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Fixed point and convergence theorems for different classes of generalized nonexpansive mappings in CAT(0) spaces

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ABSTRACT

In this paper, we prove the existence of common fixed points in CAT(0) spaces for three different classes of generalized nonexpansive mappings including a quasi-nonexpansive single valued mapping, a pointwise asymptotically nonexpansive mapping, and a multivalued mapping satisfying the conditions (E) and (C_λ) for some $\lambda \in (0, 1)$. Moreover, we introduce an iterative process for these mappings and prove $\triangle$-convergence and strong convergence theorems for such an iterative process in CAT(0) spaces.

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is called

(i) nonexpansive if $d(Tx, Ty) \le d(x, y)$ for all $x, y \in X$,
(ii) quasi-nonexpansive if the set $F(T)$ of fixed points of T is nonempty and $d(Tx, Ty) \le d(x, y)$ for all $x \in X$ and $y \in F(T)$,
(iii) pointwise asymptotically nonexpansive if there exists a sequence of functions $\alpha_n(x) \ge 1$ with $\lim_{n\to\infty} \alpha_n(x) = 1$ such that

$$d(T^n(x), T^n(y)) \le \alpha_n(x) d(x, y), \quad n \ge 1,\ x, y \in X.$$

(iv) In case when each α_n is constant, T is called asymptotically nonexpansive.

The class of pointwise asymptotically nonexpansive mappings was introduced by Kirk and Xu [1] as a generalization of the class of asymptotically nonexpansive mappings which had already been introduced by Goebel and Kirk in [2]. It is immediately clear that a nonexpansive mapping is pointwise asymptotically nonexpansive.

In [3], Garcia-Falset et al. introduced two types of generalization for nonexpansive mappings.

Definition 1.1. Let (X, d) be a metric space and $\mu \ge 1$. A mapping $T : X \to X$ is said to satisfy condition (E_μ) if

$$d(x, Ty) \le \mu\, d(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies the condition (E_μ) for some $\mu \ge 1$.

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Definition 1.2. Let (X, d) be a metric space and $\lambda \in (0, 1)$. A mapping $T : X \to X$ is said to satisfy condition (C_λ) if

$$\lambda\, d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) \le d(x, y), \quad x, y \in X.$$

Very recently, the current authors have modified these conditions to incorporate the multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying these conditions in CAT(0) spaces [4]. In this paper, we consider a CAT(0) space, and intend to prove the existence of common fixed points for three different classes of generalized nonexpansive mappings including a quasi-nonexpansive single valued mapping, a pointwise asymptotically nonexpansive mapping, and a multivalued mapping satisfying the condition (E) and (C_λ) for some $\lambda \in (0, 1)$. Moreover, we introduce an iterative process for these mappings and prove $\triangle$-convergence and strong convergence theorems for such an iterative process in CAT(0) spaces. Our result generalizes a number of recent known results; including that of Abkar and Eslamian [4], Hussain and Khamsi [5], Khan and Abbas [6], and of Dhompongsa and Panyanak [7].

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. In particular, the mapping c is an isometry and $d(x, y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $z = (1 - \alpha)x \oplus \alpha y$, where $0 \le \alpha \le 1$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of X is called convex if D includes every geodesic segment joining any two points of itself.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom.

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, $d(x, y) \le d_{R^2}(\bar{x}, \bar{y})$.

The following properties of a CAT(0) space are useful (see [8]):

(i) A CAT(0) space X is uniquely geodesic;
(ii) For any $x \in X$ and any closed convex subset $D \subset X$, there is a unique closest point to x.

Let $\{x_n\}$ be a bounded sequence in X and D be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(D, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in D\},$$

where

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x),$$

and the set

$$A = A(D, \{x_n\}) = \{x \in D : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to D. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to D. In a CAT(0) space, the asymptotic center $A = A(D, \{x_n\})$ of $\{x_n\}$ consists of exactly one point when D is closed and convex. A sequence $\{x_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of $\{x_n\}$. Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ is $\triangle$-convergent to x and given $y \in X$ with $x \ne y$,

$$\limsup_{n\to\infty} d(x, x_n) < \limsup_{n\to\infty} d(y, x_n).$$

Thus every CAT(0) space X satisfies the Opial property.

