |\nabla^2 scal|^2 \right)^{\frac{1}{2}} \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}}. \tag{2.12}$$

Then the desired estimate can be achieved choosing ϕ such that it also satisfies $\phi|_{B_{\frac{r}{2}}(x)} \equiv 1$ and $|\nabla \phi| \leq 2r^{-1}$. $\square$

For the general estimates we need elliptic equations for $\nabla^m Ric$ and $\nabla^m Rm$. These can be derived taking derivatives of the equations for Rm and Ric (1.11) and (1.12) and using the standard commutator formulas (see Lemma A.6 for details). We have

$$\Delta \left(\nabla^m Ric \right) = \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k} Ric + \nabla^{m+2} scal, \tag{2.13}$$

and

$$\Delta \left(\nabla^m Rm \right) = \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k} Rm + \nabla^{m+2} Ric. \tag{2.14}$$

Now we can proceed with the general induction argument.

Proof of (2.1). Let ϕ be any smooth cutoff function with $0 \le \phi \le 1$ and support contained in $B_r(x)$. From the Sobolev inequality and integration by parts, we obtain

$$C_S \left(\int \phi^4 |\nabla^m Ric|^4 \right)^{\frac{1}{2}} \le \underbrace{- \int \phi^2 |\nabla^m Ric| \Delta |\nabla^m Ric|}_{I} + \underbrace{\int |\nabla \phi|^2 |\nabla^m Ric|^2}_{II}.$$

Using $|T| \Delta |T| \ge \langle T, \Delta T \rangle$ (A.8) and the equation for $\nabla^m Ric$ (2.13) yields

$$I \le \underbrace{\int \phi^2 \langle \nabla^m Ric, Rm * \nabla^m Ric \rangle}_{I_1} + \underbrace{\int \phi^2 \left\langle \nabla^m Ric, \sum_{k=1}^{m-1} \nabla^k Rm * \nabla^{m-k} Ric \right\rangle}_{I_2}$$
$$+ \underbrace{\int \phi^2 \langle \nabla^m Ric, \nabla^m Rm * Ric \rangle}_{I_3} + \underbrace{\int \phi^2 \langle \nabla^m Ric, \nabla^{m+2} scal \rangle}_{I_4}.$$

Next we bound the third term. First we integrate by parts

$$I_3 = - \int \langle \nabla^m Ric, \nabla^{m-1} Rm * \phi^2 \nabla Ric \rangle - \int \langle \nabla^{m+1} Ric, \nabla^{m-1} Rm * \phi^2 Ric \rangle$$
$$- 2 \int \phi \langle \nabla^m Ric, \nabla^{m-1} Rm * \mathrm{tr}(\nabla \phi \otimes Ric) \rangle,$$

and then we use Cauchy-Schwarz followed by the δ-Cauchy inequality

$$I_3 \le \int \phi^2 |\nabla^m Ric| |\nabla^{m-1} Rm| |\nabla Ric| + \frac{\delta}{2} \int \phi^2 |\nabla^{m+1} Ric|^2$$
$$+ \left(\frac{1}{2\delta} + 1 \right) \int \phi^2 |\nabla^{m-1} Rm|^2 |Ric|^2 + \int |\nabla \phi|^2 |\nabla^m Ric|^2.$$

Now we deal with $\int \phi^2 |\nabla^{m+1} Ric|^2$. Integrating by parts we have

$$\int \phi^2 |\nabla^{m+1} Ric|^2 = - \int \phi^2 \langle \Delta(\nabla^m Ric), \nabla^m Ric \rangle - 2 \int \phi \langle \mathrm{tr}(\nabla \phi \otimes \nabla^{m+1} Ric), \nabla^m Ric \rangle.$$

The first term can be bounded as I above. We obtain

$$\int \phi^2 |\nabla^{m+1} Ric|^2 \le I_1 + I_2 + I_3 + I_4 + 2 \int \phi |\nabla \phi| |\nabla^{m+1} Ric| |\nabla^m Ric|.$$

Using the bound for I_3 and the δ-Cauchy inequality on the last term, we have

$$\int \phi^2 |\nabla^{m+1} Ric|^2 \leq I_1 + I_2 + I_4 + \left(1 + \frac{1}{\delta}\right) II + \frac{3\delta}{2} \int \phi^2 |\nabla^{m+1} Ric|^2$$
$$+ \left(1 + \frac{1}{2\delta}\right) \int \phi^2 |\nabla^{m-1} Rm|^2 |Ric|^2 + \int \phi^2 |\nabla^m Ric| |\nabla^{m-1} Rm| |\nabla Ric|.$$

Thus, choosing δ small enough, we can write

$$C \int \phi^2 |\nabla^{m+1} Ric|^2 \leq I_1 + I_2 + I_4 + II$$
$$+ \int \phi^2 |\nabla^{m-1} Rm|^2 |Ric|^2 + \int \phi^2 |\nabla^m Ric| |\nabla^{m-1} Rm| |\nabla Ric|, \quad (2.15)$$

which plugged back into the initial inequality results in

$$C \left(\int \phi^4 |\nabla^m Ric|^4 \right)^{\frac{1}{2}} \leq I_1 + I_2 + I_4 + II + \int \phi^2 |\nabla^{m-1} Rm|^2 |Ric|^2,$$

where we used that the last summand in (2.15) can be incorporated into I_2 (see inequality below). Now we choose suitably supported cutoff functions and deal with the remaining terms. By Hölder's inequality,

$$I_1 \leq \int \phi^2 |\nabla^m Ric|^2 |Rm| \leq \left(\int \phi^4 |\nabla^m Ric|^4 \right)^{\frac{1}{2}} \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}},$$

so we can regroup I_1 on the left hand side provided we choose ε_0 small enough. To bound I_2 we use Cauchy-Schwarz's inequality, Hölder's inequality and the induction hypothesis

$$
\begin{aligned}
I_2 &\leq \sum_{k=1}^{m-1} \int \phi^2 |\nabla^m Ric||\nabla^k Rm||\nabla^{m-k} Ric| \\
&\leq \sum_{k=1}^{m-1} \left(\int_{supp(\phi)} |\nabla^m Ric|^2 \right)^{\frac{1}{2}} \left(\int \phi^4 |\nabla^k Rm|^4 \right)^{\frac{1}{4}} \left(\int \phi^4 |\nabla^{m-k} Ric|^4 \right)^{\frac{1}{4}} \\
&\leq C \left[\left(r^{-2} + P_m\right)^{\frac{m}{2}} \sum_{k=1}^{m-1} \left(r^{-2} + P_m\right)^{\frac{k+1}{2}} \left(r^{-2} + P_m\right)^{\frac{m-k+1}{2}} \right] \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}} \\
&\leq C \left[\left(r^{-2} + P_m\right)^{\frac{m}{2}} \sum_{k=1}^{m-1} \left(r^{-2} + P_m\right)^{\frac{m+2}{2}} \right] \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

For I_4 we use Cauchy-Schwarz and Hölder's inequality followed by the induction hypothesis

$$
\begin{aligned}
I_4 &\leq \int \phi^2 |\nabla^m Ric||\nabla^{m+2} scal| \leq \left(\int \phi^2 |\nabla^m Ric|^2 \right)^{\frac{1}{2}} \left(\int \phi^2 |\nabla^{m+2} scal|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(r^{-2} + P_m\right)^{\frac{m}{2}} \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{4}} P_m^{\frac{m+2}{2}} \\
&\leq C \left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

The induction hypothesis can also be used to bound II

$$
II \leq C r^{-2} \left(r^{-2} + P_m\right)^m \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}} \leq C \left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}},
$$

as well as the last term

$$
\begin{aligned}
\int \phi^2 |\nabla^{m-1} Rm|^2 |Ric|^2 &\leq \left(\int \phi^2 |\nabla^{m-1} Rm|^4 \right)^{\frac{1}{2}} \left(\int \phi^2 |Ric|^4 \right)^{\frac{1}{2}} \\
&\leq C \left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Ric|^2 \right)^{\frac{1}{2}}.
\end{aligned}
$$

□

Proof of (2.2). From (2.15) and the estimates obtained in the proof of (2.1), we have

$$C\int \phi^2|\nabla^{m+1}Ric|^2 \le \left(\int_{B_r(x)}|Rm|^2\right)^{\frac{1}{2}}\left(\int \phi^4|\nabla^m Ric|^4\right)^{\frac{1}{2}} + \left(r^{-2}+P_m\right)^{m+1}\left(\int_{B_r(x)}|Ric|^2\right)^{\frac{1}{2}},$$

and the estimate follows using (2.1). □

Proof of (2.3). Let ϕ be a smooth function with $0 \le \phi \le 1$ and support contained in $B_r(x)$. We integrate by parts to obtain

$$\int \phi^2|\nabla^m Rm|^2 = \underbrace{-\int \phi^2\langle\Delta(\nabla^{m-1}Rm), \nabla^{m-1}Rm\rangle}_{I} \underbrace{-2\int \phi\langle \mathrm{tr}(\nabla\phi\otimes\nabla^m Rm), \nabla^{m-1}Rm\rangle}_{II}.$$

Using the equation for $\nabla^{m-1}Rm$ (2.14), we have

$$I = \underbrace{-\sum_{k=0}^{m-1}\int \phi^2\langle\nabla^k Rm * \nabla^{m-k-1}Rm, \nabla^{m-1}Rm\rangle}_{I_1} \underbrace{-\int \phi^2\langle\nabla^{m+1}Ric, \nabla^{m-1}Rm\rangle}_{I_2}.$$

By Hölder's inequality and the induction hypothesis,

$$\begin{aligned} I_1 &\le \sum_{k=0}^{m-1}\left(\int\phi^4|\nabla^k Rm|^4\right)^{\frac{1}{4}}\left(\int\phi^4|\nabla^{m-k-1}Rm|^4\right)^{\frac{1}{4}}\left(\int_{supp(\phi)}|\nabla^{m-1}Rm|^2\right)^{\frac{1}{2}} \\ &\le C\left[\left(r^{-2}+P_m\right)^{\frac{m-1}{2}}\sum_{k=1}^{m-1}\left(r^{-2}+P_m\right)^{\frac{k+1}{2}}\left(r^{-2}+P_m\right)^{\frac{m-k}{2}}\right]\left(\int_{B_r(x)}|Rm|^2\right)^{\frac{1}{2}} \\ &\le C\left(r^{-2}+P_m\right)^m\left(\int_{B_r(x)}|Rm|^2\right)^{\frac{1}{2}}, \end{aligned}$$

where we have chosen suitably supported cutoff functions. We can bound I_2 in a similar fashion using (2.2)

$$I_2 \le \left(\int\phi^2|\nabla^{m+1}Ric|^2\right)^{\frac{1}{2}}\left(\int\phi^2|\nabla^{m-1}Rm|^2\right)^{\frac{1}{2}} \le C\left(r^{-2}+P_m\right)^m\left(\int_{B_r(x)}|Rm|^2\right)^{\frac{1}{2}}.$$

Lastly, using the δ-Cauchy inequality , we can bound II as follows

$$II \le 2\int \phi|\nabla\phi||\nabla^m Rm||\nabla^{m-1}Rm| \le \delta\int \phi^2|\nabla^m Rm| + \frac{1}{\delta}\int |\nabla\phi|^2|\nabla^{m-1}Rm|^2.$$

Thus, choosing δ small enough, we obtain

$$C\int \phi^2|\nabla^m Rm|^2 \le \left(r^{-2}+P_m\right)^m + \int |\nabla\phi|^2|\nabla^{m-1}Rm|^2,$$

and we are done by noting that

$$\int |\nabla\phi|^2|\nabla^{m-1}Rm|^2 \le Cr^{-2}\left(r^{-2}+P_m\right)^{m-1} \le C\left(r^{-2}+P_m\right)^m,$$

where again we have chosen suitable cutoff functions and the induction hypothesis.

□

Proof of (2.4). By the Sobolev inequality and integration by parts we obtain

$$C_S\left(\int \phi^4|\nabla^m Rm|^4\right)^{\frac{1}{2}} \le \underbrace{\int \phi^2\langle\Delta(\nabla^m Rm),\nabla^m Rm\rangle}_{I} + \underbrace{\int |\nabla\phi|^2|\nabla^m Rm|^2}_{II},$$

where we also have used the inequality $|T|\,\Delta\,|T| \ge \langle T,\Delta T\rangle$ (A.8). Then, using the equation for $\nabla^m Rm$ (2.14) we can write

$$I = \underbrace{\int \phi^2\langle\nabla^{m+2}Ric,\nabla^m Rm\rangle}_{I_1} + \underbrace{2\int \phi^2\langle Rm*\nabla^m Rm,\nabla^m Rm\rangle}_{I_2} \\ + \underbrace{\sum_{k=1}^{m-1}\int \phi^2\langle\nabla^k Rm*\nabla^{m-k}Rm,\nabla^m Rm\rangle}_{I_3}.$$

Integrate by parts the first term and then use the δ-Cauchy inequality to obtain

$$\begin{aligned} I_1 &= -\int \phi^2\langle\nabla^{m+1}Ric,\nabla^{m+1}Rm\rangle - 2\int \phi\left\langle\nabla^{m+1}Ric,\mathrm{tr}(\nabla\phi\otimes\nabla^m Rm)\right\rangle \\ &\le \left(\frac{1}{2\delta}+1\right)\int \phi^2|\nabla^{m+1}Ric|^2 + \frac{\delta}{2}\int \phi^2|\nabla^{m+1}Rm|^2 + \int |\nabla\phi|^2|\nabla^m Rm|^2. \end{aligned}$$

Therefore, setting $\delta = 1$, we end up with

$$C\left(\int \phi^4|\nabla^m Rm|^4\right)^{\frac{1}{2}} \leq I_2 + I_3 + II + \int \phi^2|\nabla^{m+1}Ric|^2 + \int \phi^2|\nabla^{m+1}Rm|^2. \quad (2.16)$$

Now we bound the term $\int \phi^2|\nabla^{m+1}Rm|^2$ as in the proof of (2.3). Integrating by parts we obtain

$$\int \phi^2|\nabla^{m+1}Rm|^2 = -\int \phi^2\langle\Delta(\nabla^m Rm), \nabla^m Rm\rangle - 2\int \phi\langle \mathrm{tr}(\nabla\phi \otimes \nabla^{m+1}Rm), \nabla^m Rm\rangle.$$

By Cauchy-Schwarz and the δ-Cauchy inequality , we bound the second term

$$-2\int \phi\langle \mathrm{tr}(\nabla\phi \otimes \nabla^{m+1}Rm), \nabla^m Rm\rangle \leq \delta\int \phi^2|\nabla^{m+1}Rm|^2 + \frac{1}{\delta}\int |\nabla\phi|^2|\nabla^m Rm|^2.$$

Therefore, using the bound for I obtained above, we end up with the following

$$\begin{aligned}\int \phi^2|\nabla^{m+1}Rm|^2 \leq I_2 + I_3 + \left(\frac{1}{\delta}+1\right) II \\ + \left(\frac{1}{2\delta}+1\right)\int \phi^2|\nabla^{m+1}Ric|^2 + \frac{3\delta}{2}\int \phi^2|\nabla^{m+1}Rm|^2.\end{aligned}$$

So choosing δ small enough to regroup the term $\int \phi^2|\nabla^{m+1}Rm|^2$ and plugging back into (2.16) yields

$$C\left(\int \phi^4|\nabla^m Rm|^4\right)^{\frac{1}{2}} \leq I_2 + I_3 + II + \int \phi^2|\nabla^{m+1}Ric|^2.$$

Note that by Hölder's inequality

$$I_2 \leq 2\left(\int_{supp(\phi)} |Rm|^2\right)^{\frac{1}{2}}\left(\int \phi^4|\nabla^m Rm|^4\right)^{\frac{1}{2}},$$

which allows us to regroup this term on the left hand side by choosing ε_0 small enough. Next we choose suitably supported cutoff functions and deal with the remaining terms.

