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Central Intertwining Lifting, Suboptimization, and Interpolation in Several Variables

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suboptimization type results for the noncommutative (resp. commutative) analytic Toeplitz algebra F_n^∞ (resp. W_n^∞). The algebra F_n^∞ (resp. W_n^∞) can be viewed as a multivariable noncommutative (resp. commutative) analogue of the Hardy space H^∞. Similar results are provided for $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and $W_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, where $B(\mathcal{K}, \mathcal{K}')$ is the set of all bounded linear operators acting on Hilbert spaces. New extensions of the Sarason, Carathéodory, and Nevanlinna–Pick type interpolation results are obtained for $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and some consequences to the operator-valued analytic interpolation in the unit ball of $\mathbb{C}^n$ are considered.

In studying subalgebras of C^*-algebras, Kaftal *et al.* [KLW] discovered a joint norm control Nehari type theorem. Stated for H^∞, their theorem says that if $\delta > 1$, $f \in H^\infty$, and $\varphi \in H^\infty$ is an inner function, then there exists $h \in H^\infty$ such that

$$\|f - \varphi h\|_\infty \leqslant \delta d_\infty(f, \varphi H^\infty)$$

and

$$\|f - \varphi h\|_2 \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} d_2(f, \varphi H^\infty).$$

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This gives quantitative estimates on the trade-off between the infinity-norm and the 2-norm approximation of $f \in H^\infty$ with elements in the w^*-closed ideal φH^∞ of H^∞. This result was generalized by Foiaş and Frazho in [FFr2]. They obtained simultaneously a suboptimal solution to the operatorial two-sided Nehari optimization problem, with respect to the L^∞ norm and the L^2 norm. The $H^\infty - H^2$ optimization has played an important role in H^∞ control theory (see [FFrGK] and its references). The goal of this paper is to provide multivariable versions of the above-mentioned results. We mention that noncommutative generalizations of the Kaftal–Larson–Weiss result were considered by Pisier in [Pi1] and [Pi2].

The noncommutative analytic Toeplitz algebra F_n^∞ is the WOT-closed algebra generated by the left creation operators $S_1, \ldots, S_n$ on the full Fock space $F^2(H_n)$ on n generators and the identity (see Section 1). The algebra F_n^∞ and its norm-closed version (the noncommutative disk algebra $\mathscr{A}_n$) were introduced by the author in [Po4] in connection with a noncommutative von Neumann inequality and have been studied in several papers [Po2, Po6, Po7, Po8, Po9, ArPo1] and recently in [DP1, DP2, ArPo2, DP3, Po10, Kr, DKP, PPoS].

We established a strong connection between the algebra F_n^∞ and the function theory on the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ through the noncommutative von Neumann inequality [Po4] (see also [Po6, Po8, Po9]). In particular, we proved that there is a completely contractive homomorphism

$$\Phi\colon F_n^\infty \to H^\infty(\mathbb{B}_n), \qquad [\Phi(f)](\lambda_1, \ldots, \lambda_n) = f(\lambda_1, \ldots, \lambda_n),$$

for any $f := f(S_1, \ldots, S_n) \in F_n^\infty$ and $(\lambda_1, \ldots, \lambda_n) \in \mathbb{B}_n$. A characterization of the analytic functions in the range of the map Φ was obtained in [ArPo2], and independently in [DP3]. Moreover, it was proved that the quotient $F_n^\infty/\ker \Phi$ is an operator algebra which can be identified with $W_n^\infty := P_{F_s^2(H_n)} F_n^\infty |_{F_s^2(H_n)}$, the compression of F_n^∞ to the symmetric Fock space $F_s^2(H_n) \subset F^2(H_n)$. In [Po9, Arv, ArPo2, DP3, PPoS], a good case is made that the appropriate multivariable commutative analogue of H^∞ is the algebra W_n^∞, which was also proved to be the w^*-closed algebra generated by $B_i := P_{F_s^2(H_n)} S_i |_{F_s^2(H_n)}$, $i = 1, \ldots, n$, and the identity. Moreover, Arveson showed in [Arv] that W_n^∞ can be seen as the algebra of all analytic multipliers on $F_s^2(H_n)$.

Interpolation problems for the noncommutative analytic Toeplitz algebra F_n^∞ were first considered in [Po7], where we obtained the Carathéodory interpolation theorem in this setting. In [ArPo2], Arias and the author extended Sarason's result [S] and obtained a distance formula to an arbitrary WOT-closed ideal in F_n^∞ as well as a Nevanlinna–Pick type interpolation theorem (see [N]) for the noncommutative analytic Toeplitz algebra F_n^∞.

Using different methods, Davidson and Pitts proved these results independently in [DP3]. Let us mention that interpolation problems on the ball $\mathbb{B}_n$ were recently considered in [Po10, Po11, Po12, AMc1, AMc2, BV].

In Section 1, we consider some preliminary results concerning the structure of multianalytic operators on Fock spaces as well as $W_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, the WOT-closed operator space generated by the spatial tensor product. In Section 2, we extend a result of Foiaş and Frazho [FFr2] and obtain a multivariable central intertwining lifting theorem (see Theorem 2.1), which will play a crucial role in this paper.

The main result of this paper is the Kaftal–Larson–Weiss type theorem [KLW], obtained in Section 3, for the noncommutative analytic Toeplitz algebra F_n^∞. More precisely, let $f \in F_n^\infty$ and let J be a WOT-closed right ideal of F_n^∞. Define

$$d_\infty(f, J) := \inf\{\|f - g\|_\infty : g \in J\}$$

and

$$d_2(f, J) := \inf\{\|f - g\|_2 : g \in J\}.$$

We show that if $\delta > 1$, then there exists $\varphi \in J$ satisfying

$$\|f - \varphi\|_\infty \leqslant \delta d_\infty(f, J) \qquad \text{and} \qquad \|f - \varphi\|_2 \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} d_2(f, J).$$

Actually, we obtain a more general result for the tensor product $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ (see Theorem 3.5). This leads to new extensions of the Sarason, Carathéodory, and Nevanlinna–Pick type interpolation theorems for $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and some consequences to the operator-valued analytic interpolation in the unit ball of $\mathbb{C}^n$. The multivariable central intertwining lifting theorem is also used to obtain a Foiaş–Frazho suboptimization type result [FFr2] in our setting (see Theorem 3.7). Finally, in Section 4, we provide a multivariable commutative version of the Kaftal–Larson–Weiss theorem.

We expect the $F_n^\infty - F^2(H_n)$ (resp. $W_n^\infty - F_s^2(H_n)$) optimization to play a similar role in multivariable control theory as the $H^\infty - H^2$ optimization has played in H^∞ control theory. The central intertwining lifting for row contractions will be used in a sequel to this paper to establish a maximal principle for the noncommutative commutant lifting theorem. This principle will show that the central intertwining lifting is a maximal entropy solution for the noncommutative commutant lifting theorem. This will lead to a permanence principle (as in the case $n = 1$, [FFrG]) with applications to the Charathéodory and Nevanlinna–Pick interpolation problems in several variables.

1. MULTIANALYTIC OPERATORS ON FOCK SPACES

Let us consider the full Fock space on n generators

$$F^2(H_n) := \mathbb{C}1 \oplus \bigoplus_{m \geqslant 1} H_n^{\otimes m},$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, ..., e_n\}$ if n is finite and $\{e_1, e_2, ...\}$ if $n = \infty$. For each $i = 1, 2, ...,$ let S_i be the left creation operator with e_i, i.e., $S_i \xi := e_i \otimes \xi$, $\xi \in F^2(H_n)$. We shall denote by $\mathscr{P}$ the set of all $p \in F^2(H_n)$ which are sums of a finite number of tensor monomials; i.e., $p = a_0 + \sum a_{i_1 \cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}$, where $a_0, a_{i_1 \cdots i_k} \in \mathbb{C}$. The set $\mathscr{P}$ may be viewed as the algebra of polynomials in n noncommuting indeterminates, with $p \otimes q$, $p, q \in \mathscr{P}$, as multiplication. Define F_n^∞ as the set of all $g \in F^2(H_n)$ such that

$$\|g\|_\infty := \sup\{\|g \otimes p\|_2 : p \in \mathscr{P}, \|p\|_2 \leqslant 1\} < \infty,$$

where $\|\cdot\|_2 := \|\cdot\|_{F^2(H_n)}$. We denote by $\mathscr{A}_n$ the closure of $\mathscr{P}$ in $(F_n^\infty, \|\cdot\|_\infty)$. The Banach algebra F_n^∞ (resp. $\mathscr{A}_n$) can be viewed as a noncommutative analogue of the Hardy space H^∞ (resp. disk algebra); when $n = 1$ they coincide. It follows from [Po6] that the noncommutative analytic Toeplitz algebra F_n^∞ can be identified with the WOT-closed algebra generated by the left creation operators $S_1, ..., S_n$ and the identity.

