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ADHM construction of nested Hilbert scheme

A construção ADHM do esquema de Hilbert aninhado

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Tese apresentada ao Instituto de Matemática, Estatística e Computação Científica da Universidade Estadual de Campinas como parte dos requisitos exigidos para a obtenção do título de Doutora em matemática.

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Orientador: Marcos Benevenuto Jardim

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Resumo

O espaço de moduli das representações do quiver ADHM na categoria de espaços vetoriais complexos é isomorfo ao espaço de moduli dos feixes livres de torção no plano projetivo. Em particular, quando são considerados os feixes livres de torção de posto 1, os dados ADHM dão uma descrição do esquema de Hilbert de n pontos no plano complexo, para algum inteiro n. De modo semelhante, afirmamos e provamos uma relação entre o espaço de moduli das representações estáveis do quiver ADHM aumentado e o esquema de Hilbert aninhado.

Palavras-chave: esquemas de Hilbert, quivers, equações ADHM.

Abstract

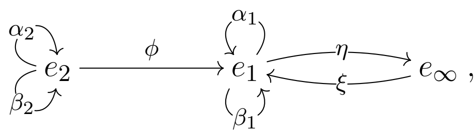
The moduli space of representations of the ADHM quiver in the category of complex vector spaces is isomorphic to moduli space of framed torsion-free sheaves on the projective plane. In particular, when we consider rank 1 torsion-free sheaves, the ADHM data describe the Hilbert scheme of n points in the complex plane, for some given integer n. Similarly, we state and prove a relation between the moduli space of stable framed representations of the enhanced ADHM quiver and the nested Hilbert scheme.

Keywords: Hilbert scheme, quivers, ADHM equations.

Contents

Introduction

The *enhanced ADHM quiver* is the one of the form:



with relations specified by the linear combination of paths

$$\alpha_1\beta_1 - \beta_1\alpha_1 + \xi\eta; \quad \alpha_1\phi - \phi\alpha_2; \quad \eta\phi; \quad \alpha_2\beta_2 - \beta_2\alpha_2; \quad \beta_1\phi - \phi\beta_2; \ .$$

The literature, in [2, Section 3], has previously considered the enhanced ADHM quivers. In particular, omitting the vertex e_2 and all above relations except the first one, we obtain the usual ADHM quiver.

It is well-known that moduli spaces of representations of the ADHM quiver in the category of complex vector spaces are isomorphic to moduli spaces of framed torsion-free sheaves on the projective plane [11, Chapter 2]. Particularly, when one considers rank 1 torsion-free sheaves, the ADHM data describe the Hilbert scheme of n points in $\mathbb{C}^2$, for some given integer n.

Similarly, this work aims is to provide a relation between the moduli space of stable framed representations of the enhanced ADHM quiver and the nested Hilbert scheme. In fact, as Theorem 3.1.1 will show, one obtains a one-to-one correspondence between the following objects:

- equivalence classes of stable framed representations of the enhanced ADHM quiver of dimension vector $(1, n_1, n_2)$.

- closed subschemes (Z_1, Z_2) of $\mathbb{C}^2$ with Hilbert polynomial n_1 and $n_1 - n_2$ respectively, and $Z_2 \subset Z_1$.

Although we only present a set-theoretical bijection in the main theorem, we can conjecture that this bijection is an isomorphism between schemes. This description intends to motivate and to be useful to the study of nested Hilbert schemes.

Furthermore, using the above correspondence, we also show that the nested Hilbert scheme with quotients supported on curves is in one-to-one correspondence with the equivalence classes of stable framed representations of the enhanced ADHM quiver which satisfies some polynomial equation.

Let us summarize the plan of the work. One of the major aims of the Chapter 1 is to introduce the concept of nested Hilbert schemes of points on a surface. For this we will touch a few aspects of the theory of representable functors and Hilbert schemes. Chapter 2 assembles basic facts about the representation of quivers, monads and ADHM data, besides introducing the definition of enhanced ADHM data according to [2, Section 3]. Finally, Chapter 3 confirms the author's result, as stated above, which relates the moduli space of stable framed representations of the enhanced ADHM quiver to the nested Hilbert scheme.

Chapter 1

Hilbert Schemes

Before starting our construction, we define the objects we shall study in this work and state some of their properties.

1.1 Representable Functors

For the convenience of the reader, we repeat parts of the relevant material from [5, Section 2.1] without proofs, thus making our exposition self-contained.

Let $\mathcal{C}$ be a category. Consider functors $\mathcal{C}^{op}$ to Set. These are the objects of a category, denoted by $\mathrm{Hom}(\mathcal{C}^{op}, Sets)$, in which the arrows are natural transformations. From now on we will refer to natural transformations of contravariant functors on $\mathcal{C}$ as morphisms.

Let X be an object of $\mathcal{C}$. There is a functor

$$h_X : \mathcal{C}^{op} \to Sets$$

to the category of sets, which sends an object U of $\mathcal{C}$ to the set:

$$h_X U = \mathrm{Hom}_{\mathcal{C}}(U, X).$$

If $\alpha : U' \to U$ is an arrow in $\mathcal{C}$, then $h_X \alpha : h_X U \to h_X U'$ is defined as composition with α (when $\mathcal{C}$ is the category of schemes over a fixed base scheme, h_X is often called the functor of points of X).

Thus, an arrow $f : X \to Y$ yields a function $h_f : h_X U \to h_Y U$ for each object

$U \in \mathcal{C}$, derived from f through composition. This defines a morphism $h_X \to h_Y$, i.e., for all arrow $\alpha : U' \to U$ the diagram

$$\begin{array}{ccc} h_X U & \xrightarrow[h_f U]{} & h_Y U \\ {\scriptstyle h_X\alpha}\downarrow & & \downarrow{\scriptstyle h_Y\alpha} \\ h_X U' & \xrightarrow[h_f U']{} & h_Y U' \end{array}$$

is commutative. Sending each object X of $\mathcal{C}$ to h_X, and each arrow $f : X \to Y$ of $\mathcal{C}$ to $h_f : h_X \to h_Y$ defines a functor $\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{op}, Sets)$.

Lemma 1.1.1 (Yoneda lemma, weak version)**.** Let X and Y be objects of $\mathcal{C}$. The function

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{C}}(h_X, h_Y)$$

which sends $f : x \to Y$ to $h_f : h_X \to h_Y$ is bijective.

In other words, the functor $\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{op}, Sets)$ is fully faithful. It fails to be an equivalence of categories, because in general it is not essentially surjective. This means that not every functor $\mathcal{C}^{op} \to Sets$ is isomorphic to a functor of the form h_X. However, if we restrict it to the full subcategory of $\mathrm{Hom}(\mathcal{C}^{op}, Sets)$ consisting of functors $\mathcal{C}^{op} \to Sets$ which are isomorphic to a functor of the form h_X, we do obtain a category which is equivalent to $\mathcal{C}$.

Definition 1.1.2. A *representable functor* on the category $\mathcal{C}$ is a functor

$$F : \mathcal{C}^{op} \to Sets$$

which is isomorphic to a functor of the form h_X for some object X of $\mathcal{C}$. If it happens, we say that F is represented by X.

Given two isomorphisms $F \simeq h_X$ and $F \simeq h_Y$, the resulting isomorphism $h_X \simeq h_Y$ comes from a unique isomorphism $X \simeq Y$ in $\mathcal{C}$, because of the weak form of Yoneda lemma. Hence two objects representing the same functor are canonically isomorphic.

The condition that a functor be representable is given a new expression with the more general version of Yoneda lemma. Let X be an object of $\mathcal{C}$ and $F : \mathcal{C}^{op} \to Sets$

a functor. Given a natural transformation $\tau : h_X \to F$, we obtain an element $\xi \in FX$, defined as the image of the identity map $id_X \in h_X X$ via the function $\tau_X : h_X X \to FX$, i.e., $\xi = \tau_X(id_X)$. This construction defines a function $\mathrm{Hom}(h_X, F) \to FX$.

Conversely, given an element $\xi \in FX$, we can define a morphism $\tau : h_X \to F$ as follows. Given an object U of $\mathcal{C}$, an element of $h_X U$ is an arrow $f : U \to X$; this arrow induces a function $Ff : FX \to FU$. We define a function $\tau_U : h_X U \to FU$ by sending $f \in h_X U$ to $Ff(\xi) \in FU$. It is straightforward to check that τ is in fact a morphism. Thus, we have defined the functions

$$\mathrm{Hom}(h_X, F) \to FX$$

and

$$FX \to \mathrm{Hom}(h_X, F).$$

Lemma 1.1.3 (Yoneda lemma)**.** These two functions are mutually inverse, and therefore, establish the bijective correspondence

$$\mathrm{Hom}(h_X, F) \simeq FX$$

Let us see how this form of Yoneda lemma implies the weak version above. Assuming that $F = h_Y$: the function $\mathrm{Hom}(X,Y) = h_X Y \to \mathrm{Hom}(h_X, h_Y)$ sends each arrow $f : X \to Y$ to

$$h_Y(id_Y) = id \circ f : X \to Y,$$

precisely the function $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(h_X, h_Y)$ in the weak form of the result.

One way to consider Yoneda lemma is as follows. The weak form states that the category $\mathcal{C}$ is embedded in the category $\mathrm{Hom}(\mathcal{C}^{op}, Sets)$. The strong version states that, given a functor $F : \mathcal{C}^{op} \to Sets$, this can be extended to the representable functor $h_F : \mathrm{Hom}(\mathcal{C}^{op}, Sets) \to Sets$: thus, every functor becomes representable, when appropriately extended (in practice, the category $\mathrm{Hom}(\mathcal{C}^{op}, Sets)$ is usually much too big, and we must restrict it appropriately). We can use Yoneda lemma to give a very important characterization of representable functors.

Definition 1.1.4. Let $F : \mathcal{C}^{op} \to Sets$ be a functor. A *universal object* for F is a pair (X, ξ) consisting of an object X of $\mathcal{C}$ and an element $\xi \in FX$ such that for each object U of $\mathcal{C}$ and each $\sigma \in FU$, there exists a unique arrow $f : U \to X$ such that $Ff(\xi) = \sigma \in FU$.

In other words: the pair (X,ξ) is an universal object if the morphism $h_X \to F$ defined by ξ is an isomorphism. Since every natural transformation $h_X \to F$ is defined by some object $\xi \in FX$, it leads to the following proposition:

Proposition 1.1.5. A functor $F : \mathcal{C}^{op} \to Sets$ is representable if and only if it has a universal object.

Proof. On the one hand, suppose that $F : \mathcal{C}^{op} \to Sets$ is a representable functor, then there exists $X \in \mathcal{C}$ such that $F \simeq h_X$. Let $\tau : h_X \to F$ be the isomorphism and take $\xi = \tau_X(id_X)$. We will show (X,ξ) is a universal object for F. Let $U \in \mathcal{C}$ and $\sigma \in FU$. Since $\tau : h_X \to F$ is an isomorphism, we know that $\tau_U : h_X(U) \to F(U)$ is also an isomorphism. Thus, since $\sigma \in FU$, there exists a unique $f \in h_X(U)$ such that $\tau_U(f) = \sigma$. Let us see that $\tau_U(f) = F(f)(\xi)$. Since τ is a natural transformation, we have the commutative diagram:

$$\begin{array}{ccc} h_X X & \xrightarrow[\tau_X]{} & FX \\ {\scriptstyle h_X(f)}\downarrow & & \downarrow{\scriptstyle F(f)} \\ h_X U & \xrightarrow[\tau_U]{} & FU \end{array}$$

Thus, $\tau_U(h_X(f)(id_X)) = F(f)(\tau_X(id_X)) \Rightarrow \tau_U(f) = F(f)(\xi)$. On the other hand, suppose (X,ξ) is a universal object for F. Define $\tau : h_X \to F$ in the following way: for each $U \in \mathcal{C}$ take $\tau_U : h_X U \to FU$ which associates each $f : U \to X$ with the object $F(f)(\xi)$, i.e., $\tau_U(f) = F(f)(\xi)$. Let us see τ is an isomorphism. Let $U \in \mathcal{C}$ and $\sigma \in FU$, by the property of the universal object, there exists a unique $f : U \to X$ such that $F(f)(\xi) = \sigma$, i.e., $\tau_U(f) = \sigma$. Thus, τ_U is an isomorphism, since U is arbitrary and we know that $\tau : h_X \to F$ is an isomorphism. F, therefore, is a representable functor. $\square$

Also, if F has a universal object (X,ξ), then F is represented by X. Yoneda lemma ensures that the natural functor $\mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{op}, Sets)$ which sends an object X to the functor h_X is an equivalence of $\mathcal{C}$ with the category of representable functors.

Example 1.1.6. Here are some examples of representable functors:

1. Let us recall that the product $X \times Y$ of two sets is the set of ordered pairs $\{(x,y) \mid x \in X, y \in Y\}$. We present two definitions of the direct product of two objects $X, Y \in Ob\mathcal{C}$ of an arbitrary category $\mathcal{C}$.

a. The direct product $X \times Y$ "is" the object Z representing the functor

$$U \mapsto (\text{ the direct product } h_X U \times h_Y U)$$

(if this functor is representable).

b. The direct product $X \times Y$ "is" an object Z together with projection morphisms $X \xleftarrow{p_X} Z \xrightarrow{p_Y} Y$ such that for any pair of morphisms $X \xleftarrow{p'_X} Z' \xrightarrow{p'_Y} Y$ there exists a unique morphism $q : Z' \to Z$ such that, $p'_X = p_X q$, and $p'_Y = p_Y q$, (again if a triple (Z, p_X, p_Y) with this property exists).