Lemma 2.1 ([9]). *Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.2 ([10]). *If D is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in D, then the asymptotic center of $\{x_n\}$ is in D.*

Lemma 2.3 ([7]). *If $\{x_n\}$ is a bounded sequence in a complete CAT(0) space X with $A(\{x_n\}) = \{x\}$, and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Theorem 2.4 ([5]). *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Suppose $f : D \to D$ is a pointwise asymptotic nonexpansive mapping. If $\{x_n\}$ is a sequence in D such that $\lim_{n\to\infty} d(fx_n, x_n) = 0$ and $\triangle - \lim_n x_n = v$. Then $v = f(v)$.*

Theorem 2.5 ([5]). *Let D be a nonempty closed convex bounded subset of a complete $CAT(0)$ space X. Let $f : D \to D$ be a pointwise asymptotic nonexpansive mapping. Then $F(f)$ is a nonempty closed and convex set.*

Lemma 2.6 ([3]). *Assume that a mapping T satisfies the condition* (E) *and has a fixed point. Then T is quasi-nonexpansive, but the converse is not true.*

Theorem 2.7 ([11]). *Let D be a nonempty bounded closed convex subset of a complete $CAT(0)$ space X. Suppose $T : D \to D$ is a quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.8 ([7]). *Let X be a $CAT(0)$ space. Then for all $x, y, z \in X$ and all $t \in [0, 1]$ we have*

(i) $d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z)$,
(ii) $d((1-t)x \oplus ty, z)^2 \le (1-t)d(x, z)^2 + td(y, z)^2 - t(1-t)d(x, y)^2$.

Let D be a subset of a CAT(0) space X. We denote by $CB(D)$, $K(D)$ and $KC(D)$ the collection of all nonempty closed bounded subsets, nonempty compact subsets, and nonempty convex compact subsets of D, respectively. The Hausdorff metric H on $CB(X)$ is defined by

$$H(A, B) := \max\left\{\sup_{x\in A} dist(x, B), \sup_{y\in B} dist(y, A)\right\},$$

for all $A, B \in CB(X)$, where $dist(x, B) = inf\{d(x, z) : z \in B\}$.

A subset $D \subset X$ is called proximal if for each $x \in X$, there exists an element $y \in D$ such that

$$d(x, y) = dist(x, D).$$

It is known that every closed convex subset of a CAT(0) space is proximal. We denote by $P(D)$ the collection of all nonempty proximal bounded subsets of D.

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T, if $x \in Tx$.

Definition 2.9. A multivalued mapping $T : X \to CB(X)$ is called

(i) nonexpansive if

$$H(Tx, Ty) \le d(x, y), \quad x, y \in X,$$

(ii) quasi-nonexpansive if $F(T) \ne \emptyset$ and $H(Tx, Tp) \le d(x, p)$ for all $x \in X$ and all $p \in F(T)$.

In the following we state the multivalued analogs of the conditions (E) and (C_λ) (see also [4]):

Definition 2.10. A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (E_μ) provided that

$$dist(x, Ty) \le \mu\, dist(x, Tx) + d(x, y), \quad x, y \in X.$$

We say that T satisfies condition (E) whenever T satisfies (E_μ) for some $\mu \ge 1$.

Definition 2.11. A multivalued mapping $T : X \to CB(X)$ is said to satisfy condition (C_λ) for some $\lambda \in (0, 1)$ provided that

$$\lambda\, dist(x, Tx) \le d(x, y) \Longrightarrow H(Tx, Ty) \le d(x, y), \quad x, y \in X.$$

Lemma 2.12 ([4]). *Let $T : X \to CB(X)$ be a multivalued nonexpansive mapping, then T satisfies the condition (E_1).*

Theorem 2.13 ([12]). *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Suppose $T : D \to K(D)$ satisfies the condition* (E). *If $\{x_n\}$ is a sequence in D such that $\lim_{n\to\infty} dist(Tx_n, x_n) = 0$ and $\triangle - \lim_n x_n = v$. Then $v \in Tv$.*

The following lemma is a consequence of Proposition 2 proved by Goebel and Kirk in [13].