Let $\mathbb{F}_n^+$ be the unital free semigroup on n generators $g_1, ..., g_n$, and let e be its neutral element. For any word $\sigma := g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+$, we define its length $|\sigma| := k$, and $|e| = 0$. If $T_i \in B(\mathscr{H})$, $i = 1, ..., n$, we set $T_\sigma := T_{i_1} \cdots T_{i_k}$ and $T_e := I_{\mathscr{H}}$. Similarly, we denote $e_\sigma := e_{i_1} \otimes \cdots \otimes e_{i_k}$ and $e_\sigma = 1$ if $\sigma = e$. Notice that $\{e_\sigma\}_{\sigma \in \mathbb{F}_n^+}$ is the canonical basis of $F^2(H_n)$.

Let $\mathscr{K}, \mathscr{K}'$ be Hilbert spaces. As in [Po2], we say that a bounded linear operator $A \in B(F^2(H_n) \otimes \mathscr{K}, F^2(H_n) \otimes \mathscr{K}')$ is multianalytic if $A(S_i \otimes I_{\mathscr{K}}) = (S_i \otimes I_{\mathscr{K}'}) A$ for any $i = 1, ..., n$. Notice that A is uniquely determined by the operator $\theta: \mathscr{K} \to F^2(H_n) \otimes \mathscr{K}'$, $\theta k := A(1 \otimes k)$, $k \in \mathscr{K}$, which is called the symbol of A, and we denote $A = A_\theta$. Moreover, A is uniquely determined by the coefficients of θ, i.e., the operators $A_{(\alpha)} \in B(\mathscr{K}, \mathscr{K}')$ given by

$$\langle A_{(\alpha)} k, k' \rangle := \langle \theta(k), e_\alpha \otimes k' \rangle, \qquad k \in \mathscr{K}, k' \in \mathscr{K}', \alpha \in \mathbb{F}_n^+.$$

Notice that $\sum_{\alpha \in \mathbb{F}_n^+} A_{(\alpha)}^* A_{(\alpha)} \leqslant \|A\| \, I_{\mathscr{K}}$. We can associate with A a unique formal Fourier expansion

$$A \sim \sum_{\alpha \in \mathbb{F}_n^+} U^* S_\alpha U \otimes A_{(\alpha)}, \tag{1.1}$$

where U is the (flipping) unitary operator on $F^2(H_n)$ mapping $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ into $e_{i_k} \otimes \cdots \otimes e_{i_2} \otimes e_{i_1}$. Since A acts like its Fourier representation on "polynomials," we will identify them for simplicity. As in [Po6], using the noncommutative von Neumann inequality, one can show that if $0 < r < 1$, then

$$A = \text{SOT} - \lim_{r \to 1} \sum_{\alpha \in \mathbb{F}_n^+} r^{|\alpha|} U^* S_\alpha U \otimes A_{(\alpha)},$$

where, for each $r \in (0, 1)$, the series converges in the uniform norm. The multianalytic operator A_θ (resp. its symbol θ) is called inner if A_θ is an isometry. According to [Po10], when $\mathcal{K} = \mathcal{K}'$ the algebra of all multianalytic operators acting on $F^2(H_n) \otimes \mathcal{K}$ coincides with $R_n^\infty \bar{\otimes} B(\mathcal{K})$, the WOT-closed algebra generated by the spatial tensor product of $R_n^\infty := U^* F_n^\infty U$ and $B(\mathcal{K})$. A similar result holds in our more general setting. Since the proof is similar to that of Theorem 1.3 from [Po12], we shall omit it. In the following, we use the notation $R_i := U^* S_i U$ for the right creation operator with e_i.

THEOREM 1.1. *The set of multianalytic operators in $B(F^2(H_n) \otimes \mathcal{K}, F^2(H_n) \otimes \mathcal{K}')$ coincides with $R_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and is equal to the WOT-closed operator space generated by $R_\alpha \otimes Z$, $\alpha \in \mathbb{F}_n^+$, $Z \in B(\mathcal{K}, \mathcal{K}')$.*

Let J be a w^*-closed, two-sided ideal of F_n^∞ and define $\mathcal{N}_J := J(1)^\perp$, the orthogonal complement of the image of J in $F^2(H_n)$. Let $\mathcal{W}(B_1, \ldots, B_n)$ be the w^*-closure of the algebra generated by $B_i := P_{\mathcal{N}_J} S_i|_{\mathcal{N}_J}$ for $i = 1, \ldots, n$, and the identity. Since $\mathcal{W}(B_1, \ldots, B_n)$ has the $\mathbb{A}_1$ property (see [ArPo2]) the w^* and WOT topologies agree on this algebra. Similarly to Proposition 4.2 from [ArPo2], one can prove the following result. The proof is based on the noncommutative commutant lifting theorem (see [Po1]), Theorem 1.1, and the observation that the operator space $P_{\mathcal{N}_J \otimes \mathcal{K}'} [F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')]|_{\mathcal{N}_J \otimes \mathcal{K}}$ coincides with the set of all the operators intertwining $P_{\mathcal{N}_J} U^* S_i U|_{\mathcal{N}_J} \otimes I_{\mathcal{K}}$ and $P_{\mathcal{N}_J} U^* S_i U|_{\mathcal{N}_J} \otimes I_{\mathcal{K}'}$, $i = 1, \ldots, n$. We shall omit it.

THEOREM 1.2. *Let J be a WOT-closed, two-sided ideal of F_n^∞. Then*

$$\mathcal{W}(B_1, \ldots, B_n) \bar{\otimes} B(\mathcal{K}, \mathcal{K}') = P_{\mathcal{N}_J \otimes \mathcal{K}'} [F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')]|_{\mathcal{N}_J \otimes \mathcal{K}}.$$

Moreover, $\mathcal{W}(B_1, \ldots, B_n) \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ is the WOT-closed operator space generated by $B_\alpha \otimes W$, $\alpha \in \mathbb{F}_n^+$, $W \in B(\mathcal{K}, \mathcal{K}')$.

Let us mention that multianalytic operators were considered in [Po2] in connection with the characteristic function associated to a row contraction and were studied in [Po2, Po3, Po5, Po7].

2. CENTRAL INTERTWINING LIFTING FOR ROW CONTRACTIONS

Let us recall from [Po1, Po2, Po5] a few results concerning the noncommutative dilation theory for n-tuples of operators (see [SzNF] for the classical case). A sequence of operators $\mathcal{T} := [T_1, \ldots, T_n]$, $T_i \in B(\mathcal{H})$, $i = 1, \ldots, n$, is called contractive (or row contractive) if $T_1 T_1^* + \cdots + T_n T_n^* \leqslant I_{\mathcal{H}}$. We say that a sequence of isometries $\mathcal{V} := [V_1, \ldots, V_n]$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$ is a minimal isometric dilation (m.i.d.) of $\mathcal{T}$ if the following properties are satisfied:

(i) $V_1 V_1^* + \cdots + V_n V_n^* \leqslant I_{\mathcal{K}}$;

(ii) $V_i^*|_{\mathcal{H}} = T_i^*$, $i = 1, \ldots, n$;

(iii) $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} V_\alpha \mathcal{H}$.

If $\mathcal{V}$ satisfies only the condition (i) and $P_{\mathcal{H}} V_i = T_i P_{\mathcal{H}}$, $i = 1, \ldots, n$, then $\mathcal{V}$ is called an isometric lifting of $\mathcal{T}$. The minimal isometric dilation of $\mathcal{T}$ is an isometric lifting and is uniquely determined up to an isomorphism [Po1]. Let us consider a realization of it on Fock spaces. As in [Po1], let us define $D_{\mathcal{T}}: \bigoplus_{j=1}^n \mathcal{H} \to \bigoplus_{j=1}^n \mathcal{H}$ by $D_{\mathcal{T}} := (I_{\bigoplus_{j=1}^n \mathcal{H}} - \mathcal{T}^* \mathcal{T})^{1/2}$, and set $\mathcal{D} := \overline{D_{\mathcal{T}}(\bigoplus_{j=1}^n \mathcal{H})}$. Let $D_i: \mathcal{H} \to F^2(H_n) \otimes \mathcal{D}$ be defined by

$$D_i h := 1 \otimes D_{\mathcal{T}}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, h, 0, \ldots, 0) \oplus 0 \oplus 0 \cdots.$$

Consider the Hilbert space $\mathcal{K} := \mathcal{H} \oplus [F^2(H_n) \otimes \mathcal{D}]$ and define $V_i: \mathcal{K} \to \mathcal{K}$ by

$$V_i(h \oplus (\xi \otimes d)) := T_i h \oplus [D_i h + (S_i \otimes I_{\mathcal{D}})(\xi \otimes d)] \tag{2.1}$$

for any $h \in \mathcal{H}$, $\xi \in F^2(H_n)$, $d \in \mathcal{D}$. Notice that

$$V_i = \begin{bmatrix} T_i & 0 \\ D_i & S_i \otimes I_{\mathcal{D}} \end{bmatrix}$$

with respect to the decomposition $\mathcal{K} = \mathcal{H} \oplus [F^2(H_n) \otimes \mathcal{D}]$. It was proved in [Po1] that $\mathcal{V} := [V_1, \ldots, V_n]$ is the minimal isometric dilation of $\mathcal{T}$.

