UNIFORM ALGEBRAS WITH UNBOUNDED FUNCTIONS*

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Foreword: Most of the theorems in this article appear without proof. References for published results and credits are included in the notes at the end. Certain papers listed in the bibliography are not discussed; however, these are given because they do embellish the general subject, or represent a direction which is ignored in this exposition.

1. *Introduction*[1]

For our purposes a function algebra on Hausdorff space X will be a uniformly closed subalgebra with 1 of the algebra $CB(X)$ of bounded continuous functions on X. In the last fifteen years the theory of *function algebras* (on compact spaces) has developed into a distinctive field, rich in applications to classical function theory and rich in questions of interest frequently overlooked by classical analysts. Much of this work needs and supplements deep results and techniques in the theory of functions of several complex variables.

Our purpose is to discuss the following more general algebras.

Definition 1.1. A *uniform algebra A* on a Hausdorff space X is a subalgebra of the algebra $C(X)$ of all continuous functions on X which contains the identity 1 and which is complete in the topology of uniform convergence on compact subsets of X.

Developments comparable to the aforementioned research in function algebras have not taken place in the study of uniform algebras. Even now these algebras are getting limited attention, although it is abundantly clear that here too there is an intimate relation to the theory of functions of several complex variables. Neglect is probably due in large measure to the primary importance in the theory of function algebras of dual arguments based on representing and orthogonal measures. Uniform algebras do not seem to be tractable yet to these methods.

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However, there are scattered results of the last decade and recently more systematic studies which we wish to describe together with some few applications to the structure of specific uniform algebras.

In what follows the reader should relate the discussion, whenever possible, to the concrete cases of algebras of holomorphic functions on subsets of $\mathbf{C}^n$.

2. *Inverse Limits of Function Algebras*[2]

Let X be a topological space. A collection $\mathscr{K}$ of compact subsets of X satisfying:

$$\bigcup_{K \in \mathscr{K}} K = X, \tag{2.1}$$

(2.2) Every compact subset L of X is contained in some $K \in \mathscr{K}$,

will be called a norming family in X. Suprema on K will be denoted by $\|\cdot\|_K$. If A is a uniform algebra on X and $\mathscr{K}$ is a norming family, then $\{\|\cdot\|_K : K \in \mathscr{K}\}$ is a family of pseudonorms giving the topology of A. If X is completely regular and if $\mathscr{K}$ can be chosen countable, then the uniform algebra $C(X)$ is metrizable. If X is σ-compact and locally compact, the existence of a countable norming family $\mathscr{K}$ is assured.

Let A be a uniform algebra on X. For $Y \subset X$ denote by A_Y the completion of $A \mid Y$ in the topology of uniform convergence on compact subsets of Y. If Y is compact, A_Y is the $\|\cdot\|_Y$-completion of $A \mid Y$. If $K \subset K'$, then $\pi(K', K): A \mid K' \to A \mid K$ extends to a continuous mapping $\bar{\pi}(K', K): A_{K'} \to A_K$ with dense range. The collection $\{A_K; \bar{\pi}(K', K): K, K' \in \mathscr{K}\}$ is called a strongly dense inverse limit system. It follows easily that the inverse limit of the function algebras A_K is the uniform algebra A. Let $\Delta(A_K)$ (or simply Δ_K) denote the set of non-zero homomorphisms of A_K into $\mathbf{C}$ and let $\Delta(A)$ denote the set of non-zero continuous homomorphisms of A into $\mathbf{C}$. Then

$$\Delta(A) = \bigcup_{K \in \mathscr{K}} \Delta(A_K) \qquad \text{(in a natural sense).} \tag{2.3}$$

Precisely, $\Delta(A)$ is an injective, direct limit with mappings dual to those of the inverse limit system.

Definition 2.1. A uniform algebra A is called a *Frechet algebra* or *$\mathscr{F}$-algebra*, if A is metrizable.

The conditions under which $\Delta(A)$ coincides with the set of all non-zero complex homomorphisms of A are not completely known, but are mostly irrelevant. It is however true, if A is a finitely generated uniform $\mathscr{F}$-algebra.