2. A generalization of the above construction enables us to define the fiber product in the category theory language. Let us recall that if $\varphi : X \to S$, and $\psi : Y \to S$ are two mappings of sets, the fiber product of X and Y over S is the following set of pairs:

$$X \times_S Y = \{(x, y) \in X \times Y \, : \varphi(x) = \psi(y)\} \subset X \times Y.$$

The object $X \times_S Y$ in the category can be represented in two ways.

a'. $X \times_S Y$ represents the functor: $U \mapsto X(U) \times_{S(U)} Y(U)$.

b'. $X \times_S Y$ "is" the ordinary product in the new category $\mathcal{C}_S$ whose objects are morphisms $\varphi : X \to S$, and morphisms from $\varphi : X \to S$ to $\psi : Y \to S$ are commutative diagrams:

$$\begin{array}{ccc} X & \xrightarrow{\chi} & Y \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & S & \end{array}$$

where $\chi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. The diagram 1.b. in the category $\mathcal{C}_S$ is represented by the following diagram in the category $\mathcal{C}$:

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{\varphi} & S \end{array}$$

This category has a universal property and it is called the Cartesian square.

1.2 General Results on Hilbert Schemes

One of the major aims of this chapter is to introduce the concept of nested Hilbert schemes of points on a surface. In this regard, we begin with a brief collection of facts on the Hilbert schemes of points on a surface. Next, we apply these facts to prove the main result of the last chapter.

Since the definition of the nested Hilbert scheme is similar to the definition of the Hilbert scheme, we begin with a general description of the latter.

Let X be a quasi-projective variety over the field $\mathbb{K}$ and consider the functor:

$$\underline{Hilb}_X : \; Sch_{\mathbb{K}}^{op} \;\to\; Sets$$

from the category of Noetherian schemes over $\mathbb{K}$ to the category of sets defined by:

Objects:

$$U \;\mapsto\; \underline{Hilb}_X(U) := \left\{ Z \subset X \times U \;\middle|\; \begin{array}{l} Z \text{ is a closed subscheme,} \\ Z \text{ is flat over } U \text{ via} \\ \pi : Z \hookrightarrow X \times U \to U \end{array} \right\}$$

Morphisms: for all $f : U \to V \in Mor_{Sch_{\mathbb{K}}}$

$$\begin{array}{llll} \underline{Hilb}_X(f): & \underline{Hilb}_X(V) & \to & \underline{Hilb}_X(U): \\ & Z & \mapsto & f^*(Z) \end{array}$$

where:

$$\begin{array}{ccc} f^*(Z) & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X \times U & \xrightarrow[id_X \times f]{} & X \times V \\ \downarrow & & \downarrow \\ U & \xrightarrow[f]{} & V \end{array}$$

Since a pullback of flat morphism is still flat and tensor product is a right exact functor this diagram is reasonable.

For each point $u \in U$, the Hilbert polynomial in U is defined by:

$$P_u(m) = \chi(\mathcal{O}_{Z_u} \otimes \mathcal{O}_X(m))$$

where $Z_u = \pi^{-1}(u)$ and $\mathcal{O}_X(m)$ is an ample line bundle on X. Since π is flat, P_u is locally constant. Define the subfunctor $\underline{Hilb}_X^P$ of $\underline{Hilb}_X$, which associates U with the subset of $\underline{Hilb}_X(U)$ with P as its Hilbert polynomial. In that case we have:

Theorem 1.2.1. The functor $\underline{Hilb}_X^P$ is representable by a projective scheme $Hilb^P(X)$.

Proof. See [7]. □

In general we have:

Definition 1.2.2. Let P be the constant polynomial given by $P(m) = n$, for all $m \in \mathbb{Z}$. We denote by $Hilb^n(X)$ the corresponding Hilbert scheme and call it *the Hilbert scheme of n points in X.*

Example 1.2.3. Consider the constant polynomial $p(t) = 1$. Then we have the canonical identification $Hilb^1(Y) = Y$ and the universal family is the diagonal $\Delta \subset Y \times Y$. In fact, take a scheme S and an element of $\underline{Hilb}_Y(S)$, named

$$\begin{array}{ccc} \Gamma & \hookrightarrow & S \times Y \\ \downarrow f & & \\ S & & \end{array}$$

Then f is an isomorphism: indeed, it is a one-to-one morphism and $\mathcal{O}_S \to f_*\mathcal{O}_\Gamma$ is an isomorphism since $f_*\mathcal{O}_\Gamma$ is an $\mathcal{O}_S$ - algebra which is locally free of rank 1 over $\mathcal{O}_S$. Therefore, we have the well-defined morphism $g = q \circ f^{-1} : S \to \Gamma \hookrightarrow S \times Y \to Y$, where $q : S \times Y \to Y$ is the projection. The morphism θ bellow factors Δ

$$\begin{array}{rccl} \theta = (gf, q): & \Gamma & \to & Y \times Y \\ & (s,y) & \mapsto & (gf(s,y), q(s,y)) \in \Delta \end{array}$$

and induces a commutative diagram:

$$\begin{array}{ccccc} & & \Gamma & & \\ & {}^{f}\swarrow & & \searrow^{\theta} & \\ S & & & & \Delta \\ & {}_{g}\searrow & & \swarrow_{\pi} & \\ & & Y & & \end{array}$$

$$\pi \circ \theta(s,y) = \pi(q(s,y), q(s,y)) = q(s,y) = gf(s,y)$$

such that $\Gamma \simeq g^*\Delta = S \times_Y \Delta$. The family Γ is induced by Δ via the morphism g. For this reason, (Y,Δ) represents $\underline{Hilb}^1(.)$ and, therefore, $Hilb^1(Y) = Y$.

It is important to notice that, in general, even if the variety X is nonsingular, the Hilbert scheme $Hilb^n(X)$ can be singular, if $n \geq 3$. The following result indicates gives the nonsingularity of the Hilbert scheme for a quasi-projective nonsingular surface.

Theorem 1.2.4. If X is a quasi-projective nonsingular surface, then the Hilbert scheme $Hilb^n(X)$ of n points on X is nonsingular.

Proof. See [6, Theorem 2.4]. □

We will restrict our attention when P is a constant polynomial and X is the affine plane $\mathbb{C}^2$. In that case, we have the following handy description:

$$Hilb^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[X,Y] \,|\, dim\mathbb{C}[X,Y]/I = n\}.$$

Indeed, a closed point of $Hilb^n(\mathbb{C}^2)$ can be identified with an ideal $I \subset \mathbb{C}[X,Y]$ of length n. This means that the quotient $\mathbb{C}[X,Y]/I$ is an n-dimensional vector space over $\mathbb{C}$.

1.3 General Results on Nested Hilbert Schemes

This section summarizes a few fundamental results concerning nested Hilbert schemes. We begin with a general definition of a quasi-projective scheme, but for our purposes a simpler characterization, presented later, will suffice.

Let X be a quasi-projective scheme defined over the field $\mathbb{K}$. Fix an integer $r \geq 1$ and a $m-$tuple of numerical polynomials

$$\mathbf{P}(\mathbf{t}) = (P_1(t), \ldots, P_m(t)), \qquad m \geq 1.$$

For every scheme S we have:

$$\underline{Hilb}_X^{\mathbf{P}(\mathbf{t})}(S) = \left\{ (\mathcal{Z}_1, \ldots, \mathcal{Z}_m) \;\middle|\; \begin{array}{l} \mathcal{Z}_1 \subset \ldots \subset \mathcal{Z}_m \subset X \times S \\ S-\text{flat closed subschemes with} \\ \text{Hilbert polynomials } \mathbf{P}(\mathbf{t}) \end{array} \right\}$$

This defines a contravariant functor:

$$\underline{Hilb}_X^{\mathbf{P}(\mathbf{t})} : \ Sch_{\mathbb{K}}^{op} \ \rightarrow \ Sets$$

called nested Hilbert functor of X relative to $\mathbf{P}(\mathbf{t})$. When $m = 1$, the nested Hilbert functor is just the ordinary Hilbert functor.

Thus, we can state the analogy to Theorem 1.2.1 for nested Hilbert schemes on $X = \mathbb{P}^r$:

Theorem 1.3.1. For every $r \geq 1$ and $\mathbf{P}(\mathbf{t})$ as above, the nested Hilbert functor $\underline{Hilb}_{\mathbb{P}^r}^{\mathbf{P}(\mathbf{t})}$ is represented by a projective scheme $Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r)$, called nested Hilbert scheme of $\mathbb{P}^r$ relative to $\mathbf{P}(\mathbf{t})$, and using a universal family:

$$\begin{array}{ccc} \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_m & \hookrightarrow & \mathbb{P}^r \times Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r) \\ & & \downarrow \\ & & Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r) \end{array}$$

Proof. [13, Theorem 4.5.1] □

If the polynomial $\mathbf{P}(\mathbf{t}) = (n_1, \ldots, n_m)$ is an $m-$tuple of positive integers, it is often written $Hilb^{n_1,\ldots,n_m}(\mathbb{P}^r)$ rather than $Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r)$.

From the definition it follows that the closed points of $Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r)$ are in one-to-one correspondence with the m-tuples $(Z_1, \ldots, Z_m)$ of closed subschemes of $\mathbb{P}^r$ such that X_i has Hilbert polynomial $P_i(t)$ and $Z_1 \subset Z_2 \subset \ldots \subset Z_m$. Moreover, it follows that $Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r)$ is a closed subscheme of

$$\prod_{i=1}^{m} Hilb^{P_i(t)}(\mathbb{P}^r).$$

We will denote the projections as:

$$pr_i : Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r) \rightarrow Hilb^{P_i(t)}(\mathbb{P}^r), \quad i = 1, \ldots, m.$$

For every subset $\mathbf{I} \subset \{1, \ldots, m\}$ with cardinality μ we can consider the μ-tuple of polynomials $\mathbf{P_I}(\mathbf{t}) = (P_{i_1}, \ldots P_{i_\mu})$ and the nested Hilbert scheme $Hilb^{\mathbf{P_I}(\mathbf{t})}(\mathbb{P}^r)$. We have natural projection morphisms:

$$pr_{\mathbf{I}} : Hilb^{\mathbf{P}(\mathbf{t})}(\mathbb{P}^r) \rightarrow Hilb^{\mathbf{P_I}(\mathbf{t})}(\mathbb{P}^r),$$

whose $pr_i's$ are special cases.

It is of interest to know whether the nested Hilbert scheme is smooth. In fact, in [3] Cheah proved that:

Theorem 1.3.2. If X is a nonsingular quasi-projective m-dimensional variety, the nested Hilbert scheme $Hilb^{n_1,\ldots,n_r}(X)$ is nonsingular precisely when either:

1. $m \leq 1$ or
2. $m = 2$ and $Hilb^{n_1,\ldots,n_r}(X)$ is equal to $Hilb^n(X)$ or $Hilb^{n-1,n}(X)$ for some n or
3. $m \geq 3$ and $Hilb^{n_1,\ldots,n_r}(X)$ is equal to one of the spaces $Hilb^1(X)$, $Hilb^2(X)$, $Hilb^3(X)$, $Hilb^{1,2}(X)$ or $Hilb^{2,3}(X)$.

Proof. [3, Lemmas 1.1, 1.2, 1.3 and 1.4] □

Interestingly, in the theorem above, there are few cases in which the nested Hilbert scheme is smooth.

Chapter 2

Representations of quivers

2.1 Representations of quivers

A quiver is just a directed graph. A more formal definition goes as follows.

Definition 2.1.1. A *quiver* is a pair $Q = (Q_0, Q_1)$, where Q_0 is a finite set of vertices and Q_1 is a finite set of arrows between them. If $a \in Q_1$ is an arrow, then ta and ha denote its tail and its head, respectively.

Let us fix a quiver Q and a base field $\mathbb{K}$. Add a finite-dimensional vector space to each vertex of Q and a linear map to each arrow (with the appropriate domain and codomain). Thus, we obtain what is known as *representation of* Q. More precisely,

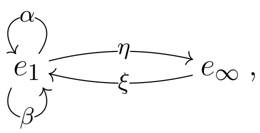
Definition 2.1.2. A *representation* $\mathbf{R}$ of Q is a collection $\{V_i | i \in Q_0\}$ of finite-dimensional $\mathbb{K}$-vector spaces together with a collection $\{V_a : V_{ta} \to V_{ha} | a \in Q_1\}$ of $\mathbb{K}$-linear maps. If $\mathbf{R}$ is a representation of Q, then its *dimension vector* or *numerical type* is the n-tuple $(\dim_{\mathbb{K}} V_i)_{i \in Q_0}$, where $n = |Q_0|$.

By path we mean a concatenation of arrows such that each ends where the next one starts. Formally,

Definition 2.1.3. A *path* in Q is a sequence $x = \rho_1 \rho_2 ... \rho_m$ of arrows such that $t\rho_i = h\rho_{i+1}$ for $1 \leq i \leq m-1$. A *relation* is a formal sum of paths which end and start on the same vertex.

Here are a few interesting examples of quiver representations:

Example 2.1.4. The *ADHM quiver* is the one of the form:



with relation

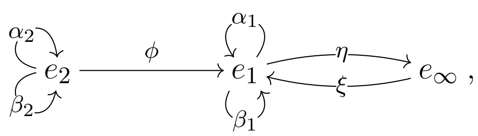
$$\alpha\beta - \beta\alpha + \xi\eta.$$

A representation of this quiver is a pair of finite-dimensional vector spaces (V, W), assigned to the vertices (e_1, e_∞), together with linear maps (A, B, I, J) assigned to the arrows $(\alpha, \beta, \xi, \eta)$, which satisfies the so-called *ADHM equation*

$$[A, B] + IJ = 0. \tag{2.1.1}$$

The numerical type of this representation is the pair $(\dim V, \dim W) \in (\mathbb{Z}_{\geq 0})^2$.