By Hölder's inequality and the induction hypothesis, we have

$$\begin{aligned} I_3 &\leq \sum_{k=1}^{m-1} \left(\int \phi^4 |\nabla^k Rm|^4 \right)^{\frac{1}{4}} \left(\int \phi^4 |\nabla^{m-k} Rm|^4 \right)^{\frac{1}{4}} \left(\int_{supp(\phi)} |\nabla^m Rm|^2 \right)^{\frac{1}{2}} \\ &\leq C \left[\left(r^{-2} + P_m\right)^{\frac{m}{2}} \sum_{k=1}^{m-1} \left(r^{-2} + P_m\right)^{\frac{k+1}{2}} \left(r^{-2} + P_m\right)^{\frac{m-k+1}{2}} \right] \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

By the induction hypothesis,

$$\begin{aligned} II &\leq \int |\nabla \phi|^2 |\nabla^m Rm|^2 \leq C r^{-2} \left(r^{-2} + P_m\right)^m \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}} \\ &\leq C \left(r^{-2} + P_m\right)^{m+1} \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

and, finally, the term $\int \phi^2 |\nabla^{m+1} Ric|^2$ can be bounded with (2.2). □

2.2 L^∞ bounds for $|Rm|$

Now that we have established L^2 and L^4 estimates, we can use Lemma 1.12 to obtain the L^∞ estimates in Theorem 2.1. Here is where we need to use the upper volume growth assumption.

Proposition 2.8. *There are constants $C = C(m, r, S_0, C_S, V_0) < \infty$, $\varepsilon_0 = \varepsilon_0(C_S) > 0$ such that if $\left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\sup_{B_{\frac{r}{8}}(x)} |\nabla^m Rm| \leq C \left(\int_{B_r(x)} |Rm|^2 \right)^{\frac{1}{4}}$$

Proof. For the sake of an induction argument, we start with $m = 0$ case. Recall that we have the elliptic inequality for Rm (1.16)

$$\Delta |Rm| \geq -|Rm|^2 - |\nabla^2 Ric|,$$

so we can use Lemma 1.12 to obtain

$$\sup_{B_{\frac{r}{8}}(x)} |Rm| \leq C\left(r^{-2} + \left\|\nabla^2 scal\right\|_{L^2(B_r(x))}\right)\left(\|Rm\|_{L^2(B_r(x))} + r^3\left\|\nabla^2 Ric\right\|_{L^4\left(B_{\frac{r}{4}}(x)\right)}\right),$$

where we also used $\mathrm{Vol}(B_r(x)) \leq V_0 r^4$. Thus, using (2.1) to bound the $\nabla^2 Ric$ term, we have

$$\sup_{B_{\frac{r}{8}}(x)} |Rm| \leq C\,\|Rm\|^{\frac{1}{2}}_{L^2(B_r(x))}.$$

In general, from the equation for $\nabla^m Rm$ (2.14) we have

$$\Delta|\nabla^m Rm| \geq -2|\nabla^m Rm||Rm| - \underbrace{\left(\sum_{k=1}^{m-1}|\nabla^k Rm||\nabla^{m-k}Rm| + |\nabla^{m+2}Ric|\right)}_{S}.$$

So this time Lemma 1.12 yields

$$\sup_{B_{\frac{r}{8}}(x)} |\nabla^m Rm| \leq C\left(r^{-2} + \left\|\nabla^2 scal\right\|_{L^2(B_r(x))}\right) \left(\|\nabla^m Rm\|_{L^2\left(B_{\frac{r}{4}}(x)\right)} + r^3\,\|S\|_{L^4\left(B_{\frac{r}{4}}(x)\right)}\right).$$

Note that for each m we can adjust the cutoff functions appearing in the proof of Proposition 2.3 in order to modify the radius of the geodesic balls at will. This allows us to conclude that $\sum_{k=1}^{m-1}|\nabla^k Rm||\nabla^{m-k}Rm| \leq C\,\|Rm\|_{L^2(B_r(x))}$ by the induction hypothesis even though it appears in a bigger geodesic ball. Using (2.1) again we can bound the term S

$$\|S\|_{L^4\left(B_{\frac{r}{4}}(x)\right)} \leq C\left(\|Rm\|_{L^2(B_r(x))}\left(\mathrm{Vol}\,B_{\frac{r}{4}}(x)\right)^{\frac{1}{4}} + \|Ric\|^{\frac{1}{2}}_{L^2(B_r(x))}\right),$$

and by (2.3) we have $\|\nabla^m Rm\|_{L^2\left(B_{\frac{r}{4}}(x)\right)} \leq C\,\|Rm\|^{\frac{1}{2}}_{L^2(B_r(x))}$ so the result follows immediately. $\square$

2.2.1 L^∞ bounds for $|Rm|$ with constant scalar curvature

Here we provide the proof for the statement regarding constant scalar curvature in the ε-regularity Theorem 2.1. Here we don't need the upper volume growth assumption because it can be derived from the simpler equation for Ric. As mentioned above, this can be found in [TV05a].

Lemma 2.9. *There are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $V_0 = V_0(C_S) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then we have*

$$\operatorname{Vol} B_s(y) \leq V_0 s^4 \text{ for all } B_s(y) \subset B_{\frac{r}{4}}(x). \tag{2.17}$$

Proof. First note that by scale invariance it is enough to show the result for $r = 1$. If we assume constant scalar curvature, the inequality for Ric (1.15) is

$$\Delta |Ric| \geq -|Rm||Ric|,$$

and the bound (2.9) is $\left(\int_{B_{\frac{1}{2}}(x)} |Rm|^4\right)^{\frac{1}{2}} \leq C\varepsilon_0$. Therefore, we can use Lemma 1.10 to obtain $\sup_{B_{\frac{1}{4}}(x)} |Ric| \leq C$ for some constant $C = C(C_S)$. Now the desired volume estimate follows from Bishop-Gromov volume comparison

$$\operatorname{Vol}(B_s(y)) \leq V_{-C}(s) \leq \frac{V_{-C}(1)}{V_0(1)} V_0(s) \leq C(C_S)s^4,$$

where $V_{-\Lambda}(t)$ denotes the volume of a geodesic ball of radius t in the space form of constant curvature $-\Lambda$. $\square$

Now that we have an upper volume growth estimate, the rest of the proof is similar to the non-constant scalar curvature case. The only difference is that the bounds line up nicely.

Proposition 2.10. *There are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(m, C_S) < \infty$ such that if $\left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}} \leq \varepsilon_0$, then*

$$\sup_{B_{\frac{r}{8}}(x)} |\nabla^m Rm| \leq \frac{C}{r^{m+2}} \left(\int_{B_r(x)} |Rm|^2\right)^{\frac{1}{2}}$$

Proof. We can argue by induction as before. We have (1.16)

$$\Delta|Rm| \geq -|Rm|^2 - |\nabla^2 Ric|,$$

so we can use Lemma 1.12 and the L^4-bound for Rm (2.9) to obtain

$$\sup_{B_{\frac{r}{8}}(x)} |Rm| \leq Cr^{-2}\left(\|Rm\|_{L^2(B_r(x))} + r^3 \left\|\nabla^2 Ric\right\|_{L^4\left(B_{\frac{r}{4}}(x)\right)}\right),$$

where we also used the volume estimate (2.17) from above. By (2.1), we have $\|\nabla^2 Ric\|_{L^4} \leq Cr^{-3}\|Ric\|_{L^2(B_r(x))}$ which yields the $m = 0$ case. In general, from the equation for $\nabla^m Rm$ (2.14) we obtain the inequality

$$\Delta|\nabla^m Rm| \geq -2|\nabla^m Rm||Rm| - \underbrace{\left(\sum_{k=1}^{m-1} |\nabla^k Rm||\nabla^{m-k} Rm| + |\nabla^{m+2} Ric|\right)}_{\mathcal{S}}.$$

Thus, Lemma 1.12 implies

$$\sup_{B_{\frac{r}{8}}(x)} |\nabla^m Rm| \leq Cr^{-2}\left(\|\nabla^m Rm\|_{L^2\left(B_{\frac{r}{4}}(x)\right)} + r^3 \|S\|_{L^4\left(B_{\frac{r}{4}}(x)\right)}\right).$$

Using the induction hypothesis we have

$$\sum_{k=1}^{m-1} |\nabla^k Rm||\nabla^{m-k} Rm| \leq Cr^{-m-4}\|Rm\|^2_{L^2(B_r(x))},$$

so using (2.1) and (2.17) we obtain

$$\|S\|_{L^4\left(B_{\frac{r}{4}}(x)\right)} \leq Cr^{-m-3}\|Rm\|_{L^2(B_r(x))}.$$

The only bound left is obtained using (2.3)

$$\|\nabla^m Rm\|_{L^2\left(B_{\frac{r}{4}}(x)\right)} \leq Cr^{-m} \|Rm\|_{L^2(B_r(x))} .$$

$\square$

Chapter 3

Proof of the main results

In this chapter we give the details for the proof of Theorem 1.4, our gap theorem for "almost" scalar-flat half-conformally flat manifolds, and for the proof of Proposition 1.6, the removable singularity result for Hodge-harmonic self-dual 2-forms on compact Riemannian orbifolds with isolated singularities.

To gain some intuition about the problem, we first look into the simpler situation considered in [LeB86], where the scalar curvature is assumed to be non-negative.

Proposition 3.1 (cf. [LeB86, Proposition 2]). *Let (M, g) be a closed, anti-self-dual 4-manifold with $scal \geq 0$. Then we have $b_+(M) \leq 3$ and if $b_+(M) \neq 0$, we also have $scal \equiv 0$.*

As discussed in Section 1.3.1, we know that $b_+(M)$ can be realized as the dimension of the space of Hodge-harmonic self-dual 2-forms on (M, g), which we denoted by

$\mathcal{H}_g^+(M)$. Recall that for 2-forms on 4-manifolds, the Böchner formula reads [Bou81]

$$\Delta_H \omega = -\Delta \omega - 2W(\omega, \cdot) + \frac{scal}{3}\omega. \tag{3.1}$$

Now suppose that $\eta \in \mathcal{H}_g^+(M)$. Since the metric g is anti-self-dual, the Weyl tensor acts trivially on self-dual 2-forms. This implies

$$W\eta = W^+\eta + W^-\eta = 0.$$

Therefore, the Böchner formula simplifies to

$$\Delta \eta = \frac{scal}{3}\eta, \tag{3.2}$$

hence, we can take the inner product with η and integrate by parts to obtain

$$0 = \int_M |\nabla \eta|^2 + \int_M \frac{scal}{3}|\eta|^2.$$

Since $scal \geq 0$, it follows that $\nabla \eta = 0$ and that $|\eta|$ is constant. If $b_+(M) \neq 0$, we can actually choose η to be non-trivial, which then forces *scal* to vanish everywhere. Further, if we have $\eta, \nu \in \mathcal{H}_g^+(M)$, then $\langle \eta, \nu \rangle$ must be constant as well. In particular, if η and ν are L^2-orthogonal, they are also point-wise orthogonal. Finally, observe that Λ^+ is a rank 3 vector bundle, so we conclude that $b_+(M) \leq 3$.

Of course, this argument breaks down if we allow *scal* to take negative values. However, if we are able to take a limit as $scal \to 0$, any Hodge-harmonic self-dual 2-form would satisfy the *limiting* equation $\Delta \eta = 0$ in some appropriate sense. Therefore, one expects their behavior to be similar to the case $scal \equiv 0$ we just illustrated. This is the guiding idea in our strategy to prove Theorem 1.4, which we restate here for the convenience of the reader.

Theorem 3.2 (Theorem 1.4)**.** *Fix any $m \geq 2$ and let (M, g) be a closed, unit-volume, anti-self-dual 4-manifold with $\pi_1(M) = 0$. Suppose that there are constants $E_0 < \infty$, $V_0 < \infty$, $S_0 < \infty$, and $C_S > 0$ such that*

$$\left(\int_M |Rm|^2 dV\right)^{\frac{1}{2}} \leq E_0, \tag{3.3}$$

$$\|scal\|_{W^{m+2,4}(M)} \leq S_0, \tag{3.4}$$

$$\mathrm{Vol}\left(B_r(x)\right) \leq V_0 r^4 \text{ for all } x \in M \text{ and } r > 0, \tag{3.5}$$

and

$$C_S\left(M\right) \leq C_S. \tag{3.6}$$

Then there is a constant $\delta_0 = \delta_0(E_0, V_0, S_0, C_S) > 0$ such that if $\|scal\|_{L^1(M)} < \delta_0$, then we have $b_+(M) \leq 3$.