Let $T_i \in B(\mathcal{H})$, $T'_i \in B(\mathcal{H}')$, $i = 1, \ldots, n$, be operators such that $\mathcal{T} := [T_1, \ldots, T_n]$ and $\mathcal{T}' = [T'_1, \ldots, T'_n]$ are row contractions. Let $\mathcal{V} := [V_1, \ldots, V_n]$ be the minimal isometric dilation of $\mathcal{T}$ on a Hilbert space $\mathcal{K} \supseteq \mathcal{H}$, and $\mathcal{V}' := [V'_1, \ldots, V'_n]$ be the minimal isometric dilation of $\mathcal{T}'$ on a Hilbert space $\mathcal{K}' \supseteq \mathcal{H}'$. Let $A \in B(\mathcal{H}, \mathcal{H}')$ be a contraction ($\|A\| \leqslant 1$) satisfying $AT_i = T'_i A$, $i = 1, \ldots, n$. A contractive intertwining lifting of A is a contraction $B \in B(\mathcal{K}, \mathcal{K}')$ satisfying $BV_i = V'_i B$, $i = 1, \ldots, n$, and $P_{\mathcal{H}'} B = A P_{\mathcal{H}}$.

The noncommutative commutant lifting theorem [Po1] (see also [Po5]) states that there always exists a contractive intertwining lifting B of A with $\|B\| = \|A\|$ (see [SzNF] for the classical case).

Following the classical result (see [FFr1]), we proved in [Po5] that all the contractive intertwining dilations of A may be canonically parametrized by the closed unit ball of all multianalytic operators R from $F^2(H_n) \otimes \mathcal{R}_1$ to $F^2(H_n) \otimes \mathcal{R}_2$ for some suitable Hilbert spaces $\mathcal{R}_1, \mathcal{R}_2$. The dilation corresponding to the center of the ball ($R = 0$) is called the central dilation and will play an important role in our investigation.

Let us remark that if $\mathcal{W} := [W_1, ..., W_n]$ is an isometric lifting of $\mathcal{T}$ on the Hilbert space $\mathcal{G} \supseteq \mathcal{H}$, then $\mathcal{W}$ admits a reducing decomposition $W_i = V_i \oplus U_i$, $i = 1, ..., n$, on $\mathcal{G} = \mathcal{K} \oplus \mathcal{K}_0$, where $\mathcal{V} := [V_1, ..., V_n]$ is the m.i.d. of $\mathcal{T}$ and $\mathcal{K} = \bigvee_{\alpha \in \mathbb{F}_n^+} W_\alpha \mathcal{H}$. To see this, notice that $\mathcal{K}$ is invariant under each W_i. Using the lifting property, we infer that $T_i^* = W_i^*|_{\mathcal{H}}$. Therefore $\mathcal{K}$ is also invariant under W_i^*, $i = 1, ..., n$, and $V_i = W_i|_{\mathcal{K}}$. According to [Po1], $\mathcal{V}$ is the m.i.d. of $\mathcal{T}$. As in the classical case [FFr1], using this decomposition, one can extend the noncommutative commutant lifting theorem [Po5] from minimal isometric dilations to isometric liftings of $\mathcal{T}$ and $\mathcal{T}'$.

For any contraction $X \in B(\mathcal{K}, \mathcal{K}')$, let us define the defect operator D_X by setting $D_X := (I_{\mathcal{K}} - X^*X)^{1/2}$ and its defect space $\mathcal{D}_X := \overline{D_X \mathcal{K}}$. In what follows we obtain an explicit central intertwining lifting for row contractions.

THEOREM 2.1. *Let $A \in B(\mathcal{H}, \mathcal{H}')$ be a strict contraction ($\|A\| < 1$) satisfying $AT_i = T_i'A$, $i = 1, ..., n$, where $T_i \in B(\mathcal{H})$, $T_i' \in B(\mathcal{H}')$, $i = 1, ..., n$, $\mathcal{T} := [T_1, ..., T_n]$ is an isometry, and $\mathcal{T}' := [T_1', ..., T_n']$ is a row contraction. Let $\mathcal{L} := \mathcal{H} \ominus [T_1\mathcal{H} \oplus \cdots \oplus T_n\mathcal{H}]$ be the wandering subspace determined by $\mathcal{T}$. Then there is a contractive intertwining lifting B_c of A satisfying*

$$\|B_c \ell\| \leqslant \frac{\|A\ell\|}{\sqrt{1 - \|A\|^2}} \quad \text{for any} \quad \ell \in \mathcal{L}. \tag{2.2}$$

In particular, if $\mathcal{L}$ is finite dimensional, then

$$\|B_c|_{\mathcal{L}}\|_2 \leqslant \frac{\|A|_{\mathcal{L}}\|_2}{\sqrt{1 - \|A\|^2}}, \tag{2.3}$$

where $\|\cdot\|_2$ is the Hilbert–Schmidt norm.

Proof. Since any isometric lifting $\mathcal{W}' := [W_1', ..., W_n']$ of $\mathcal{T}'$ admits a decomposition $W_i' = V_i' \oplus U_i'$, $i = 1, ..., n$, where $\mathcal{V}' := [V_1', ..., V_n']$ is the minimal isometric dilation of $\mathcal{T}'$, we can assume without loss of generality that $\mathcal{W}' = \mathcal{V}'$. Let $\mathcal{V}' := [V_1', ..., V_n']$ be the minimal isometric dilation of

$\mathcal{T}' := [T'_1, \ldots, T'_n]$ on the Hilbert space $\mathcal{K}' := \mathcal{H}' \oplus [F^2(H_n) \otimes \mathcal{D}']$ where $\mathcal{D}' := \overline{D_{\mathcal{T}'}(\bigoplus_{j=1}^n \mathcal{H}')}$. As in (2.1) we have

$$V'_i[h \oplus (\xi \otimes d)] := T'_i h \oplus [D'_i h + (S_i \otimes I_{\mathcal{D}'})(\xi \otimes d)] \tag{2.4}$$

for any $h \in \mathcal{H}'$, $\xi \in F^2(H_n)$, and $d \in \mathcal{D}'$. Define the following subspaces:

$$\mathcal{M} := \bigvee_{i=1}^n D_A T_i \mathcal{H}, \qquad \text{and}$$

$$\mathcal{N} := \left\{ D_{\mathcal{T}'}\left(\bigoplus_{i=1}^n Ah_i \right) \oplus \left(\bigoplus_{i=1}^n D_A h_i \right) : h_i \in \mathcal{H} \right\}^{-}.$$

Notice that $\mathcal{M} \subseteq \mathcal{D}_A \subseteq \mathcal{H}$ and $\mathcal{N} \subseteq \mathcal{D}' \oplus (\bigoplus_{i=1}^n \mathcal{D}_A)$. A straightforward computation shows that the operator $W: \mathcal{M} \to \mathcal{N}$ defined by

$$W\left(\sum_{i=1}^n D_A T_i h_i \right) := D_{\mathcal{T}'}\left(\bigoplus_{i=1}^n Ah_i \right) \oplus \left(\bigoplus_{i=1}^n D_A h_i \right) \tag{2.5}$$

for any $h_i \in \mathcal{H}$, $i = 1, \ldots, n$, is unitary. Let us consider the orthogonal projections $P_{\mathcal{D}'}: \mathcal{D}' \oplus (\bigoplus_{j=1}^n \mathcal{D}_A) \to \mathcal{D}'$, and $P_i: \mathcal{D}' \oplus (\bigoplus_{j=1}^n \mathcal{D}_A) \to \mathcal{D}_A$ defined by $P_i[d \oplus (\bigoplus_{j=1}^n d_j)] := d_i$.