Lemma 2.14. *Let $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in a $CAT(0)$ space X, and let $0 < \lambda < 1$. If for every natural number n we have $z_{n+1} = \lambda w_n \oplus (1-\lambda)z_n$ and $d(w_{n+1}, w_n) \le d(z_{n+1}, z_n)$, then $\lim_{n\to\infty} d(w_n, z_n) = 0$.*

Lemma 2.15 ([14]). *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to 0, then $\lim_{n\to\infty} a_n = 0$.

3. A common fixed point

Definition 3.1. Let D be a nonempty subset of a CAT(0) space X. Two mappings $t : D \to D$, and $T : D \to 2^D$ are said to be commuting, if $t(T(x)) \subset T(t(x))$ for all $x \in D$.

We now state the main result of this section.

Theorem 3.2. *Let D be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $f : D \to D$ be a pointwise asymptotically nonexpansive mapping, and $g : D \to D$ be a quasi-nonexpansive single valued mapping, and let $T : D \to KC(D)$ be a multivalued mapping satisfying the conditions* (E) *and* (C_λ) *for some $\lambda \in (0, 1)$. If f, g and T are pairwise commuting, then they have a common fixed point, i.e., there exists a point $z \in D$ such that $z = f(z) = g(z) \in T(z)$.*

Proof. Using Theorem 2.7, it follows that $Fix(g)$ is a nonempty closed convex subset of D. Since f and g commute, we have $f : Fix(g) \to Fix(g)$. Hence by Theorem 2.5, $Fix(f) \cap Fix(g) \neq \emptyset$. Also we have for each $x \in Fix(g)$, $T(x) \cap Fix(g) \neq \emptyset$. Indeed let $x \in Fix(g)$ and let $y \in T(x)$ be the unique closest point to x from $T(x)$, since g and T commute, we have $g(y) \in T(x)$. Further, quasi-nonexpansiveness of g implies that $d(g(y), x) \le d(y, x)$. Now by the uniqueness of y as the closest point to x, we get $g(y) = y$. Therefore $T(x) \cap Fix(g) \neq \emptyset$ for $x \in Fix(g)$. Now we show that for each $x \in Fix(g) \cap Fix(f)$,

$$T(x) \cap Fix(g) \cap Fix(f) \neq \emptyset.$$

To see this, Let $x \in Fix(g) \cap Fix(f)$ and $y \in T(x) \cap Fix(g)$, then $g(f(y)) = f(g(y)) = f(y)$, hence $f(y) \in Fix(g)$. Also by commutativity of f and T we have $f(y) \in T(x)$. Therefore $f : T(x) \cap Fix(g) \to T(x) \cap Fix(g)$. Since $T(x) \cap Fix(g)$ is a closed convex subset of D and f is a pointwise asymptotically nonexpansive mapping, by Theorem 2.5 we have $T(x) \cap Fix(g) \cap Fix(f) \neq \emptyset$ for each $x \in Fix(g) \cap Fix(f)$. Now we find an approximate fixed point sequence in $Fix(f) \cap Fix(g)$, for T. Take $x_0 \in Fix(f) \cap Fix(g)$, since $T(x_0) \cap Fix(f) \cap Fix(g) \neq \emptyset$, we can choose $y_0 \in T(x_0) \cap Fix(f) \cap Fix(g)$. Define

$$x_1 = (1 - \lambda)x_0 \oplus \lambda y_0.$$

Since $Fix(g) \cap Fix(f)$ is convex, we have $x_1 \in Fix(g) \cap Fix(f)$. Let $y_1 \in T(x_1)$ be chosen in such a way that

$$d(y_0, y_1) = dist(y_0, T(x_1)).$$

Next we show that $y_1 \in Fix(f) \cap Fix(g)$. Indeed, by quasi-nonexpansiveness of g we have $d(g(y_1), y_0) \le d(y_1, y_0)$ and hence by the uniqueness of y_1 as the unique closest point to y_0 we get $g(y_1) = y_1$. Also we see that $y_1 \in Fix(f)$. Indeed, we consider the sequence $\{f^n(y_1)\}$. Since T and f are commuting, we know that $f^n(y_1) \in T(x_1)$ for all n. Since $T(x_1)$ is compact, the sequence $\{f^n(y_1)\}$ has a convergent subsequence with $\lim_{k\to\infty} f^{n_k}(y_1) = z \in T(x_1)$, then we have

$$\begin{aligned} d(z, y_0) &= \lim_{k\to\infty} d(f^{n_k}(y_1), y_0) = \lim_{k\to\infty} d(f^{n_k}(y_1), f^{n_k}(y_0)) \\ &\le \lim_{k\to\infty} \alpha_{n_k}(y_0) d(y_1, y_0) \le dist(y_0, T(x_1)) = d(y_0, y_1). \end{aligned}$$

