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Improbability of Nonconvergence in a Cubic Root-Finding Method

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Certain iterative methods for finding the roots of real functions can be profitably studied as dynamical systems on the real line. While many features, such as rates of convergence, of a root-finding method are local phenomena, other features are inherently nonlocal. This paper is concerned with one such feature, namely the structure of the set of initial conditions for which the method fails to converge.

In the case of Newton's Method,

$$x_{k+1} = Nf(x_k), \qquad \text{where} \quad Nf(x) = x - [f(x)/f'(x)],$$

the structure of the set of nonconvergent points has been studied by several authors. B. Barna ([1], see also [3, 9]) has shown that if f is a polynomial with all roots real, then the nonconvergent set is a Cantor set of measure zero. Recently D. Saari and J. Urenko [5, 8] have extended Barna's results to a much larger class of functions.

The purpose of this note is to show that similar results hold for the Euler method

$$x_{k+1} = Ef(x_k), \text{ where } Ef(x) = x - [f(x)/f'(x)][1 + f(x)\,f''(x)/2(f'(x))^2].$$

This algorithm has cubic convergence to simple roots of f; it is one of a family of root-finding algorithms whose description can be found in [6].

Throughout this paper we will assume that f satisfies the following properties. They are adapted from a similar list of properties used in [8] to study the structure of the set of nonconvergent points of Newton's method. The set of functions that satisfy all of these properties includes all polynomials with only real roots as well as other functions, including those described in [8, p. 44]. We give a brief discussion of these properties in the last section of this paper; the reader should consult [5, 8] for more details.

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(1.1) f is C^∞, and f has at least one nonvanishing derivative at each point. This ensures that if f, f' both vanish at p then we can define $Nf(p) = Ef(p) = p$ and thereby obtain functions that are smooth on the complement of the set $C = \{x \mid f'(x) = 0 \text{ and } f(x) \neq 0\}$.

(1.2) The zero set of f is finite, contains at least three points, and meets every connected component of the complement of C.

By (1.1) and (1.2), C is finite.

(1.3) If S, L are, respectively, the smallest and largest roots of f, then $ff'' > 0$ on $(-\infty, S) \cup (L, \infty)$.

By (1.2), f' is nonzero on the complement of $[S, L]$.

(1.4) There is a constant A in $(0, 2.5)$ with $A(f'')^2 - f'f''' \geqslant 0$.

Let $NC(Ef)$ denote the nonconvergent set for Ef, defined to be the set of x satisfying: all iterates $(Ef)^k(x)$ are defined ($k = 0, 1, \ldots$), but the sequence of iterates does not converge to a root of f.

THEOREM. *If f satisfies (1.1)–(1.4) then $NC(Ef)$ is a Cantor set of Lebesgue measure zero, and the restriction of Ef to $NC(Ef)$ is conjugate to a one-sided subshift of finite type.*

PRELIMINARIES

The theorem will be established by reducing questions about the Euler algorithm to questions about Newton's Method. We continue to assume that f satisfies (1.1)–(1.4).

(2.1) If $f(p) = 0$ then $Nf(p) = p$ and $(Nf)'(p) \in [0, 1)$; Nf has a vertical asymptote at each point of C.

(2.2) $(Nf)' = ff''/(f')^2$.

(2.3) If $c \in C$, then $(Nf)'(x)$ tends to $-\infty$ as x approaches c.

The proofs of (2.1)–(2.3) are straightforward calculations; further details can be found in [8].

Ef AND ITS RELATION TO Nf

(3.1) $Ef(x) = x$ if and only if either $f(x) = 0$ or $(Nf)'(x) = -2$.

(3.2) $(Ef)' = \frac{1}{2}[3(f'')^2 - f'f'''][f^2/(f')^4]$.

(3.3) Since f satisfies (1.4), $(Ef)' \geqslant 0$.

A calculation based on (1.1) yields

(3.4) If $f(p)=0$, then $(Ef)'(p)<1$,

so that Ef has an attracting fixed point at each root of f. Next we verify that the other fixed points of f are repelling.

LEMMA. *If* $|(Nf)'(x)|\geqslant k$, *then* $(Ef)'(x)\geqslant k^2(3-A)/2$.

Proof. $(Ef)' = \frac{1}{2}[3(f'')^2-f'f'''][(Nf)']^2/(f'')^2 \geqslant \frac{1}{2}k^2[3-(f'f''')/(f'')^2] \geqslant \frac{1}{2}k^2(3-A)$ by (1.4). ▮

Combining the lemma with (3.1) and (2.3) we see that

(3.5) If $Ef(q)=q$ and $f(q)\neq 0$, then $(Ef)'(q)\geqslant 2(3-A)>1$.