Define $B_c: \mathcal{H} \to \mathcal{H}' \oplus [F^2(H_n) \otimes \mathcal{D}']$ by setting

$$B_c h := Ah \oplus \sum_{\sigma \in \mathbb{F}_n^+} e_\sigma \otimes (P_{\mathcal{D}'} W P_{\mathcal{M}})\, E_\sigma D_A h, \qquad h \in \mathcal{H}, \tag{2.6}$$

where $P_{\mathcal{M}}$ is the orthogonal projection on $\mathcal{M}$, $E_e := I_{\mathcal{D}_A}$, and

$$E_\sigma := (P_{i_k} W P_{\mathcal{M}})(P_{i_{k-1}} W P_{\mathcal{M}}) \cdots (P_{i_1} W P_{\mathcal{M}}) \qquad \text{for} \quad \sigma := g_{i_1} \cdots g_{i_k} \in \mathbb{F}_n^+.$$

Using the definition of W in (2.5), we infer that

$$(P_j W P_{\mathcal{M}})\, D_A T_i h = \delta_{ij} D_A h \qquad \text{for any} \quad i, j \in \{1, 2, \ldots, n\}. \tag{2.7}$$

According to (2.6) and (2.7), we deduce that

$$\begin{aligned} B_c T_i h &= AT_i h \oplus \Bigg[1 \otimes D_{\mathcal{T}'}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, Ah, 0, \ldots, 0) \\ &\quad + \sum_{\sigma \in \mathbb{F}_n^+, \sigma \neq e} e_\sigma \otimes (P_{\mathcal{D}'} W P_{\mathcal{M}})\, E_\sigma D_A T_i h \Bigg] \\ &= AT_i h \oplus \Bigg[1 \otimes D_{\mathcal{T}'}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, Ah, 0, \ldots, 0) \\ &\quad + \sum_{\gamma \in \mathbb{F}_n^+} e_{g_i \gamma} \otimes (P_{\mathcal{D}'} W P_{\mathcal{M}})\, E_\gamma D_A h \Bigg]. \end{aligned}$$

On the other hand, using (2.4) and (2.6), we have

$$V_i' B_c h = T_i' A h \oplus \Bigg[1 \otimes D_{\mathcal{T}'}(\underbrace{0, \ldots, 0}_{i-1 \text{ times}}, Ah, 0, \ldots, 0)$$
$$+ (S_i \otimes I_{\mathcal{D}'}) \Bigg(\sum_{\sigma \in \mathbb{F}_n^+} e_\sigma \otimes (P_{\mathcal{D}'} W P_{\mathcal{M}})\, E_\sigma D_A h \Bigg) \Bigg].$$

Since $AT_i = T_i' A$ for any $i = 1, 2, \ldots, n$, it is clear that $B_c T_i = V_i' B_c$ for any $i = 1, 2, \ldots, n$. Now, we prove that

$$\|B_c h\|^2 \leqslant \|Ah\|^2 + \|P_{\mathcal{M}} D_A h\|^2 \qquad \text{for any} \quad h \in \mathcal{H}. \tag{2.8}$$

Due to (2.6) and the definition of E_σ, we have:

$$\begin{aligned}
\|B_c h\|^2 &= \|Ah\|^2 + \sum_{\sigma \in \mathbb{F}_n^+} \|(P_{\mathcal{D}'} W P_{\mathcal{M}})\, E_\sigma D_A h\|^2 \\
&= \|Ah\|^2 + \sum_{\sigma \in \mathbb{F}_n^+} \left(\|W P_{\mathcal{M}} E_\sigma D_A h\|^2 - \sum_{i=1}^n \|(P_i W P_{\mathcal{M}})\, E_\sigma D_A h\|^2 \right) \\
&\leqslant \|Ah\|^2 + \sum_{\sigma \in \mathbb{F}_n^+} \left(\|E_\sigma P_{\mathcal{M}} D_A h\|^2 - \sum_{i=1}^n \|E_{\sigma g_i} D_A h\|^2 \right) \\
&= \|Ah\|^2 + \lim_{m \to \infty} \sum_{\sigma \in \mathbb{F}_n^+,\, |\sigma| \leqslant m} \left(\|E_\sigma P_{\mathcal{M}} D_A h\|^2 - \sum_{i=1}^n \|E_{\sigma g_i} P_{\mathcal{M}} D_A h\|^2 \right) \\
&= \|Ah\|^2 + \|P_{\mathcal{M}} D_A h\|^2 - \lim_{m \to \infty} \sum_{\sigma \in \mathbb{F}_n^+,\, |\sigma| = m+1} \|E_\sigma P_{\mathcal{M}} D_A h\|^2 \\
&\leqslant \|Ah\|^2 + \|P_{\mathcal{M}} D_A h\|^2.
\end{aligned}$$

Therefore (2.8) is proved. Now, it is clear that B_c is a contractive intertwining lifting of A, i.e., $\|B_c\| \leqslant 1$, $B_c T_i = V_i' B_c$, $i = 1, \ldots, n$, and $P_{\mathcal{H}} B_c = A$. Define $X := D_A[T_1, T_2, \ldots, T_n]$. Since $T_1, T_2, \ldots, T_n$ are isometries with orthogonal ranges, we have

$$\begin{aligned}
X^* X &= [T_i^* (I - A^* A)\, T_j]_{n \times n} = [\delta_{ij} I_{\mathcal{H}} - T_i^* A^* A T_j]_{n \times n} \\
&= D^2_{[AT_1, \ldots, AT_n]}.
\end{aligned} \tag{2.9}$$

Since $[AT_1, \ldots, AT_n]$ is a strict contraction, $D_{[AT_1, \ldots, AT_n]}$ is invertible. Hence X^*X is invertible on $\bigoplus_{i=1}^n \mathscr{H}$. On the other hand, for any $h_i \in \mathscr{H}$, we have

$$\|D_A(T_1h_1 + \cdots + T_nh_n)\|^2 \geqslant \frac{1}{\|D_A^{-1}\|^2} \|T_1h_1 + \cdots + T_nh_n\|^2 = \frac{1}{\|D_A^{-1}\|^2} \left\| \bigoplus_{i=1}^n h_i \right\|^2.$$

This shows that the range of X is closed and equal to $\mathscr{M}$. Define $P := X(X^*X)^{-1} X^*$ and notice that $P^2 = P$, $P = P^*$, and range $P = \mathscr{M}$. Using (2.9), we infer that

$$P_{\mathscr{M}} = D_A[T_1, \ldots, T_n]\, D_{[AT_1, \ldots, AT_n]}^{-2} \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} D_A. \tag{2.10}$$

Notice that

$$\begin{aligned} D_{[AT_1, \ldots, AT_n]}^2 &= [\delta_{ij} I_{\mathscr{H}} - T_i^* A^* A T_j]_{n\times n} = [\delta_{ij} I_{\mathscr{H}} - A^* T_i'^* T_j' A]_{n\times n} \\ &= [\delta_{ij}(I_{\mathscr{H}} - A^*A)]_{n\times n} + \left(\bigoplus_{i=1}^n A^*\right) D_{\mathscr{T}'}^2 \left(\bigoplus_{i=1}^n A\right) \\ &\geqslant \bigoplus_{i=1}^n D_A^2. \end{aligned}$$

Hence

$$D_{[AT_1, \ldots, AT_n]}^{-2} \leqslant \bigoplus_{i=1}^n D_A^{-2}. \tag{2.11}$$

According to [Po1], $\mathscr{L} = \bigcap_{i=1}^n \ker T_i^*$. Using (2.8), (2.10), and (2.11), we deduce that, for any $\ell \in \mathscr{L}$,

$$\begin{aligned} \|B_c \ell\|^2 &\leqslant \|A\ell\|^2 + \left\langle D_{[AT_1, \ldots, AT_n]}^{-2} \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} D_A^2 \ell, \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} D_A^2 \ell \right\rangle \\ &= \|A\ell\|^2 + \left\langle D_{[AT_1, \ldots, AT_n]}^{-2} \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} A^*A\ell, \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} A^*A\ell \right\rangle \\ &\leqslant \|A\ell\|^2 + \left\langle \left(\bigoplus_{i=1}^n D_A^{-2}\right) \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} A^*A\ell, \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} A^*A\ell \right\rangle. \end{aligned}$$

Since $\bigoplus_{i=1}^n D_A^{-2} \leqslant (1/(1-\|A\|^2))\, I_{\bigoplus_{i=1}^n \mathscr{H}}$, we infer that

$$\|B_c\ell\|^2 \leqslant \|A\ell\|^2 + \frac{1}{1-\|A\|^2} \left\| \begin{bmatrix} T_1^* \\ \vdots \\ T_n^* \end{bmatrix} A^*A\ell \right\|^2$$

$$\leqslant \|A\ell\|^2 + \frac{\|A\|^2\, \|A\ell\|^2}{1-\|A\|^2}$$

$$= \frac{\|A\ell\|^2}{1-\|A\|^2}.$$

Therefore, relation (2.2) holds. Assume now that $\mathscr{L}$ is finite dimensional and let $\{\ell_1, ..., \ell_k\}$ be an orthonormal basis for $\mathscr{L}$. Using (2.2), we have

$$\|B_c|_{\mathscr{L}}\|_2^2 = \sum_{i=1}^k \|B_c\ell_i\|^2 \leqslant \sum_{i=1}^k \frac{\|A\ell_i\|^2}{1-\|A\|^2} = \frac{\|A|_{\mathscr{L}}\|_2^2}{1-\|A\|^2}.$$

This completes the proof. ∎

Some consequences of this theorem will be presented in the next sections.