We give a brief outline of the proof. For the sake of contradiction, suppose that the conclusion of the theorem doesn't hold. That is, no matter how small we choose δ_0, there is always some anti-self-dual 4-manifold, $(M_{\delta_0}, g_{\delta_0})$, satisfying the hypothesis of the theorem but with $b_+\left(M_{\delta_0}\right) \geq 4$. In other words, we have a so called *contradicting sequence* of closed, unit-volume, anti-self-dual 4-manifolds, (M_i, g_i), with $\pi_1\left(M_i\right) = 0$ and such that

$$\left(\int_{M_i} |Rm_i|^2 dV_i\right)^{\frac{1}{2}} \leq E_0,$$

$$\left\|\nabla^2 scal\right\|_{W^{m,4}(M_i)} \leq S_0,$$

$$\mathrm{Vol}\left(B_r(x)\right) \leq V_0 r^4 \text{ for all } x \in M_i \text{ and } r > 0,$$

and

$$C_S\left(M_i\right) \leq C_S,$$

but with $b_+(M_i) \geq 4$ and $\|scal_i\|_{L^1(M_i)} \to 0$ as $i \to \infty$. The first step is to use the ε-regularity theorem for anti-self-dual metrics, Theorem 2.1, and the upper volume growth (3.5) to prove that the sequence, (M_i, g_i), must converge to a limit space, (M_∞, g_∞), in the Gromov-Hausdorff sense. Further, we have that M_∞ is a $\mathcal{C}^{m+1,\alpha}$ Riemannian manifold except for finitely many singular points, $\mathcal{S}$, and the convergence is uniform on compact subsets of $M_\infty \setminus \mathcal{S}$ in the $\mathcal{C}^{m,\alpha}$-topology. Next, since $b_+\left(M_i\right) \geq 4$, we can pick an L^2-orthonormal set of at least 4 Hodge-harmonic self-dual 2-forms on each M_i. Then, we can use the equation (3.2) to prove regularity estimates that allow us to show that this L^2-orthonormal set converges to a point-wise orthogonal set of 4 self-dual 2-forms on the regular part of the limit, $M_\infty \setminus \mathcal{S}$, a contradiction with $\dim\left(\Lambda^+\right) = 3$.

The rest of the chapter is organized as follows. In the first section, we prove all the necessary regularity estimates concerning Hodge-harmonic self-dual 2-forms. The second section is dedicated to describe the limit spaces and finalizing the proof of Theorem 1.4. In the third and last section we prove Proposition 1.6, the removable singularity result for orbifolds.

3.1 Analysis of Hodge-harmonic self-dual 2-forms

In this section we study the analysis of Hodge-harmonic self-dual 2-forms. More specifically, we prove uniform L^∞-bounds and an ε-regularity result for regions with small curvature in the L^2 sense. The main tools used in the proofs are the equation coming from the Böchner formula, (3.2), the regularity theory discussed in Section 1.4 and bounds on the (local) Sobolev constant.

Lemma 3.3. *Fix $q \geq 2$ and let (M, g) be a closed, anti-self-dual 4-manifold. Suppose that η is a Hodge-harmonic self-dual 2-form in $L^q(M)$. If $C_S(M) \leq C_S < \infty$, then there is some constant $C = C(q, C_S) < \infty$ such that*

$$\sup_M |\eta| \leq C \left(\|scal\|^2_{L^4(M)} + \frac{1}{\mathrm{Vol}(M,g)^{\frac{1}{2}}} \right)^{\frac{2}{q}} \left(\int_M |\eta|^q \right)^{\frac{1}{q}}. \tag{3.7}$$

If $C_S\left(B_r(x)\right) \geq C_S > 0$, there is a constant $C_1 = C_1(q, C_S) < \infty$ such that

$$\sup_{B_{\frac{r}{2}}(x)} |\eta| \leq C_1 \left(\|scal\|^2_{L^4(B_r(x))} + r^{-2} \right)^{\frac{2}{q}} \left(\int_{B_r(x)} |\eta|^q \right)^{\frac{1}{q}}. \tag{3.8}$$

Proof. Recall that the Böchner formula (3.1) on Hodge-harmonic self-dual 2-forms reduces to $\Delta \eta = \frac{scal}{3}\eta$. Then we can use the inequality (A.8), $|T|\,\Delta\,|T| \geq \langle T, \Delta T\rangle$, to obtain

$$\Delta|\eta| \geq -\frac{1}{3}|scal||\eta|. \tag{3.9}$$

At this point we can directly apply Lemma 1.10 to produce the local bound (3.8). For the global bound, we can just proceed as in the proof of Lemma 1.10. By the Sobolev inequality (1.7), for any $p \geq 2$ we have

$$\left(\int_M |\eta|^{2p} \right)^{\frac{1}{2}} \leq C_S \frac{p^2}{4} \int_M |\eta|^{p-2} \left|\nabla\,|\eta|\right|^2 + \frac{1}{\mathrm{Vol}(M,g)^{\frac{1}{2}}} \int_M |\eta|^p .$$

The first term can be integrated by parts and bounded using (3.9)

$$p^2 \int_M |\eta|^{p-2} |\nabla |\eta||^2 = \frac{p^2}{p-1} \int_M \left\langle \nabla |\eta|^{p-1}, \nabla |\eta| \right\rangle = -\frac{p^2}{p-1} \int_M |\eta|^{p-1} \Delta |\eta|$$
$$\leq \frac{2p}{3} \int_M |scal| \, |\eta|^p ,$$

where we also used $\frac{p}{p-1} \leq 2$ for any $p \geq 2$. Now we continue using Hölder's inequality and the δ-Cauchy inequality

$$p \int_M |scal| \, |\eta|^p \leq p \left(\int_M |\eta|^{2p} \right)^{\frac{1}{4}} \left(\int_M |scal|^4 \right)^{\frac{1}{4}} \left(\int_M |\eta|^p \right)^{\frac{1}{2}}$$
$$\leq \frac{\delta}{2} \left(\int_M |\eta|^{2p} \right)^{\frac{1}{2}} + \frac{p^2}{2\delta} \left(\int_M |scal|^4 \right)^{\frac{1}{2}} \int_M |\eta|^p .$$

Therefore, choosing δ sufficiently small in terms of C_S, we arrive at

$$\left(\int_M |\eta|^{2p} \right)^{\frac{1}{2}} \leq p^2 C(C_S) \left(\|scal\|^2_{L^4(M)} + \frac{1}{\mathrm{Vol}(M,g)^{\frac{1}{2}}} \right) \int_M |\eta|^p , \tag{3.10}$$

which can be iterated just as in Lemma 1.10. $\square$

We can also obtain global *uniform* L^∞ estimates in terms of the local Sobolev constant of geodesic balls of a definite size.

Corollary 3.4. *Fix some $q \geq 2$ and let (M, g) be a closed, anti-self-dual 4-manifold satisfying $\|scal\|_{L^4(M)} \leq S_0$. Suppose that there are constants $C_S > 0$ and $r_0 > 0$ such that $C_S(B_{r_0}(x)) > C_S$ for all $x \in M$. Then there is a constant $C = C(q, r_0, S_0, C_S) < \infty$ such that for*

$$\sup_M |\eta| \leq C \, \|\eta\|_{L^q(M)} ,$$

where η is any Hodge-harmonic self-dual 2-from in $L^q(M)$.

Proof. Given any $x \in M$ we can always write $|\eta(x)| \leq \sup_{B_{\frac{r_0}{2}}(x)} |\eta|$ and use (3.8). $\square$

Next we deal with higher order covariant derivatives. We start with a bound for the L^2-norm of $\nabla\eta$.

Lemma 3.5. *Let (M,g) be a closed, anti-self-dual 4-manifold. Suppose that η is a Hodge-harmonic self-dual 2-form in $L^2(M)$. If $C_S(B_r(x)) \geq C_S > 0$, there are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(C_S) < \infty$ such that if $\|scal\|_{L^2(B_r(x))} < \varepsilon_0$, then we have*

$$\int_{B_{\frac{r}{2}}(x)} |\nabla\eta|^2 \leq C\left(\|scal\|_{L^2(B_r(x))} + 1\right) r^{-2} \int_{B_r(x)} |\eta|^2. \tag{3.11}$$

If $C_S(M) \leq C_S < \infty$, then there is a constant $C_1 = C_1(C_S)$ such that

$$\int_M |\nabla\eta|^2 \leq C_1 \|scal\|_{L^2(M)} \left(\|scal\|^2_{L^4(M)} + \frac{1}{\mathrm{Vol}(M,g)^{\frac{1}{2}}}\right) \int_M |\eta|^2 .$$

Proof. We prove (3.11) first. Let ϕ be a cutoff function with $0 \leq \phi \leq 1$ and support contained in $B_r(x)$. We integrate by parts and use the equation for η (3.2) to obtain

$$\int \phi^2|\nabla\eta|^2 = -2\int \phi\langle \mathrm{tr}(\nabla\phi\otimes\nabla\eta),\eta\rangle - \frac{1}{3}\int \phi^2 scal|\eta|^2.$$

Then, using the δ-Cauchy inequality we can bound

$$\int \phi^2|\nabla\eta|^2 \leq \frac{1}{\delta}\int |\nabla\phi|^2|\eta|^2 + \delta\int \phi^2|\nabla\eta|^2 + \frac{1}{3}\int \phi^2|scal||\eta|^2,$$

so choosing $\delta = \frac{1}{2}$ yields

$$C\int \phi^2|\nabla\eta|^2 \leq \int |\nabla\phi|^2|\eta|^2 + \int \phi^2|scal||\eta|^2.$$

The second term can be bounded using Hölder's inequality followed by the Sobolev inequality

$$\int \phi^2|scal||\eta|^2 \leq \left(\int_{supp(\phi)} |scal|^2\right)^{\frac{1}{2}} \left(\int \phi^4|\eta|^4\right)^{\frac{1}{2}} \leq C\|scal\|_{L^2(B_r(x))}\int |\nabla(\phi|\eta|)|^2.$$

Integrating by parts and then using (3.9), we obtain

$$\int |\nabla(\phi|\eta|)|^2 = -\int \phi^2|\eta|\Delta|\eta| + \int |\nabla\phi|^2|\eta|^2 \leq \frac{1}{3}\int \phi^2|scal||\eta|^2 + \int |\nabla\phi|^2|\eta|^2.$$

Therefore, choosing ε_0 small enough we obtain

$$\int \phi^2|scal||\eta|^2 \leq C \left\|scal\right\|_{L^2(B_r(x))} \int |\nabla\phi|^2|\eta|^2,$$

which yields

$$\int \phi^2|\nabla\eta|^2 \leq C \left(\left\|scal\right\|_{L^2(B_r(x))} + 1\right) \int |\nabla\phi|^2|\eta|^2.$$

Then (3.11) follows choosing the cutoff ϕ so that $\phi\big|_{B_{\frac{r}{2}}(x)} \equiv 1$ and $|\nabla\phi| \leq cr^{-1}$.

If $C_S(M) \leq C_S < \infty$, we can proceed slightly differently. Integrating by parts and using Hölder's inequality

$$\int_M |\nabla\eta|^2 \leq \frac{1}{3}\int_M |scal|\,|\eta|^2 \leq \frac{1}{3}\left(\int_M |scal|^2\right)^{\frac{1}{2}} \left(\int_M |\eta|^4\right)^{\frac{1}{2}},$$

so we can use the estimate in (3.10) with $p = 2$ to obtain the desired bound. □

Much like before, we can obtain global bounds depending on the local Sobolev constant of geodesic balls of definite size.

Corollary 3.6. *Let (M, g) be a closed, anti-self-dual 4-manifold with* $\mathrm{Vol}(M, g) \leq V$*. If there is some $r_0 > 0$ such that $C_S(B_{r_0}(x)) > C_S$ for all $x \in M$, then there are constants $\varepsilon_0 = \varepsilon_0(C_S) > 0$ and $C = C(r_0, C_S, V) < \infty$ such that if $\left\|scal\right\|_{L^2(M)} \leq \varepsilon_0$, then we have*

$$\left\|\nabla\eta\right\|_{L^2(M)} \leq C \left\|\eta\right\|_{L^2(M)},$$

where η is any Hodge-harmonic self-dual 2-form in $L^2(M)$.

Proof. Choose ε_0 as in Lemma 3.5. Then consider a maximally disjoint family of $\frac{r_0}{4}$-balls, $\{B_{\frac{r_0}{4}}(x_i)\}_{i=1}^N$. Since we have a lower volume growth (3.16), a simple covering argument yields $N \leq V v_0^{-1} r_0^{-4} 4^4$. Then, using (3.11), we obtain

$$\int_M |\nabla\eta|^2 \leq \sum_{i=1}^N \int_{B_{\frac{r_0}{2}}(x_i)} |\nabla\eta|^2 \leq C \|\eta\|^2_{L^2(M)},$$

as desired. □

The next step is to obtain L^2-bounds for an arbitrary number of covariant derivatives. From the equation for Hodge-harmonic self-dual 2-forms (3.2), we can produce similar equations for each $\nabla^k\eta$ (see Lemma A.7 for details)

$$\Delta\left(\nabla^k\eta\right) = \sum_{l=0}^k \nabla^l Rm * \nabla^{k-l}\eta. \tag{3.12}$$

Note that in general we don't have control over the curvature terms appearing in the equation above, so we are not going to be able to obtain global bounds as before. However, due to the ε-regularity Theorem 2.1, we are able to derive local estimates in regions with small L^2-norm for the curvature.