As usual, $\Delta(A)$ is topologized by the weakest topology renderiug the maps

$$\hat{f}:\Delta(A)\to \mathbf{C} \text{ given by } \hat{f}(\phi)=\phi(f) \tag{2.4}$$

continuous. $\Delta(A)$ is closed in the weak* topology $\sigma(A^*,A)$. Should A separate the points of X, then

$$j: X\to\Delta(A) \text{ given by } j(x)=\delta_x \tag{2.5}$$

with

$$\delta_x(f)=f(x) \text{ for all } f\in A$$

identifies X with a subset of $\Delta(A)$. j is continuous, but need not be open, so in general we must distinguish topologically between X and jX.

We want to observe that if A is a uniform algebra on X, then $B(A)=\{f\in A: \| f\|_X=\sup|f(x)|<\infty\}$ is a function algebra on X.

Definition 2.2. Let A be a uniform algebra on X. For compact $K\subset X$, we put

$$\operatorname{hull}_A(K)=\{\phi\in\Delta(A): |\hat{f}(\phi)|\leqq \| f\|_K \text{ for all } f\in A\}. \tag{2.6}$$

For an arbitrary subset $Y\subset X$, we put

$$\operatorname{hull}_A(Y)=\bigcup_K \operatorname{hull}_A K \tag{2.7}$$

where K is any compact subset of Y.

Theorem 2.1. *If Y is a σ-compact locally compact subspace of X, then $M(A_Y)=\operatorname{hull}_A(Y)$, both setwise and topologically.*

Theorem 2.2. *If $\mathscr{K}$ is a norming family in X for the uniform $\mathscr{F}$-algebra A on X, then $\mathscr{H}=\{\operatorname{hull}_A(K): K\in\mathscr{K}\}$ is a norming family for $\hat{A}$. $\hat{A}$ is a uniform $\mathscr{F}$-algebra on $\Delta(A)$ and A is topologically isomorphic to $\hat{A}$.*

3. *Natural Algebras and A-Holomorphic Functions*[3]

Let X be a Hausdorff space and A denote a subalgebra of $C(X)$ with the topology of uniform convergence on compact subsets of X. Following Rickart we state:

Definition 3.1. A pair $[X,A]$ is a *system* if A determines the topology of X, i.e., if the weakest topology rendering all functions in A continuous on X is the given topology of the space X.

Definition 3.2. $[X,A]$ is a *natural algebra*, if $[X,A]$ is a system and the space $\Delta(A)$ of continuous complex homomorphisms of A is X.

One analogue of the Šilov boundary is the Rickart boundary $\partial[X,A]$.

Definition 3.3. The *Rickart boundary* $\partial[X,A]$ of a system $[X,A]$ is

$$\{x \in X\colon \text{for all compact sets } K \subset X,\ x \notin K \rightarrow x \notin \operatorname{int}(\operatorname{hull}_A K)\}.$$

For locally compact spaces X, Definition 3.3 is equivalent to the following:

(3.1) $x \in \partial[X,A]$ iff there exists a compact set $L \subset X$ such that for all compact sets $K \supset L$, x is in the Šilov boundary $\delta(A_K)$ of the function algebra A_K.

The points $x \in X$ such that, for all compact sets K in X, $x \notin K$ implies $x \notin \operatorname{hull}_A K$ are called independent points and every local peak point of A in X is an independent point.

Theorem 3.1. *Let* $[X,A]$ *be a natural algebra and let* $X_0 = \operatorname{hull}_A(X \backslash \partial[X,A])$. *Then* $[X_0, A|X_0]$ *is a natural algebra and* $\partial[X_0, A|X_0] = \varnothing$.

The relation between X and X_0 in case $(X \backslash \partial[X,A])^- = X$ sheds some light on this normalization procedure. X is the space of continuous complex homomorphisms of the algebra $A|X_0$ with the topology of uniform convergence on the trace on X_0 of compact subsets of X.

It should be noted that local independent points (i.e., independent points of $[U, A|U]$ where U is open in Σ) are global independent points (i.e., independent points of $[\Sigma, A]$).