3. KAFTAL–LARSON–WEISS THEOREM ON FOCK SPACES AND INTERPOLATION

In this section we use our central intertwining lifting theorem for row contractions to obtain Kaftal–Larson–Weiss and Foiaş–Frazho type results for the noncommutative analytic Toeplitz algebra F_n^∞, as well as for $F_n^\infty \bar{\otimes} B(\mathscr{K}, \mathscr{K}')$, the WOT-closed operator space generated by the spatial tensor product.

THEOREM 3.1. *Let $\mathscr{K}, \mathscr{K}'$ be Hilbert spaces and let $\mathscr{N}' \subseteq F^2(H_n) \otimes \mathscr{K}'$ be a subspace with the property that $\mathscr{N}'$ is invariant under each $R_1^* \otimes I_{\mathscr{K}'}, ..., R_n^* \otimes I_{\mathscr{K}'}$. If $\delta > 1$ and $\psi \in F_n^\infty \bar{\otimes} B(\mathscr{K}, \mathscr{K}')$, then there exists $\varphi \in F_n^\infty \bar{\otimes} B(\mathscr{K}, \mathscr{K}')$ with $P_{\mathscr{N}'}\varphi = 0$ and such that*

(3.1) $$\|\psi - \varphi\| \leqslant \delta\, \|P_{\mathscr{N}'}\psi\|,$$

and

(3.2) $$\|(\psi - \varphi)(1 \otimes k)\| \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}}\, \|P_{\mathscr{N}'}\psi(1 \otimes k)\|,$$

for any $k \in \mathcal{K}$. If $\mathcal{K}$ is finite dimensional, then

$$\|(\psi-\varphi)|_{1\otimes\mathcal{K}}\|_2 \leqslant \frac{\delta}{\sqrt{\delta^2-1}}\,\|P_{\mathcal{N}'}\psi|_{1\otimes\mathcal{K}}\|_2, \tag{3.3}$$

where $\|\cdot\|_2$ is the Hilbert–Schmidt norm.

Proof. If $P_{\mathcal{N}'}\psi=0$, then we take $\varphi=\psi$, and the inequalities (3.1), (3.2), and (3.3) are satisfied. Now, assume that $P_{\mathcal{N}'}\psi\neq 0$. Let us define $A\in B(F^2(H_n)\otimes\mathcal{K}, (U\otimes I_{\mathcal{K}'})\,\mathcal{N}')$ by setting

$$A := P_{(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'}(U^*\otimes I_{\mathcal{K}'})\,\psi_1(U\otimes I_{\mathcal{K}}), \tag{3.4}$$

where

$$\psi_1 := \delta^{-1}\,\|P_{\mathcal{N}'}\psi\|^{-1}\,\psi. \tag{3.5}$$

Since $(U^*\otimes I_{\mathcal{K}'})\,P_{\mathcal{N}'}(U\otimes I_{\mathcal{K}'}) = P_{(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'}$, it is clear that $\|A\|=\delta^{-1}<1$. According to Theorem 1.1, $(U^*\otimes I_{\mathcal{K}'})\,\psi_1(U\otimes I_{\mathcal{K}})$ intertwines $S_i\otimes I_{\mathcal{K}}$ with $S_i\otimes I_{\mathcal{K}'}$ for any $i=1, ..., n$. Since $(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'$ is invariant under each $S_i^*\otimes I_{\mathcal{K}'}, ..., S_n^*\otimes I_{\mathcal{K}'}$, we infer that

$$A(S_i\otimes I_{\mathcal{K}}) = T'_i A,$$

where $T'_i := P_{(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'}(S_i\otimes I_{\mathcal{K}'})|_{(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'}$ for any $i=1, ..., n$. Notice that $[S_1\otimes I_{\mathcal{K}'}, ..., S_n\otimes I_{\mathcal{K}'}]$ is an isometric lifting of $[T'_1, ..., T'_n]$. According to Theorem 2.1, there exists $B_c\in B(F^2(H_n)\otimes\mathcal{K}, F^2(H_n)\otimes\mathcal{K}')$, a contractive intertwining lifting of A, satisfying the following properties:

(i) $B_c(S_i\otimes I_{\mathcal{K}}) = (S_i\otimes I_{\mathcal{K}'})\,B_c$, for any $i=1, ..., n$;
(ii) $\|B_c(1\otimes k)\| \leqslant \frac{1}{\sqrt{1-\|A\|^2}}\,\|A(1\otimes k)\|$ for any $k\in\mathcal{K}$;
(iii) $P_{(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'}B_c = A$.

According to (i) and Theorem 1.1, there exists $\Phi\in F_n^\infty\,\bar{\otimes}\,B(\mathcal{K}, \mathcal{K}')$ such that $B_c = (U\otimes I_{\mathcal{K}'})\,\Phi(U\otimes I_{\mathcal{K}})$. Using (iii) and (3.4), we infer that

$$P_{(U\otimes I_{\mathcal{K}'})\,\mathcal{N}'}(U^*\otimes I_{\mathcal{K}'})(\Phi-\psi_1)(U\otimes I_{\mathcal{K}}) = 0. \tag{3.6}$$

which is equivalent to

$$P_{\mathcal{N}'}(\Phi-\psi_1) = 0. \tag{3.7}$$

Hence $\Phi - \psi_1 = -g$ for some $g \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ with $P_{\mathcal{N}'} g = 0$. Notice that

$$\|\psi_1 - g\| = \|\Phi\| = \|B_c\| \leqslant 1. \tag{3.8}$$

Taking into account (3.5), the relation (3.8) becomes

$$\|\psi - \varphi\| \leqslant \delta \, \|P_{\mathcal{N}'} \psi\|,$$

where $\varphi := \delta \, \|P_{\mathcal{N}'} \psi\| \, g$. Hence, we deduce relation (3.1) with $\varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ satisfying $P_{\mathcal{N}'} \varphi = 0$. On the other hand, according to (ii) we have, for any $k \in \mathcal{K}$,

$$\begin{aligned} \|[\psi_1 - g](1 \otimes k)\| &= \|\Phi(1 \otimes k)\| = \|B_c(1 \otimes k)\| \\ &\leqslant \frac{1}{\sqrt{1 - \|A\|^2}} \|A(1 \otimes k)\|. \end{aligned} \tag{3.9}$$

From the definition of A, we deduce that

$$\begin{aligned} \|A(1 \otimes k)\| &= \|P_{(U \otimes I_{\mathcal{K}'}) \mathcal{N}'} (U^* \otimes I_{\mathcal{K}'}) \, \psi_1 (U \otimes I_{\mathcal{K}})(1 \otimes k)\| \\ &= \|P_{\mathcal{N}'} \psi_1 (1 \otimes k)\|. \end{aligned}$$

Using (3.9), we infer that

$$\|(\psi_1 - g)(1 \otimes k)\| \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} \|P_{\mathcal{N}'} \psi_1 (1 \otimes k)\|.$$

This inequality together with (3.5) implies

$$\|(\psi - \varphi)(1 \otimes k)\| \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} \|P_{\mathcal{N}'} \psi (1 \otimes k)\|, \qquad k \in \mathcal{K}, \tag{3.10}$$

which is equivalent to (3.2). Now it is easy to see that (3.3) is a consequence of (3.10). This completes the proof. ∎

A first consequence of Theorem 3.1 is the following Sarason type result for $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$.