Now by the uniqueness of y_1 as the closest point to y_0 we have $z = y_1$, consequently $\lim_{k\to\infty} f^{n_k}(y_1) = y_1$ and so $f(y_1) = y_1$; (note that $f(y_1) \in f(T(x_1)) \subset T(f(x_1)) = T(x_1)$). Similarly, put $x_2 = (1 - \lambda)x_1 \oplus \lambda y_1$, again we choose $y_2 \in T(x_2)$ in such a way that

$$d(y_1, y_2) = dist(y_1, T(x_2)).$$

By the same argument, as stated in above we get $y_2 \in Fix(f) \cap Fix(g)$. In this way we will find a sequence $\{x_n\}$ in $Fix(f) \cap Fix(g)$ such that $x_{n+1} = (1 - \lambda)x_n \oplus \lambda y_n$ where $y_n \in T(x_n) \cap Fix(f) \cap Fix(g)$ and

$$d(y_{n-1}, y_n) = dist(y_{n-1}, T(x_n)).$$

Therefore for every natural number $n \ge 1$ we have

$$\lambda\, d(x_n, y_n) = d(x_n, x_{n+1})$$

from which it follows that

$$\lambda\, dist(x_n, T(x_n)) \le \lambda d(x_n, y_n) = d(x_n, x_{n+1}), \quad n \ge 1.$$

Since T satisfies the condition (C_λ) we have

$$H(T(x_n), T(x_{n+1})) \le d(x_n, x_{n+1}), \quad n \ge 1,$$

hence for each $n \ge 1$ we have

$$d(y_n, y_{n+1}) = dist(y_n, T(x_{n+1})) \le H(T(x_n), T(x_{n+1})) \le d(x_n, x_{n+1}).$$

We now apply Lemma 2.14 to conclude that $\lim_{n\to\infty} d(x_n, y_n) = 0$ where $y_n \in T(x_n)$. By Lemma 2.1, the bounded sequence $\{x_n\}$ in $Fix(f) \cap Fix(g)$ has a $\triangle$-*convergent* subsequence, hence by passing to a subsequence we can assume that $\triangle - \lim_n x_n = v$. We note that by Lemma 2.2, $v \in Fix(f) \cap Fix(g)$. For each $n \geq 1$, we choose $z_n \in T(v)$ such that

$$d(x_n, z_n) = dist(x_n, T(v)).$$

By the same argument, we obtain that $z_n \in Fix(f) \cap Fix(g)$ for all natural numbers $n \geq 1$. Since $T(v)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = w \in T(v)$. Because $z_{n_k} \in Fix(f) \cap Fix(g)$ for all n, and $Fix(f) \cap Fix(g)$ is closed, we obtain that $w \in Fix(f) \cap Fix(g)$. By the condition (E), we have for some $\mu \geq 1$,

$$dist(x_{n_k}, T(v)) \leq \mu\, dist(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, v).$$

Note also that

$$\begin{aligned} d(x_{n_k}, w) &\leq d(x_{n_k}, z_{n_k}) + d(z_{n_k}, w) \\ &\leq \mu\, dist(x_{n_k}, T(x_{n_k})) + d(x_{n_k}, v) + d(z_{n_k}, w). \end{aligned}$$

These entail

$$\limsup_{k\to\infty} d(x_{n_k}, w) \leq \limsup_{k\to\infty} d(x_{n_k}, v).$$

It now follows from the Opial property of CAT(0) space X that $v = w \in T(v)$. Consequently, $v = fv = gv \in T(v)$. □

As a result we obtain the following theorem which improves and generalizes a result of Hussain and Khamsi [5].