Definition 3.4. Let $[X,A]$ be a natural system and let Y be a subset of X. $H_{A,1}(Y)$ is defined to be the algebra of continuous functions on Y which are locally approximated on Y by functions in A. The elements of this algebra are called *A-holomorphic functions on Y of class* 1.

Clearly this process of local approximation can be iterated transfinitely. For any ordinal ν, Let $H_{A,\nu}(Y)$ be the algebra of continuous functions locally approximable on Y by functions in $(\bigcup_{\alpha<\nu} H_{A,\alpha}(Y))$. For cardinality reasons, the process terminates at some ordinal μ.

Definition 3.5. $H_{A,\nu}(Y)$ is called the algebra of *A-holomorphic functions on Y of class* ν. $H_A(Y) = H_{A,\mu}(Y)$ denotes the algebra of all A-holomorphic functions on Y.

Natural algebras $[X,A]$ are known with the property that

$$[X,A] \subsetneqq H_{A,1}(X) \subsetneqq H_{A,2}(X). \tag{3.2}$$

Suppose X is a compact space, $[X,A]$ is a natural algebra and $\bar{H}_{A,1}(X)$ is the uniform closure of $H_{A,1}(X)$ in $C(X)$. If B is a uniformly closed subalgebra of $C(X)$ such that $A \subset B \subset \bar{H}_{A,1}(X)$, then is the homomorphism space of B still X? In particular, is $[X, \bar{H}_{A,1}(X)]$ a natural algebra? Let M be the homomorphism space of B.

$$\delta(A) = \delta(\bar{H}_{A,1}(X)) = \delta(B) = \delta(\bar{H}_{B,1}(M)) \tag{3.3}$$

follows from Rossi's Local Maximum Modulus Theorem. $\sigma \in M$ implies $\sigma\big|_A = \delta_x$ for some unique point $x \in X$. Define $\rho : M \to X$ by setting $\rho(\sigma) = x$. ρ is a continuous map and $\rho(\delta_x) = x$. Define $\tilde{h} \in C(M)$ by $\tilde{h}(\sigma) = h(\rho(\sigma))$, for any continuous function $h \in C(X)$. Then

(i) $L \subset X \Rightarrow \|h\|_L = \|\tilde{h}\|_{\rho^{-1}L}$;

(ii) $f \in A \Rightarrow \hat{f} = \tilde{f}$ ($\hat{f}$ denotes f as an element of $\hat{B}$).

Suppose $g \in B$. Then there exists $g_n \in H_{A,1}(X)$ such that $g_n \to g$ on X. Thus $g_n = \lim f_{nj_\nu}$ uniformly on V_{nj}, $1 \leqq j \leqq m_n$, where $f_{nj_\nu} \in A$ and $V_{n1}, \cdots, V_{nm_n}$ cover X. By (i), $\{\tilde{f}_{nj_\nu}\}_{\nu=1}^{\infty}$ is a uniform Cauchy sequence in $C(\rho^{-1}V_{nj})$. For $\sigma \in \rho^{-1}V_{nj}$, we have $\lim_\nu \tilde{f}_{nj_\nu}(\sigma) = \lim_\nu f_{nj_\nu}(\rho(\sigma)) = g_n(\rho(\sigma)) = \tilde{g}_n(\sigma)$. Hence, by (ii), $\tilde{g}_n = \lim \hat{f}_{nj_\nu}$ uniformly on $\rho^{-1}V_{nj}$, $1 \leqq j \leqq m_n$, $n = 1, 2, \cdots$. Since ρ is onto, $\rho^{-1}V_{n1}, \cdots, \rho^{-1}V_{nm_n}$ cover M. So $\tilde{g}_n \in H_{B,1}(M)$. Again (i) shows that $\{\tilde{g}_n\}_{n=1}^{\infty}$ is uniformly Cauchy on M. A check of pointwise limits will show $\tilde{g}_n \to \tilde{g}$ uniformly on M or equivalently $\tilde{g} \in \bar{H}_{B,1}(M)$. Suppose now that $\sigma \in M \backslash X$ so that $\rho(\sigma) \neq \sigma$. Then there exists $g \in B$ such that $\hat{g}(\sigma) \neq \hat{g}(\rho(\sigma)) = g(\rho(\sigma)) = \tilde{g}(\sigma)$. But $\hat{g} \in \bar{H}_B(M)$, so $\hat{g} - \tilde{g} \in \bar{H}_{B,1}(M)$. This is a contradiction, since $\hat{g} = \tilde{g} = g$ on $X \supset \delta(\bar{H}_{B,1}(M)) = \delta(A)$, by 3.3, implies $\tilde{g} = \hat{g}$ on M.