Example 2.1.5. The *enhanced ADHM quiver* is the one of the form:



with relations

$$\alpha_1\beta_1 - \beta_1\alpha_1 + \xi\eta; \quad \alpha_1\phi - \phi\alpha_2; \quad \eta\phi; \quad \alpha_2\beta_2 - \beta_2\alpha_2; \quad \beta_1\phi - \phi\beta_2; \ .$$

A representation of this quiver consists of a triple (V_1, V_2, W) of finite-dimensional vector spaces and linear maps $(A_1, B_1, I, J, A_2, B_2, f)$ which satisfy the following equations and are assigned, respectively, to the vertices (e_1, e_2, e_∞) and the arrows $(\alpha_1, \beta_1, \xi, \eta, \alpha_2, \beta_2, \phi)$

$$\begin{cases} [A_1, B_1] + IJ = 0 \\ A_1 f - f A_2 = 0 \\ B_1 f - f B_2 = 0 \\ [A_2, B_2] = 0 \\ Jf = 0 \end{cases} \tag{2.1.2}$$

named *enhanced ADHM equations*. The numerical type of this representation is the triple

$(\dim W, \dim V_1, \dim V_2) \in (\mathbb{Z}_{\geq 0})^3$.

Remark 2.1.6. The enhanced ADHM quiver was conceived for the first time in [2] in a slightly different way. The authors disregard the existence of a map B_2, although they take account of the existence of a map $g : V_1 \to V_2$ (see Section 3 of [2] for more details).

Definition 2.1.7. Given $\mathbf{R}$ and $\mathbf{R}'$ two representations of the same quiver Q, a *morphism* $f : \mathbf{R} \to \mathbf{R}'$ is a collection of $\mathbb{K}$-linear maps

$$\{f_i : V_i \to W_i : i \in Q_0\}$$

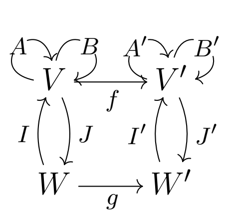
such that the diagram commutes for every arrow $a \in Q_1$:

$$\begin{array}{ccc} V_{ta} & \xrightarrow{\Phi_a} & V_{ha} \\ {\scriptstyle f_{ta}}\downarrow & & \downarrow{\scriptstyle f_{ha}} \\ W_{ta} & \xrightarrow[\Psi_a]{} & W_{ha} \end{array}$$

i.e., $f_{ha} \circ \Phi_a = \Psi_a \circ f_{ta}$, for all $a \in Q_1$. A morphism $f : \mathbf{R} \to \mathbf{R}'$ is an isomorphism if f_i is invertible for every $i \in Q_0$.

According to the previous definition, we obtain what will be referred to as the category of representations of a quiver Q over a field $\mathbb{K}$, denoted by $\mathcal{A} = \mathrm{Rep}_{\mathbb{K}}(Q)$. It is the category whose objects are representations of Q with the morphisms as defined above.

Example 2.1.8. Let Q be the quiver of Example 2.1.4. A morphism between the representations of Q, $\mathbf{R} = (V, W, (A, B, I, J))$ and $\mathbf{R}' = (V', W', (A', B', I', J'))$, is a pair (f, g) of linear maps such that



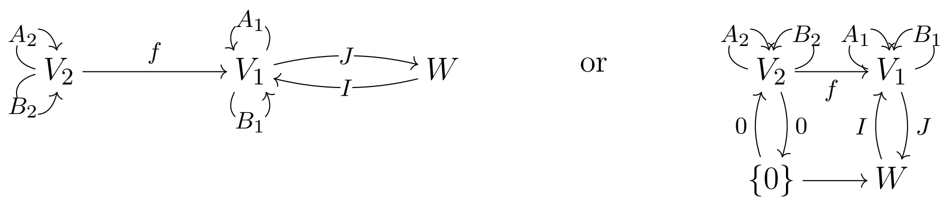
$$\Longrightarrow \quad \begin{cases} fA = A'f \\ fI = I'g \\ fB = B'f \\ gJ = J'f \end{cases}$$

or simply,

$$Hom_{\mathcal{A}}(\mathbf{R}, \mathbf{R}') = \left\{ (f, g) \in Hom(V, V') \oplus Hom(W, W') \;\middle|\; \begin{array}{cc} fA = A'f & fI = I'g \\ fB = B'f & gJ = J'f \end{array} \right\}.$$

Similarly, a morphism between two representations $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f))$ and $\mathbf{R}' = (V_1', V_2', W', (A_1', B_1', I', J', A_2', B_2', f'))$ of the enhanced ADHM quiver is a triple $(\xi_1, \xi_2, \xi_\infty)$ of linear maps between the vector spaces (V_1, V_2, W) and (V_1', V_2', W'), respectively, satisfying obvious compatibility conditions with the morphisms attached to the arrows.

Remark 2.1.9. Sometimes it is useful to look at a representation of the enhanced ADHM quiver as a pair of representations of the ADHM quiver plus a morphism between them, i.e., given $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f))$ a representation of an enhanced ADHM quiver, we can write the representations of ADHM quivers $\mathbf{R_1} = (V_1, W, (A_1, B_1, I, J))$, $\mathbf{R_2} = (V_2, \{0\}, (A_2, B_2, 0, 0))$ and a map $(f, 0)$ between them.



(2.1.3)

2.2 Monads

We start by recalling the concept of monad, whose best general reference is [12], precisely §3 and §4 of Chapter 2.

Let X be a smooth projective variety.

Definition 2.2.1. A *monad* on X is a complex of locally free sheaves on X:

$$M_\bullet : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

such that β is surjective and α is injective. The sheaf $\mathcal{H}(M_\bullet) = \ker \beta / \mathrm{im}\ \alpha$ is called the *cohomology of the monad* $M_\bullet$.

It is important to observe that a morphism between two monads is simply a morphism of complexes.

Theorem 2.2.2. If $E = H(M_\bullet)$ and $E' = H(M_\bullet')$ are, respectively, the cohomology sheaves of two monads:

$$M_\bullet : 0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

$$M'_\bullet : 0 \longrightarrow A' \xrightarrow{\alpha'} B' \xrightarrow{\beta'} C' \longrightarrow 0$$

over a smooth projective variety X. The mapping

$$h : \mathrm{Hom}(M_\bullet, M'_\bullet) \to \mathrm{Hom}(E, E')$$

which associates the induced homomorphism of cohomology sheaves with each homomorphism of monads, is bijective, if the following hypotheses are satisfied:

$$\mathrm{Hom}(B, A') = \mathrm{Hom}(C, B') = 0$$

$$H^1(X, C^* \otimes A') = H^1(X, B^* \otimes A') = H^1(X, C^* \otimes B') = H^2(X, C^* \otimes A') = 0.$$

Proof. See [12, Lemma II.4.1.3]. □

Definition 2.2.3. A monad on $\mathbb{P}^n$ of the form:

$$0 \longrightarrow V_1 \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\alpha} V_2 \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\beta} V_3 \otimes \mathcal{O}_{\mathbb{P}^n}(1) \longrightarrow 0$$

where V_k, $k = 1, 2, 3$, is a vector space, is called a linear monad.

The next result, which is a special case of Theorem 2.2.2, proved to be extremely useful in Section 3.1.

Corollary 2.2.4. If E and E' are, respectively, the cohomology sheaves of two linear monads $M_\bullet$ and $M'_\bullet$, then the map which associates the induced homomorphism of cohomology sheaves with each homomorphism of monads, is bijective.

2.3 ADHM data

In this section, we review some of the recent results about ADHM data. Let V and W be complex vector spaces with dimension n and r, respectively. The ADHM data is the set given by:

$$\mathbb{B} = \mathbb{B}(r, n) := \mathrm{End}(V)^{\oplus 2} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W).$$

An element of $\mathbb{B}$ is a datum $\mathtt{X} = (A, B, I, J)$ with $A, B \in \mathrm{End}(V)$, $I \in \mathrm{Hom}(W, V)$ and $J \in \mathrm{Hom}(V, W)$. We let $GL(V)$ act on $\mathbb{B}$ by defining:

$$[\mathtt{X}] = g.\mathtt{X} := (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}),$$

for $g \in GL(V)$ and $\mathtt{X} = (A, B, I, J) \in \mathbb{B}$.

Definition 2.3.1. A datum $\mathtt{X} = (A, B, I, J)$ is said to be

i. *stable* if there is no subspace $S \subsetneq V$ preserved by A, B and containing the image of I;

ii. *costable* if there is no subspace $\{0\} \neq S \subset V$ preserved by A, B and contained in $\ker J$;

iii. *regular* if it is both stable and costable.

A representation of the ADHM quiver $\mathbf{S} = (V, W, \mathtt{X})$ is said to be *stable (respectively* costable) if $\mathtt{X}$ is stable (respectively costable).

Definition 2.3.2. Fix homogeneous coordinates $(x : y : z)$ in $\mathbb{P}^2$. For any representation $\mathbf{R} = (V, W, (A, B, I, J))$ of the ADHM quiver, define the complex

$$E^{\bullet}_{\mathbf{R}} : V \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} (V \oplus V \oplus W) \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where

$$\alpha = \begin{pmatrix} zA + x1 \\ zB + y1 \\ zJ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -zB - y1 & zA + x1 & zI \end{pmatrix}.$$

Any complex on $\mathbb{P}^2$ obtained in this way will be called an *ADHM complex.*

Remark 2.3.3. Note that the ADHM equation is equivalent to $\beta\alpha = 0$. Indeed, we have:

$$\beta\alpha = \begin{pmatrix} -zB - y1 & zA + x1 & zI \end{pmatrix} \begin{pmatrix} zA + x1 \\ zB + y1 \\ zJ \end{pmatrix} = z^2([A, B] + IJ).$$

Hence, $\beta\alpha = 0 \Leftrightarrow [A, B] + IJ = 0$.

Consider now the category of ADHM complex, denoted by $\mathcal{K}om_{\text{ADHM}}(\mathbb{P}^2)$, subcategory of complexes on $\mathbb{P}^2$. The assignment:

$$\begin{array}{rcl} \mathbb{F}: \mathcal{A} & \to & \mathcal{K}om_{\text{ADHM}}(\mathbb{P}^2) \\ \mathbf{R} & \mapsto & E^{\bullet}_{\mathbf{R}} \end{array},$$

provides a functor, where $\mathcal{A}$ is the category of representations of the ADHM quiver. The functor acts on morphisms as follows: let $(f,g) \in Hom_{\mathcal{A}}(\mathbf{R},\mathbf{R}')$ be a morphism between representations $\mathbf{R}$ and $\mathbf{R}'$; then we have the morphism $f^{\bullet}$ between the corresponding ADHM complexes $E^{\bullet}_{\mathbf{R}}$ and $E^{\bullet}_{\mathbf{R}'}$:

$$\begin{array}{ccccc} V\otimes\mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha} & (V\oplus V\oplus W)\otimes\mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta} & V\otimes\mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow{\scriptstyle f\otimes 1} & & \downarrow{\scriptstyle (f\oplus f\oplus g)\otimes 1} & & \downarrow{\scriptstyle f\otimes 1} \\ V'\otimes\mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha'} & (V'\oplus V'\oplus W')\otimes\mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta'} & V'\otimes\mathcal{O}_{\mathbb{P}^2}(1) \end{array}.$$

Thus, $\mathbb{F}((f,g))$ is the map $f^{\bullet} = (f\otimes 1, (f\oplus f\oplus g)\otimes 1, f\otimes 1)$.

Proposition 2.3.4. Given the functor

$$\begin{array}{rccc} \mathbb{F}: & \mathcal{A} & \longrightarrow & \mathcal{K}om_{\text{ADHM}}(\mathbb{P}^2) \\ & \mathbf{R} & \longmapsto & E^{\bullet}_{\mathbf{R}} \\ & (f,g) & \longmapsto & (f\otimes 1,(f\oplus f\oplus g)\otimes 1, f\otimes 1) \end{array}.$$

It follows that:

i. $\mathbb{F}$ is an exact functor.

ii. $\mathbb{F}$ is fully faithful.