Theorem 3.3. *Let D be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $f : D \to D$ be a pointwise asymptotically nonexpansive mapping, and let $T : D \to KC(D)$ be a multivalued mapping satisfying the conditions* (E) *and* (C_λ) *for some $\lambda \in (0, 1)$. If f and T commute, then they have a common fixed point, i.e., there exists a point $z \in D$ such that $z = f(z) \in T(z)$.*

Theorem 3.4. *Let D be a nonempty closed convex bounded subset of a complete CAT(0) space X. Let $g : D \to D$ be a quasi-nonexpansive single valued mapping, and let $T : D \to KC(D)$ be a multivalued mapping satisfying the conditions* (E) *and* (C_λ) *for some $\lambda \in (0, 1)$. If g and T commute, then they have a common fixed point, i.e., there exists a point $z \in D$ such that $z = g(z) \in T(z)$.*

4. A convergence theorem

In this section we introduce the following iteration process.

(A): Let X be a CAT(0) space, D be a nonempty convex subset of X and $T : D \to CB(D)$ and $f, g : D \to D$ be three given mappings. Then, for $x_1 \in D$ define three sequences $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ by

$$\begin{aligned} w_n &= (1 - c_n)x_n \oplus c_n g x_n, \quad n \geq 1, \\ y_n &= (1 - b_n)x_n \oplus b_n z_n, \quad n \geq 1, \\ x_{n+1} &= (1 - a_n)x_n \oplus a_n f^n y_n, \quad n \geq 1, \end{aligned}$$

where $z_n \in T(w_n)$, and $a_n, b_n, c_n \in [0, 1]$. In the sequel, $\mathcal{F} = F(T) \bigcap F(g) \bigcap F(f)$ is the set of all common fixed points of the mappings f, g and T.

Definition 4.1. A mapping $T : D \to CB(D)$ is said to satisfy condition (I) if there is a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0, \infty)$ such that

$$dist(x, Tx) \geq \varphi(dist(x, F(T))).$$

Let $T : D \to CB(D)$ and $f, g : D \to D$ be three given mappings. The mappings T, f, g are said to satisfy condition (II) if there exists a nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$, $\varphi(r) > 0$ for $r \in (0, \infty)$, such that

$$\max\{dist(x, Tx), d(x, fx), d(x, gx)\} \geq \varphi(dist(x, \mathcal{F})).$$

Theorem 4.2. *Let D be a nonempty closed convex subset of a complete CAT(0) space X. Let $T : D \to CB(D)$ be a quasi-nonexpansive multivalued mapping satisfying the condition* (E), *$g : D \to D$ be a quasi-nonexpansive single valued mapping and $f : D \to D$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $\mathcal{F} \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the sequence defined by the iterative process* (A), *and $a_n, b_n, c_n \in [a, b] \subset (0, 1)$ for all $n \geq 1$. If the mappings f, g and T satisfy the condition* (II), *then $\{x_n\}$ converges strongly to a common fixed point of f, g and T.*

Proof. Let $p \in \mathcal{F}$. Then, using (A) and Lemma 2.8 we have

$$\begin{aligned} d(w_n, p)^2 &= d((1-c_n)x_n \oplus c_n g x_n, p)^2 \\ &\le (1-c_n)d(x_n, p)^2 + c_n d(gx_n, p)^2 - c_n(1-c_n)d(x_n, gx_n)^2 \\ &\le (1-c_n)d(x_n, p)^2 + c_n d(x_n, p)^2 - c_n(1-c_n)d(x_n, gx_n)^2 \\ &= d(x_n, p)^2 - c_n(1-c_n)d(x_n, gx_n)^2 \end{aligned}$$

and

$$\begin{aligned} d(y_n, p)^2 &= d((1-b_n)x_n \oplus b_n z_n, p)^2 \\ &\le (1-b_n)d(x_n, p)^2 + b_n d(z_n, p)^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &= (1-b_n)d(x_n, p)^2 + b_n dist(z_n, T(p))^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &\le (1-b_n)d(x_n, p)^2 + b_n H(T(w_n), T(p))^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &\le (1-b_n)d(x_n, p)^2 + b_n d(w_n, p)^2 - b_n(1-b_n)d(x_n, z_n)^2 \\ &\le d(x_n, p)^2 - b_n(1-b_n)d(x_n, z_n)^2 - b_n c_n(1-c_n)d(x_n, gx_n)^2. \end{aligned}$$