This intriguingly simple argument of F. Quigley was expanded by him to show that: if $A \subset B \subset \bar{H}_A(X)$, then the homomorphism space of B is X.

Let $[X, A]$ be a natural algebra on a Hausdorff space X and suppose Ω is a subset of X. Let $B \subset H_A(\Omega)$.

Definition 3.6. A set $G \subset \Omega$ is said to be *B-convex* if for every compact set $K \subset G$, $\mathrm{hull}_B K$ is compact and contained in G. If Ω is $H_A(\Omega)$-convex, then Ω is said to be *A-holomorphically convex.*

Open Ω contained in X are A-holomorphically convex if and only if the system $[\Omega, H_A(\Omega)]$ is natural. This fact follows from a generalization of the Oka convexity theorem. We wish to state this result to show the remarkable connection of this field with the theory of several complex variables.

Theorem 3.2. *Let Ω be an open subset of X. If G is any A-holomorphically convex subset of Ω, then*

$$\tilde{G} = \{(\sigma, h_\lambda(\sigma)) : \sigma \in G \text{ and } H_A(\Omega) = \{h_\lambda : \lambda \in \Lambda\}\}$$

is $A \otimes P$-convex in $X \times \mathbf{C}^\Lambda$, where P is the algebra of polynomials on

$C^{\wedge}$. *If* $\tilde{G} \subset X \times \mathbf{C}^{\wedge}$ *is* $A \otimes P$*-convex in* $X \times \mathbf{C}^{\wedge}$, *then* G *is* A*-holomorphically convex.*

For certain open sets Ω in X which are A-holomorphically convex, it is possible to construct a function $g \in H_A(\Omega)$ such that $\| g \|_{U \cap \Omega} = \infty$ for every open set U that meets the boundary of Ω. Also, the notion of A-analytic varieties can be formulated in much the same manner as in several complex variables, their A-convexity studied and uniform algebras defined on them.

Much of the arguments depend upon the following local maximum modulus result.

Theorem 3.3. *If* $[\Sigma, A]$ *is a natural algebra, then each independent point of* $H_A(\Sigma)$ *is an independent point of* $[\Sigma, A]$.

4. *Ringed Spaces Defining Uniform Algebras*[4]

Whereas inverse limits are the natural bridge from function algebras to uniform algebras, the natural structure to bridge the theory of functions of several complex variables and uniform algebras seems to be the ringed space.

Using the work of Quigley we formulate the connection as follows:

Definition 4.1. A pair (X, A) is a *ringed space* if it is a subsheaf of rings with identity of the sheaf of germs of continuous functions on the space X. Let $\Gamma(U, A)$ denote the sections over an open subset $U \subset X$ relative to A. If $\phi \in \Gamma(U, A)$, then there exists an open neighborhood V of $u \in U$ and a continuous function g on V such that $\phi(u) = g_u$, the germ of g at u. By defining $\phi(u)(u) = g(u)$ we obtain uniquely a continuous function on U. For $\Gamma(U, A)$, let $A(U)$ denote all functions obtained in this fashion. $f \in A(U)$ is called an (X, A)*-holomorphic function on* U. If X is locally compact, then call (X, A) a *compact-open complete ringed space* provided that $A(U)$ is closed in $C(U)$ in the compact-open topology on $C(U)$ for all open subsets $U \subset X$.