Corollary 3.2. *Let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces and let $\mathcal{N}' \subseteq F^2(H_n) \otimes \mathcal{K}'$ be a subspace with the property that $\mathcal{N}'$ is invariant under each $R_1^* \otimes I_{\mathcal{K}'}, \ldots, R_n^* \otimes I_{\mathcal{K}'}$. If $\psi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, then*

$$\inf\{\|\psi + \varphi\| : \varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}'), P_{\mathcal{N}'} \varphi = 0\} = \|P_{\mathcal{N}'} \psi\|. \tag{3.11}$$

Moreover, there exists $\varphi_0 \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ *with* $P_{\mathcal{N}'}\varphi_0 = 0$ *such that*

$$\|\psi + \varphi_0\| = \|P_{\mathcal{N}'}\psi\|.$$

If $\mathcal{K}$ *is finite dimensional, then*

(3.12)
$$\inf\{\|(\psi+\varphi)|_{1\otimes\mathcal{K}}\|_2 : \varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}'),\ P_{\mathcal{N}'}\varphi = 0\} = \|P_{\mathcal{N}'}\psi|_{1\otimes\mathcal{K}}\|_2,$$

where $\|\cdot\|_2$ *is the Hilbert–Schmidt norm.*

Proof. Using (3.1) of Theorem 3.1 as $\delta \to 1$, it is easy to see that

$$\inf\{\|\psi+\varphi\| : \varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}'),\ P_{\mathcal{N}'}\varphi = 0\} \leqslant \|P_{\mathcal{N}'}\psi\|. \tag{3.13}$$

On the other hand, for any $\phi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, with $P_{\mathcal{N}'}\phi = 0$, we have

$$\|\psi+\phi\| \geqslant \|P_{\mathcal{N}'}(\psi+\phi)\| = \|P_{\mathcal{N}'}\psi\|. \tag{3.14}$$

Combining (3.13) with (3.14), we obtain (3.11). Now let us apply the non-commutative commutant lifting theorem to the operator A given by (3.4) and satisfying $A(S_i \otimes I_{\mathcal{K}}) = T_i' A$, $i = 1, ..., n$. We find a multianalytic operator $B \in B(F^2(H_n) \otimes \mathcal{K}, F^2(H_n) \otimes \mathcal{K}')$ such that $\|A\| = \|B\|$ and $P_{(U \otimes I_{\mathcal{K}'})\mathcal{N}'} B = A$. According to Theorem 1.1, $B = (U \otimes I_{\mathcal{K}'}) f(U \otimes I_{\mathcal{K}})$ for some $f \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$. It is clear that $P_{\mathcal{N}'}(f - \psi) = 0$ and $\|f\| = \|P_{\mathcal{N}'}\psi\|$. Setting $\psi_0 := f - \psi$, the first part of the theorem follows.

If $\mathcal{K}$ is finite dimensional, then relation (3.12) can be proved similarly if one uses (3.3) of Theorem 3.1 as $\delta \to \infty$. This completes the proof. ∎

As mentioned in the Introduction, if $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, then $\lambda \mapsto \Phi(\lambda)$ is a $\mathcal{B}(\mathcal{K}, \mathcal{K}')$-valued bounded analytic function on $\mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$. For each $j = 1, ..., k$, let $\lambda_j := (\lambda_{j1}, ..., \lambda_{jn}) \in \mathbb{B}_n$ and, for $\alpha := g_{j_1} g_{j_2}, ..., g_{j_m}$ in $\mathbb{F}_n^+$, let $\lambda_{j\alpha} := \lambda_{jj_1}\lambda_{jj_2}, ..., \lambda_{jj_m}$ and $\lambda_e := 1$. Define $z_{\lambda_j} \in F^2(H_n)$, $j = 1, ..., k$, by

$$z_{\lambda_i} := \sum_{\alpha \in \mathbb{F}_n^+} \bar{\lambda}_{i\alpha} e_\alpha,$$

and notice that $\langle z_{\lambda_i}, z_{\lambda_j}\rangle = 1/(1 - \langle \lambda_j, \lambda_i\rangle)$. For any element $\phi := \sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes A_{(\alpha)}$ in $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, we have

$$\phi^*(z_{\lambda_j} \otimes k') = z_{\lambda_j} \otimes \phi(\lambda_j)^* k' \tag{3.15}$$

for any $k' \in \mathcal{K}'$, $\beta \in \mathbb{F}_n^+$, and $j = 1, ..., k$. Indeed, it is enough to observe that

$$\langle \phi(e_\beta \otimes k), z_{\lambda_j} \otimes k' \rangle = \langle e_\beta \otimes k, z_{\lambda_j} \otimes \phi(\lambda_j)^* k' \rangle,$$

for any $k \in \mathcal{K}$.

Another consequence of Theorem 3.1 is the following interpolation problem of Nevanlinna–Pick type for $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, the noncommutative analytic Toeplitz algebra, which was obtained in [ArPo2, DP3, Po10], when $\mathcal{K} = \mathcal{K}'$. We only sketch the proof.

COROLLARY 3.3. *Let $\lambda_1, ..., \lambda_k$ be k distinct points in $\mathbb{B}_n$ and let $B_j \in B(\mathcal{K}, \mathcal{K}')$, $j = 1, ..., k$, where $\mathcal{K}, \mathcal{K}'$ are Hilbert spaces. Then there exists Φ in $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, such that $\|\Phi\| \leqslant 1$ and $\Phi(\lambda_j) = B_j$, $j = 1, 2, ..., k$, if and only if the operator matrix*

$$\left[\frac{I_{\mathcal{K}'} - B_i B_j^*}{1 - \langle \lambda_i, \lambda_j \rangle} \right]_{i, j = 1, 2, ..., k} \tag{3.16}$$

is positive semidefinite.

Proof. Let $\mathcal{N}' := \text{span}\{z_{\lambda_j} : j = 1, ..., k\} \otimes \mathcal{K}'$ and notice that if $\phi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, then $P_{\mathcal{N}'}\phi = 0$ if and only if $\phi(\lambda_j) = 0$ for any $j = 1, ..., k$. Applying Corollary 3.2 to an element $\psi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ with $\psi(\lambda_j) = B_j$, $j = 1, ..., k$, we find Φ in $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, such that $\|\Phi\| \leqslant 1$ and $\Phi(\lambda_j) = B_j$, $j = 1, 2, ..., k$, if and only if $\|P_{\mathcal{N}'}\psi\| \leqslant 1$. Using (3.15), one can show that the latter inequality is equivalent to (3.16). ∎

Notice that if (3.16) holds then there exists Φ in $H^\infty(\mathbb{B}_n) \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ such that $\|\Phi\| \leqslant 1$ and $\Phi(\lambda_j) = B_j$, $j = 1, 2, ..., k$.

Let $\mathcal{P}_m$ be the set of all polynomials in $\mathcal{P}$ of degree $\leqslant m$. Using Corollary 3.2 in the particular case when $\mathcal{N}' := \mathcal{P}_m \otimes \mathcal{K}'$ and $\psi := p$, we obtain the following extension of the Carathéodory interpolation problem to $F_n^\infty \bar{\otimes} \mathcal{B}(\mathcal{K}, \mathcal{K}')$.

COROLLARY 3.4. *Let $p := \sum_{|\alpha| \leqslant m} S_\alpha \otimes B_{(\alpha)} \in F_n^\infty \bar{\otimes} \mathcal{B}(\mathcal{K}, \mathcal{K}')$. Then there exists $f = \sum_{\alpha \in \mathbb{F}_n^+} S_\alpha \otimes W_{(\alpha)} \in F_n^\infty \bar{\otimes} \mathcal{B}(\mathcal{K}, \mathcal{K}')$ with $\|f\| \leqslant 1$ and $W_{(\alpha)} = B_{(\alpha)}$ if $|\alpha| \leqslant m$ if and only if*

$$\|P_{\mathcal{P}_m \otimes \mathcal{K}'} p|_{\mathcal{P}_m \otimes \mathcal{K}}\| \leqslant 1.$$

In what follows we obtain a multivariable noncommutative analogue of the Kaftal–Larson–Weiss theorem [KLW] on Fock spaces. If $f \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, then $\|f\|$ denotes the uniform norm of f and, if $\mathcal{K}$ is finite dimensional, $\|f\|_2$ stands for the Hilbert–Schmidt norm of $f|_{1\otimes\mathcal{K}}$. Let d_∞ and d_2 be the corresponding metrics on $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$. Let us remark that if f has the Fourier representation $\sum_{\alpha\in\mathbb{F}_n^+} S_\alpha \otimes X_{(\alpha)}$, then $\|f\|_2 = \sum_{\alpha\in\mathbb{F}_n^+} \operatorname{trace}[X_{(\alpha)}^* X_{(\alpha)}]$.