Proof. We start our proof by showing that $\mathbb{F}$ is exact. Given any exact sequence of representations of the ADHM quiver

$$0 \to \mathbf{Z} \to \mathbf{S} \to \mathbf{Q} \to 0, \tag{2.3.1}$$

we will show that the corresponding sequence of ADHM complexes:

$$0 \to E^{\bullet}_{\mathbf{Z}} \to E^{\bullet}_{\mathbf{S}} \to E^{\bullet}_{\mathbf{Q}} \to 0, \tag{2.3.2}$$

is still exact. The sequence (2.3.1) can be rewritten as:

$$\begin{array}{ccccccccc} & & A_{\mathbf{Z}} \circlearrowleft\ \circlearrowright B_{\mathbf{Z}} & & A_{\mathbf{S}} \circlearrowleft\ \circlearrowright B_{\mathbf{S}} & & A_{\mathbf{Q}} \circlearrowleft\ \circlearrowright B_{\mathbf{Q}} & & \\ 0 & \longrightarrow & V_{\mathbf{Z}} & \xrightarrow{f_1} & V_{\mathbf{S}} & \xrightarrow{f_2} & V_{\mathbf{Q}} & \longrightarrow & 0\ . \\ & & I_{\mathbf{Z}} \uparrow\ \downarrow J_{\mathbf{Z}} & & I_{\mathbf{S}} \uparrow\ \downarrow J_{\mathbf{S}} & & I_{\mathbf{Q}} \uparrow\ \downarrow J_{\mathbf{Q}} & & \\ 0 & \longrightarrow & W_{\mathbf{Z}} & \xrightarrow[g_1]{} & W_{\mathbf{S}} & \xrightarrow[g_2]{} & W_{\mathbf{Q}} & \longrightarrow & 0 \end{array} \tag{2.3.3}$$

Notice that this sequence provides the following exact sequences:

$$0 \to V_{\mathbf{Z}} \to V_{\mathbf{S}} \to V_{\mathbf{Q}} \to 0 \qquad (I)$$

$$0 \to V_{\mathbf{Z}} \oplus V_{\mathbf{Z}} \oplus W_{\mathbf{Z}} \to V_{\mathbf{S}} \oplus V_{\mathbf{S}} \oplus W_{\mathbf{S}} \to V_{\mathbf{Q}} \oplus V_{\mathbf{Q}} \oplus W_{\mathbf{S}} \to 0 \quad (II)$$

For every open set $U \subset \mathbb{P}^2$, the module $\mathcal{O}_{\mathbb{P}^2}(U)$ is free, hence it is flat. Tensoring the sequences (I) and (II) with $\mathcal{O}_{\mathbb{P}^2}(U)$ for every open set $U \subset \mathbb{P}^2$ we obtain the following exact sequences:

$$0 \to V_{\mathbf{Z}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V_{\mathbf{S}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V_{\mathbf{Q}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to 0 \qquad (III)$$

$$0 \to (V_{\mathbf{Z}} \oplus V_{\mathbf{Z}} \oplus W_{\mathbf{Z}}) \otimes \mathcal{O}_{\mathbb{P}^2} \to (V_{\mathbf{S}} \oplus V_{\mathbf{S}} \oplus W_{\mathbf{S}}) \otimes \mathcal{O}_{\mathbb{P}^2} \to \Big(V_{\mathbf{Q}} \oplus V_{\mathbf{Q}} \oplus W_{\mathbf{S}}\Big) \otimes \mathcal{O}_{\mathbb{P}^2} \to 0 \quad (IV)$$

$$0 \to V_{\mathbf{Z}} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to V_{\mathbf{S}} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to V_{\mathbf{Q}} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \to 0 \qquad (V)$$

Combining (III), (IV) and (V), the result is the exact sequence of ADHM complex:

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E^{\bullet}_{\mathbf{Z}}: & & V_{\mathbf{Z}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & (V_{\mathbf{Z}} \oplus V_{\mathbf{Z}} \oplus W_{\mathbf{Z}}) \otimes \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & V_{\mathbf{Z}} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E^{\bullet}_{\mathbf{S}}: & & V_{\mathbf{S}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & (V_{\mathbf{S}} \oplus V_{\mathbf{S}} \oplus W_{\mathbf{S}}) \otimes \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & V_{\mathbf{S}} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ E^{\bullet}_{\mathbf{Q}}: & & V_{\mathbf{Q}} \otimes \mathcal{O}_{\mathbb{P}^2}(-1) & \longrightarrow & \Big(V_{\mathbf{Q}} \oplus V_{\mathbf{Q}} \oplus W_{\mathbf{Q}}\Big) \otimes \mathcal{O}_{\mathbb{P}^2} & \longrightarrow & V_{\mathbf{Q}} \otimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 & & 0 \end{array}$$

and $\mathbb{F}$ is an exact functor, as claimed.

To check that $\mathbb{F}$ is faithful, we need to show, for every $\mathbf{R},\mathbf{R}'\in\mathcal{A}$, the map

$$\mathbb{F}: Hom(\mathbf{R},\mathbf{R}') \longrightarrow Hom(E^{\bullet}_{\mathbf{R}},E^{\bullet}_{\mathbf{R}'})$$

is injective. Indeed, given $(f,g)\in Hom(\mathbf{R},\mathbf{R}')$ such that $\mathbb{F}(f,g)=0$, we have

$$(f\otimes 1,(f\oplus f\oplus g)\otimes 1, f\otimes 1)=0.$$

Immediately, $f=g=0$ and (f,g) is the zero morphism between representations of the ADHM quiver.

There remains to prove that $\mathbb{F}$ is full. For this purpose, it suffices to show that for every $\mathbf{R},\mathbf{R}'\in\mathcal{A}$ the map:

$$\mathbb{F}: Hom(\mathbf{R},\mathbf{R}') \longrightarrow Hom(E^{\bullet}_{\mathbf{R}},E^{\bullet}_{\mathbf{R}'})$$

is surjective. Given $E^{\bullet}_{\mathbf{R}}$ and $E^{\bullet}_{\mathbf{R}'}$ two ADHM complexes and (F,G,H) a map between them, we need to find a map $(f,g)\in Hom(\mathbf{R},\mathbf{R}')$ such that $\mathbb{F}(f,g)=(F,G,H)$.The proof derives from the analysis of the commutativity of the diagram:

$$\begin{array}{ccccc} V\otimes\mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha} & (V\oplus V\oplus W)\otimes\mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta} & V\otimes\mathcal{O}_{\mathbb{P}^2}(1) \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle H} \\ V'\otimes\mathcal{O}_{\mathbb{P}^2}(-1) & \xrightarrow{\alpha'} & (V'\oplus V'\oplus W')\otimes\mathcal{O}_{\mathbb{P}^2} & \xrightarrow{\beta'} & V'\otimes\mathcal{O}_{\mathbb{P}^2}(1) \end{array} \tag{2.3.4}$$

Highlight that the sheaf morphisms F,G,H are given by vector spaces maps, which we will denote similarly, $F,H:V\to V'$ and $G:V\oplus V\oplus W\to V'\oplus V'\oplus W'$. Since $G\in \mathrm{Hom}(V\oplus V\oplus W,V'\oplus V'\oplus W')$ we can write G into block form:

$$G=\begin{pmatrix} G_1 & G_2 & G_3 \\ G_4 & G_5 & G_6 \\ G_7 & G_8 & G_9 \end{pmatrix}.$$

The commutativity of the diagram (2.3.4) gives us the equations:

$$\begin{aligned} \alpha' F &= G\alpha \\ \beta' G &= H\beta \end{aligned} \tag{2.3.5}$$

and, using the description of $\alpha, \alpha', \beta, \beta'$ given by the ADHM construction, we obtain relations for the maps involved in G. Namely, from the first equation of (2.3.5):

$$\alpha' F = \begin{pmatrix} zA' + x1 \\ zB' + y1 \\ zJ' \end{pmatrix} F = \begin{pmatrix} zA'F + xF \\ zB'F + yF \\ zJ'F \end{pmatrix}$$

$$G\alpha = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_4 & G_5 & G_6 \\ G_7 & G_8 & G_9 \end{pmatrix} \begin{pmatrix} zA + x1 \\ zB + y1 \\ zJ \end{pmatrix} = \begin{pmatrix} zG_1A + xG_1 + zG_2B + yG_2 + zG_3J \\ zG_4A + xG_4 + zG_5B + yG_5 + zG_6J \\ zG_7A + xG_7 + zG_8B + yG_8 + zG_9J \end{pmatrix}$$

we obtain $G_1 = G_5 = F$ and $G_2 = G_4 = G_7 = G_8 = 0$.

The same reasoning applies to the second equation of (2.3.5). Replacing G with $\begin{pmatrix} F & 0 & G_3 \\ 0 & F & G_6 \\ 0 & 0 & G_9 \end{pmatrix}$, gives us:

$$\beta' G = \begin{pmatrix} -zB' - y1 & zA' + x1 & zI \end{pmatrix} \begin{pmatrix} F & 0 & G_3 \\ 0 & F & G_6 \\ 0 & 0 & G_9 \end{pmatrix} =$$

$$= \begin{pmatrix} -zB'F - yF & zA'F + xF & -zB'G_3 - yG_3 + zA'G_6 + xG_6 + zIG_9 \end{pmatrix}$$

$$H\beta = H \begin{pmatrix} -zB - y1 & zA + x1 & zI \end{pmatrix} = \begin{pmatrix} -zHB - yH & zHA + xH & zHI \end{pmatrix}.$$

And this provides $G_9 = F = H$ and $G_3 = G_6 = 0$.

Hence, $G = \begin{pmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}$, the map $(f,g) = (F,F)$ satisfies the condition $\mathbb{F}(f,g) = (F,G,H)$ and the proof is complete. □

For further references, we list some properties of the ADHM complex proved in [9, Section 5].

Lemma 2.3.5. Let us fix a representation $\mathbf{R} = (V, W, (A, B, I, J))$, and the corresponding ADHM complex $E^\bullet_{\mathbf{R}}$ as above. Then:

1. The sheaf map α is injective. The fiber maps α_P are injective for every $P \in \mathbb{P}^2$ if and only if $\mathbf{R}$ is costable.

2. If $\mathbf{R}$ is stable, then $\mathcal{H}^1(E^{\bullet}_{\mathbf{R}}) = 0$, and $\mathcal{H}^0(E^{\bullet}_{\mathbf{R}})$ is a torsion free sheaf whose restriction to l_∞ is trivial of rank $r = dimW$ and second Chern class $c = dimV$.

3. For any $\mathbf{R} \in \mathcal{A}$, $H^0(\mathcal{H}^0(E^{\bullet}_{\mathbf{R}})(-1)) = 0$ holds.

Proof. [9, Lemmas 5.2, 5.3 and 5.4] □

Lemma 2.3.6. For any representation $\mathbf{R} = (V, W, (A, B, I, J))$, and the corresponding ADHM complex $E^{\bullet}_{\mathbf{R}}$ as above, we have:

$$\text{supp } \mathcal{H}^1(E^{\bullet}_{\mathbf{R}}) = \{p \in \mathbb{P}^2 : \beta_{(p)} \text{ is not surjective } \} \subset \mathbb{P}^2 \setminus l_\infty$$

Proof. To prove these statements we need to do some remarks: Let $\beta : \mathcal{F} \to \mathcal{G}$ be a sheaf morphism and $\beta_p : \mathcal{F}_p \to \mathcal{G}_p$ the induced morphism on the stalks. We know that $(\mathcal{F}/\mathcal{F}')_p = \mathcal{F}_p/\mathcal{F}'_p$ and $(\text{im } \beta)_p = \text{im } \beta_p$. So $(\text{coker } \beta)_p = (\mathcal{G}/\text{im } \beta)_p = \mathcal{G}_p/(\text{im } \beta)_p = \mathcal{G}_p/\text{im } \beta_p = \text{coker } \beta_p$. Moreover, the Nakayama lemma says: the map on the stalks $\beta_p : \mathcal{F}_p \to \mathcal{G}_p$ is surjective precisely when the map on the fibers $\beta_{(p)} : \mathcal{F}(p) \to \mathcal{G}(p)$ is surjective. So, let $p \in \mathbb{P}^2$ then:

$$\mathcal{H}^1(E^{\bullet}_{\mathbf{R}})_p \neq 0 \Leftrightarrow (\text{coker } \beta)_p \neq 0 \Leftrightarrow \text{coker } \beta_p \neq 0 \Leftrightarrow \beta_p \text{ is nonsurjective } \Leftrightarrow \beta_{(p)} \text{ is nonsurjective.}$$

So, $\text{supp}\,\mathcal{H}^1(E^{\bullet}_{\mathbf{R}}) = \{p \in \mathbb{P}^2 : \beta_{(p)} \text{ is not surjective }\}$. To prove the other statement, let $p = (x : y : 0) \in l_\infty$, then:

$$\beta_{(p)} = \begin{pmatrix} -y\mathbf{1} & x\mathbf{1} & 0 \end{pmatrix}$$

is always surjective. Then $p \notin \text{supp}\,\mathcal{H}^1(E^{\bullet}_{\mathbf{R}})$ and $\text{supp}\,\mathcal{H}^1(E^{\bullet}_{\mathbf{R}}) \cap l_\infty = \emptyset$. □

2.4 Nakajima's description of Hilbert scheme of points

It is well-known that as a set, the Hilbert scheme of n points on $\mathbb{C}^2$ is given by:

$$\text{Hilb}^n(\mathbb{C}^2) = \{I \subset \mathbb{C}[X, Y] \,|\, dim\mathbb{C}[X, Y]/I = n\}.$$

The existence of its schematic structure is a special case of the general result of Grothendieck [7]. Another explicit construction of the Hilbert scheme of points on the affine plane is given by Nakajima [11, Chapter 1]. In this section, let us recall some details of this construction which gives the ADHM description of the Hilbert scheme of n points in $\mathbb{C}^2$. Namely,

Theorem 2.4.1. Let $\widetilde{H} \overset{\text{def.}}{=} \left\{ (A,B,I) \;\middle|\; \begin{array}{l} (i)\,[A,B]=0, \\ (ii)\,(A,B,I) \text{ is stable} \end{array} \right\}$, where $A,B \in \mathrm{End}(\mathbb{C}^n)$ and $I \in \mathrm{Hom}(\mathbb{C},\mathbb{C}^n)$. Define an action of $GL(\mathbb{C}^n)$ on $\widetilde{H}$ by:

$$g.(A,B,I) = (gAg^{-1}, gBg^{-1}, gI),$$

for $g \in GL(\mathbb{C}^n)$, and consider the quotient space $H \overset{\text{def.}}{=} \widetilde{H}/GL(\mathbb{C}^n)$. Then H is a nonsingular variety and represents the functor $\underline{Hilb}_X^P$ for $X = \mathbb{C}^2$ and $P = n$.

Proof. [11, Theorem 1.9] □

Although the proof of this theorem is not given here, we point out some remarkable steps of the proof which are useful later.

According to the hypotheses of Theorem 2.4.1, set $\mathtt{X} = (A,B,I)$ an ADHM datum and define the map $\Phi_{\mathtt{X}}$ by:

$$\begin{array}{rrl} \Phi_{\mathtt{X}}: & \mathbb{C}[X,Y] \to & \mathbb{C}^n \\ & p(X,Y) \mapsto & p(A,B)I(1) \end{array}.$$

Furthermore, denote the class of the ADHM datum $\mathtt{X} \in \widetilde{H}$ by $[\mathtt{X}] \in H = \widetilde{H}/GL(\mathbb{C}^n)$. Thus, in order to prove the Theorem 2.4.1, it is necessary to consider the map

$$\begin{array}{rrl} \Psi: & \widetilde{H}/GL(\mathbb{C}^n) \to & \mathrm{Hilb}^n(\mathbb{C}^2) \\ & [\mathtt{X}] \mapsto & \ker \Phi_{\mathtt{X}} \end{array}$$

which associates the ideal $\ker \Phi_{\mathtt{X}}$ with the class $[\mathtt{X}]$ of a stable ADHM datum $\mathtt{X} = (A,B,I)$.