Lemma 3.7. *Fix some integer $k \geq 1$ and let (M, g) be a closed anti-self-dual 4-manifold. Suppose that there are finite constants V_0 and S_0 such that* $\mathrm{Vol}(B_r(x)) \leq V_0 r^4$ *and $||\nabla^2 scal||_{W^{k,4}(B_r(x))} \leq S_0$. If $C_S(B_r(x)) \geq C_S > 0$, then there are constants $\varepsilon_0 = \varepsilon_0(C_S)$ and $C = C(k, r, C_S, S_0, V)$ such that if $\|Rm\|_{L^2(B_r(x))} < \varepsilon_0$, then we have*

$$\int_{B_{\frac{r}{2^k}}(x)} |\nabla^k\eta|^2 \leq C \int_{B_r(x)} |\eta|^2, \tag{3.13}$$

where η is any Hodge-harmonic self-dual 2-form in $L^2(M)$.

Proof. Since we have the estimate for $k = 1$, (3.11), we argue by induction. Let ϕ be a smooth cutoff function with $0 \le \phi \le 1$ and support contained in $B_{r2^{-k+1}}(x)$. Integrating by parts and using (3.12), we have

$$\int \phi^2 |\nabla^k \eta|^2 = -2 \int \phi \left\langle \mathrm{tr}(\nabla\phi \otimes \nabla^k \eta), \nabla^{k-1}\eta \right\rangle - \int \phi^2 \left\langle \sum_{l=0}^{k-1} \nabla^l Rm * \nabla^{k-l}\eta, \nabla^k \eta \right\rangle .$$

We bound the first term with the δ-Cauchy inequality

$$-2 \int \phi \left\langle \mathrm{tr}(\nabla\phi \otimes \nabla^k \eta), \nabla^{k-1}\eta \right\rangle \le \frac{1}{2} \int \phi^2 |\nabla^k \eta|^2 + 2 \int |\nabla \phi|^2 |\nabla^{k-1}\eta|^2.$$

By Theorem 2.1 and Hölder's inequality, we can bound the second term

$$-\int \phi^2 \left\langle \sum_{l=0}^{k-1} \nabla^l Rm * \nabla^{k-l}\eta, \nabla^k \eta \right\rangle \le C \sum_{l=0}^{k-1} \left(\int \phi^2 |\nabla^{k-l}\eta|^2 \right)^{\frac{1}{2}} \left(\int \phi^2 |\nabla^{k-l}\eta|^2 \right)^{\frac{1}{2}},$$

where C is a constant depending on k, C_S, S_0, V_0 and r. Thus, using the induction hypothesis and combining these two estimates, we obtain

$$\frac{1}{2} \int \phi^2 |\nabla^k \eta|^2 \le 2 \int |\nabla \phi|^2 |\nabla^{k-1}\eta|^2 + C \int_{B_r(x)} |\eta|^2,$$

and the result follows choosing ϕ so that $\phi \,|_{B_{r2^{-k}}(x)} \equiv 1$ and $|\nabla \phi| \le c_k r^{-1}$ where c_k is some constant only depending on k. $\square$

Now we can use this lemma to prove L^∞ estimates for $\nabla^k \eta$ in regions where the L^2-norm of Rm is small.

Lemma 3.8. *Fix some integer $k \ge 0$ and let (M, g) be a closed anti-self-dual 4-manifold. Suppose that there are finite constants V_0 and S_0 such that* $\mathrm{Vol}(B_r(x)) \le$

V_0r^4 *and* $||\nabla^2 scal||_{W^{k,4}(B_r(x))} \leq S_0$. *If* $C_S\left(B_r(x)\right) \geq C_S > 0$, *there are constants* $\varepsilon_0 = \varepsilon_0(C_S)$ *and* $C = C(k, r, C_S, S_0, V_0)$ *such that if* $\|Rm\|_{L^2(B_r(x))} < \varepsilon_0$, *then we have*

$$\sup_{B_{\frac{r}{2^{k+1}}}(x)} \left|\nabla^k \eta\right| \leq C \left(\int_{B_r(x)} |\eta|^2\right)^{\frac{1}{2}}, \tag{3.14}$$

where η *is any Hodge-harmonic self-dual 2-form in* $L^2(M)$.

Proof. We argue by induction. The case $k = 0$ is already proved in (3.8), so assume that we have obtained the bound for all non-negative integers up to $k-1$. For the general case we use an argument similar to Lemma 1.12 combined with the estimates in Theorem 2.1. From (3.12) and the inequality $|T|\,\Delta\,|T| \geq \langle T, \Delta T\rangle$ (A.8) it follows that

$$\Delta\left|\nabla^k \eta\right| \geq -\left|\nabla^k \eta\right| |Rm| - \sum_{l=1}^{k} \left|\nabla^l Rm\right| \left|\nabla^{k-l}\eta\right|.$$

If ε_0 is small enough, we know by (2.4) that for $l = 0, \ldots, k$

$$\left(\int_{B_{\frac{r}{2}}(x)} \left|\nabla^l Rm\right|^4\right)^{\frac{1}{2}} \leq C_l,$$

where C_l is a constant depending on l, C_S, S_0 and r. Further, by the induction hypothesis, we also have

$$\sup_{B_{\frac{r}{2^{k-l+1}}}(x)} \left|\nabla^{k-l}\eta\right| \leq C_{k-l} \|\eta\|_{L^2(B_r(x))},$$

which implies

$$\begin{aligned}\left\|\sum_{l=1}^{k} \left|\nabla^l Rm\right| \left|\nabla^{k-l}\eta\right|\right\|_{L^4\left(B_{\frac{r}{2^k}}(x)\right)} &\leq \sum_{l=1}^{k} C_{k-l} \|\eta\|_{L^2(B_r(x))} \left\|\nabla^l Rm\right\|_{L^4\left(B_{\frac{r}{2^k}}(x)\right)} \\ &\leq \sum_{l=1}^{k} C_{k-l} \|\eta\|_{L^2(B_r(x))} C_l \leq C_k \|\eta\|_{L^2(B_r(x))}.\end{aligned}$$

Finally, by Lemma 1.12 we obtain

$$\sup_{B_{\frac{r}{2^{k+1}}}(x)} \left|\nabla^k \eta\right| \leq C \left(\left\|\nabla^k \eta\right\|_{L^2\left(B_{\frac{r}{2^k}}(x)\right)} + \|\eta\|_{L^2\left(B_{\frac{r}{2^k}}(x)\right)} \right),$$

where now C depends on k, C_S, S_0, r and V_0. We only need to use (3.13) in order to obtain the desired estimate. □

3.2 Proof of Theorem 1.4

In this section we give the details for the proof of Theorem 1.4, our gap theorem for the scalar curvature of anti-self-dual 4-manifolds. To set the stage, recall that we use a contradiction argument. More explicitly, from now on we consider a sequence of closed, unit-volume, anti-self-dual 4-manifolds, (M_i, g_i), with $\pi_1(M_i) = 0$ and satisfying the property that there are constants $E_0 < \infty$, $V_0 < \infty$, $S_0 < \infty$ and $C_S < \infty$ such that

$$\left(\int_{M_i} |Rm_i|^2 dV_i \right)^{\frac{1}{2}} \leq E_0,$$

$$\|scal\|_{W^{m+2,4}(M_i)} \leq S_0,$$

$$\operatorname{Vol} B_r(x) \leq V_0 r^4 \text{ for all } x \in M_i \text{ and } r > 0,$$

and

$$C_S\,(M) \leq C_S,$$

while simultaneously satisfying $\|scal_i\|_{L^1(M_i)} \to 0$ and $b_+(M_i) \geq 4$. As discussed in the outline given at the beginning of the chapter, there are two aspects to consider. First we need to study the convergence of the sequence, (M_i, g_i), to a limit space M_∞

and then we need to understand how this convergence interacts with the analysis of Hodge-harmonic self-dual 2-forms that realize $b_+(M_i)$ as explained in Section 1.3.1.

Before we continue we make a few general "geometric" observations. Earlier, we proved in Lemma 1.7 that a bound on $C_S(M)$ implies a bound on the local Sobolev constant $C_S(\Omega)$ as long as $\mathrm{Vol}(\Omega) \leq c^2 \,\mathrm{Vol}(M, g)$ for some constant $0 < c < 1$. By our assumptions on volume, if we take $c = \frac{1}{2}$, this means that there is a constant $C(C_S) > 0$ such that

$$C_S\left(B_r(x)\right) \geq C(C_S), \tag{3.15}$$

as long as $r \leq r_0 = \sqrt{2} V_0^{-\frac{1}{4}}$. It is also well known that the local Sobolev constant controls the lower volume growth of small geodesic balls (see Lemma A.9 for details). On this account, there is a constant $v_0 = v_0(C_S) > 0$ such that

$$\mathrm{Vol}\left(B_r(x)\right) \geq v_0 r^4, \tag{3.16}$$

for all $x \in M_i$ and $r \leq r_0$. Lastly, the upper volume growth (3.5) implies a lower bound on the diameter, $\mathrm{diam}(M, g) \geq d_0(V_0)$.

3.2.1 Convergence theory for anti-self-dual 4-manifolds

In this section we show that the sequence, (M_i, g_i), or possibly a subsequence, converges in the Gromov-Hausdorff topology to a compact length space, (M_∞, g_∞), with the following structure. There is a finite number of *singular* points, $\mathcal{S}$, such that $M_\infty \setminus \mathcal{S}$ is a $\mathcal{C}^{m+1,\alpha}$ anti-self-dual Riemannian manifold and the convergence is uniform on compact subsets of $M_\infty \setminus \mathcal{S}$ in the $\mathcal{C}^{m,\alpha}$-topology. On top of that, there is a constant $N = N(E_0, C_S)$ such that $|\mathcal{S}| \leq N$. This kind of argument has appeared many times in the literature and it is well understood that the key ingredients are an ε-regularity theorem like Theorem 2.1 and an upper volume growth (3.5). Roughly speaking, the finite L^2-norm of Rm combined with the ε-regularity theorem tells us that, on most of M_i, we have uniformly bounded curvature, then the upper volume growth guarantees that those regions were the curvature becomes unbounded are small. For the most part, our arguments are adaptations of those appearing in [And89, BKN89, TV05b, Nak88].

Right from the start we can prove that we have *some* sort of Gromov-Hausdorff convergence

Lemma 3.9. *There is a compact length space (X_∞, d_∞) such that*

$$(M_i, g_i) \longrightarrow (X_\infty, d_\infty) \tag{3.17}$$

in the Gromov-Hausdorff topology.

Proof. Consider, $\left\{B_{\frac{r}{2}}\left(x_i^k\right)\right\}_{k=1}^{N_i}$, a maximal family of disjoint geodesic $\frac{r}{2}$-balls contained in M_i. In particular, the family $\left\{B_r\left(x_i^k\right)\right\}_{k=1}^{N_i}$ covers M_i. If we assume $r < 2r_0$, where r_0 is as in (3.15), by the lower volume growth (3.16) we obtain

$$1 = \mathrm{Vol}(M_i, g_i) \geq \sum_{k=1}^{N_i} \mathrm{Vol}\left(B_{\frac{r}{2}}\left(x_i^k\right)\right) \geq N_i C(v_0) r^4,$$

which gives the uniform upper bound on the cardinality of the family, $N_i \leq C(v_0) r^{-4}$. Therefore, the result follows from Gromov's precompactness theorem [Pet98, Proposition 44]. $\square$

Next we improve the regularity of the limit space. As mentioned above, on of the most important aspects is to understand how curvature concentrates as measured by the L^2-norm. From now on, we fix $\varepsilon_0 > 0$ to be the same constant that appears in the ε-regularity Theorem 2.1. Given $r > 0$, we can define the *good set* and *bad set* of M_i. Respectively,

$$\mathcal{G}_{i,r} = \left\{ x \in M_i : \left(\int_{B_r(x)} |Rm_i|^2 \right)^{\frac{1}{2}} < \varepsilon_0 \right\}, \tag{3.18}$$

and

$$\mathcal{B}_{i,r} = \left\{ x \in M_i : \left(\int_{B_r(x)} |Rm_i|^2 \right)^{\frac{1}{2}} \geq \varepsilon_0 \right\}. \tag{3.19}$$

We want to use the local Sobolev constant, so we further assume that $r \leq r_0$, where r_0 is as in (3.15). Clearly, the good and bad sets are disjoint and constitute a cover, $M_i = \mathcal{G}_{i,r} \bigsqcup \mathcal{B}_{i,r}$. The key for the regular convergence is to take limits of these good sets, where we have control of the geometry thanks to Theorem 2.1.

Lemma 3.10. *Fix some $0 < r \leq r_0$. Then there is a $C^{m+1,\alpha}$ Riemannian manifold, $(\mathcal{G}_{\infty,r}, g_{\infty,r})$, such that $\mathcal{G}_{i,r} \to \mathcal{G}_{\infty,r}$ uniformly on compact subsets in the $C^{m,\alpha}$-topology as $i \to \infty$. If $m \geq 2$, then $g_{\infty,r}$ is an anti-self-dual metric with $scal_\infty = 0$.*

Proof. Using Theorem 2.1, we have constants $C_k = C_k(r, S_0, C_S, V_0)$ such that

$$\sup_{\mathcal{G}_{i,r}} |\nabla^k Rm| \leq C_k, \tag{3.20}$$

for $k = 0, \ldots, m$. By the work in [CGT82], this curvature estimate combined with the lower volume growth (3.16) implies a lower bound on the injectivity radius,

$\mathrm{inj}(M_i, g_i) \geq \iota(v_0, C_0) > 0$. Therefore, we can apply Gromov's compactness theorem [And89, Theorem 2.2] to conclude that there is a $C^{m+1,\alpha}$ manifold, $(\mathcal{G}_{\infty,r}, g_{\infty,r})$, with the desired properties, after possibly passing to a subsequence. That is, there are diffeomorphisms, $\psi_{i,r} : \mathcal{G}_{\infty,r} \to \mathcal{G}_{i,r}$, such that $\psi_{i,r}^* g_i \to g_{\infty,r}$ uniformly on compact subsets in the $C^{m,\alpha}$-topology. To see that $g_{\infty,r}$ is an anti-self-dual metric when $m \geq 2$, recall that the self-dual part of the Weyl tensor is just an expression in terms of up to two derivatives of the metric, and since the g_i converges to $g_{\infty,r}$ in the $\mathcal{C}^{m,\alpha}$-topology with $m \geq 2$, we have $W_i^+ \to W_\infty^+$. Since we are assuming $W_i^+ \equiv 0$, it follows that $W_\infty^+ = 0$. By the same token, we also have $scal_i \to scal_\infty$ but since $\|scal_i\|_{L^1(\mathcal{G}_{i,r})} \to 0$, Fatou's lemma implies that $\int_{\mathcal{G}_{\infty,r}} |scal_\infty| = 0$ and we end up with $scal_\infty = 0$ on $\mathcal{G}_{\infty,r}$. □

Remark 3.11. *Although not quite relevant to this dissertation, one can show that* $(\mathcal{G}_{\infty,r}, g_{\infty,r})$ *is a smooth manifold instead of just* $\mathcal{C}^{m+1,\alpha}$. *This is because* $scal_\infty \equiv 0$ *allows for Theorem 2.1 to be used with any* m.

