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End-extension と standardizable extension について

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Abstract

We consider end-extension and standardizable extension in nonstandard universes. We have treelike order of nonstandard universes of radius $\leq \aleph_1$ about end-extension (Lemma 4) and about standardizable extension (Theorem 5).

1 Nonstandard Universe

Definitions 1 (superstructure, base set). Given a set X , we define the iterated power set $V_n(X)$ over X recursively by

$$V_0(X) = X, \quad \text{and} \quad V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)).$$

The *superstructure* $V(X)$ is the union $\bigcup_{n < \omega} V_n(X)$. The set X is said to be a *base set* if $\emptyset \notin X$ and each element of X is disjoint from $V(X)$. We call a *set in* $V(X)$ an element of $V(X) \setminus X$.

Definition 2 (nonstandard universe). A *nonstandard universe* is a triple $\langle V(X), V(Y), \star \rangle$ such that:

1. X and Y are infinite base sets.
2. (Transfer Principle) The symbol \star is a bounded elementary embedding from $V(X)$ into $V(Y)$: which is

$$V(X) \models \varphi(a_1, \dots, a_n) \quad \text{if and only if} \quad V(Y) \models \varphi(\star a_1, \dots, \star a_n)$$

holds for any bounded formula $\varphi(x_1, \dots, x_n)$ and $a_1, \dots, a_n \in V(X)$.

3. $\star X = Y$.
4. For every infinite subset A of X , $\{\star a \mid a \in A\}$ is a proper subset of $\star A$.

Definitions 3 (standard, internal). For $a \in V(*X)$, we call a *standard* if there is an $x \in V(X)$ such that $a = *x$.

For $a \in V(*X)$, we call a *internal* if there is an $x \in V(X)$ such that $a \in *x$. We denote by $*V(X)$ the set of all internal elements in $V(*X)$.

From now on, we denote a nonstandard universe by single $*V(X)$.

Definitions 4 (norm, radius). The *norm (of standardness)* of an internal element a is a cardinal defined by

$$\text{nos}(a) = \min \{|x| \mid a \in *x\}.$$

The *radius* of $*V(X)$ is a cardinal defined by

$$\text{rad}(*V(X)) = \min \{\kappa \mid \forall y \in *V(X) \text{ nos}(y) < \kappa\}.$$

For further detail, we refer to [1]. We shall consider bounded an elementary embedding e from $*^1V(X)$ into $*^2V(X)$.

2 End-extension and Standardizable extension

Let \mathbb{N} be a structure of standard natural numbers in $V(X)$, which is isomorphic to ω .

Definition 5 (end-extension). An elementary embedding $e: *^1V(X) \rightarrow *^2V(X)$ is an *end-extension* if any initial segment of $*^1\mathbb{N}$ is not extended by e :

$$\forall n \in *^1\mathbb{N} \forall m_2 \in *^2\mathbb{N} \exists m_1 \in *^1\mathbb{N} \ *^2V(X) \models m_2 \leq e(n) \Rightarrow m_2 = e(m_1).$$

We say a set A in $*^1V(X)$ is *finite in $*^1V(X)$* or *\star -finite* if there is a bijection from an initial segment of $*^1\mathbb{N}$ onto A inside $*^1V(X)$.

Lemma 1. Any (\star_1) -finite set in $*^1V(X)$ is not extended by an end-extension $e: *^1V(X) \rightarrow *^2V(X)$.

Proof. Let σ be a bijection from an initial segment I of $*^1\mathbb{N}$ onto A in $*^1V(X)$. Since e is an end-extension, for an element $a \in e(A)$, there is $n \in I$ such that $e(n) = (e(\sigma))^{-1}(a)$. Then we have $e(\sigma(n)) = (e(\sigma))(e(n)) = a$. \square

Definitions 6 (standardization, standardizable extension). A set A_1 in $*^1V(X)$ is a *standardization* a set A_2 in $*^2V(X)$ by $e: *^1V(X) \rightarrow *^2V(X)$ if

$$\forall x \in *^1V(X) \ [[*^1V(X) \models x \in A_1] \Leftrightarrow [*^2V(X) \models e(x) \in A_2]].$$

We say e is κ -*standardizable* if every set of power less than $*^2\kappa$ in $*^2V(X)$ has its standardization by e . We say e is *standardizable* if e is $|V(X)|$ -standardizable.