Let $\mathbf{S} = (V,W,(A,B,I,J))$ be a stable representation of the ADHM quiver of numerical type $(r,n) \in (\mathbb{Z}_{\geq 0})^2$.

Proposition 2.4.2. If $r = 1$, then:

i. $J = 0$.

ii. The sheaf $\mathcal{I} = \ker\beta/\text{im }\alpha$ is isomorphic to the ideal $I = \{f(X,Y) \in \mathbb{C}[X,Y] \,|\, f(A,B) = 0\}$.

Proof. [11, Proposition 2.8] □

From now on we assume that $\dim W = 1$. From the above proposition $J = 0$, and the ADHM datum $\mathtt{X} = (A,B,I,0)$ will be denoted by $\mathtt{X} = (A,B,I)$ to shorten notation.

Moreover, we can do a construction similar to that of the previous Theorem for both complex vector space V with dimensions n and representations of the ADHM quiver of numerical type $(1,n)$. Explicitly, we can state:

Lemma 2.4.3. Let $\mathbf{R} = (V,\mathbb{C},(A,B,I))$ be a stable representations of the ADHM quiver of numerical type $(1,n)$. If $\mathtt{X} = (A,B,I)$ then the map:

$$\begin{aligned} \Phi_{\mathtt{X}}: \quad \mathbb{C}[X,Y] &\to \quad V \\ p(X,Y) &\mapsto \quad p(A,B)I(1) \end{aligned}$$

is a surjective linear map. In particular, $\mathbb{C}[X,Y]/\ker\Phi_{\mathtt{X}}$ is isomorphic to V.

Proof. First note that the linear map $\Phi_{\mathtt{X}}$ is well-defined, since $[A,B] = 0$. Observe also that $\text{im } I \subset \text{im } \Phi_{\mathtt{X}}$ since the elements of $\text{im } I$ consist of vectors of the form $\alpha I(1)$, for some constant $\alpha \in \mathbb{C}$. Moreover, the image $\text{im } \Phi_{\mathtt{X}}$ of the map $\Phi_{\mathtt{X}}$ is invariant under A and B, since they commute. By stability of the datum $\mathtt{X}$ we must have $\text{im } \Phi_{\mathtt{X}} = V$. Hence, $\Phi_{\mathtt{X}}$ is surjective. Straightforwardly, $\ker\Phi_{\mathtt{X}} \subset \mathbb{C}[X,Y]$ is an ideal. Hence, we get the isomorphism $\mathbb{C}[X,Y]/\ker\Phi_{\mathtt{X}} \simeq V$. □

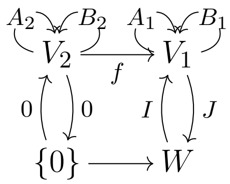
Lemma 2.4.4. Let $\mathbf{R} = (V_1,\mathbb{C},(A_1,B_1,I_1))$ and $\mathbf{S} = (V_2,\mathbb{C},(A_2,B_2,I_2))$ be two stable representations of the ADHM quiver of numerical type $(1,n)$. If $\mathtt{X} = (A_1,B_1,I_1)$ and $\mathtt{Y} = (A_2,B_2,I_2)$ satisfy $[\mathtt{X}] = [\mathtt{Y}]$, then $\ker\Phi_{\mathtt{X}} \simeq \ker\Phi_{\mathtt{Y}}$.

Proof. Since $[\mathtt{X}] = [\mathtt{Y}]$, there exists an element $g \in GL(V)$ such that $\mathtt{Y} = g.\mathtt{X}$. Then $\mathtt{Y} = (A_2,B_2,I_2) = (gA_1g^{-1}, gB_g^{-1}, gI_1)$. Now, for any polynomial $p \in \mathbb{C}[X,Y]$ we have $p(A_2,B_2)I_2(1) = p(gA_1g^{-1}, gB_1g^{-1})(gI_1(1)) = gp(A_1,B_1)g^{-1}gI_1(1) = gp(A_1,B_1)I_1(1)$, in other words, $\Phi_{\mathtt{Y}} = g\Phi_{\mathtt{X}}$. Since g is invertible, then it follows that $\ker\Phi_{\mathtt{Y}} = \ker\Phi_{\mathtt{X}}$, which is the desired conclusion. □

It is clear that in Lemma 2.4.3 $\ker\Phi_{\mathtt{X}}$ belongs to $Hilb^n(\mathbb{C}^2)$ and in Lemma 2.4.4 the map Ψ is well-defined.

2.5 Enhanced ADHM data

Let $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f))$ be a representation of numerical type $(r, n_1, n_2) \in (\mathbb{Z}_{\geq 0})^3$ of the enhanced ADHM quiver:



with relations:

$$\begin{cases} [A_1, B_1] + IJ = 0 \\ A_1 f - f A_2 = 0 \\ B_1 f - f B_2 = 0 \\ [A_2, B_2] = 0 \\ Jf = 0. \end{cases} \tag{2.5.1}$$

In order to construct moduli spaces of framed representations of the enhanced ADHM quiver, we need to introduce some concepts.

Definition 2.5.1. A *framed representation* of the enhanced ADHM quiver of numerical type $(r, n_1, n_2) \in (\mathbb{Z}_{\geq 0})^3$ is a pair $(\mathbf{R}, h)$ consisting of a representation $\mathbf{R}$ and an isomorphism $h : W \tilde{\to} \mathbb{C}^r$. Two framed representations $(\mathbf{R}, h)$ and $(\mathbf{R}', h')$ are *isomorphic* if there is an isomorphism of the form $(\xi_1, \xi_2, \xi_\infty) : \mathbf{R} \tilde{\to} \mathbf{R}'$ such that $h'\xi_\infty = h$.

Definition 2.5.2. Given a representation of an enhanced ADHM quiver $\mathbf{R}$ of numerical type $(r, n_1, n_2) \in (\mathbb{Z}_{\geq 0})^3$ and a triple $\theta = (\theta_1, \theta_2, \theta_\infty) \in \mathbb{Q}^3$ satisfying the relation

$$n_1\theta_1 + n_2\theta_2 + r\theta_\infty = 0,$$

we say that $\mathbf{R}$ is *θ-(semi)stable* if the following conditions hold

(*i*) Any subrepresentation $\mathbf{R}' \subset \mathbf{R}$ of numerical type $(0, n_1', n_2')$ satisfies

$$n_1'\theta_1 + n_2'\theta_2 (\leq) < 0,$$

(*ii*) Any subrepresentation $\mathbf{R}' \subset \mathbf{R}$ of numerical type (r, n_1', n_2') satisfies

$$n_1'\theta_1 + n_2'\theta_2 + r\theta_\infty (\leq) < 0.$$

The following lemma establishes the existence of generic stability parameters for any given dimension vector (r, n_1, n_2).

Lemma 2.5.3. Let $\theta = (\theta_1, \theta_2, \theta_\infty) \in \mathbb{Q}^3$ with $\theta_2 > 0$ and $\theta_1 + n_2\theta_2 < 0$. For every representation $\mathbf{R}$ of numerical type $(r, n_1, n_2) \in \mathbb{Z}^3_{>0}$ fixed, the following statements are equivalent:

a) $\mathbf{R}$ is θ-stable;

b) $\mathbf{R}$ is θ-semistable;

c) the following conditions are satisfied:

(S.1) $f : V_2 \to V_1$ is injective.

(S.2) The data $\mathbf{S} = (V_1, W, (A_1, B_1, I, J))$ are stable.

Proof. The affirmation a)⇒b) is straightforward by definition.

b)⇒ c) Let $\mathbf{R}$ be a θ-semistable representation. Suppose that f is not injective, so $\ker f \subset V_2$ is preserved by A_2 and B_2, i.e., $A_2(\ker f) \subset \ker f$ and $B_2(\ker f) \subset \ker f$. The reason is that, for all $u \in \ker f$, we have $f(A_2(u)) = A_1(f(u)) = A_1(0) = 0$ and $f(B_2(u)) = B_1(f(u)) = B_1(0) = 0$. Then we can construct a subrepresentation $\mathbf{R}' = (\{0\}, \ker f, \{0\}, (0,0,0,0, A_2, B_2, 0))$ with $n_1' = 0$, $n_2' = \dim\ker f$ and $r' = 0$. The θ-semistability implies $n_2'\theta_2 \leq 0$. Since $\theta_2 > 0$, we have $0 < \dim\ker f \leq 0$. Therefore, f must be injective. If the condition $(S.2)$ is not true, there is a proper nontrivial subspace $\{0\} \subset V_1' \subset V_1$ such that:

$$\mathbf{R}' = \left(V_1', \{0\}, W, (A_1|_{V_1'}, B_1|_{V_1'}, I, J|_{V_1'}, 0, 0, 0)\right)$$

determines a proper nontrivial subrepresentation of $\mathbf{R}$ with $r' = r$. Since,

$$n_1\theta_1 + n_2\theta_2 + r\theta_\infty = 0 \Rightarrow r\theta_\infty = -n_1\theta_1 - n_2\theta_2,\ (n_1' - n_1) < 0 \text{ and } \theta_1 + n_2\theta_2 < 0$$

we have

$$n_1'\theta_1 + n_2'\theta_2 + r\theta_\infty' = n_1'\theta_1 + r\theta_\infty = n_1'\theta_1 - n_1\theta_1 - n_2\theta_2 = (n_1' - n_1)\theta_1 - n_2\theta_2 > 0.$$

This is a contradiction.

c)⇒ a) Let $\mathbf{R}$ be a subrepresentation satisfying the conditions $(S.1)$ and $(S.2)$, and suppose

$$\mathbf{R}' = \left(V_1', V_2', W', (A_1', B_1', I', J', A_2', B_2', f')\right) \subset \mathbf{R}$$

is a nontrivial proper subrepresentation of $\mathbf{R}$. We must consider two possibilities:

- $r' = r \Rightarrow W' = W$. In that case, the condition $(S.2)$ implies I is not identically zero. Otherwise, V_1' violates the stability condition, because $A_1(V_1'), B_1(V_1') \subset V_1'$ and $\{0\} = \text{im } I \subset V_1'$. Thus, $\text{im } I \neq \{0\}$ and $n_1' > 0$. Similarly, if $n_1' < n_1$, the data $(V_1', A_1', B_1', I', J')$ violates the condition $(S.2)$. Therefore, $n_1' = n_1$. Since $\mathbf{R}'$ has to be a proper subrepresentation, $n_2' < n_2$. Then:

$$n_1'\theta_1 + n_2'\theta_2 + r\theta_\infty = n_1\theta_1 + n_2'\theta_2 + r\theta_\infty = n_2'\theta_2 - n_2\theta_2 = (n_2' - n_2)\theta_2 < 0.$$

- $r' = 0 \Rightarrow W' = 0$. If $n_1 = 0$ as $f|_{V_2'} \subset V_1'$ then $V_2' \subset \ker f = \{0\}$ so $n_2' = 0$. This is impossible, since $\mathbf{R}'$ is assumed nontrivial. Therefore, $n_1' \geq 1$. Note that $\theta_1 > 0$ is impossible, for in that case $0 < \theta_1 \leq \theta_1 + n_2\theta_2 < 0$. Thus, $\theta_1 \leq 0$ and, in this case, we have:

$$n_1'\theta_1 \leq \theta_1 \Rightarrow n_1'\theta_1 + n_2'\theta_2 \leq \theta_1 + n_2'\theta_2 \leq \theta_1 + n_2\theta_2 < 0.$$

Therefore, $\mathbf{R}$ is θ-stable.

□

This lemma asserts that there exists a special stability chamber, which is determined by the inequalities $\theta_2 > 0$ and $\theta_1 + n_2\theta_2 < 0$, within which θ-semistability is equivalent to θ-stability, and to the conditions (S.1) and (S.2) as previously stated.

Definition 2.5.4. Under the assumptions of Lemma 2.5.3, a representation (respectively a framed representation) of the enhanced ADHM quiver which satisfies the conditions (S.1) and (S.2) will be called *stable representation* (respectively *stable framed representation*).

In the next section, moduli spaces of θ-semistable framed quiver representations will be constructed employing GIT techniques, similarly to [10].

2.6 Moduli spaces of stable enhanced representations

We wish to investigate a relation between the moduli space of stable framed representations of an enhanced ADHM quiver and the nested Hilbert scheme $Hilb^{n_1-n_2,n_2}(\mathbb{C}^2)$ which was introduced in the first section.

For this purpose, consider V_1, V_2, W three vector spaces of dimensions n_1, n_2 and $r \in \mathbb{Z}_{>0}$, respectively, and the reductive algebraic group $G = GL(V_1) \times GL(V_2)$. If,

$$\mathbb{X}(r,n_1,n_2) = \mathrm{End}(V_1)^{\oplus 2} \oplus \mathrm{Hom}(W,V_1) \oplus \mathrm{Hom}(V_1,W) \oplus \mathrm{End}(V_2)^{\oplus 2} \oplus \mathrm{Hom}(V_2,V_1),$$

there exists a G-action on $\mathbb{X}(r,n_1,n_2)$ given by:

$$(g_1,g_2) \times (A_1,B_1,I,J,A_2,B_2,f) \to \left(g_1A_1g_1^{-1}, g_1B_1g_1^{-1}, g_1I, Jg_1^{-1}, g_2A_2g_2^{-1}, g_2B_2g_2^{-1}, g_1fg_2^{-1}\right).$$

The closed points of $\mathbb{X}(r,n_1,n_2)$ will be denoted by $\mathtt{X} = (A_1,B_1,I,J,A_2,B_2,f)$, the action of $(g_1,g_2) \in G$ on a point $\mathtt{X} \in \mathbb{X}(r,n_1,n_2)$ will be denoted by $(g_1,g_2).\mathtt{X}$, the orbit of a point $\mathtt{X} \in \mathbb{X}(r,n_1,n_2)$ will be denoted by $G.\mathtt{X}$ and, finally, the stabilizer of a given point $\mathtt{X}$ will be denoted by $G_{\mathtt{X}} \subset G$.