Theorem 4.1. *Let* X *be a locally connected Hausdorff space and* (X, A) *a ringed space. Then a necessary and sufficient condition that* $A(U)$ *be quasi-analytic (vanishing on an open subset implies vanishing everywhere) for all open connected subsets* $U \subset X$ *is that* A *be a Hausdorff sheaf.*

In particular, such Hausdorff sheaves have no zero divisors. This theorem makes possible the notion of A-analytic continuation and domains of existence in the framework of ringed spaces. Results have been obtained along those lines.

It is perfectly clear how the concept of ringed space provides a generic geometric context for the algebras which were previously touched upon. What is more, it is adaptable to a discussion of generalized meromorphic functions.

Definition 4.2. Let (X, A) be a ringed space. Let E_x be the set of elements of the stalk A_x above x in A that are not zero divisors. A_x can be embedded in a quotient ring B_x with identity in which the units are precisely the elements in E_x. Let F_x be the set of germs in E_x which do not belong to the closure of the zero subsheaf of A. Each F_x is multiplicatively closed. Let M_x be the set of all $b \in B_x$ such that $b = st^{-1}$ where $s \in A_x$ and $t \in F_x$. In a natural fashion, $M = \bigcup_{x \in X} M_x$ becomes a sheaf of rings called the *sheaf of fractions of A*. Let U be open in X and $\phi \in \Gamma(U, M)$. Let $S\phi = \{u \in U$: for all $s \in A_x$, and $t \in F_x$ with $\phi(u) = st^{-1}$ we have $t(u) = 0\}$. $S\phi$ is closed and nowhere dense in U. With ϕ we associate a function $F\phi \in C(U \setminus S\phi)$ called the *meromorphic function on U associated with ϕ*.

Theorem 4.2. *Let (X, A) be a ringed space. Let M be the sheaf of fractions of A. For each $x \in X$, let $L_x = \{b \in M : b = st^{-1}$ and $t(x) \neq 0\}$. L_x is contained in M_x, $L = \bigcup L_x$ is a sheaf of local rings which can be considered a subsheaf in which we can embed A of the sheaf of germs of continuous functions. If $\phi \in \Gamma(U, L)$, then $M\phi = \varnothing$. Thus $L(U) \subset C(U)$.*

5. *Holomorphic Behavior in Uniform Algebras*[5]

The preceding three sections focused on general perspectives in the study of uniform algebras. But just as specific properties of the boundary value algebra on the unit disk furnished much of the motivation to seek like properties in general function algebras, most probably an analogously pregnant source for uniform algebra is provided by the uniform algebra of holomorphic functions of open subsets on the complex plane. Since one of the more striking results about the boundary value algebras is Wermer's Maximality Theorem, perhaps a similar maximality theorem can be invented at the outset.

Definition 5.1. A uniform algebra A on a Hausdorff space X is said to be

(i) *Liouville*, if the only bounded functions in A are the constant functions (i.e. $B(A) = \mathbf{C}$);

(ii) *Montel*, if bounded subsets of A are relatively compact;

(iii) *Without topological zero divisors*, if there exists a norming family $\mathscr{K}$ for A such that $A|K$ for $K \in \mathscr{K}$ is without topological zero divisors in the sense of normed algebras (i.e., there does not exist a sequence $\{u_n\}$ of elements of norm 1 corresponding to any element $u \neq 0$ such that $\| u_n u \| \to 0$.).

Properties (i), (ii), and (iii) have been isolated precisely because no function algebra enjoys these properties and because algebras of holomorphic functions always are Montel and without topological zero divisors (certain interesting ones are Liouville). Also, Liouville algebras and function algebras are, in one sense at least, totally dissimilar.

Theorem 5.1. *If A is a uniform algebra on an open connected subset G of $\mathbf{C}$ with continuous homomorphism space G and containing the identity function $\mathscr{Z}: z \to z$, then the following statements are equivalent:*

(0) $A = \mathrm{Hol}(G)$;

(1) *A is Montel;*

(2) *A is without topological zero divisors.*

The next theorem shows that it is essential for the continuous homomorphism space of A to be G.