It follows that

$$\begin{aligned} d(x_{n+1}, p)^2 &= d((1-a_n)x_n \oplus a_n f^n y_n, p)^2 \\ &\le (1-a_n)d(x_n, p)^2 + a_n d(f^n y_n, p)^2 - a_n(1-a_n)d(x_n, f^n y_n)^2 \\ &\le (1-a_n)d(x_n, p)^2 + a_n k_n^2 d(y_n, p)^2 - a_n(1-a_n)d(x_n, f^n y_n)^2 \\ &\le (1-a_n)d(x_n, p)^2 + a_n k_n^2 d(x_n, p)^2 - a_n k_n^2 b_n(1-b_n)d(x_n, z_n)^2 \\ &\quad - a_n k_n^2 b_n c_n(1-c_n)d(x_n, gx_n)^2 - a_n(1-a_n)d(x_n, f^n y_n)^2 \\ &\le (1+(k_n^2-1))d(x_n, p)^2 - a_n(1-a_n)d(x_n, f^n y_n)^2 \\ &\quad - a_n k_n^2 b_n(1-b_n)d(x_n, z_n)^2 - a_n k_n^2 b_n c_n(1-c_n)d(x_n, gx_n)^2. \end{aligned}$$

So that we obtain

$$d(x_{n+1}, p)^2 \le (1+(k_n^2-1))d(x_n, p)^2.$$

Therefore by Lemma 2.15 $\lim_{n\to\infty} d(x_n, p)$ exists. Now we set

$$M = \sup\{d(x_n, p) : n \ge 1\}.$$

We also have

$$\begin{aligned} a^3(1-b)d(x_n, gx_n)^2 &\le a_n k_n^2 b_n c_n(1-c_n)d(x_n, gx_n)^2 \\ &\le d(x_n, p)^2 - d(x_{n+1}, p)^2 + M(k_n^2-1). \end{aligned}$$

The assumption $\sum_{n=1}^{\infty}(k_n-1) < \infty$ implies that $\sum_{n=1}^{\infty}(k_n^2-1) < \infty$ (note that $\{k_n\}$ converges to 1, hence $\{1+k_n\}$ is bounded). Therefore we get

$$\sum_{n=1}^{\infty} a^3(1-b)d(x_n, gx_n)^2 \le d(x_1, p)^2 + M(k_n^2-1) < \infty.$$

Thus $\lim_{n\to\infty} d(x_n, gx_n)^2 = 0$, and hence $\lim_{n\to\infty} d(x_n, gx_n) = 0$. By a similar argument we obtain

$$\lim_{n\to\infty} d(x_n, z_n) = \lim_{n\to\infty} d(f^n y_n, x_n) = 0.$$

Hence we obtain $dist(x_n, Tw_n) \le d(x_n, z_n) \to 0$ as $n \to \infty$. Also we have

$$\lim_{n\to\infty} d(x_n, w_n) = \lim_{n\to\infty} c_n d(gx_n, x_n) = 0$$

and

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} a_n d(f^n y_n, x_n) = 0.$$

By the condition (E), we have for some $\mu \ge 1$,

$$\begin{aligned} dist(x_n, Tx_n) &\le d(x_n, w_n) + dist(w_n, Tx_n) \\ &\le d(x_n, w_n) + \mu\, dist(Tw_n, w_n) + d(x_n, w_n) \\ &\le d(x_n, w_n) + \mu\, dist(x_n, Tw_n) + \mu\, d(x_n, w_n) + d(x_n, w_n) \\ &= (\mu+2)d(x_n, w_n) + \mu\, dist(x_n, Tw_n). \end{aligned}$$

Hence $dist(x_n, Tx_n) \to 0$ as $n \to \infty$. We also have

$$\begin{aligned} d(x_n, f^n x_n) &\le d(x_n, f^n y_n) + d(f^n y_n, f^n x_n) \\ &\le d(x_n, f^n y_n) + k_n d(y_n, x_n) \\ &\le d(x_n, f^n y_n) + k_n b_n d(x_n, z_n) \to 0, \quad n \to \infty. \end{aligned}$$