Furthermore, denote by $\mathbb{X}_0(r,n_1,n_2)$ the subscheme of $\mathbb{X}(r,n_1,n_2)$ defined by the algebraic equations:

$$\begin{cases} [A_1,B_1] + IJ = 0 \\ A_1f - fA_2 = 0 \\ B_1f - fB_2 = 0 \\ [A_2,B_2] = 0 \\ Jf = 0. \end{cases} \tag{2.6.1}$$

First observe that, $\mathbb{X}_0(r,n_1,n_2)$ is preserved by the G-action. Note also each representation $\mathbf{R} = (V_1,V_2,W,(A_1,B_1,I,J,A_2,B_2,f))$ corresponds to a unique point $\mathtt{X} = (A_1,B_1,I,J,A_2,B_2,f) \in \mathbb{X}_0(r,n_1,n_2)$ and, two framed representations are isomorphic if and only if the corresponding points in $\mathbb{X}_0(r,n_1,n_2)$ are in the same G-orbit.

Next, we recover some standard facts on GIT quotients for a reductive algebraic group G acting on a vector space $\mathbb{X}(r,n_1,n_2)$, whose best reference is Sections 2, 3 and 4 of [10].

Definition 2.6.1. Given an algebraic character $\chi : G \to \mathbb{C}^{\times}$ we have:

(a) A point $\mathtt{X}_0$ is called χ-semistable if there exists a polynomial function $p(\mathtt{X})$ on $\mathbb{X}(r,n_1,n_2)$ satisfying $p((g_1,g_2).\mathtt{X}) = \chi(g_1,g_2)^l p(\mathtt{X})$ for some $l \in \mathbb{Z}_{\geq 1}$, and such that $p(\mathtt{X}_0) \neq 0$.

(b) A point $\mathtt{X}_0$ is called χ-stable if it is χ-semistable and, further, $\dim(G.\mathtt{X}_0) = \dim(G/\Delta)$, where $\Delta \subset G$ is the subgroup acting trivially on $\mathbb{X}(r,n_1,n_2)$, and the action of G on $\{\mathtt{X} \in \mathbb{X}(r,n_1,n_2) \,|\, p(\mathtt{X}) \neq 0\}$ is closed.

Another way of stating (a) and (b) is given by the next Lemma:

Lemma 2.6.2. Let $\chi : G \to \mathbb{C}^\times$ be an algebraic character and suppose G acts on the direct product $\mathbb{X}_0(r,n_1,n_2) \times \mathbb{C}$ by:

$$(g_1,g_2) \times (\mathtt{X}, z) \to ((g_1,g_2).\mathtt{X}, \chi(g_1,g_2)^{-1} z).$$

Then $\mathtt{X} \in \mathbb{X}(r,n_1,n_2)$ is:

(a′) χ-semistable if and only if the closure of the orbit $G.(\mathtt{X}, z)$ is disjoint from the zero section $\mathbb{X}(r,n_1,n_2) \times \{0\}$, for any $z \neq 0$.

(b′) χ-stable if and only if the orbit $G.(\mathtt{X}, z)$ is closed in complement of the zero section, and the stabilizer $G_{(\mathtt{X},z)}$ is a finite index subgroup of Δ.

Proof. [10, Lemma 2.2] □

We can form the quasi-projective scheme:

$$\mathcal{M}^{ss}_\theta(r,n_1,n_2) = \mathbb{X}_0(r,n_1,n_2)//_\chi G := \operatorname{Proj}\left(\bigoplus_{n\geq 0} A\left(\mathbb{X}_0(r,n_1,n_2)\right)^{G,\chi^n}\right),$$

where

$$A\left(\mathbb{X}_0(r,n_1,n_2)\right)^{G,\chi^n} := \{f \in A\left(\mathbb{X}_0(r,n_1,n_2)\right) \,|\, f(g.\mathtt{X}) = \chi(g)^n f(\mathtt{X}), \forall g \in G\}.$$

Notice that $\mathcal{M}^{ss}_\theta(r,n_1,n_2)$ is projective over $\operatorname{Spec}\left(\mathbb{X}_0(r,n_1,n_2)^G\right)$, and it is quasi-projective over $\mathbb{C}$. Geometric invariant theory tells us that $\mathcal{M}^{ss}_\theta(r,n_1,n_2)$ is the space of χ-semistable orbits; moreover, it contains an open subscheme $\mathcal{M}^{s}_\theta(r,n_1,n_2) \subseteq \mathcal{M}^{ss}_\theta(r,n_1,n_2)$ consisting of χ-stable orbits.

We can now state the analogue of [10, Prop. 3.1, Thm. 4.1].

Proposition 2.6.3. Suppose $\theta = (\theta_1, \theta_2) \in \mathbb{Z}^2$, and let $\chi_\theta : G \to \mathbb{C}^\times$ be the character:

$$\chi_\theta(g_1, g_2) = \det(g_1)^{-\theta_1} \det(g_2)^{-\theta_2}.$$

Then a representation $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f))$ of an enhanced ADHM quiver, of dimension vector $(r, n_1, n_2) \in (\mathbb{Z}_{>0})^3$, is θ-(semi)stable if and only if the corresponding closed point $\mathtt{X} \in \mathbb{X}_0$ is χ_θ-(semi)stable.

Proof. See [10, Prop. 3.1, Thm. 4.1]. □

It follows that $\mathcal{M}_\theta^{ss}(r, n_1, n_2)$ parameterizes S-equivalence classes of θ-semistable framed representations, while $\mathcal{M}_\theta^{s}(r, n_1, n_2)$ parameterizes isomorphism classes of θ-stable framed representations.

The GIT quotient $\mathbb{X}_0(r, n_1, n_2)//_\chi G$ is isomorphic to the moduli space of S-equivalence classes of θ-semistable quiver representations $\mathcal{M}_\theta^{ss}(r, n_1, n_2)$.

From the above statements, we are able to introduce:

Definition 2.6.4. The moduli space of stable framed representations is:

$$\mathcal{M}(r, n_1, n_2) = \left\{ \begin{array}{l} \text{stable framed representations of} \\ \text{enhanced ADHM quiver of fixed} \\ \text{numerical type } (r, n_1, n_2) \in \mathbb{Z}^3_{>0} \end{array} \right\} \Big/ \; G \;\; \simeq \mathbb{X}_0(r, n_1, n_2)//_\chi G.$$

Remark 2.6.5. The Lemma 2.5.3 is a restatement of [2, Lemma 3.1] for enhanced ADHM quiver with $B_2 = 0$. There, the moduli space of stable framed representations of the enhanced ADHM quiver with $B_2 = 0$ was denoted by $\mathcal{N}(r, n_1, n_2)$. The details of the construction of this moduli space were presented (see Section 3 of [2]).

Theorem 2.6.6. The moduli space $\mathcal{N}(r, n_1, n_2)$ of stable framed representations of the enhanced ADHM quiver of fixed numerical invariants $(r, n_1, n_2) \in \mathbb{Z}^3_{>0}$ and $B_2 = 0$ is a smooth, quasi-projective variety of dimension $(2n_1 - n_2)r$.

Proof. See [2, Theorem 3.2]. □

Chapter 3

Enhanced ADHM Quiver and Nested Hilbert Schemes

Our aim in this chapter is to establish a bijection between the moduli space of stable framed representations of the enhanced ADHM quiver $\mathcal{M}(1, n_1, n_2)$ and the closed points of the nested Hilbert scheme $\mathrm{Hilb}^{n_1-n_2,n_2}(\mathbb{C}^2)$.

3.1 Correspondence between enhanced ADHM data and nested 0-dimensional ideals

Theorem 3.1.1. There exists a one-to-one correspondence between the following sets:

- equivalence classes of stable framed representations of the enhanced ADHM quiver of dimension vector $(1, n_1, n_2)$.
- closed subschemes (Z_1, Z_2) of $\mathbb{C}^2$ with Hilbert polynomial n_1 and $n_1 - n_2$ respectively, and $Z_2 \subset Z_1$.

The proof is constructive and we have divided it into a sequence of steps and lemmas.

Step 1: In the first step we will see how to construct closed points of $\mathrm{Hilb}^{n_1-n_2,n_2}(\mathbb{C}^2)$ out of representations of an enhanced ADHM quiver. For this purpose, we define the map:

$$\varphi : \mathcal{M}(1, n_1, n_2) \longrightarrow \mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2).$$

Consider $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, 0, A_2, B_2, f))$, a stable representation of dimension vector $(1, n_1, n_2) \in \mathbb{Z}^3_{>0}$, with $n_1 > n_2$, of the enhanced quiver:

$$\circlearrowright_{A_2, B_2} V_2 \xrightarrow{\ f\ } V_1 \circlearrowleft_{A_1, B_1} \xleftarrow{\ I\ } W \, .$$

Remark 3.1.2. According to Proposition 2.4.2, when $r = 1$, we have $J = 0$, and again we will delete this term of the tuple.

Define $V := V_1/\mathrm{im}\ f$, and notice that the linear maps (A_1, B_1, I) yield linear maps

$$\widetilde{A}_1 : V \to V \qquad \widetilde{B}_1 : V \to V \qquad \widetilde{I} : W \to V$$

such that,

$$\widetilde{A}_1 \circ \pi = \pi \circ A_1 \qquad \widetilde{B}_1 \circ \pi = \pi \circ B_1 \qquad \widetilde{I} = \pi \circ I,$$

where $\pi : V_1 \to V$ is the canonical projection.

Lemma 3.1.3. The datum $\mathtt{Y} = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I})$ is stable and satisfies $[\widetilde{A}_1, \widetilde{B}_1] = 0$.

Proof. We first examine the equation $[\widetilde{A}_1, \widetilde{B}_1]$. Let $y = \pi(x) \in V$ and compute:

$$\left([\widetilde{A}_1, \widetilde{B}_1]\right)(\pi(x)) = [\widetilde{A}_1(\pi(x)), \widetilde{B}_1(\pi(x))] = [\pi(A_1(x)), \pi(B_1(x))] = \pi\left([A_1(x), B_1(x)]\right) = 0$$

since $[A_1, B_1] = 0$.

Now, let us prove the stability of $\mathtt{Y} = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I})$. Suppose that there exists a subspace $\widetilde{S} \subset V$ such that $\widetilde{A}_1(\widetilde{S})$, $\widetilde{B}_1(\widetilde{S})$, $\widetilde{I}(W) \subset \widetilde{S}$. Define $S = \pi^{-1}(\widetilde{S}) \subset V_1$. Notice that S is nontrivial subspace of V_1. Indeed, if $S = \{0\}$ then given $y \in \widetilde{S} \subset V$ there exists $x \in V_1$ such that $y = \pi(x)$. Since $\pi(x) = y \in \widetilde{S}$ then $x \in \pi^{-1}(\widetilde{S}) = S = \{0\}$. Thus, $x = 0$, $y = \pi(x) = 0$ and $\widetilde{S} = \{0\}$, a contradiction. We claim that $A_1(S)$, $B_1(S)$, $I(S) \subset S$. Indeed, let $x \in S = \pi^{-1}(\widetilde{S})$ then $y = \pi(x) \in \widetilde{S}$. Since $\widetilde{S}$ is preserved by $\widetilde{A_1}$, $\widetilde{B_1}$ we have $\pi(A_1(x)) = \widetilde{A_1}(\pi(x)) \in \widetilde{S}$, $\pi(B_1(x)) = \widetilde{B_1}(\pi(x)) \in \widetilde{S}$ and $A_1(x), B_1(x) \in \pi^{-1}(\widetilde{S}) = S$. As $\widetilde{S}$ contains the image of $\widetilde{I}$, given $w \in W$ we have $\pi(I(w)) = \widetilde{I}(w) \in \widetilde{S}$. Consequently, $I(w) \in \pi^{-1}(\widetilde{S}) = S$. The stability of (A_1, B_1, I) forces $V_1 = \pi^{-1}(\widetilde{S}) = S$, hence $\widetilde{S} = V$ and $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{I})$ is stable. $\square$

We now have the ADHM data $\mathtt{X} = (A_1, B_1, I)$ and $\mathtt{Y} = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I})$ which are stable and satisfy the ADHM equation. Applying the Lemma 2.4.3 yields the surjective linear maps:

$$\Phi_{\mathtt{X}} : \mathbb{C}[X,Y] \to V_1 \quad \text{and} \quad \Phi_{\mathtt{Y}} : \mathbb{C}[X,Y] \to V \ .$$

Thus, $\dim(\mathbb{C}[X,Y]/\ker\Phi_{\mathtt{X}}) = n_1$ and $\dim(\mathbb{C}[X,Y]/\ker\Phi_{\mathtt{Y}}) = n_1 - n_2$. Since f is injective and π surjective, we can construct short exact sequences:

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & V_2 & & \\
 & & & & & & \downarrow f & & \\
0 & \longrightarrow & \ker\Phi_{\mathtt{X}} & \longrightarrow & \mathbb{C}[X,Y] & \xrightarrow{\Phi_{\mathtt{X}}} & V_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \mathbf{1} & & \downarrow \pi & & \\
0 & \longrightarrow & \ker\Phi_{\mathtt{Y}} & \longrightarrow & \mathbb{C}[X,Y] & \xrightarrow{\Phi_{\mathtt{Y}}} & V & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

From Snake Lemma, we get the short exact sequence:

$$0 \longrightarrow \ker\Phi_{\mathtt{X}} \longrightarrow \ker\Phi_{\mathtt{Y}} \longrightarrow V_2 \longrightarrow 0 \,.$$

Thus, indeed, $(\ker\Phi_{\mathtt{Y}}, \ker\Phi_{\mathtt{X}}) \in \mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2)$ and $\varphi(\mathbf{R}) = (\ker\Phi_{\mathtt{Y}}, \ker\Phi_{\mathtt{X}})$.