Theorem 5.2. *If G is an open connected subset of $\mathbf{C}$, then the uniform closure $P(G^n)$ in $C(G^n)$ of the polynomials on $\mathbf{C}^n$ is topologically isomorphic to a separating uniform subalgebra A of $C(G)$.*

Thus, if $G = \mathrm{hull}_{\mathrm{Hol}(\mathbf{C})} G \equiv \hat{G}$, then $G^n = (G^n)^\wedge$ and $\mathrm{Hol}(G^n) = P(G^n)$. Hence, $\mathrm{Hol}(G^n) \cong A \subset C(G)$ and $\mathrm{Hol}(G) \subsetneq A$. But 5.1(ii) and 5.1(iii) are preserved by topological isomorphisms. (In the proof of Theorem 5.2, A is finitely generated by $f_1, f_2, \cdots, f_{n-1}$, $\mathscr{Z}$ where $f_i (1 \leqq i \leqq n-1)$ are Peano functions.)

What now is the situation for uniform algebras on domains in $\mathbf{C}^n$? For simplicity, we restrict the discussion to uniform algebras on polydisks with point evaluations thereon giving the continuous homomorphisms.

Definition 5.2. A uniform algebra A on a polydisk $\mathbf{P}$ in $\mathbf{C}^n$ is said to satisfy *Condition* (S) if constant extensions of restrictions of functions in A to coordinate slices are again in A (if $z^0 \in \mathbf{P}$ and $f \in A$, then the functions $z \to f(z^0{}_1, z^0{}_2, \cdots, z^0{}_{k-1}, z_k, z^0{}_{k+1}, \cdots, z^0{}_n)$ are in A, $1 \leqq k \leqq n$).

Theorem 5.1′. *Let A be a uniform algebra on an open polydisk $\mathbf{P}$ in $\mathbf{C}^n$ with continuous homomorphism space $\mathbf{P}$ and containing the coordinate functions $\mathscr{Z}^1, \mathscr{Z}^2, \cdots, \mathscr{Z}^n$. If A satisfies Condition (S), then the following statements are equivalent:*

(0) $A = \mathrm{Hol}(\mathbf{P})$;

(1) *A is a Montel algebra;*

(2) *A is without topological zero divisors.*

Turning to the Liouville property, we consider the following:

Definition 5.3. Let A be a uniform algebra with σ-compact continuous homomorphism space $\Delta(A)$ contained in $\mathbf{C}$ and containing the identity function $\mathscr{Z}$. Then A is said to satisfy

Condition (p): if there exists a neighborhood D of p in $\Delta(A)$ such that for all $a \in D$ there is an $\varepsilon_a > 0, f_a \in A$, and $g_a = (\mathscr{Z}^1 - a)h_a \in \mathrm{Hol}(W)$ where W is a neighborhood of $\{(z, f_a(z)) : z \in \Delta(A)\}$ such that $\operatorname{Re} g_a(z, f_a(z)) < 0$ whenever $z \in \Delta(A)$ and $0 < |z - a| < \varepsilon_a$.

In particular, if $p \in D \subset \Delta(A)$ and D is open in the complex plane and if there is a function $f \in A \cap C^{(1)}(D)$ such that $\partial f / \partial \overline{\mathscr{Z}} \neq 0$ on D, then Condition (p) is satisfied: Let

$$R(z) = f(z) - f(a) + \frac{\partial f}{\partial \mathscr{Z}}(a)(z - a) + \frac{\partial f}{\partial \overline{\mathscr{Z}}}(a)(\bar{z} - a) \quad \text{for} \quad z \in D.$$

Then there exists $\varepsilon_a > 0$ such that

$$\left| \left(\frac{\partial f}{\partial \overline{\mathscr{Z}}}(a)\right)^{-1} R(z) \right| < |z - a| \text{ for } 0 < |z - a| \leqq \varepsilon_a .$$

Set $f_a = f$ and $g_a(z, w) = -(z - a) \left(\frac{\partial f}{\partial \overline{\mathscr{Z}}}(a)\right)^{-1} \left(w - f(a) - \frac{\partial f}{\partial \mathscr{Z}}(a)(z - a)\right)$.