Next we see how the volume of the bad set is controlled.

Lemma 3.12. *There is a constant* $C = C(C_S, V_0, E_0)$ *such that* $\mathrm{Vol}(\mathcal{B}_{i,r}) \leq Cr^4$.

Proof. Consider, $\{B_r(x_i^k)\}_{k=1}^N$, a maximal family of disjoint geodesic r-balls with centers in the bad set, $x_i^k \in \mathcal{B}_{i,r}$. By definition, we have

$$N\varepsilon_0^2 \leq \sum_{k=1}^N \int_{B_r(x_i^k)} |Rm_i|^2 \leq \int_{M_i} |Rm_i|^2,$$

and since we have finite L^2-norm for $|Rm|$ (3.3), we obtain $N \leq E_0^2 \varepsilon_0^{-2}$. Note that, doubling the radii of the geodesic balls, we can cover the bad set. Therefore, using the upper volume growth (3.5), we have that $\mathrm{Vol}(\mathcal{B}_{i,r}) \leq \sum_{k=1}^{N} \mathrm{Vol}(B_{2r}(x_{i,k})) \leq NV_0(2r)^4$. Recall that ε_0 only depended on C_S so the result follows. $\square$

In the following, we finish the construction of M_∞. Observe that in the proof of Lemma 3.12 we actually showed that the bad set $\mathcal{B}_{i,r}$ can always be covered by at most N geodesic balls of radius $2r$, where N only depends on C_S and E_0. Therefore, we can restrict to a subsequence of (M_i, g_i) where this covering number is always the same. We keep track of the centers of these geodesic balls with the set $\mathcal{S}_i = \{x_i^k\}_{k=1}^N$ and let $\mathcal{S}_{i,r} = \bigcup_{k=1}^N B_{2r}(x_i^k)$ the aforementioned covering of $\mathcal{B}_{i,r}$. This leads us to define the *even better* set, $\mathcal{E}_{i,r} = M_i \setminus \mathcal{S}_{i,r} \subset \mathcal{G}_{i,r}$. Now take a sequence of increasingly smaller scales, $r_j \to 0$, for each fixed j we can use Lemma 3.10 to obtain a converging sequence, $\mathcal{E}_{i,r_j} \to \mathcal{E}_{\infty,r_j}$, in the $\mathcal{C}^{m,\alpha}$-topology. Since we have $\mathcal{E}_{i,r_j} \subset \mathcal{E}_{i,r_{j+1}}$ because $r_{j+1} < r_j$, taking further subsequences we obtain nested limit manifolds, $\mathcal{E}_{\infty,r_j} \subset \mathcal{E}_{\infty,r_{j+1}} \subset \ldots$ and we can define the limit manifold $\mathcal{E}_\infty = \bigcup_{j=1}^\infty \mathcal{E}_{\infty,r_j}$, which carries a scalar-flat anti-self-dual metric, g_∞, induced from the scalar-flat anti-self-dual metrics g_{∞,r_j}.

Finally, we define M_∞ as the metric completion of $\mathcal{E}_\infty$ with respect to g_∞. Using Lemma 3.12, it is easy to prove that $M_\infty \setminus \mathcal{E}_\infty$ is a finite set of points which we call *singular* and denote by $\mathcal{S}$. Since convergence in the $\mathcal{C}^{m,\alpha}$-topology implies convergence in the Gromov-Hausdorff topology, it is straightforward to see that M_i converges to M_∞ in the Gromov-Hausdorff topology as well. In particular, (M_∞, g_∞) is isometric to the limit length space, (X_∞, d_∞), from (3.17). This implies that (M_∞, g_∞) is a compact length such that g_∞ restricts to an anti-self-dual scalar-flat metric on $M_\infty \setminus \mathcal{S}$ and has all the convergence properties claimed above.

Based on arguments in [CQY07, Lemma 3.2] and [TV05b, Proposition 7.2], we are able to prove that the regular part of the limit space is connected. This will be useful later on when we study the convergence of the Hodge-harmonic self-dual 2-forms representing $b_+(M_i)$.

Lemma 3.13. $\pi_1(M_i) = 0$ *implies that* $M_\infty \setminus \mathcal{S}$ *is connected.*

Proof. We prove something slightly stronger. Given a singular point $s \in \mathcal{S}$, we prove that $B_r(s) \setminus \{s\}$ must be connected for all sufficiently small $r > 0$. Suppose to the contrary that there is some $r_0 > 0$ with the property that $B_{r_0}(s) \setminus \{s\}$ has at least two connected components, $B^1_{r_0}(s)$ and $B^2_{r_0}(s)$. Let v denote the volume of the smallest of these components. Then we can intersect the annulus $A_{\rho,r_0}(s)$ with each $B^i_{r_0}(s)$ to obtain two connected components, $A^1_{\rho,r_0}(s)$ and $A^2_{\rho,r_0}(s)$. Moreover, the upper volume growth (3.5) implies that for all small ρ, we have $\mathrm{Vol}\left(A^k_{2\rho,r_0}(s)\right) \geq \frac{1}{2}v$. Next, let $\{x_i\}$ be a sequence of points in M_i converging to s in the Gromov-Hausdorff topology. If we choose some $r < \rho$, then, by Lemma 3.10, the annulus $A_{\rho,r_0}(s) \subset \mathcal{E}_{\infty,r}$ is diffeomorphic to the annulus $A_{\rho,r_0}(x_i) \subset \mathcal{E}_{i,r} \subset M_i$ for large enough i. This implies that $A_{\rho,r_0}(x_i)$ also has two connected components, $A^1_{\rho,r_0}(x_i)$ and $A^2_{\rho,r_0}(x_i)$, satisfying $\mathrm{Vol}\left(A^k_{2\rho,r_0}(x_i)\right) \geq \frac{1}{3}v$. We claim that this is impossible. Indeed, the assumption $\pi_1(M_i) = 0$ implies that there is no path connecting $A^1_{\rho,r_0}(x_i)$ and $A^2_{\rho,r_0}(x_i)$ that is contained in $M_i \setminus B_{r_0}(x_i)$. It follows that the subannulus $A^1_{\rho,2\rho}(x_i)$ separates M_i into exactly two connected components, $A \cup B = M_i \setminus A^1_{\rho,2\rho}(x_i)$. Without loss of generality we may assume that $\mathrm{Vol}(A) \leq \mathrm{Vol}(B)$. As the final step, consider a smooth cutoff function, ϕ, with $0 < \phi < 1$ and such that $\phi\big|_A \equiv 1$, $\phi\big|_B \equiv 0$ and $|\nabla \phi| \leq 2\rho^{-1}$ on $A^1_{\rho,2\rho}(x_i)$. Using the Sobolev inequality (1.7), Hölder's inequality and the upper volume growth (3.5), we obtain

$$\begin{aligned}
\left(\int_{M_i} \phi^4\right)^{\frac{1}{2}} &\leq C_S \int_{M_i} |\nabla \phi|^2 + \int_{M_i} \phi^2 \\
&\leq 4 C_S \rho^{-2} \,\mathrm{Vol}(A^1_{\rho,2\rho}(x_i)) + \int_A \phi^2 + \int_{A^1_{\rho,2\rho}(x_i)} \phi^2 \\
&\leq C(C_S, V_0)\rho^2 + \left(\int_{M_i} \phi^4\right)^{\frac{1}{2}} \mathrm{Vol}(A)^{\frac{1}{2}} + 16 V_0 \rho^4.
\end{aligned}$$

Since we are assuming unit-volume for M_i, it follows that $\mathrm{Vol}(A) \leq \frac{1}{2}$. So we obtain

$$\left(\int_{M_i} \phi^4\right)^{\frac{1}{2}} \leq C(C_S, V_0)(\rho^2 + \rho^4).$$

On the other hand, we constructed our annuli to have a definite amount of volume

$$\left(\frac{v}{3}\right)^{\frac{1}{2}} \leq \mathrm{Vol}(A)^{\frac{1}{2}} \leq \left(\int_{M_i} \phi^4\right)^{\frac{1}{2}},$$

which results in $C(C_S, V_0, v) \leq \rho^2 + \rho^4$, a contradiction because we can choose ρ as small as necessary. □

Remark 3.14. *If we assume that the scalar curvature is constant, the resulting convergence theory can be significantly improved. This was done by Tian-Viaclovsky in [TV05a, TV05b, TV08] (cf. [And05]), where they prove that the singularities are actually of (multi-)orbifold type.*

3.2.2 Final arguments for Theorem 1.4

Recall that our sequence satisfies $b_+(M_i) \geq 4$, which is the dimension of Hodge-harmonic self-dual 2-forms as it was shown in Section 1.3.1. Thus, we can choose a L^2-orthonormal set of at least 4 such forms, $\{\eta_i^k\}_{k=1}^4$. That is $\int_{M_i} \langle \eta_i^k, \eta_i^l \rangle = \delta_{kl}$. From the discussion in Section 3.2.1, now we better grasp on the possible degenerations in the geometry of (M_i, g_i). Now, it is time to examine the interplay between this geometric convergence and the convergence of $\{\eta_i^k\}_{k=1}^4$. The first step is to prove that each $\{\eta_i^k\}$ converges to some Hodge-harmonic self-dual 2-form, η^k, along the regular part, $\mathcal{E}_{i,r} \subset (M_i.g_i) \longrightarrow \mathcal{E}_{\infty,r} \subset M_\infty$ as $i \to \infty$.

Lemma 3.15. *For each $k = 1, \dots, 4$ there is some Hodge-harmonic self-dual 2-form on $M_\infty \backslash \mathcal{S}$, denoted by η^k, such that $\eta_i^k \longrightarrow \eta^k$ uniformly on compact subsets of $M_\infty \backslash \mathcal{S}$ in the $\mathcal{C}^{m,\alpha}$-topology. Further, we also have $\Delta \eta^k = 0$.*

Proof. Fix $r \leq r_0 = \sqrt{2} V_0^{-\frac{1}{4}}$ so that the bound on the local Sobolev constant (3.15) is active and consider the even better set $\mathcal{E}_{i,r}$. By definition, any point x in $\mathcal{E}_{i,r}$ satisfies $\|Rm\|_{L^2(B_r(x))} \leq \varepsilon_0$ and since $\left\|\eta_i^k\right\|_{L^2(M_i)} = 1$, from Lemma 3.8 we obtain

$$\sup_{\mathcal{E}_{i,r}} \left|\nabla^l \eta_i^k\right| \leq C_l,$$

for each $l = 0, \dots, m$ where C_l is a constant depending only on l, r, C_S, S_0 and V_0. Recall that that the convergence $\mathcal{E}_{i,r} \to \mathcal{E}_{\infty,r}$ comes equipped with diffeomorphisms, $\psi_{i,r} : \mathcal{E}_{\infty,r} \to \mathcal{E}_{i,r}$, such that $\psi_{i,r}^* g_i \to g_{\infty,r}$ uniformly on compact subsets in the $\mathcal{C}^{m,\alpha}$-topology. It follows that $\left\|\psi_{i,r}^* \eta_i^k\right\|_{\mathcal{C}^{m,\alpha}}$ is uniformly bounded, therefore, from Arzelà-Ascoli we obtain a 2-form, η^k, such that $\eta_i^k \to \eta^k$ uniformly on compact subsets in the $\mathcal{C}^{m,\alpha'}$-topology for $\alpha' < \alpha$, but α is arbitrary between 0 and 1, so we do obtain $\mathcal{C}^{m,\alpha}$ for any $0 < \alpha < 1$.

Next, we verify the properties that were claimed about η^k. Since $m \geq 2$, the convergence is good enough so that the usual quantities that can be defined in terms of the Riemannian metric pass to the limit. We have $*\eta_i^k \to \eta^k$ and $\Delta_H \eta_i^k \to \Delta_H \eta^k$, so η^k is Hodge-harmonic and self-dual. We also have $\Delta\eta_i^k \to \Delta\eta^k$ and $scal_i \to scal_\infty$, but in Lemma 3.10 we proved $scal_\infty = 0$, so the equation $\Delta\eta_i^k = \frac{scal_i}{3}\eta_i^k$ becomes $\Delta\eta^k = 0$ in the limit.

To finish the proof, let $r_j \to 0$ and take diagonal subsequences. $\square$

The goal for the rest of the proof is to show that that the set $\{\eta^k\}_{k=1}^4$ must be point-wise linearly independent on $M_\infty \setminus \mathcal{S}$. The difficulty lies in the fact that, at this point, we don't really know what happens with the convergence of η_i^k on the singular set. In the following we deal this problem. The first thing is to observe that each η^k must be parallel on $M_\infty \setminus \mathcal{S}$.