Hence

$$\begin{aligned} d(x_n, fx_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, f^{n+1}x_{n+1}) + d(f^{n+1}x_{n+1}, f^{n+1}x_n) + d(f^{n+1}x_n, fx_n) \\ &\le d(x_{n+1}, x_n) + d(x_{n+1}, f^{n+1}x_{n+1}) + k_{n+1} d(x_{n+1}, x_n) + k_1 d(f^n x_n, x_n) \end{aligned}$$

which implies that

$$\lim_{n\to\infty} d(fx_n, x_n) = 0.$$

Note that by our assumption $\lim_{n\to\infty} dist(x_n, \mathcal{F}) = 0$. Hence there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and a sequence $\{p_k\}$ in $\mathcal{F}$ such that $d(x_{n_k}, p_k) < \frac{1}{2^k}$ for all k. By using Lemma 2.8 and a similar argument as above we obtain

$$d(x_{n+1}, p) \le d(x_n, p) + \theta_n,$$

where $\theta_n = (k_n - 1)d(x_n, p)$. (Note that $\sum_{n=1}^{\infty} \theta_n < \infty$.) Therefore for each $p \in \mathcal{F}$ we get

$$\begin{aligned} d(x_{n_{k+1}}, p) &\le d(x_{n_{k+1}-1}, p) + \theta_{n_{k+1}-1} \\ &\le d(x_{n_{k+1}-2}, p) + \theta_{n_{k+1}-2} + \theta_{n_{k+1}-1} \\ &\le \cdots \\ &\le d(x_{n_k}, p) + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}. \end{aligned}$$

This implies that

$$d(x_{n_{k+1}}, p) \le d(x_{n_k}, p_k) + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i} \le \frac{1}{2^k} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}.$$

Now, we show that $\{p_k\}$ is a Cauchy sequence in D. Note that

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i} < \frac{1}{2^{k-1}} + \sum_{i=1}^{n_{k+1}-n_k-1} \theta_{n_k+i}. \end{aligned}$$

This implies that $\{p_k\}$ is a Cauchy sequence, hence converges to $q \in D$. Since

$$dist(p_k, T(q)) \le H(T(p_k), T(q)) \le d(p_k, q)$$

and $p_k \to q$ as $n \to \infty$, it follows that $dist(q, T(q)) = 0$, so that $q \in F(T)$. We also have

$$d(p_k, g(q)) \le d(p_k, q),$$

hence $d(q, g(q)) = 0$ and thus $q \in F(g)$. Also, by the continuity of f we have

$$d(p_k, f(p_k)) \to d(f(q), q), \quad k \to \infty.$$

Hence $d(f(q), q) = 0$ which implies that $q \in fq$. Therefore $q \in \mathcal{F}$ and $\{x_{n_k}\}$ converges strongly to q. Since $\lim_{n\to\infty} d(x_n, q)$ exists, we conclude that $\{x_n\}$ converges strongly to q. This completes the proof. □

Theorem 4.3. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $T : D \to K(D)$ be a quasi-nonexpansive multivalued mapping satisfying the condition (E), $g : E \to E$ be a single valued mapping satisfying the condition (E) and $f : D \to D$ be an asymptotically nonexpansive mapping with the sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Assume that $\mathcal{F} \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in \mathcal{F}$. Let $\{x_n\}$ be the sequence defined by the iterative process (A), and $a_n, b_n, c_n \in [a, b] \subset (0, 1)$ for all $n \ge 1$. Then $\{x_n\}$ is $\triangle$-convergent to a common fixed point of f, g and T.*

Proof. As in the proof of Theorem 4.2, we have

$$\lim_{n\to\infty} dist(Tx_n, x_n) = \lim_{n\to\infty} d(fx_n, x_n) = \lim_{n\to\infty} d(gx_n, x_n) = 0.$$

Now we let $W_w(x_n) := \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $W_w(x_n) \subset \mathcal{F}$. Let $u \in W_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemmas 2.1 and 2.2 there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\triangle - \lim_n v_n = v \in D$. Since

$$\lim_{n\to\infty} dist(Tv_n, v_n) = \lim_{n\to\infty} d(fv_n, v_n) = \lim_{n\to\infty} d(gv_n, v_n) = 0,$$

it follows from Theorems 2.13 and 2.4 that $v \in \mathcal{F}$, moreover the limit $\lim_{n\to\infty} d(x_n, v)$ exists by Theorem 4.2. Hence $u = v \in \mathcal{F}$ by Lemma 2.3. This shows that $W_w(x_n) \subset \mathcal{F}$. Next we show that $W_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(\{u_n\}) = \{u\}$ and let $A(\{x_n\}) = \{x\}$. Since $u \in W_w(x_n) \subset \mathcal{F}$ and $d(x_n, v)$ converges, by Lemma 2.3 we have $x = u$. □