Theorem 5.3. *Let A be a uniform algebra on a σ-compact $\Delta(A)$ in $\mathbf{C}$ and let $\mathscr{Z} \in A$. If there exists $p \in \Delta(A)$ such that A satisfies Condition (p), then A is not a Liouville algebra.*

The proof of this theorem required a modification of a result of Rossi which depends upon the solution of a Cousin II problem in $\mathbf{C}^n$. From Theorem 5.4 we get the following characterization of the uniform algebra of entire functions.

Theorem 5.4. *Let A be a Liouville algebra on $\mathbf{C}$ with $\Delta(A) = \mathbf{C}$ and let $\mathscr{Z} \in A$. If A is generated by continuously differentiable functions, then $A = \mathrm{Hol}(\mathbf{C})$.*

For polydisks in $\mathbf{C}^n$, a Theorem 5.4′ again follows, if Condition (S) is assumed on A. Furthermore, as the next example shows, in this case Condition (S) cannot be omitted without some further assumption.

Example 5.1. The uniform algebra of all continuous functions on $\mathbf{C}^n$ whose restrictions to complex lines through the origin are entire functions has $\mathbf{C}^n$ as continuous homomorphism space, contains the coordinate functions, is Liouville, but is strictly larger than $\mathrm{Hol}(\mathbf{C}^n)$.

Theorem 5.5. *If A is a singly generated Liouville algebra of continuous functions on* $\mathbf{C}$, *then A is topologically isomorphic to* $\mathrm{Hol}(\mathbf{C})$. (Caution: There are singly generated Liouville algebras A with $\Delta(A)$ setwise equal to $\mathbf{C}$, which are, however, strictly larger than $\mathrm{Hol}(\mathbf{C})$!).

It could very well be the case that every separating Liouville algebra of continuous functions on $\mathbf{C}$ is topologically isomorphic to $\mathrm{Hol}(\mathbf{C}^n)$ for some n. On the other hand, Example 5.1 might be a counterexample.

6. *Uniform Algebras and Several Complex Variables*[6]

In order to obtain Theorem 5.4 without any assumption about the existence of continuously differentiable generators, it would suffice to know that locally independent points of a separating uniform algebra A in metrizable $\Delta(A)$ are peak points for functions in $B(A)$. This generalization of Rossi's Local Peak Point Theorem seems to be inaccessible and, as in the case of the Local Peak Point Theorem for function algebras, dependent upon methods of several complex variable theory. A natural first step is to prove a generalized Arens-Calderon theorem for uniform algebras. It is via the Arens-Calderon theorem anyhow that several complex variables is introduced into the study of function algebras.

Definition 6.1. Let A be a uniform algebra and $\underline{a} = (a_1, \cdots, a_n) \in A^n$. The joint spectrum $\sigma_A(\underline{a})$ of $\underline{a}$ relative to A is

$$\{\underline{a}(\xi) = (a_i(\xi)) \in \mathbf{C}^n \colon \xi \in \Delta(A)\}.$$

Definition 6.2. Let A, $\underline{a}$ be as in Definition 6.1 and let S be a subset of $\mathbf{C}^n$. We define

(i) $\mathcal{O}(S) = \lim\limits_{\substack{D \supset S \\ D \text{ open}}} \{\mathcal{O}(D); r^{D'}{}_D\}$ where $\mathcal{O}(D)$ is the algebra of all holomorphic functions on D and $r^{D'}{}_D$ are restriction maps. Thus $\mathcal{O}(S)$ is a direct limit of $\mathcal{O}(D)$ for $D \supset S$, D open in $\mathbf{C}^n$;

(ii) $\mathscr{H}(\sigma_A(\underline{a})) = \varprojlim\limits_{\sigma_{A_\nu}(\underline{a}) \subset \sigma_A(\underline{a})} \{\mathcal{O}(\sigma_{A_\nu}(\underline{a})); r^\nu_\mu\}$ where "lim" denotes an inverse limit, A is an inverse limit of the Banach algebras A_ν, and r^ν_μ are again restriction maps.