Lemma 3.16. *$\nabla\eta^k \equiv 0$ on $M_\infty \setminus \mathcal{S}$.*

Proof. Integrating by parts equation (3.2), $\Delta\eta_i^k = \frac{scal_i}{3}\eta_i^k$, followed by Hölder's inequality, we obtain

$$\int_{M_i} \left|\nabla\eta_i^k\right|^2 = -\frac{1}{3}\int_{M_i} scal \left|\eta_i^k\right|^2 \leq \frac{1}{3}\sup_{M_i}\left|\eta_i^k\right|^2 \|scal\|_{L^1(M_i)} .$$

From the discussion surrounding (3.15), we can take $r_0 = \sqrt{2}V_0^{-\frac{1}{4}}$ in Corollary 3.4 to obtain a constant $C = C(C_S, V_0)$ such that $\sup_{M_i}\left|\eta_i^k\right|^2 \leq C$. Since $\|scal\|_{L^1(M_i)} \to 0$, we conclude that $\int_{M_i} \left|\nabla \eta_i^k\right|^2 \to 0$. Recall that from Lemma 3.10 we have $\left|\nabla \eta_i^k\right| \to \left|\nabla \eta^k\right|$, so Fatou's lemma implies

$$\int_{\mathcal{E}_{\infty,r}} \left|\nabla \eta^k\right|^2 \leq \lim_{i\to\infty} \int_{\mathcal{E}_{i,r}} \left|\nabla \eta_i^k\right|^2 \leq \lim_{i\to\infty} \int_{M_i} \left|\nabla \eta_i^k\right|^2 = 0,$$

and the claim follows by letting $r \to 0$. $\square$

From this we obtain $\nabla \left\langle \eta^k, \eta^l \right\rangle = \left\langle \nabla \eta^k, \eta^l \right\rangle + \left\langle \eta^k, \nabla \eta^l \right\rangle = 0$, hence, the functions $\left\langle \eta^k, \eta^l \right\rangle$ are constant on each connected component of $M_\infty \setminus \mathcal{S}$. We proved in Lemma 3.13 that $M_\infty \setminus \mathcal{S}$ is actually connected so let $c_{kl} = \left\langle \eta^k, \eta^l \right\rangle$ if $k \neq l$ and $c_k = \left|\eta^k\right|^2$ the constant value that these functions take on $M_\infty \setminus \mathcal{S}$. The main issue that could arise now would be for the singularities to concentrate all the L^2-norm of our Hodge-harmonic self-dual 2-forms, $\left\{\eta_i^k\right\}_{k=1}^4$. In other words, we might have that $\eta^k \equiv 0$ or that they "fold" into each other and don't become orthogonal. The final step is to prove that these phenomena don't actually occur.

Lemma 3.17. *$c_k > 0$ and $c_{kl} = 0$ for $k \neq l$.*

Proof. Fix some small $r > 0$. Recall that $M_i = \mathcal{E}_{i,r} \bigsqcup \mathcal{S}_{i,r}$. We can take $r_0 = \sqrt{2} V_0^{-\frac{1}{4}}$ in Corollary 3.4 to obtain a constant $C = C(C_S, V_0)$ such that $\sup_{M_i} \left|\eta_i^k\right| \leq C$. Thus, using the proof of Lemma 3.12, we have

$$\int_{\mathcal{S}_{i,r}} \left|\eta_i^k\right|^2 \leq C(C_S, V_0, E_0) r^4.$$

Therefore, since $\left\|\eta_i^k\right\|_{L^2(M_i)} = 1$, we obtain

$$\int_{\mathcal{E}_{i,r}} \left|\eta^k\right|^2 = 1 - \int_{\mathcal{S}_{i,r}} \left|\eta_i^k\right|^2 \geq 1 - C(C_S, V_0, E_0) r^4.$$

On the other hand, by dominated convergence and Lemmas 3.10 and 3.15, we also have

$$c_k \operatorname{Vol}(\mathcal{E}_{\infty,r}) = \int_{\mathcal{E}_{\infty,r}} \left|\eta^k\right|^2 = \lim_{i\to\infty} \int_{\mathcal{E}_{i,r}} \left|\eta_i^k\right|^2 \leq 1.$$

Combining these, we reach

$$1 \geq c_k \operatorname{Vol}(\mathcal{E}_{\infty,r}) \geq 1 - C(C_S, V_0, E_0) r^4,$$

which is valid for any small $r > 0$. Letting $r \to 0$ implies $c_k > 0$, where we are also using Lemma 3.12 and $\operatorname{Vol}(M_i, g_i) = 1$ to verify that $\operatorname{Vol}(\mathcal{E}_{\infty,r}) > 0$. In fact, we obtain $c_k = \operatorname{Vol}(M_\infty)^{-1} = 1$. We can proceed in a similar fashion with c_{kl}. We obtain

$$\left|\int_{\mathcal{S}_{i,r}} \langle \eta_i^k, \eta_i^l \rangle\right| \leq C(C_S, V_0, E_0) r^4,$$

and using $\langle \eta_i^k, \eta_i^l \rangle_{L^2(M_i)} = 0$,

$$\left|\int_{\mathcal{E}_{i,r}} \langle \eta_i^k, \eta_i^l \rangle\right| \leq \left|\int_{\mathcal{S}_{i,r}} \langle \eta_i^k, \eta_i^l \rangle\right| \leq C(C_S, V_0, E_0) r^4.$$

Finally,

$$c_{kl} \operatorname{Vol}(\mathcal{E}_{\infty,r}) = \int_{\mathcal{E}_{\infty,r}} \langle \eta_i^k, \eta_i^l \rangle = \lim_{i\to\infty} \int_{\mathcal{E}_{i,r}} \langle \eta_i^k, \eta_i^l \rangle \leq C(C_S, V_0, E_0) r^4,$$

and letting $r \to 0$ results in $c_{kl} = 0$. □

Remark 3.18. *To prove this lemma we only really need* $\left|\int_{\mathcal{S}_{i,r}} \langle \eta_i^k, \eta_i^l \rangle\right| \to 0$ *as* $r \to 0$.

To finish the proof of Theorem 1.4, just pick any $x_\infty \in M_\infty \setminus \mathcal{S}$ and apply Lemma 3.17 to conclude that $\left\{\eta^k(x_\infty)\right\}_{k=1}^4$ is a linearly independent subset of $\Lambda^+_{x_\infty}$. A contradiction, since Λ^+ is a rank 3 bundle.

Some remarks about Theorem 1.4

As mentioned before, if we assume constant scalar curvature the assumptions can be simplified and the ε-regularity Theorem 2.1 takes Tian-Viaclovsky's original form [TV05a, Theorem 3.1]. This way, if we restrict our attention to constant scalar curvature metrics we can restate Theorem 1.4 as follows.

Theorem 3.19. *Let (M, g) be a simply connected, unit-volume, closed anti-self-dual 4-manifold with constant scalar curvature. Suppose that there are constants $E_0 < \infty$ and $C_S < \infty$ such that $\int_M |Rm|^2 \leq E_0$ and $C_S\,(M) \leq C_S$. Then there is a constant $\delta = \delta(E_0, C_S) > 0$ such that if $|scal| < \delta_0$, then $b_+(M) \leq 3$.*

Proof. Clearly, if we have constant scalar curvature and unit-volume, the condition $|scal| < \delta_0$ implies $\|scal\|_{L^1(M)} < \delta_0$ as well. Moreover, by [TV05b, Theorem 1.2], there is a constant $V_0 = V_0(E_0, C_S)$ such that $\mathrm{Vol}\,(B_r(x)) \leq V_0 r^4$ for all $x \in M$ and $r > 0$. At this point we can just apply Theorem 1.4. $\square$

It turns out that a lower bound for *Ric* encodes the majority of the assumptions in Theorem 1.4 and, in addition, we don't need to assume that our manifolds are simply-connected. We have the following formulation of Theorem 1.4.

Theorem 3.20. *Let (M, g) be a closed, anti-self-dual 4-manifold. Suppose that there are constants $\Lambda > 0$, $S_0 < \infty$, $v > 0$ and $D < \infty$ such that $Ric \geq -3\Lambda^2$, $\|\nabla^2 scal\|_{W^{4,4}(M)} \leq S_0$, $\mathrm{Vol}(M, g) \geq v$ and $\mathrm{diam}(M, g) \leq D$. Then there is a constant $\delta_0 = \delta_0(v, D, \Lambda, S_0) > 0$ such that if $|scal| < \delta$, then we have $b_+(M) \leq 3$.*

Proof. First of all, by Bishop-Gromov's volume comparison, we have an upper volume growth

$$\mathrm{Vol}\,(B_r(x)) \le V_{-\Lambda}(r) \le \frac{V_{-\Lambda}(D)}{V_0(D)} V_0(r) \le C(D,\Lambda) r^4, \tag{3.21}$$

where $V_{-\Lambda}(r)$ denotes the volume of a geodesic ball of radius r in the space form of constant curvature $-\Lambda$. It is also well-known [Cro80] that the Sobolev constant can be bounded in terms of v, D and Λ. In fact, the local Sobolev constant can also be bounded. Inded, using Bishop-Gromov's volume comparison again, we have

$$v \le \mathrm{Vol}\,(M,g) \le V_{-\Lambda}(\mathrm{diam}(M,g)),$$

which implies a lower bound on the diameter, $\mathrm{diam}(M,g) \ge d_0(v,\Lambda)$. Therefore, we can use [And92, Theorem 4.1] to obtain that, for any $r \le \min\left\{\frac{1}{4}d_0, 1\right\}$ and x in M, we have

$$C_S\,(B_r(x)) \ge C(\Lambda)\left(\frac{\mathrm{Vol}(B_r(x))}{V_{-\Lambda}(r)}\right)^{\frac{1}{4}} \ge C(v,\Lambda) > 0,$$

where we also used Bishop-Gromov's volume comparison on the last inequality. Next, observe that we may assume $|scal| \le 1$, which combined with the lower bound on Ric, implies a two-sided bound, $|Ric| \le C(\Lambda)$. Therefore, the work of Cheeger-Naber [CN15, Theorem 1.13] gives a bound for $\|Rm\|_{L^2(M)}$.

Now the only missing piece is the simply-connected condition in Theorem 1.4. However, that assumption was only necessary to prove Lemma 3.13, that the regular part, $M_\infty \backslash \mathcal{S}$, is connected. In the presence of bounds for Ric, we can adapt the proof of this lemma using the Cheeger-Gromoll splitting theorem (see [And89, BKN89]). We briefly give the details here using the notation from Section 3.2.1.

Let s be singular point in $\mathcal{S}$ and suppose that for some small $r_0 > 0$ the metric ball $B_{r_0}(s) \setminus \{s\}$ has at least two connected components, $B^1_{r_0}(s)$ and $B^2_{r_0}(s)$. Then we can intersect the annulus $A_{\rho,r_0}(s)$ with each $B^i_{r_0}(s)$ to obtain two connected components, $A^1_{\rho,r_0}(s)$ and $A^2_{\rho,r_0}(s)$. Next, let $\{x_i\}$ be a sequence of points in M_i converging to s in the Gromov-Hausdorff topology. We can also assume that $\{x_i\}$ satisfies

$$\sup_{B_{r_0}(x_i)} |Rm_i| \leq |Rm_i|(x_i) \to \infty.$$

Choose some $r < \rho$, by Lemma 3.10, the annulus $A_{\rho,r_0}(s) \subset \mathcal{E}_{\infty,r}$ is diffeomorphic to the annulus $A_{\rho,r_0}(x_i) \subset \mathcal{E}_{i,r} \subset M_i$ for large enough i. This implies that $A_{\rho,r_0}(x_i)$ also has two connected components, $A^1_{\rho,r_0}(x_i)$ and $A^2_{\rho,r_0}(x_i)$. Since M_∞ is a length space, if we choose r_0 sufficiently small, we can take a minimizing segment, γ, with endpoints on $\partial B_\rho(s) \cap A^1_{\rho,r_0}(s)$ and $\partial B_\rho(s) \cap A^2_{\rho,r_0}(s)$ and $\gamma(0) = s$. Then, there is a corresponding minimizing geodesic segment, γ_i, with endpoints on $\partial B_\rho(x_i) \cap A^1_{\rho,r_0}(x_i)$ and $\partial B_\rho(x_i) \cap A^2_{\rho,r_0}(x_i)$ and that goes through x_i. Finally, we can rescale the metric, $\tilde{g}_i = |Rm_i|(x_i) g_i$, so that $|\widetilde{Rm_i}|(x_i) \equiv 1$. Notice that before rescaling there is some $r_0 = r_0(v, \Lambda)$ such that

$$r^4 v_0(v, \Lambda) \leq \mathrm{Vol}(B_r(x)) \leq V_0(v, \Lambda) r^4,$$

for all $r \leq r_0$. Therefore, after rescaling we still have two-sided volume growth estimates. Since the curvature is bounded, this implies that there exists some $r_1 = r_1(v, \Lambda)$ such that $\widetilde{\mathcal{E}}_{i,r} = \widetilde{M}_i$ for all $r \leq r_1$. Consequently, from Lemma 3.10 we conclude that the sequence, $(B_{r_0}(x_i), \tilde{g}_i, x_i)$, converges uniformly on compact subsets in the $\mathcal{C}^{2,\alpha}$-topology to a complete, smooth Ricci-flat 4-manifold with a line,

$(\widetilde{M}_\infty, \tilde{g}_\infty, x_\infty)$. By Cheeger-Gromoll's splitting theorem, $\widetilde{M}_\infty$ is isometric to $\mathbb{R}^l \times N$ where N is a compact Ricci-flat manifold of dimension at most $4-l \leq 3$. This implies that $\widetilde{M}_\infty$ is flat, a contradiction with $|\widetilde{Rm_i}|(x_\infty) \equiv 1$. □

3.3 Proof of Proposition 1.6

In this section we prove the removable singularity result for Hodge-harmonic self-dual 2-forms on compact orbifolds with isolated singularities. For convenience of the reader, we recall the statement of Proposition 1.6.