Lemma 2. *If e is ω -standardizable then e is an end-extension.*

Proof. Let n be an element of ${}^*1\mathbb{N}$ and let m_2 be an element of ${}^*2\mathbb{N}$ such that ${}^*2V(X) \models m_2 \leq e(n)$. Let A be the standardization of the initial segment $\{k \mid k \leq m_2\}$ of ${}^*2\mathbb{N}$. Then we have $e(\max A) = m_2$. \square

Corollary 3. *Any finite set in ${}^*1V(X)$ is not extended by an ω -standardizable extension $e: {}^*1V(X) \rightarrow {}^*2V(X)$.*

We cannot prove the converse implication of the previous lemma (see [4]).

Fact 1. There is an end-extension which is not ω -standardizable, if continuum hypothesis holds.

3 Ordering of Nonstandard Universes by Standardization

In this section, we consider relation of two elementary embeddings $e_1: {}^*1V(X) \rightarrow {}^*V(X)$ and $e_2: {}^*2V(X) \rightarrow {}^*V(X)$.

Lemma 4. *Suppose $\text{rad}({}^*V(X)) \leq \aleph_1$. If e_1 and e_2 are end-extensions then there is either an end-extension $e: {}^*1V(X) \rightarrow {}^*2V(X)$ such that $e_2 \circ e = e_1$ or $e: {}^*2V(X) \rightarrow {}^*1V(X)$ such that $e_1 \circ e = e_2$*

Proof. Since both e_1 and e_2 are end-extensions, $e_1 \upharpoonright {}^*1\mathbb{N} \subseteq e_2 \upharpoonright {}^*2\mathbb{N}$ or $e_2 \upharpoonright {}^*2\mathbb{N} \subseteq e_1 \upharpoonright {}^*1\mathbb{N}$. Without loss of generality, we can assume $e_1 \upharpoonright {}^*1\mathbb{N} \subseteq e_2 \upharpoonright {}^*2\mathbb{N}$. Let $f: {}^*1\mathbb{N} \rightarrow {}^*2\mathbb{N}$ be a map satisfying $e_2 \circ f = e_1 \upharpoonright {}^*1\mathbb{N}$.

Let a in an element of ${}^*1V(X)$. Since $\text{rad}({}^*V(X)) \leq \aleph_1$, there are a countable set R_a in $V(X)$ such that $a \in {}^*1R$ and a bijection $\sigma_a: \mathbb{N} \rightarrow R_a$. Defining e by $e(a) = {}^*2\sigma_a(f({}^*1\sigma_a^{-1}(a)))$, we have completed the proof. \square

Theorem 5. *Suppose $\text{rad}({}^*V(X)) \leq \aleph_1$. If e_1 and e_2 are standardizable then there is either a standardizable $e: {}^*1V(X) \rightarrow {}^*2V(X)$ such that $e_2 \circ e = e_1$ or $e: {}^*2V(X) \rightarrow {}^*1V(X)$ such that $e_1 \circ e = e_2$.*

Proof. By the previous lemma, we only check e is standardizable. The standardization of a set A in ${}^*2V(X)$ by e is the standardization of the set $e_2(A)$ by e_1 . \square

In the case of ultrapowers, standardizable extension corresponds to Rudin-Frolík order [2, 3, 4] of ultrafilters [5]. On ultrafilters over countable sets, Rudin-Frolík order is treelike: every initial segment are comparably ordered. So the following question rise.

Question 2. The assumption $\text{rad}({}^*V(X)) \leq \aleph_1$ in the previous theorem is required?

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