We now intend to remove the restriction that $T(p) = \{p\}$ for each $p \in \mathcal{F}$. We define the following iteration process.
(B): Let $T : D \to P(D)$ be a multivalued mapping and

$$P_T(x) = \{y \in Tx : \|x - y\| = dist(x, Tx)\}$$

and $f, g : D \to D$. Then, for $x_1 \in D$, we define sequences $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ by

$$\begin{aligned} w_n &= (1 - c_n)x_n \oplus c_n g x_n, \quad n \geq 1, \\ y_n &= (1 - b_n)x_n \oplus b_n z_n, \quad n \geq 1, \\ x_{n+1} &= (1 - a_n)x_n \oplus a_n f^n y_n, \quad n \geq 1, \end{aligned}$$

where $z_n \in P_T(w_n)$, and $a_n, b_n, c_n \in [0, 1]$.

Theorem 4.4. *Let D be a nonempty closed convex subset of a complete $CAT(0)$ space X. Let $T : D \to CB(D)$ be a multivalued mapping such that P_T is a quasi-nonexpansive mapping satisfying the condition* (E)*, $g : D \to D$ be a quasi-nonexpansive single valued mapping and $f : D \to D$ be an asymptotically nonexpansive mapping with sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{x_n\}$ be the sequence defined by the iterative process* (B)*, and $a_n, b_n, c_n \in [a, b] \subset (0, 1)$ for all $n \geq 1$. If $\mathcal{F} \neq \emptyset$ and the mappings f, g and T satisfy the condition* (II)*, then $\{x_n\}$ converges strongly to a common fixed point of f, g and T.*

Proof. Let $p \in \mathcal{F}$. Then $P_T(p) = \{p\}$. Also we have

$$d(z_n, p)^2 \leq dist(z_n, P_T(p))^2 \leq H(P_T(w_n), P_T(p))^2.$$

By a similar argument as in the proof of Theorem 4.2 we obtain the desired result. □

References

[1] W.A. Kirk, H.K. Xu, Asymptotic pointwise contractions, Nonlinear Anal. 69 (2008) 4706–4712.
[2] K. Goebel, W.A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Amer. Math. Soc. 35 (1972) 171–174.
[3] J. Garcia-Falset, E. Llorens-Fuster, T. Suzuki, Fixed point theory for a class of generalized nonexpansive mappings, J. Math. Anal. Appl. 375 (2011) 185–195.
[4] A. Abkar, M. Eslamian, Common fixed point results in CAT(0) spaces, Nonlinear Anal. 74 (2011) 1835–1840.
[5] N. Hussain, M.A. Khamsi, On asymptotic pointwise contractions in metric spaces, Nonlinear Anal. 71 (2009) 4423–4429.
[6] S.H. Khan, M. Abbas, Strong and △-convergence of some iterative scheme in CAT(0) space, Comput. Math. Appl. 61 (2011) 109–116.
[7] S. Dhompongsa, B. Panyanak, On △-convergence theorems in CAT(0) space, Comput. Math. Appl. 56 (2008) 2572–2579.
[8] M. Bridson, A. Haefliger, Metric Spaces of Nonpositive Curvature, Springer-Verlag, Berlin, 1999.
[9] W.A. Kirk, B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal. 68 (2008) 3689–3696.
[10] S. Dhompongsa, W.A. Kirk, B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal. 8 (2007) 35–45.
[11] P. Chaoha, A. Phon-on, A note on fixed point sets in CAT(0) spaces, J. Math. Anal. Appl. 320 (2006) 983–987.
[12] A. Abkar, M. Eslamian, Strong and △-convergence theorems of some iterative process for multivalued mappings in CAT(0) spaces, J. Nonlinear Anal. Optim. 2 (2011) 67–74.
[13] K. Goebel, W.A. Kirk, Iteration processes for nonexpansive mappings, Contemp. Math. 21 (1983) 115–123.
[14] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301–308.