Proposition 3.21 (Proposition 1.6)**.** *Let (X, g) be a compact, oriented, smooth Riemannian orbifold with isolated singularities. Also assume that the orbifold metric is anti-self-dual with non-negative scalar curvature. Let X_R denote the regular set of the orbifold. If η is a Hodge-harmonic self-dual 2-form in $L^2(X_R)$, then η must be parallel. In particular, it can be extended across orbifold singularities. Further, if $\eta \not\equiv 0$, then the orbifold is scalar-flat.*

There are two main ingredients in the proof: an improved Kato inequality [Sea91], and a lemma due to Sibner [Sib85], which allows us to integrate by parts even in the presence of isolated singularities. We quote Sibner's lemma as it appears in [CW11, Lemma 2.5].

Lemma 3.22 ([Sib85, Lemma 2.1])**.** *Assume that we have two-sided Euclidean volume growth and bounds for the local Sobolev constant. Let $u \geq 0$ be a smooth function* *defined on $B \setminus \{x\}$ satisfying, $\Delta u \geq -fu$, where f is a non-negative function in $L^{\frac{n}{2}}(B \setminus \{x\})$. Then there are constants ε_0 and C, depending on the assumptions, such that if $\phi \in \mathcal{C}_c^\infty(B)$ with $\int_{supp(\phi)} f^{\frac{n}{2}} \leq \varepsilon_0$, then we have*

$$\int_B \phi^2 \left|\nabla u^q\right|^2 \leq C \int_B \left|\nabla \phi\right|^2 u^{2q},$$

whenever $q > \frac{n}{2(n-2)}$.

We would like to use this inequality with $q = 1$ but in dimension 4 the requirement is $q > 1$. However, this issue can be avoided using elliptic inequalities with fractional powers of u. For example, if we know that $\Delta u^{\frac{1}{2}} \geq -fu^{\frac{1}{2}}$, we can then use $q = 2$ and obtain

$$\int_B \phi^2 \left|\nabla u\right|^2 \leq C \int_B \left|\nabla \phi\right|^2 u^4,$$

which is what we wanted in the first place. In geometric situations, these improved elliptic inequalities can often be achieved using the so called improved Kato inequalities. Recall that, given any tensor T, the standard Kato inequality says that

$$\left|\nabla \left|T\right|\right| \leq \left|\nabla T\right|,$$

wherever $|T| \neq 0$. Then, an improved Kato inequality would take the form

$$(1+\delta)\left|\nabla \left|T\right|\right| \leq \left|\nabla T\right|,$$

for some $\delta > 0$. If we have extra information about the tensor T and its algebraic properties, there is a chance that such an inequality can be reached. For example, in [BKN89, Lemma 4.9] it was proved that the inequality holds whenever T is the curvature tensor of an Einsten metric or, if dimension is 4, when T is the curvature tensor of a half-conformally-flat metric. More generally, in [Bra00, CGH00], they developed a theory of improved Kato inequalities when T is in the kernel of certain first order elliptic operators. Oversimplifying, the problem of finding such an inequality reduces to a Lagrange multiplier problem, minimizing the quantity $|\langle \nabla T, T\rangle|$ subject to the algebraic constraints of T.

For the case of Hodge-harmonic self-dual 2-forms on 4-manifolds, Seaman gave a purely geometric proof in [Sea91]. We include the details here for the sake of completeness.

Lemma 3.23 ([Sea91, Theorem 1])**.** *Let (M, g) be a closed, anti-self-dual 4-manifold and suppose η is a Hodge-harmonic self-dual 2-form. Then we have*

$$\frac{3}{2}\left|\nabla|\eta|\right|^2 \leq |\nabla\eta|^2, \tag{3.22}$$

which holds point-wise wherever $|\eta| \neq 0$, and in the distribution sense otherwise.

Proof. In the region determined by $|\eta| \neq 0$, we can conformally change the metric as follows, $\tilde{g} = |\eta| g$. Recall that the Hodge-star operator is by $\alpha \wedge *\beta = \langle \alpha, \beta\rangle\, dV$. Therefore, acting on 2-forms it is conformally invariant, $\tilde{*} = *$, which implies that η is self-dual and Hodge-harmonic with respect to $\tilde{g}$ as well. Thus, from the Böchner formula we still have $\widetilde{\Delta}\eta = \frac{1}{3}\widetilde{scal}$. Since $|\eta|_{\tilde{g}} \equiv 1$, we can compute

$$0 = \widetilde{\Delta}|\eta|^2_{\tilde{g}} = |\nabla\eta|^2_{\tilde{g}} + \left\langle \eta, \widetilde{\Delta}\eta \right\rangle_{\tilde{g}} = |\nabla\eta|^2_{\tilde{g}} + \frac{1}{3}\widetilde{scal},$$

whence, $\widetilde{scal} \leq 0$. From [Bes08, Theorem 1.159], we have the following expression for the scalar curvature after the conformal change of the metric

$$\widetilde{scal} = |\eta|^{-1}\left(scal - 3\frac{|\eta|\Delta|\eta| - |\nabla|\eta||^2}{|\eta|^2} - \frac{3}{2}\frac{|\nabla|\eta||^2}{|\eta|^2}\right).$$

Now we can use the identity (A.7), $|\eta|\Delta|\eta| = |\nabla\eta|^2 - |\nabla|\eta||^2 + \frac{1}{3}scal|\eta|^2$, to obtain

$$0 \geq \widetilde{scal} = 3\frac{\frac{3}{2}|\nabla|\eta||^2 - |\nabla\eta|^2}{|\eta|^3},$$

which implies $\frac{3}{2}|\nabla|\eta||^2 \leq |\nabla\eta|^2$. □

Once we have the improved Kato inequality, we can easily derive an elliptic inequality with fractional exponents.

Lemma 3.24. *Assume η is a Hodge-harmonic self-dual 2-form on an anti-self-dual 4-manifold. Then we have*

$$\Delta|\eta|^{\frac{1}{2}} \geq -\frac{1}{6}|scal||\eta|^{\frac{1}{2}}. \tag{3.23}$$

Proof. It is a straightforward computation

$$\begin{aligned}
\Delta|\eta|^{\frac{1}{2}} &= -\frac{1}{4}|\eta|^{-\frac{3}{2}}|\nabla|\eta||^2 + \frac{1}{2}|\eta|^{-\frac{1}{2}}\Delta|\eta| \\
&= -\frac{1}{4}|\eta|^{-\frac{3}{2}}|\nabla|\eta||^2 + \frac{1}{2}|\eta|^{-\frac{3}{2}}\left(|\nabla\eta|^2 - |\nabla|\eta||^2 + \frac{1}{3}scal|\eta|^2\right) \\
&= \frac{1}{2}|\eta|^{-\frac{3}{2}}\left(|\nabla\eta|^2 - \frac{3}{2}|\nabla|\eta||^2\right) + \frac{1}{6}scal|\eta|^{\frac{1}{2}} \\
&\geq -\frac{1}{6}|scal||\eta|^{\frac{1}{2}},
\end{aligned}$$

where we used the identity (A.7), $|\eta|\Delta|\eta| + |\nabla|\eta||^2 = |\nabla\eta|^2 + \langle\eta, \Delta\eta\rangle$, on the second line. □

As outlined above, we can combine this inequality with Sibner's lemma to perform integration by parts even in the presence of isolated singularities.

Proof of Proposition 1.6. On the regular part of the orbifold, the Hodge-harmonic self-dual 2-form η satisfies the equation, $\Delta\eta = \frac{scal}{3}\eta$. In particular, the improved inequality (3.23) still holds. Then we can use Siebner's lemma to carry out the usual Moser iteration process (cf. Lemma 1.10), we only need to be careful about the integration by parts. Let ϕ be a cutoff function with $0 \leq \phi \leq 1$ and supported in some $B_r(x)$. The local Sobolev inequality still holds for orbifolds [Nak93, Far01], so we have

$$C_S \left(\int \phi^4 |\eta|^{2p} \right)^{\frac{1}{2}} \leq 2 \int |\nabla\phi|^2 |\eta|^p + 2 \int \phi^2 \left| \nabla |\eta|^{\frac{p}{2}} \right|^2 .$$

Since we have $\Delta|\eta|^{\frac{1}{2}} \geq -\frac{1}{6}|scal||\eta|^{\frac{1}{2}}$ and $p > 1$, Sibner's Lemma 3.22 implies

$$2 \int \phi^2 \left| \nabla |\eta|^{\frac{p}{2}} \right|^2 \leq C \int |\nabla\phi|^2 |\eta|^p ,$$

as long as $\int_{B_r(x)} |scal|^2 < \varepsilon_0$. By volume comparison for orbifolds [Bor92], there is some constant $V_0(X, g) > 0$ such that $\mathrm{Vol}(B_s(x)) \leq V_0 s^4$ for all $s > 0$. Therefore we can choose $r < r_0(X, g)$ so that the condition $\int_{B_r(x)} |scal|^2 < \varepsilon_0$ is always satisfied. Then we have the inequality

$$\left(\int \phi^4 |\eta|^{2p} \right)^{\frac{1}{2}} \leq C(X, g) \int |\nabla\phi|^2 |\eta|^p ,$$

and the rest of the argument in Lemma 1.10 can be followed without change to obtain

$$\sup_{B_{\frac{r}{2}}(x)} |\eta| \leq C r^{-2} \left(\int_{B_r(x)} |\eta|^2 \right)^{\frac{1}{2}} ,$$

even in the presence of singularities. Hence, since we can cover X with geodesic balls of radius r_0, there is some constant $C = C(X, g)$ such that

$$\sup_X |\eta| \leq C \|\eta\|_{L^2(X)} .$$

Let $S = \{s_i\}_{i=1}^N$ denote the singular set of X and let $S^r = \{x \in X : \mathrm{dist}(x, S) < r\}$. Since the singularities are finite and isolated, we can choose $r > 0$ small enough so that S^r is a disjoint union of geodesic balls, $S^r = \bigsqcup_{i=1}^N B_r(s_i)$. For the next step we choose a cutoff function, ϕ, with $0 \leq \phi \leq 1$ and such that $\phi\,|_{S^r} \equiv 0$, $\phi\,\big|_{X\backslash S^{2r}} \equiv 1$ and $|\nabla\phi| \leq 24r^{-1}$ on the transition region. Integrating by parts

$$\int_{X\backslash S^r} \phi^2|\nabla\eta|^2 = -2\int_{X\backslash S^r} \phi\,\langle \mathrm{tr}(\nabla\phi \otimes \nabla\eta), \eta\rangle - \frac{1}{3}\int_{X\backslash S^r} \phi^2 scal|\eta|^2,$$

so rearranging and using Cauchy's inequality, we obtain

$$\int_{X\backslash S^r} \left(\frac{1}{2}\phi^2|\nabla\eta|^2 + \frac{1}{3}\phi^2 scal|\eta|^2\right) \leq 2\int_{X\backslash S^r} |\nabla\phi|^2|\eta|^2. \tag{3.24}$$

Therefore, since $scal \geq 0$ and we have a bound for $\sup_X |\eta|$, it follows that

$$\begin{aligned}\int_{X\backslash S^{2r}} |\nabla\eta|^2 \leq \int_{X\backslash S^{2r}} \left(|\nabla\eta|^2 + scal|\eta|^2\right) &\leq \frac{24}{r^2}\sum_{i=1}^N \int_{A_{r,2r}(s_i)} |\eta|^2 \\ &\leq C\,\|\eta\|_{L^2(X)}\,Nr^2,\end{aligned}$$

where we also used volume comparison for orbifolds. Now, for each singularity we choose ψ to be a cutoff function supported in $B_{4r}(s_i)$ with $0 \leq \psi \leq 1$ and such that $\psi\,\big|_{B_{2r}(s_i)} \equiv 1$ and $|\nabla\psi| \leq \frac{1}{2}r^{-1}$ on the transition region. Note that, choosing r smaller if necessary, we can assume that $B_{4r}(s_i)$ are also disjoint. We can use the bound for $\sup_X |\eta|$ and Lemma 3.22 to obtain

$$\int_{B_{4r}(s_i)} \psi^2|\nabla\eta|^2 \leq C\int_{B_{4r}(s_i)} |\nabla\psi|^2|\eta|^2 \leq C\,\|\eta\|_{L^2(X)}\,r^2,$$

provided that $\int_{B_r(s_i)} |scal|^2 < \varepsilon_0$, which will hold for small enough r as argued before.

Hence, combining the last two estimates yields

$$\int_X |\nabla \eta|^2 \leq C \|\eta\|_{L^2(X)} r^2,$$

for all sufficiently small $r > 0$. Letting $r \to 0$, we obtain that $\nabla \eta \equiv 0$. In particular, passing to a local orbifold cover of any singular point, η can be trivially extended. Now suppose that $\eta \not\equiv 0$, which means that $|\eta|$ is a non-zero constant everywhere. Thus, the inequality (3.24) above implies

$$\int_{X \setminus S^{2r}} scal \leq C\left(N, (X, g), |\eta|\right) r^2.$$

On the other hand

$$\int_{S^{2r}} scal \leq \sum_{i=1}^{N} \int_{B_{2r}(s_i)} scal \leq C \left(\max_X scal\right) r^4,$$

and we conclude that $\int_X scal \leq Cr^2 \to 0$ as $r \to 0$. Since $scal \geq 0$, it follows that $scal \equiv 0$. $\square$

Appendix A

The purpose of this appendix is to prove some elementary results that might be unfamiliar to the uninitiated reader. We include several useful identities; elliptic equations for $\nabla^m Rm$, $\nabla^m Ric$ and $\nabla^m \eta$; a lemma regarding the relation between the Weyl tensor and the Hodge-star; and, finally, a lemma about how the Sobolev constant provides control on the lower volume growth of small geodesic balls.

Some useful identities

Lemma A.1 (Divergence formulas)**.** *Let (M, g) be a Riemannian n-manifold. Then we have the following divergence formulas.*

$$R_{ijkl,i} = R_{jk,l} - R_{jl,k}, \tag{A.1}$$

$$R_{km,k} = \frac{1}{2} scal_{,m}, \tag{A.2}$$

$$W_{ijkl,i} = (n-3)\left(S_{jk,l} - S_{jl,k}\right), \tag{A.3}$$

where $S = \frac{1}{n-2}\left(Ric - \frac{scal}{2(n-1)} g\right)$ is the Schouten tensor.