THEOREM 3.5. *Let J be a WOT-closed right ideal in F_n^∞, let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces, and let $\delta > 1$. If $\psi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and $\mathcal{K}$ is finite dimensional, then there exists $\varphi \in J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ such that*

$$\|\psi - \varphi\| \leqslant \delta d_\infty[\psi, J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')] \tag{3.17}$$

and

$$\|\psi - \varphi\|_2 \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} d_2(\psi, J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')). \tag{3.18}$$

Proof. Denote $\mathcal{N}_J := \overline{J(1)}^\perp$. Since J is a WOT-closed right ideal in F_n^∞, $U\mathcal{N}_J$ is invariant to each $S_1^*, ..., S_n^*$, so we can apply Theorem 3.1, when $\mathcal{N}' := \mathcal{N}_J \otimes \mathcal{K}'$. Therefore, there exists $\varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ with

$$P_{\mathcal{N}_J \otimes \mathcal{K}'} \varphi = 0, \tag{3.19}$$

such that

$$\|\psi - \varphi\| \leqslant \delta \, \|P_{\mathcal{N}_J \otimes \mathcal{K}'} \psi\|, \tag{3.20}$$

and (3.3) holds. According to Corollary 3.2, we have

$$\|P_{\mathcal{N}_J \otimes \mathcal{K}'} \psi\| = \inf\{\|\psi + \varphi\| : \varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}'), P_{\mathcal{N}_J \otimes \mathcal{K}'} \varphi = 0\}.$$

To complete the proof of (3.17), it remains to show that if $f \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, then $P_{\mathcal{N}_J \otimes \mathcal{K}'} f = 0$ if and only if $f \in J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$. One implication is trivial. Let us assume that

$$P_{\mathcal{N}_J \otimes \mathcal{K}'} f = 0. \tag{3.21}$$

Since f can be identified with an operator matrix $[f_{ij}] \in B(F^2(H_n) \otimes \mathcal{K}, F^2(H_n) \otimes \mathcal{K}')$ with entries in F_n^∞, the relation (3.21) is equivalent to $P_{\mathcal{N}_J \otimes \mathcal{K}'} f_{ij} = 0$. Now assume that $f_{ij} \notin J$ and let ${}^\perp J$ be the preannihilator of J in the predual of F_n^∞. Since $({}^\perp J)^* = F_n^\infty / J$, there exists $\Phi \in {}^\perp J$ such that

$$\Phi(f_{ij}) \neq 0. \tag{3.22}$$

On the other hand, since F_n^∞ has the $\mathbb{A}_1$ property, similarly to Proposition 1.1 of [ArPo2], one can find $\psi_1, \psi_2 \in F^2(H_n)$ with $\psi_2 \in \mathcal{N}_J$ such that

$$\Phi(f_{ij}) = \langle f\psi_1, \psi_2 \rangle = \langle P_{\mathcal{N}_J} f\psi_1, \psi_2 \rangle = 0,$$

which contradicts (3.22). Therefore $f_{ij} \in J$ and $f \in J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$. Now, using (3.3) and Corollary 3.2, we infer (3.18). The proof is complete. ∎

Notice that $\inf_{\delta > 1} \max\{\delta, \delta/\sqrt{\delta^2 - 1}\} = \sqrt{2}$. If J is a WOT-closed right ideal in F_n^∞ and $f \in F_n^\infty$, then, according to Theorem 3.5, there exists $\varphi \in J$ such that

$$\|f - \varphi\| \leqslant \sqrt{2}\, d_\infty(f, J) \qquad \text{and} \qquad \|f - \varphi\|_2 \leqslant \sqrt{2}\, d_2(f, J).$$

This is a Pisier type result (see [Pi2]). From the proof of Theorem 3.5 we can also deduce the following result.

Corollary 3.6. *Let J be a WOT-closed right ideal in F_n^∞ and let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces. Then the map*

$$\Phi\colon F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')/J \bar{\otimes} B(\mathcal{K}, \mathcal{K}') \to B(\mathcal{N}_J) \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$$

defined by

$$\Phi[\varphi + J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')] = P_{\mathcal{N}_J \otimes \mathcal{K}'}\varphi$$

is an isometry. Moreover, there is $\psi_0 \in J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ such that

$$\|\varphi + \psi_0\| = \|P_{\mathcal{N}_J \otimes \mathcal{K}'}\varphi\|.$$

Let J be a w^*-closed, two-sided ideal of F_n^∞ and $\mathcal{W}(B_1, ..., B_n)$ be the w^*-closure of of the algebra generated by $B_i := P_{\mathcal{N}_J} S_i|_{\mathcal{N}_J}$ for, $i = 1, ..., n$, and the identity. Using Theorem 1.2 and Corollary 3.6, we infer that $F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')/J \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ is canonically isomorphic to $\mathcal{W}(B_1, ..., B_n)$.

In what follows we obtain a noncommutative analogue of the Foiaş–Frazho suboptimization theorem [FFr2] for $F^\infty \bar{\otimes} \mathcal{B}(\mathcal{K}, \mathcal{K}')$. We need to recall a Beurling type characterization of the invariant subspaces under each $S_i \otimes I_{\mathcal{K}}$, $i = 1, ..., n$, which was obtained in [Po2]. The theorem states that a subspace $\mathcal{M} \subseteq F^2(H_n) \otimes \mathcal{K}$ is invariant under each $S_i \otimes I_{\mathcal{K}}$, $i = 1, ...,$ n, if and only if there exists a Hilbert space $\mathcal{G}$ and an inner multianalytic operator $\Phi \in B(F^2(H_n) \otimes \mathcal{G}, F^2(H_n) \otimes \mathcal{K})$ such that $\mathcal{M} = \Phi(F^2(H_n) \otimes \mathcal{G})$.

Let $\Psi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and let $\Theta \in F_n^\infty \bar{\otimes} B(\mathcal{G}, \mathcal{K}')$ be inner, i.e., an isometry. Define

$$d_\infty(\Psi) := \inf\{\|\Psi + \Theta\Phi\| : \Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{G})\}$$

and, if $\mathcal{K}$ is finite dimensional, then

$$d_2(\Psi) := \inf\{\|\Psi + \Theta\Phi\|_2 : \Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{G})\},$$

where $\|\Psi + \Theta\Phi\|_2$ is the Hilbert–Schmidt norm of $(\Psi + \Theta\Phi)|_{1 \otimes \mathcal{K}}$.

THEOREM 3.7. *Let $\Psi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and let $\Theta \in F_n^\infty \bar{\otimes} B(\mathcal{G}, \mathcal{K}')$ be inner. Then*

$$d_\infty(\Psi) = \|P_{\mathcal{M}}\Psi\| \qquad \text{and} \qquad d_2(\Psi) = \|P_{\mathcal{M}}\Psi\|_2,$$

where $\mathcal{M} := [F^2(H_n) \otimes \mathcal{K}'] \ominus \Theta[F^2(H_n) \otimes \mathcal{G}]$. Moreover, there exists $\Phi_0 \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{G})$ such that

$$d_\infty(\Psi) = \|\Psi + \Theta\Phi_0\|.$$

If in addition $\mathcal{K}$ is finite dimensional and $\delta > 1$, then there exists $\Phi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{G})$ satisfying

$$\|\Psi + \Theta\Phi\| \leqslant \delta d_\infty(\Psi) \tag{3.23}$$

and

$$\|\Psi + \Theta\Phi\|_2 \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} d_2(\Psi). \tag{3.24}$$

Proof. According to Theorem 2.2 of [Po2], a subspace $\mathcal{N}' \subseteq F^2(H_n) \otimes \mathcal{K}'$ has the property that $\mathcal{M}' := (U \otimes I_{\mathcal{K}'})\, \mathcal{N}'$ is invariant under $S_i^* \otimes I_{\mathcal{K}'}$, $i = 1, \ldots, n$, if and only if there exists an inner multianalytic operator $X \in B(F^2(H_n) \otimes \mathcal{G}, F^2(H_n) \otimes \mathcal{K}')$ for some Hilbert space $\mathcal{G}$, such that $\mathcal{M}'^{\perp} = X[F^2(H_n) \otimes \mathcal{G}]$. Using Theorem 1.1, we find an inner operator $\Theta \in F_n^\infty \bar{\otimes} B(\mathcal{G}, \mathcal{K}')$ such that

$$\mathcal{N}' = [F^2(H_n) \otimes \mathcal{K}'] \ominus \Theta[F^2(H_n) \otimes \mathcal{G}]. \tag{3.25}$$