It is necessary to note that the definition of the map φ is unambiguous. As we have mentioned earlier, each representation $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f))$ corresponds to a unique point $\mathtt{Z} = (A_1, B_1, I, J, A_2, B_2, f) \in \mathbb{X}_0(r, n_1, n_2)$ and, two framed representations are isomorphic if and only if the corresponding points in $\mathbb{X}_0(r, n_1, n_2)$ are in the same G-orbit.

Let $\mathbf{R}$ and $\mathbf{R}'$ be two isomorphic stable framed representations of the enhanced ADHM quiver of numerical type $(1, n_1, n_2)$. There exists $(g_1, g_2) \in G = GL(V_1) \times GL(V_2)$ such that $\mathtt{Z}' = (g_1, g_2).\mathtt{Z}$, where $\mathtt{Z}, \mathtt{Z}' \in \mathbb{X}_0(r, n_1, n_2)$. Suppose $\mathtt{Z} = (A_1, B_1, I, A_2, B_2, f)$ then,

$$\mathtt{Z}' = \left(g_1 A_1 g_1^{-1}, g_1 B_1 g_1^{-1}, g_1 I, g_2 A_2 g_2^{-1}, g_2 B_2 g_2^{-1}, g_1 f g_2^{-1}\right).$$

It now suffices to note that the pairs of points

- $\mathtt{X} = (A_1, B_1, I)$ and $\mathtt{X}' = g_1.\mathtt{X} = \left(g_1 A_1 g_1^{-1}, g_1 B_1 g_1^{-1}, g_1 I\right)$

- $\mathtt{Y} = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I})$ and $\mathtt{Y}' = g_1.\mathtt{Y} = \left(g_1 \widetilde{A}_1 g_1^{-1}, g_1 \widetilde{B}_1 g_1^{-1}, g_1 \widetilde{I}\right)$

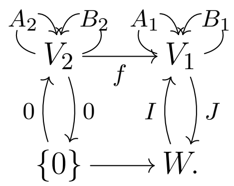
satisfies the conditions of Lemma 2.4.4. Then $\ker \Phi_{\mathtt{X}} \simeq \ker \Phi_{\mathtt{X}'}$, $\ker \Phi_{\mathtt{Y}} \simeq \ker \Phi_{\mathtt{Y}'}$ and, consequently,

$$\varphi(\mathbf{R}) = (\ker \Phi_{\mathtt{Y}}, \ker \Phi_{\mathtt{X}}) = (\ker \Phi_{\mathtt{Y}'}, \ker \Phi_{\mathtt{X}'}) = \varphi(\mathbf{R}').$$

Therefore the map φ is well-defined.

Remark 3.1.4. Although up to this point of the proof, the condition $r = 1$ has been essential, we will provide an "alternative proof", whose the condition over $\dim W$ appears only at the conclusion. The "alternative proof" becomes more interesting if we realize that, based on the methods which we have presented here, we are able to state similar link using, for instance, the usual ADHM construction of instantons or the ADHM construction of perverse coherent sheaves (see, respectively, [4], [8] and [1] for further details).

"Alternative proof": Consider $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f))$ a stable representation of dimension vector $(r, n_1, n_2) \in \mathbb{Z}^3_{>0}$, with $n_1 > n_2$, of the enhanced quiver:



Again, let $V := V_1/imf$ and the linear maps

$$\widetilde{A}_1 : V \to V \qquad \widetilde{B}_1 : V \to V \qquad \widetilde{I} : W \to V \qquad \widetilde{J} : V \to W$$

such that

$$\widetilde{A}_1 \circ \pi = \pi \circ A_1 \qquad \widetilde{B}_1 \circ \pi = \pi \circ B_1 \qquad \widetilde{I} = \pi \circ I \qquad \widetilde{J} \circ \pi = J,$$

where $\pi : V_1 \to V$ is the canonical projection. Similarly to Lemma 3.1.3, we can prove that the datum $\mathtt{Y} = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I}, \widetilde{J})$ is stable and satisfies $[\widetilde{A}_1, \widetilde{B}_1] + \widetilde{I}\widetilde{J} = 0$.

Moreover, since f is injective and π surjective, we can construct a short exact sequence of representations of the ADHM quiver

$$\begin{array}{ccccccccc} & & {\scriptstyle A_2}\circlearrowleft\circlearrowright{\scriptstyle B_2} & & {\scriptstyle A_1}\circlearrowleft\circlearrowright{\scriptstyle B_1} & & {\scriptstyle \widetilde{A}_1}\circlearrowleft\circlearrowright{\scriptstyle \widetilde{B}_1} & & \\ 0 & \longrightarrow & V_2 & \xrightarrow{f} & V_1 & \xrightarrow{\pi} & V & \longrightarrow & 0 \\ & & {\scriptstyle 0}\uparrow\downarrow{\scriptstyle 0} & & {\scriptstyle I}\uparrow\downarrow{\scriptstyle J} & & {\scriptstyle \widetilde{I}}\uparrow\downarrow{\scriptstyle \widetilde{J}} & & \\ 0 & \longrightarrow & \{0\} & \longrightarrow & W & \longrightarrow & W & \longrightarrow & 0 \end{array} \tag{3.1.1}$$

or simply

$$0 \to \mathbf{Z} \to \mathbf{S} \to \mathbf{Q} \to 0,$$

if we take $\mathbf{Z} = (V_2, \{0\}, (A_2, B_2, 0))$, $\mathbf{S} = (V_1, W, (A_1, B_1, I, J))$, $\mathbf{Q} = (V, W, (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I}, \widetilde{J}))$ in the category of representations of the ADHM quiver.

As we have already seen, according to Proposition 2.3.4 there exists an exact functor $\mathbb{F} : \mathcal{A} \to \mathcal{K}om_{ADHM}(\mathbb{P}^2)$ which associates an ADHM complex with each representation of the ADHM quiver an ADHM complex. Applying this functor to the sequence 3.1.1, we obtain a short exact sequence of complexes on $\mathbb{P}^2$:

$$0 \to E^\bullet_{\mathbf{Z}} \to E^\bullet_{\mathbf{S}} \to E^\bullet_{\mathbf{Q}} \to 0, \tag{3.1.2}$$

from which we obtain a long sequence of cohomology sheaves:

$$\begin{gathered} 0 \to \mathcal{H}^{-1}(E^\bullet_{\mathbf{Z}}) \to \mathcal{H}^{-1}(E^\bullet_{\mathbf{S}}) \to \mathcal{H}^{-1}(E^\bullet_{\mathbf{Q}}) \to \mathcal{H}^{0}(E^\bullet_{\mathbf{Z}}) \to \\ \to \mathcal{H}^{0}(E^\bullet_{\mathbf{S}}) \to \mathcal{H}^{0}(E^\bullet_{\mathbf{Q}}) \to \mathcal{H}^{1}(E^\bullet_{\mathbf{Z}}) \to \mathcal{H}^{1}(E^\bullet_{\mathbf{S}}) \to \mathcal{H}^{1}(E^\bullet_{\mathbf{Q}}) \to 0. \end{gathered}$$

We can simplify this sequence:

Lemma 3.1.5. Applying the previous conditions, we have:

1. $\mathcal{H}^{-1}(E^\bullet_{\mathbf{Z}}) = \mathcal{H}^{-1}(E^\bullet_{\mathbf{S}}) = \mathcal{H}^{-1}(E^\bullet_{\mathbf{Q}}) = 0$,

2. $\mathcal{H}^{1}(E^\bullet_{\mathbf{S}}) = \mathcal{H}^{1}(E^\bullet_{\mathbf{Q}}) = 0$,

3. $\mathcal{H}^{0}(E^\bullet_{\mathbf{Z}}) = 0$.

Proof. 1. This is true, for the maps $\alpha', \alpha, \alpha''$ which appear, respectively, in the ADHM complexes $E^\bullet_{\mathbf{Z}}$, $E^\bullet_{\mathbf{S}}$ and $E^\bullet_{\mathbf{Q}}$, are injective (see Lemma 2.3.5(i));

2. We conclude from Lemma 2.3.5(ii), since $\mathbf{S}$ and $\mathbf{Q}$ are stable representations;

3. Indeed,

$$E^{\bullet}_{\mathbf{Z}} : V_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha'} (V_2 \oplus V_2) \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta'} V_2 \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

and take $P = (x : y : 0) \in l_\infty$ we can write:

$$\alpha'_P = \begin{pmatrix} x\mathbf{1} \\ y\mathbf{1} \end{pmatrix} \qquad \beta'_P = \begin{pmatrix} -y\mathbf{1} & x\mathbf{1} \end{pmatrix}$$

then $ker\beta'_P = 0,\ \forall P \in l_\infty$. Since $\mathcal{H}^0(E^{\bullet}_{\mathbf{Z}}) = ker\beta'/\text{im }\alpha'$, the stalks of this sheaf vanish at P. Hence, the support of this sheaf is 0-dimensional scheme, because it does not intersect l_∞. In particular, $H^0(\mathcal{H}^0(E^{\bullet}_{\mathbf{Z}})) \simeq H^0(\mathcal{H}^0(E^{\bullet}_{\mathbf{Z}})(-1))$ and the right hand side vanishes by Lemma 2.3.5(iii). Since $\mathcal{H}^0(E^{\bullet}_{\mathbf{Z}})$ is supported at finitely many points, it follows that $\mathcal{H}^0(E^{\bullet}_{\mathbf{Z}}) = 0$.

□

Thus, we have :

$$0 \to \mathcal{H}^0(E^{\bullet}_{\mathbf{S}}) \to \mathcal{H}^0(E^{\bullet}_{\mathbf{Q}}) \to \mathcal{H}^1(E^{\bullet}_{\mathbf{Z}}) \to 0. \tag{3.1.3}$$

Assuming that $r = 1$, there exists according to item ii. of Proposition 2.4.2, 0-dimensional subschemes $Z_1, Z_2 \subset \mathbb{P}^2 \setminus l_\infty \cong \mathbb{C}^2$ such that:

1. $\mathcal{H}^0(E^{\bullet}_{\mathbf{S}}) = ker\beta/\text{im }\alpha = \mathcal{I}_{Z_1}$
2. $\mathcal{H}^0(E^{\bullet}_{\mathbf{Q}}) = ker\beta''/\text{im }\alpha'' = \mathcal{I}_{Z_2}$
3. $Z_2 \subset Z_1$ are 0-dimensional subschemes of length $n_1 - n_2$ and n_1, respectively.

We can rewrite the sequence (3.1.3) and obtain:

$$0 \to \mathcal{I}_{Z_1} \to \mathcal{I}_{Z_2} \to \mathcal{H}^1(E^{\bullet}_{\mathbf{Z}}) \to 0. \tag{3.1.4}$$

Thus we obtain a point $(Z_2, Z_1) \in \text{Hilb}^{n_1 - n_2, n_1}(\mathbb{C}^2)$.

Remark 3.1.6. From the sequence (3.1.4) it follows that $\mathcal{H}^1(E^{\bullet}_{\mathbf{Z}}) \simeq \mathcal{I}_{Z_2}/\mathcal{I}_{Z_1}$

Step 2: We will do now the inverse construction, i.e., we will build a representation of the enhanced ADHM quiver out of a closed point $\text{Hilb}^{n_1 - n_2, n_2}(\mathbb{C}^2)$ using the map:

$$\psi : \text{Hilb}^{n_1 - n_2, n_1}(\mathbb{C}^2) \to \mathcal{M}(1, n_1, n_2).$$

Let $(Z_1, Z_2) \in \mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2)$, so $Z_1 \subset Z_2 \subset \mathbb{C}^2$ are subschemes of length $n_1 - n_2$ and n_2 respectively. We know that there exists a short exact sequence

$$0 \to \mathcal{I}_{Z_2} \to \mathcal{I}_{Z_1} \to \mathcal{Q} \to 0, \tag{3.1.5}$$

Since $\mathbb{C}^2$ is Noetherian and $Z_1, Z_2 \subset \mathbb{C}^2$ are closed subschemes then $\mathcal{I}_{Z_1}, \mathcal{I}_{Z_2}$ are coherent torsion-free sheaves. Then we can follow the Nakajima construction [11, pag. 19-23] and find monads:

$$E_2^\bullet : V_2 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha_2} (V_2 \oplus V_2 \oplus W) \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta_2} V_2 \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where

$$\alpha_2 = \begin{pmatrix} zA_1 + x1 \\ zB_1 + y1 \\ zJ \end{pmatrix} \quad \text{e} \quad \beta_2 = \begin{pmatrix} -zB_1 - y1 & zA_1 + x1 & zI \end{pmatrix},$$

and

$$E_1^\bullet : V_1 \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha_1} (V_1 \oplus V_1 \oplus W') \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta_1} V_1 \otimes \mathcal{O}_{\mathbb{P}^2}(1)$$

where

$$\alpha_1 = \begin{pmatrix} z\widetilde{A}_1 + x1 \\ z\widetilde{B}_1 + y1 \\ zJ \end{pmatrix} \quad \text{e} \quad \beta_1 = \begin{pmatrix} -z\widetilde{B}_1 - y1 & z\widetilde{A}_1 + x1 & z\widetilde{I} \end{pmatrix},$$

such that

i. $\mathcal{I}_{Z_2} \simeq \mathcal{H}^0(E_2^\bullet)$

ii. $\mathcal{I}_{Z_1} \simeq \mathcal{H}^0(E_1^\bullet)$

iii. $\dim V_2 = c_2(\mathcal{I}_{Z_2}) = n_1$

iv. $\dim V_1 = c_2(\mathcal{I}_{Z_1}) = n_1 - n_2$

v. $\dim W = \dim W' = 1$ (since $\dim \widetilde{W} = 2c_2(\mathcal{I}_{Z_2}) + \mathrm{rank}\,(\mathcal{I}_{Z_2})$ and $\widetilde{W} = V_2 \oplus V_2 \oplus W$ then $\dim W = \mathrm{rank}\,(\mathcal{I}_{Z_2}) = 1$. The same thing happens to W').