Proof. Recall the 2nd Bianchi identity

$$R_{ijkl,m} + R_{ijlm,k} + R_{ijmk,l} = 0, \tag{A.4}$$

which readily implies (A.1)

$$R_{ijkl,i} = -R_{ijik,l} - R_{ijli,k} = R_{jk,l} - R_{jl,k}.$$

Tracing (A.1) with respect to the indeces j and k, we obtain (A.2)

$$R_{il,i} = scal_{,l} - R_{jl,j}.$$

Finally, we use the Schouten tensor to rewrite the curvature decomposition (1.9)

$$\begin{aligned} Rm &= W + \frac{1}{n-2}\mathring{Ric} \owedge g + \frac{scal}{2n(n-1)} g \owedge g \\ &= W + \frac{1}{n-2} Ric \owedge g - \frac{scal}{2(n-1)(n-2)} g \owedge g \\ &= W + S \owedge g, \end{aligned} \tag{A.5}$$

where we used $\mathring{Ric} = Ric - \frac{scal}{n} g$. In coordinate form, we have

$$W_{ijkl} = R_{ijkl} - (S_{jk}g_{il} + S_{il}g_{jk} - S_{ik}g_{jl} - S_{jl}g_{ik})$$

so using (A.1) we obtain

$$\begin{aligned} W_{ijkl,i} &= R_{jk,l} - R_{jl,k} - (S_{jk,l} + S_{il,i}g_{jk} - S_{ik,i}g_{jl} - S_{jl,k}) \\ &= (n-3)\left(S_{jk,l} - S_{jl,k}\right) + \frac{scal_{,l}}{2(n-1)} g_{jk} - \frac{scal_{,k}}{2(n-1)} g_{jl} - \left(S_{il,i}g_{jk} - S_{ik,i}g_{jl}\right), \end{aligned}$$

where we used $Ric = (n-2)S + \frac{scal}{2(n-1)} g$. Now, (A.2) implies that $S_{il,i} = \frac{scal_{,l}}{2(n-1)}$ so (A.3) follows trivially. $\square$

Lemma A.2 (Kato's inequality)**.** *Let T be any tensor defined on any Riemannian manifold, (M, g). Then we have*

$$|\nabla |T|| \leq |\nabla T|, \tag{A.6}$$

which holds point-wise wherever $|T| \neq 0$.

Proof. By Cauchy-Schwarz, we have

$$\left|\nabla |T|^2\right| = |2 \langle T, \nabla T\rangle| \leq 2 |T| |\nabla T|.$$

On the other hand, $\left|\nabla |T|^2\right| = 2 |T| |\nabla |T||$, so it follows that

$$|T| |\nabla |T|| \leq |T| |\nabla T|,$$

which implies (A.6) wherever $|T| \neq 0$. □

From this inequality we can derive a couple other identities.

Lemma A.3. *Let T be any tensor on any Riemannian n-manifold, (M, g). Then we have*

$$|T| \Delta |T| + |\nabla |T||^2 = \langle T, \Delta T\rangle + |\nabla T|^2, \tag{A.7}$$

and

$$|T| \Delta |T| \geq \langle T, \Delta T\rangle. \tag{A.8}$$

Proof. We can compute $\Delta |T|^2$ as follows

$$\Delta |T|^2 = \Delta (|T| |T|) = 2 |T| \Delta |T| + 2 |\nabla |T||^2,$$

and we also have

$$\Delta |T|^2 = \Delta\langle T, T\rangle = 2\langle T, \Delta T\rangle + 2 |\nabla T|^2 .$$

This implies (A.7). Then (A.8) follows using Kato's inequality (A.6)

$$|T| \Delta |T| = \langle T, \Delta T\rangle + |\nabla T|^2 - |\nabla |T||^2 \geq \langle T, \Delta T\rangle.$$

□

Equations for $\nabla^m Rm$, $\nabla^m Ric$ and $\nabla^m \eta$

First we need some formulas to commute covariant derivatives.

Lemma A.4 (Commutator formulas)**.** *Let (M, g) be a smooth Riemannian manifold and suppose T is a (p, q)-tensor. Then we have the Ricci identities to commute covariant derivatives*

$$\begin{aligned} T^{j_1 \dots j_p}{}_{i_1 \dots i_q, kl} - T^{j_1 \dots j_p}{}_{i_1 \dots i_q, lk} =& R_{klmi_1} T^{j_1 \dots j_p}{}_{m \dots i_q} + \cdots + R_{klmi_q} T^{j_1 \dots j_p}{}_{i_1 \dots m} \\ & - R_{klmj_1} T^{m \dots j_p}{}_{i_1 \dots i_q} - \cdots - R_{klmj_p} T^{j_1 \dots m}{}_{i_1 \dots i_q}. \end{aligned} \tag{A.9}$$

In short hand notation, $[\nabla_k, \nabla_l]T = Rm * T$.

Proof. This is a straightforward computation. Given a tensor T, the Riemannian curvature tensor acts on T as follows

$$R(X, Y)T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T,$$

which we can write in coordinates and unravel to obtain (A.9). □

We have the basic elliptic equations for Rm and Ric.

Lemma A.5. *Let (M, g) be a smooth Riemannian manifold. Then we have*

$$\Delta Rm = Rm * Rm + L(\nabla^2 Ric),$$

and

$$\Delta Rm = Rm * Ric + L(\nabla^2 scal).$$

Proof. We use the second Bianchi identity, (A.4), to compute in coordinates

$$\begin{aligned} R_{ijkl,mm} &= -R_{ijlm,km} - R_{ijmk,lm} = -R_{lmij,km} - R_{mkij,lm} \\ &= -R_{mlij,mk} - R_{kmij,ml} + Rm * Rm \\ &= R_{mljm,ik} + R_{mlmi,jk} + R_{kmjm,il} + R_{kmmi,jl} + Rm * Rm \\ &= R_{lj,ik} - R_{li,jk} - R_{kj,il} + R_{ki,jl} + Rm * Rm, \end{aligned}$$

where on the second line we used, $R_{ijkl,mn} - R_{ijkl,nm} = Rm * Rm$. The equation for Ric is already proved in Section 1.3.2. □

Now we can use induction on the order of the covariant derivative.

Lemma A.6. *Let (M, g) be a smooth Riemannian manifold. Then we have*

$$\Delta\left(\nabla^m Rm\right) = \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k} Rm + L_m\left(\nabla^{m+2} Ric\right), \tag{A.10}$$

and

$$\Delta\left(\nabla^m Ric\right) = \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k} Ric + L_m\left(\nabla^{m+2} scal\right). \tag{A.11}$$

Proof. We argue by induction. The case $m = 0$ is exactly the content of Lemma A.5. So suppose that the equation (A.10) is satisfied up to order $m - 1$. Using (A.9) twice and the induction hypothesis, we can compute

$$\begin{aligned}
\Delta\left(\nabla^m Rm\right) &= \nabla\left(\Delta\nabla^{m-1}Rm\right) + \nabla\left(Rm * \nabla^{m-1}Rm\right) + Rm * \nabla^m Rm \\
&= \nabla\left(\sum_{k=0}^{m-1} \nabla^k Rm * \nabla^{m-1-k}Rm + L_{m-1}\left(\nabla^{m-1}Ric\right)\right) \\
&\quad + \nabla\left(Rm * \nabla^{m-1}Rm\right) + Rm * \nabla^m Rm \\
&= \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k}Rm + L_m\left(\nabla^{m+2}Ric\right).
\end{aligned}$$

The proof for the equation for $\nabla^m Ric$ is exactly the same. □

Simlarly, we have elliptic equations for Hodge-harmonic self-dual 2-forms and their covariant derivatives.

Lemma A.7. *Suppose η is a Hodge-harmonic self-dual 2-form on a 4-dimensional anti-self-dual 4-manifold. Given any integer $m \geq 0$, we have the following equation*

$$\Delta\left(\nabla^m \eta\right) = \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k}\eta.$$

Proof. If $m = 0$, this equation is a consequence of the Böchner forumula as it was explained in the discussion surrounding (3.2). We argue by induction for the general case. Using the commutator formulas (A.9) twice and the induction hypothesis, we obtain

$$\begin{aligned}
\Delta\left(\nabla^m \eta\right) &= \nabla\left(\Delta\left(\nabla^{m-1}\eta\right)\right) + Rm * \nabla^m \eta + \nabla\left(Rm * \nabla^{m-1}\eta\right) \\
&= \nabla\left(\sum_{k=0}^{m-1} \nabla^k Rm * \nabla^{m-1-k}\eta\right) + Rm * \nabla^m \eta + \nabla\left(Rm * \nabla^{m-1}\eta\right) \\
&= \sum_{k=0}^{m} \nabla^k Rm * \nabla^{m-k}\eta,
\end{aligned}$$

as desired. □

Two lemmas

Here we prove the lemma that allow us to decompose the Weyl curvature into self-dual and anti-self-dual parts, $W = W^+ + W^-$.

Lemma A.8 (W and Hodge-$*$ commute)**.** *Let (M, g) be a 4-manifold. Then, as operators acting on 2-forms, the Weyl tensor, W, and the Hodge-star operator, $*$, commute with each other. In particular, we have $W(\Lambda^{\pm}) \subset \Lambda^{\pm}$.*

Proof. Recall the decomposition of 2-forms, $\Lambda = \Lambda^+ \oplus \Lambda^-$. This implies that given any 2-form ω, we can write it as, $\omega = \omega^+ + \omega^-$, where $\omega^{\pm} \in \Lambda^{\pm}$. By linearity, it is enough to work with a basis of (anti)-self-dual 2-forms. If $\{e_1, \dots, e_4\}$ is an orthonormal basis of covectors, one can check that

$$\{e_1 \wedge e_2 + e_3 \wedge e_4, \quad e_1 \wedge e_3 - e_2 \wedge e_4, \quad e_1 \wedge e_4 + e_2 \wedge e_4\},$$

is an orthogonal basis for Λ^+. Flipping the signs we obtain an orthogonal basis for Λ^-. Here we do the computations for $e_1 \wedge e_2 + e_3 \wedge e_4$, but the rest are analogous. By self-duality we have

$$W * (e_1 \wedge e_2 + e_3 \wedge e_4) = W(e_1 \wedge e_2 + e_3 \wedge e_4).$$

Recall that the Weyl tensor acts on 2-forms as follows, $W(\omega)_{ij} = \frac{1}{2} W_{klij}\omega_{kl}$. From this coordinate expression we obtain

$$\begin{aligned} W(e_1 \wedge e_2 + e_3 \wedge e_4) = {} & (W_{1212} + W_{3412})e_1 \wedge e_2 + (W_{1213} + W_{3413})e_1 \wedge e_3 \\ & + (W_{1214} + W_{3414})e_1 \wedge e_4 + (W_{1223} + W_{3423})e_2 \wedge e_3 \\ & + (W_{1224} + W_{3424})e_2 \wedge e_4 + (W_{1234} + W_{3434})e_3 \wedge e_4. \end{aligned}$$

On the other hand

$$
\begin{aligned}
*W(e_1 \wedge e_2 + e_3 \wedge e_4) = & (W_{1212} + W_{3412})e_3 \wedge e_4 - (W_{1213} + W_{3413})e_2 \wedge e_4 \\
& + (W_{1214} + W_{3414})e_2 \wedge e_3 + (W_{1223} + W_{3423})e_1 \wedge e_4 \\
& - (W_{1224} + W_{3424})e_1 \wedge e_3 + (W_{1234} + W_{3434})e_1 \wedge e_2,
\end{aligned}
$$

and we only need to check that these last two expressions are the same. We can do this using the symmetries of W and the fact that it is traceless. For example, looking at the terms with $e_1 \wedge e_2$, we need

$$
W_{1212} + W_{3412} = W_{1234} + W_{3434},
$$

which is equivalent to $W_{1212} = W_{3434}$. Since W is traceless, we have

$$
0 = g^{ij} W_{1i1j} = W_{1212} + W_{1313} + W_{1414},
$$

and also

$$
0 = g^{ij} W_{i2j2} = W_{1212} + W_{3232} + W_{4242}.
$$

Adding them together we obtain, $2W_{1212} = -W_{1313} - W_{1414} - W_{3232} - W_{4242}$. Similarly, tracing $g^{ij}W_{3i3j}$ and $g^{ij}W_{i4j4}$, we obtain

$$
2W_{3434} = -W_{3131} - W_{3232} - W_{1414} - W_{2424}.
$$

Now, using the curvature symmetries of W, we can swap indices to obtain

$$
2W_{1212} = -W_{1313} - W_{1414} - W_{3232} - W_{4242} = 2W_{3434},
$$

as we wanted. The rest of components can be computed in a similar fashion. $\square$

The last lemma gives a lower volume growth for small geodesic balls in terms of the local Sobolev constant.

Lemma A.9. *Let (M, g) be a Riemannian 4-manifold and suppose that $C_S(B_r(x)) \geq C_S > 0$. Then there is a constant $v_0 = v_0(C_S) > 0$ such that*

$$\mathrm{Vol}\left(B_r(x)\right) \geq v_0 r^4.$$

Proof. It is standard that the local Sobolev constant is equivalent to the following isoperimetric constant [Li12, Theorem 9.5]

$$I(B) = \inf_{\substack{\Omega \subset B \\ \partial\Omega \cap \partial B = \emptyset}} \frac{|\partial\Omega|}{|\Omega|^{\frac{3}{4}}}.$$

Therefore, for any geodesic ball, $B_s(x)$, with $s < r$, we have $\frac{|\partial B_s(x)|}{|B_s(x)|^{\frac{3}{4}}} \geq C(C_S)$. Now we can integrate this inequality with respect to s

$$C(C_S)r \leq \int_0^r \frac{|\partial B_s(x)|}{|B_s(x)|^{\frac{3}{4}}} ds = \mathrm{Vol}\left(B_r(x)\right)^{\frac{1}{4}},$$

and the result follows. □

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