Now let us prove that if $\Lambda \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$, then

$$P_{\mathcal{N}'}\Lambda = 0 \tag{3.26}$$

if and only if there exists $H \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{G})$ such that

$$\Lambda = \Theta H. \tag{3.27}$$

It is clear that (3.27) implies (3.26). Conversely, assume (3.26) holds. This implies

$$\Lambda[F^2(H_n) \otimes \mathcal{K}] \subseteq \Theta[F^2(H_n) \otimes \mathcal{G}].$$

Hence, for each $k \in \mathcal{K}$, there exists a unique $f_k \in F^2(H_n) \otimes \mathcal{G}$ such that

$$\Lambda(1 \otimes k) = \Theta f_k. \tag{3.28}$$

Define the linear operator Q: $1 \otimes \mathcal{K} \to F^2(H_n) \otimes \mathcal{G}$ by $Q(1 \otimes k) := f_k$. Due to (3.28) and since Θ is an isometry we infer that Q is a bounded operator. Notice that for each $k \in \mathcal{K}$, $\alpha \in \mathbb{F}_n^+$ we have

$$\begin{aligned}(U^* \otimes I_{\mathcal{K}'})\, \Lambda(U \otimes I_{\mathcal{K}})(e_\alpha \otimes k) \\ &= (S_\alpha \otimes I_{\mathcal{K}'})(U^* \otimes I_{\mathcal{K}'})\, \Theta Q(1 \otimes k) \\ &= (U^* \otimes I_{\mathcal{K}'})\, \Theta(U \otimes I_{\mathcal{G}})(S_\alpha \otimes I_{\mathcal{G}})(U^* \otimes I_{\mathcal{G}})\, Q(1 \otimes k).\end{aligned}$$

Hence we infer that

$$\begin{aligned}\sum_{\alpha \in \mathbb{F}_n^+} (S_\alpha \otimes I_{\mathcal{G}})(U^* \otimes I_{\mathcal{G}})\, Q(1 \otimes k) \\ &= (U^* \otimes I_{\mathcal{G}})\, \Theta^* \Lambda(U \otimes I_{\mathcal{K}})\Big(\sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes k_\alpha\Big)\end{aligned}$$

for any $\sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes k_\alpha \in F^2(H_n) \otimes \mathcal{K}$. Therefore, M: $F^2(H_n) \otimes \mathcal{K} \to F^2(H_n) \otimes \mathcal{G}$ defined by

$$M\Big(\sum_{\alpha \in \mathbb{F}_n^+} e_\alpha \otimes k_\alpha\Big) := \sum_{\alpha \in \mathbb{F}_n^+} (S_\alpha \otimes I_{\mathcal{G}})(U^* \otimes I_{\mathcal{G}})\, Q(U \otimes I_{\mathcal{K}})(1 \otimes k)$$

is a multianalytic operator. According to Theorem 1.1, we can see that the operator $H := (U^* \otimes I_{\mathcal{G}})\, \Theta^* M(U \otimes I_{\mathcal{K}})$ is in $F_n^\infty \bar{\otimes} B(\mathcal{G}, \mathcal{K})$. On the other hand, it is clear that $\Lambda = \Theta H$. Applying Theorem 3.1 and Corollary 3.2 to the subspace $\mathcal{N}'$ given by (3.25), we complete the proof. ∎

4. KAFTAL–LARSON–WEISS THEOREM ON SYMMETRIC FOCK SPACES

Let W_n^∞ be the w^*-closed algebra generated by $B_i := P_{F_s^2(H_n)} S_i|_{F_s^2(H_n)}$, $i = 1, \ldots, n$, and the identity, where $F_s^2(H_n) \subset F^2(H_n)$ is the symmetric Fock space. The commutative Toeplitz algebra W_n^∞ was recently studied in [Po9, Arv, ArPo2, DP3, and PPoS]. It can be viewed as a multivariable commutative analogue of the classical H^∞.

In what follows we obtain a Kaftal–Larson–Weiss type result for the tensor product $W_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$. Let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces and let $\mathcal{E}'$ be a subspace of $F_s^2(H_n) \otimes \mathcal{K}'$. We associate with $\mathcal{E}'$ the operator space

$$E' := \{g \in W_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}') : P_{\mathcal{E}'} g = 0\}.$$

For every $f \in W_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ define

$$d_\infty(f, E') := \inf\{\|f + g\| : g \in E'\}$$

and, if $\mathcal{K}$ is finite dimensional, then

$$d_2(f, E') := \inf\{\|f + g\|_2 : g \in E'\},$$

where $\|f + g\|_2$ is the Hilbert–Schmidt norm of $(f + g)|_{1 \otimes \mathcal{K}}$.

THEOREM 4.1. *Let $\mathcal{E}' \subseteq F_s^2(H_n) \otimes \mathcal{K}'$ be an invariant subspace under each $B_1^* \otimes I_{\mathcal{K}'}, \ldots, B_n^* \otimes I_{\mathcal{K}'}$, and let $\delta > 1$. If $f \in W_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ and $\mathcal{K}$ is finite dimensional, then there exists $g \in E'$ such that*

$$\|f - g\| \leqslant \delta d_\infty(f, E') \qquad \text{and} \tag{4.1}$$

$$\|f - g\|_2 \leqslant \frac{\delta}{\sqrt{\delta^2 - 1}} d_2(f, E'). \tag{4.2}$$

Proof. Since $F_s^2(H_n) \otimes \mathcal{K}'$ is an invariant subspace under each $S_1^* \otimes I_{\mathcal{K}'}, \ldots, S_n^* \otimes I_{\mathcal{K}'}$, it is easy to see that $\mathcal{E}'$ has the same property and $(U \otimes I_{\mathcal{K}'})\, \mathcal{E}' = \mathcal{E}'$. A particular case of Theorem 1.2 shows that there exists $\psi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ such that

$$f = P_{F_s^2(H_n) \otimes \mathcal{K}'} \psi|_{F_s^2(H_n) \otimes \mathcal{K}}. \tag{4.3}$$

Applying Theorem 3.1 to ψ and $\mathcal{E}'$, we find $\varphi \in F_n^\infty \bar{\otimes} B(\mathcal{K}, \mathcal{K}')$ with $P_{\mathcal{E}'} \varphi = 0$ such that

$$\|\psi - \varphi\| \leqslant \delta\, \|P_{\mathcal{E}'} \psi\| \tag{4.4}$$

and

$$(4.5) \qquad \|(\psi-\varphi)(1\otimes k)\| \leqslant \frac{\delta}{\sqrt{\delta^2-1}} \|P_{\mathscr{E}'}\psi(1\otimes k)\|$$

for any $k\in\mathscr{K}$. Notice that $g := P_{F_s^2(H_n)\otimes\mathscr{K}'}\varphi|_{F_s^2(H_n)\otimes\mathscr{K}}$ is in E'. Using (4.3) and (4.4), we infer that

$$\begin{aligned}\|f-g\| &= \|P_{F_s^2(H_n)\otimes\mathscr{K}'}(\psi\varphi)|_{F_s^2(H_n)\otimes\mathscr{K}}\| \\ &\leqslant \|\psi-\varphi\| \leqslant \delta\,\|P_{\mathscr{E}'}\psi\| \\ &\leqslant \delta\,\|P_{\mathscr{E}'}P_{F_s^2(H_n)\otimes\mathscr{K}'}\psi|_{F_s^2(H_n)\otimes\mathscr{K}}\| \\ &= \delta\,\|P_{\mathscr{E}'}f\|.\end{aligned}$$

Notice also that, using (4.3) and (4.5), we have

$$\begin{aligned}\|(f-g)(1\otimes k)\| \leqslant \|(\psi-\varphi)(1\otimes k)\| &\leqslant \frac{\delta}{\sqrt{\delta^2-1}}\|P_{\mathscr{E}'}\psi(1\otimes k)\| \\ &= \frac{\delta}{\sqrt{\delta^2-1}}\|P_{\mathscr{E}'}^{F_s^2(H_n)\otimes\mathscr{K}'}f(1\otimes k)\|.\end{aligned}$$

Now, as in the proof of Corollary 3.2 we infer that $d_2(f,E') = \|P_{\mathscr{E}'}f|_{1\otimes\mathscr{K}}\|_2$ and $d_\infty(f,E') = \|P_{\mathscr{E}'}f\|$. Therefore (4.1) and (4.2) hold. The proof is complete. ∎

If we drop the condition that $\mathscr{K}$ is finite dimensional, then we can prove that $d_\infty(f,E') = \|P_{\mathscr{E}'}f\|$ and find g satisfying (4.1) and (4.5).

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