Theorem 6.1. *Let A be a uniform algebra and $\underline{a} = (a_1, a_2, \cdots, a_n) \in A^n$. There is a continuous representation $\Lambda_a \colon \mathscr{H}(\sigma_A(\underline{a})) \to A$ such that $\Lambda_a(\mathscr{Z}_j) = a_j$, $\mathscr{Z}_j$ coordinate functions, $1 \leqq j \leqq n$, and $\Lambda_a(1) = 1$. More particularly, there is a continuous representation $\Theta_a \colon \mathcal{O}(\sigma_A(\underline{a})) \to A$ satisfying the same conditions. (Continuity is with respect to the inverse limit topology and direct topology, respectively.)*

Even with this generalized Arens-Calderon result it is not clear how to get a generalization of Rossi's theorem.

7. *Some Open Questions*[7]

A. Does the process of iterated local approximation terminate after a finite number of steps? In particular, if functions locally in A are in A, is $H_{A,1} = H_{A,2}$ or $\bar{H}_{A,1} = \bar{H}_{A,2}$?

B. Are independent points of a natural algebra $[X, A]$ local peak points or strong boundary points of A in X?

C. If A and B are separating function algebras on a compact metrizable space X and the minimal boundary of A and that of B coincide, do the homomorphism spaces of A and B coincide?

D. Describe precisely the points of X in $\text{hull}_A(X \setminus \partial[X, A]) \setminus (X \setminus \partial[X, A])$.

E. If $B(A) \subset [X, A]$, a natural algebra, separates the points of X, what is the relation between $\delta B(A)$ and $\partial[X, A]$? Inclusion is clear.

F. Let A be a separating function algebra on a compact space X. Is $H_A(\Delta_A)$ relatively maximal in $C(\Delta_A)$?

G. In Theorem 5.4′ replace Condition (S) by a more reasonable assumption.

H. Is every separating Liouville algebra of continuous functions on $\mathbf{C}$ topologically isomorphic to $\text{Hol}(\mathbf{C}^n)$ for some n?

I. Is every n-generated Liouville algebra of continuous functions on $\mathbf{C}^n$ topologically isomorphic to $\text{Hol}(\mathbf{C}^n)$?

J. Are the local peak points of a separating uniform algebra A with metrizable continuous homomorphism space $\Delta(A)$ peak points for functions in $B(A)$ on $\Delta(A)$?

NOTES

1. Point separation on X by functions in A is usually incorporated into the definitions of uniform and function algebras.

2. The concepts introduced in this section have their general origin in the papers [*1*] and [7] of Arens and Michael, respectively, on locally multiplicatively convex topological algebras. Here they are formulated for uniform algebras only.

3. Numbered definitions and theorems all appear in the work of Rickart [*10, 11, 12*] although often with different but equivalent formulations. Definition 3.3 follows Quigley's formulation. This concept of boundary was introduced earlier by Quigley in [*8*]. Condition (3.1) is discussed by Meyers in [*5*]. The elegant proof of the naturalness of the algebra of $H_A(X)$ based on the natural algebra $[X, A]$ is due to Quigley. A slightly different version of the result with a proof based on Theorem 3.2 can be found in [*12*]. For the most extensive discussion of algebras of A-holomorphic functions the reader should consult [*10*].

4. With the exception of the general theory of ringed spaces, all concepts which are defined in this section were developed by Quigley in [*9*], as is the theory alluded to in the comments after Theorem 4.1. Additional results of this type are in the work [*16, 17*] of his student, M. Schauck. Also Theorems 4.1 and 4.2 together with detailed ramifications are established in [*9*].

5. The investigation of this section was motivated by [*2, 3*] where appear the proof of Theorem 5.5 and the comment of the Caution which follows. All the other theorems of this section are the work of my student, W. Meyers, and can be found with ramifications in

[5]. It was Meyers' work which stimulated me to prepare this expository outline and which constituted the major portion of my talk at the Rice University Conference on Complex Analysis.

6. In an attempt to understand an Arens-Calderon type theorem stated in Rosenfeld [13], Meyers conducted a comprehensive seminar [6] on spectral theory as developed by Waelbroeck [18]. Theorem 6.1 is an outgrowth of that seminar.

7. There is no claim to originality for this list of open questions. In one form or another all of them have come up in conversations among Quigley, Meyers, and myself at Tulane University.

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