(3.6) If $c\in C$ then $(Ef)'(x)$ tends to ∞ as x approaches c.

Using (3.3) and (3.4) we obtain

(3.7) If I is an open interval containing a root p of f but no other fixed points of Ef, then for each x in I, the sequence of iterates $(Ef)^k(x)$ converges to p.

(3.8) ff' has a root between any two repelling fixed points of Ef (use (3.5)).

Claim. If f has d distinct roots, then there is a set X, composed of $2d-2$ pairwise disjoint compact intervals I_j and satisfying (3.9)–(3.12).

(3.9) $NC(Ef)=\bigcap_{n\geqslant 0}(Ef)^n(X)$.

(3.10) If $Ef(I_k)$ intersects I_j then it contains I_j.

(3.11) Ef is C^2 on each I_j.

(3.12) $(Ef)'\geqslant 2(3-A)>1$ on X.

Once the claim has been established, the first part of the theorem can be proved by following the arguments of [8], and the second part (dealing with the conjugacy to a subshift) is standard [3–5, 9]. We begin by noting

(3.13) $NC(Ef)$ is contained in the interval $[S, L]$,

which follows from (3.7), (1.3), and (3.1). Let the set C be as in (1.1). We will say that a bounded connected component in the complement of C is a *band* for Ef, and we will call the two unbounded components of the complement of C the *extreme bands* for Ef. By (1.2) each band or extreme band contains exactly one root of f, so there are $d-2$ bands.

Suppose $J=(\alpha, \beta)$ is a band. Let z denote the root of f in J. By (2.1) and (2.3), Ef has repelling fixed points u, v in J with $u<z<v$. By (3.8), u, v, z are all of the fixed points of Ef in J. Let S, L be as in (1.3). Then (3.6) and (3.7) imply that there are points $a<u$, $b>v$ in J with $Ef(a)=S$ and

$Ef(b)=L$. It now follows from (3.3), (3.7), and (3.13) that $NC(Ef)\cap J$ is contained in $[a,u]\cup[v,b]$. Since $(Nf)'(u)=-2$ and $(Nf)'\neq -2$ on (α,u), (2.3) shows that $(Nf)'\leqslant -2$ on $[a,u]$, and so by the lemma, $(Ef)'\geqslant 2(3-A)$ on $[a,u]$. Similarly on $[v,b]$. Thus if we let the intervals $[a,u]$, $[v,b]$ be two of the components of X, we see that (3.11) and (3.12) are satisfied on these intervals.

Next consider one of the extreme bands, say (γ,∞). This interval contains L, and by (3.4) and (3.6), Ef has a unique repelling fixed point q in (γ,L) and no fixed points in (L,∞). As in the previous case, we can show that there is a point t in (γ,q) with $Ef(t)=S$, $(Nf)'\leqslant -2$ on $[t,q]$, and $NC(Ef)\cap(\gamma,\infty)\subset[t,q]$. Consequently, if we let $[t,q]$ be one of the intervals in X, then (3.11) and (3.12) hold on this interval. The situation in the other extreme band is analogous. Thus we have defined X and verified (3.11) and (3.12); (3.9) and (3.10) are easily checked, and so the claim is established.

Remarks on Conditions (1.1)–(1.4)

In order to apply Urenko's argument to show that $NC(Ef)$ has measure zero, we need to know that Ef is C^2 on each of the intervals in X. This requires that f be at least C^4. If we allowed the possibility of all the derivatives of f vanishing at a point, two problems might arise. The first is that the set C might contain entire intervals. Second, there may be problems with the smoothness of Nf and Ef at multiple roots of f; in particular, (2.1) and (3.4) might fail. It is possible to overcome these problems by slightly altering some of our arguments, so in fact we could weaken (1.1) to

(4.1) f is C^4 and C is finite.

There are various conditions on f that will ensure (1.3). Among these is the condition from [8, 5]:

(4.2) (i) the roots of f' lie in $[S,L]$, and

(ii) if I is the smallest interval containing all the roots of f', then I contains all the roots of f''.

Also sufficient is

(4.3) (i) $[S,L]$ contains all the roots of f'', and

(ii) $f''(x)$ is bounded away from 0 as $|x|$ tends to infinity.

Either (4.2) or (4.3) implies that $(Nf)'$ is positive on the complement of $[S,L]$; the proof of the theorem requires only that $(Nf)'$ be bigger than -2.

Finally, (1.4) with $A = 1$ is a well-known inequality; Laguerre established that it is satisfied by polynomials with only real roots (the inequality is trivial if f is linear, and if f, g both satisfy the inequality, so does their product). Reference [7] contains a discussion of functions that satisfy Laguerre's inequality. Also if f has negative Schwartzian derivative, then f satisfies (1.4) with $A = 1.5$ [2].

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