So we can rewrite the exact sequence (3.1.5) as follows:

$$0 \longrightarrow \mathcal{H}^0(E_2^\bullet) \xrightarrow{\widetilde{f}} \mathcal{H}^0(E_1^\bullet) \longrightarrow \mathcal{Q} \longrightarrow 0\,. \tag{3.1.6}$$

Moreover, those monads give us the following stable representations:

$$\mathbf{Q} = (V_1, W', (\widetilde{A}_1, \widetilde{B}_1, \widetilde{I})) \text{ and } \mathbf{S} = (V_2, W, (A_2, B_2, I)),$$

which satisfy the ADHM equation.

According to Theorem 2.2.4, we have:

$$\mathrm{Hom}(E_2^\bullet, E_1^\bullet) \simeq \mathrm{Hom}(\mathcal{H}(E_2^\bullet), \mathcal{H}(E_1^\bullet)),$$

and since $\widetilde{f} \in \mathrm{Hom}(\mathcal{H}(E_2^\bullet), \mathcal{H}(E_1^\bullet))$, there exists a corresponding element $f \in \mathrm{Hom}(E_2^\bullet, E_1^\bullet)$. From Proposition 2.3.4, $\mathbb{F} : \mathcal{A} \to \mathcal{K}om_{ADHM}(\mathbb{P}^2)$ is a fully faithful functor, so $\mathbb{F}$ yields a map (ξ_1, ξ_2) between the representations $\mathbf{S}$ and $\mathbf{Q}$.

Lemma 3.1.7. Considering the previous conditions, the map $(\xi_1, \xi_2) : \mathbf{S} \to \mathbf{Q}$ is surjective.

Proof. Since $\dim W = \dim W' = 1$, we have $\dim \mathrm{im}\ \xi_2 = 0$ or $\dim \mathrm{im}\ \xi_2 = 1$. We claim that $\xi_2 \neq 0$. Indeed, suppose $\xi_2 = 0$, as $\xi_1 I = \widetilde{I}\xi_2 = 0$ we have $I(W) \subset \ker \xi_1$. Notice $\ker \xi_1$ is invariant under A_1 and B_1 because for each $x \in \ker \xi_1$ we have:

$$\xi_1(A_1(x)) = \widetilde{A_1}(\xi_1(x)) = 0,$$

$$\xi_1(B_1(x)) = \widetilde{B_1}(\xi_1(x)) = 0.$$

Hence, $\ker \xi_1 = V_2$, since (A_1, B_1, I) is stable and, consequently, $\xi_1 = 0$. However, $\widetilde{f} \neq 0$ and as it depends on ξ_1 and ξ_2, it is impossible to have $\xi_1 = \xi_2 = 0$. Thus, $\dim \mathrm{im}\ \xi_2 = 1$ and $\mathrm{im}\ \xi_2 = W'$, then ξ_2 is surjective. Consider now the subspace $\mathrm{im}\ \xi_1 \subset V_1$ and notice that $\mathrm{im}\ \xi_1$ is preserved by $\widetilde{A}_1$ and $\widetilde{B}_1$. Indeed, let $y \in \mathrm{im}\ \xi_1$, then there exists $x \in V_2$ such that $y = \xi_1(x)$. Thus,

$$\widetilde{A}_1(y) = \widetilde{A}_1(\xi_1(x)) = \xi_1(A_1(x)) \in \mathrm{im}\ \xi_1,$$

$$\widetilde{B}_1(y) = \widetilde{B}_1(\xi_1(x)) = \xi_1(B_1(x)) \in \mathrm{im}\ \xi_1.$$

Moreover, $\mathrm{im}\ \widetilde{I} \subset \mathrm{im}\ \xi_1$. In fact, let $y \in \mathrm{im}\ \widetilde{I}$ then, there exists $x' \in W'$ such that $\widetilde{I}(x') = y$. Since ξ_2 is surjective, there exists $x \in W$ such that $x' = \xi_2(x)$. Thus,

$$y = \widetilde{I}(x') = \widetilde{I}(\xi_2(x)) = \xi_1(I(x)) \in \mathrm{im}\ \xi_1$$

However, $(\widetilde{A}_1, \widetilde{B}_1, \widetilde{I}, \widetilde{J})$ is stable, then im $\xi_1 = V_1$ and, therefore, ξ_1 is also surjective. □

Consider now the subrepresentation $\mathbf{Z}$ of $\mathbf{S}$ given by:

$$\mathbf{Z} = (N, \{0\}, (A_2, B_2, 0, 0))$$

such that $N = \ker \xi_1$, $A_2 = A_1|_N$ and $B_2 = B_1|_N$. Notice that, since $[A_1, B_1] = 0$ we have:

$$[A_2, B_2] = [A_1|_N, B_1|_N] = [A_1, B_1]|_N = 0,$$

i.e., the maps A_2 and B_2 commutes. Moreover, $(A_2, B_2, 0)$ is stable. Indeed, suppose that there exists a subspace $S \subset N$ such that $A_2(S)$, $B_2(S) \subset S$. Then $A_1(S)$, $B_1(S) \subset S$ and im $I \subset S$.

Hence, we construct a stable representation of the enhanced ADHM quiver:

$$\overset{A_2}{\underset{B_2}{\circlearrowright}} N \xrightarrow{\ i\ } \overset{A_1}{\underset{B_1}{\circlearrowright}} V_2 \xleftarrow{\ I\ } W\ , \tag{3.1.7}$$

as required.

It is easy to check that the maps previously constructed:

$$\varphi : \mathcal{M}(1, n_1, n_2) \to \mathrm{Hilb}^{n_1 - n_2, n_1}(\mathbb{C}^2) \quad \text{and} \quad \psi : \mathrm{Hilb}^{n_1 - n_2, n_1}(\mathbb{C}^2) \to \mathcal{M}(1, n_1, n_2) \tag{3.1.8}$$

are mutually inverse. This finishes the proof of Theorem 3.1.1. Therefore, our construction provides a set-theoretical bijection between the set of closed points on $\mathrm{Hilb}^{n_1 - n_2, n_1}(\mathbb{C}^2)$ and points of $\mathcal{M}(1, n_1, n_2)$.

3.2 Nested Hilbert schemes with quotients supported on curves

For any stable representation of an enhanced quiver $\mathbf{R} = (V_1, V_2, W, (A_1, B_1, I, J, A_2, B_2, f)) \in \mathcal{M}(r, n_1, n_2)$ and any polynomial $F(X, Y) \in \mathbb{C}[X, Y]$, consider the following sets:

$$\Sigma = \left\{ (x : y : 1) \in \mathbb{P}^2 \,|\, F(x, y) = 0 \right\}$$

and

$$\mathcal{M}(r,n_1,n_2)_\Sigma = \{\mathbf{R} \in \mathcal{M}(r,n_1,n_2) \,|\, F(A_2,B_2) = 0\}.$$

Remark 3.2.1. This definition is a somewhat straightforward generalization of $\mathcal{N}(r,n_1,n_2)$, cited in Remark 2.6.5. Indeed, just take $F(X,Y) = Y$ and we will have, $\mathcal{N}(r,n_1,n_2) = \mathcal{M}(r,n_1,n_2)_\Sigma$.

We have proved the existence of a set-theoretical bijection:

$$\varphi : \mathcal{M}(1,n_1,n_2) \to \mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2); \tag{3.2.1}$$

our aim now is to find a similar set-theoretical bijection between the sets: $\mathcal{M}(1,n_1,n_2)_\Sigma$ and $\mathrm{Hilb}_\Sigma^{n_1-n_2,n_1}(\mathbb{C}^2)$, where:

$$\mathrm{Hilb}_\Sigma^{n_1-n_2,n_1}(\mathbb{C}^2) = \{(Z_1,Z_2) \in \mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2) \,|\, \mathrm{supp}\ \ (\mathcal{I}_{Z_1}/\mathcal{I}_{Z_2}) \subset \Sigma\}.$$

Theorem 3.2.2. For any representation of the ADHM quiver of type $\mathbf{Z} = (N,\{0\},(A_2,B_2,0,0))$ it holds that:

1. $\mathrm{supp}\ \ \mathcal{H}^1(E^\bullet_{\mathbf{Z}}) = \left\{ (x:y:1) \in \mathbb{P}^2 \;\middle|\; \begin{array}{c} x \text{ and } y \text{ are eigenvalues of } A_2 \text{ and } B_2 \\ \text{relative to the same eigenvector} \end{array} \right\}.$
2. If there exists $F(X,Y) \in \mathbb{C}[X,Y]$, such that, $F(A_2,B_2) = 0$ then

$$\mathrm{supp}\ \ \mathcal{H}^1(E^\bullet_{\mathbf{Z}}) = \left\{ (x:y:1) \in \mathbb{P}^2 \;\middle|\; \begin{array}{c} x \text{ and } y \text{ are eigenvalues of } A_2 \text{ and } B_2 \text{ relative} \\ \text{to the same eigenvector and } (x:y:1) \in \Sigma \end{array} \right\} \subset \Sigma.$$

Proof. Notice that:

$$\beta_{(p)} \text{ is surjective } \iff {}^t\beta_{(p)} : N^* \to N^* \oplus N^* \text{ is injective,}$$

where

$$\beta_{(p)} = \begin{pmatrix} -(B_2 + y\mathbf{1}) & A_2 + x\mathbf{1} \end{pmatrix} \quad \text{and} \quad {}^t\beta_{(p)} = \begin{pmatrix} -({}^tB_2 + y\mathbf{1}) \\ {}^tA_2 + x\mathbf{1} \end{pmatrix}.$$

If there exists $p = (x:y:1) \in \mathbb{P}^2$ such that $\beta_{(p)}$ is not surjective then ${}^t\beta_{(p)}$ is not injective.

Therefore, there exists $\phi \in N^*$ such that ${}^t\beta_{(p)}\phi = 0 \Rightarrow {}^tA_2(-\phi) = x(-\phi)$ and ${}^tB_2(-\phi) = y(-\phi)$, i.e., x and y are eigenvalues of ${}^tA_2$ and ${}^tB_2$, respectively, relative to the same eigenvector ϕ. The same happens to A_2 and B_2. Conversely, take $p = (x : y : 1)$ such that x and y are eigenvalues of A_2 and B_2, respectively, relative to the same eigenvector. Then there exists $\phi \in N^*$ such that ${}^tA_2\phi = x\phi$ and ${}^tB_2\phi = y\phi$. Thus, ${}^t\beta_{(p)}$ is not injective and $\beta_{(p)}$ is not surjective.

Let $F(X,Y) = \sum a_{ij}X^iY^j \in \mathbb{C}[X,Y]$ and $(x : y : 1) \in \operatorname{supp}\ \mathcal{H}^1(E^{\bullet}_{\mathbf{Z}})$, then there exists $v \in N \setminus \{0\}$ such that $A_2v = xv$ and $B_2v = yv$. Thus,

$$F(A_2,B_2)v = \sum a_{ij}A_2^iB_2^jv = \sum a_{ij}A_2^iy^jv = \sum a_{ij}y^jA_2^iv = \sum a_{ij}y^jx^iv = \sum a_{ij}x^iy^jv = F(x,y)v.$$

Since $v \neq 0$ and $F(A_2,B_2) = 0$ we have $F(x,y) = 0$. □

Consequently, we have proved more than we have intended to. Given the map

$$\varphi_\Sigma : \mathcal{M}(1,n_1,n_2)_\Sigma \to \mathrm{Hilb}_\Sigma^{n_1-n_2,n_1}(\mathbb{C}^2),$$

which associates a pair of schemes $(Z_1,Z_2) \in \mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2)$ which quotient $\mathcal{I}_{Z_1}/\mathcal{I}_{Z_2}$ is supported in the curve Σ with each stable representation $\mathbf{R} = (V_1,V_2,W,(A_1,B_1,I,A_2,B_2,f))$ of an enhanced quiver with $F(A_2,B_2) = 0$, we can state that:

Corollary 3.2.3. There exists an one-to-one correspondence between the sets $\mathcal{M}(1,n_1,n_2)_\Sigma$ and $\mathrm{Hilb}_\Sigma^{n_1-n_2,n_1}(\mathbb{C}^2)$.

Proof. The correspondence is given by the map:

$$\varphi_\Sigma : \mathcal{M}(1,n_1,n_2)_\Sigma \to \mathrm{Hilb}_\Sigma^{n_1-n_2,n_1}(\mathbb{C}^2).$$

Notice that φ_Σ is a restriction of the map 3.2.1. Thus, it suffices to apply Theorems 3.1.1 and 3.2.2 together with the observation that $\mathcal{H}^1(E^{\bullet}_{\mathbf{Z}}) \simeq \mathcal{I}_{Z_2}/\mathcal{I}_{Z_1}$ from Remark 3.1.6. □

As an immediate consequence of the results previously proved, we observe that, at least for the case $F(X,Y) = Y$, the subset $\mathrm{Hilb}_\Sigma^{n_1-n_2,n_1}(\mathbb{C}^2)$ is a nonsingular subset of $\mathrm{Hilb}^{n_1-n_2,n_1}(\mathbb{C}^2)$ according to [2, Theorem 3.2].

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