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Generalized Hyperfunctions on the Circle

Vincent Valmorin

Département Math-Info, Université des Antilles et de la Guyane, Campus de Fouillole

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We give an embedding of the space $\mathscr{B}(\mathbb{T})$ of hyperfunctions on the unit circle $\mathbb{T}$ in a differential algebra $\mathscr{H}(\mathbb{T})$ whose elements are called generalized hyperfunctions. This allows us to define the product of two hyperfunctions without any restriction. We also define pointvalues of a hyperfunction: these pointvalues are elements of an algebra $\mathscr{C}$ whose set of invertible elements is denoted $\mathscr{C}^*$. In Section 2 we recall and make precise some basic results on classical spaces of functions on $\mathbb{T}$. Section 3 is devoted to our main results: we characterize the set $\mathscr{H}^*(\mathbb{T})$ of invertible elements of $\mathscr{H}(\mathbb{T})$, and, since a generalized hyperfunction may vanish at all classical points without being zero, we give a vanishing theorem. We conclude our work with the study of the Cauchy problem: $u' + fu + gu^2 = 0$; $u(z_0) = \mu$, where $f, g \in \mathscr{H}(\mathbb{T})$, $z_0 \in \mathbb{T}$, and $\mu \in \mathscr{C}^*$, by giving an existence theorem for a solution $u \in \mathscr{H}^*(\mathbb{T})$. © 2001 Academic Press

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1. INTRODUCTION

The starting point of this paper is an open problem posed by Oberguggenberger in [6], on the existence of embeddings of the space of hyperfunctions in differential algebras of new generalized functions. This is still an open problem in the non-periodic situation, even in the one dimensional case. In the one dimensional periodic case, a first answer was recently given by the author in a talk at the Erwin Schrödinger Institut of Viena [10]. The given construction of such an algebra was based on the use of real periodic analytic functions, and, as in the usual constructions [7–9], derivatives of all orders are considered. In the analytic case, this leads to some complications in the applications, in proving, for example, the

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analiticity of solutions to differential equations. In order to overcome these difficulties and to give at the same time a simple construction, we consider here the problem in its natural setting: that of holomorphic functions of the unit circle, the set of complex numbers of modulus one. At the same time, bounds of infraexponential type are replaced by ones of exponential type in order to be able to solve in a natural way simple differential problems as in the last section of this paper.

Our construction follows those given in the n-dimensional case in [7–9] by the author, for the embedding of the space of periodic distributions in an algebra of new generalized functions. Here we take the algebra of holomorphic functions on the circle as the basic algebra instead of the one of smooth functions on the unit circle. This enables one to use Laurent series representations in this space and in the attached space of hyperfunctions. As it happens in the case of algebras of new generalized functions with an embedding of the space of distributions [7–9], such an algebra can permit us to investigate in a good setting linear or nonlinear differential problems involving periodicity and high singularities. In our case, we may consider singularities which can be interpreted, for example, as products of hyperfunctions.

2. BASIC FUNCTIONS ON THE CIRCLE

In this section we refer to [1, Chap. 1; 4, Chap. 4] and mainly to [5].

2.1. *Smooth Functions on the Circle*

Let f be a function defined on the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then f is smooth if and only if the function $\tilde{f}$ defined on $\mathbb{R}$ by $\tilde{f}(t) = f(e^{it})$ is smooth. We denote by $\mathscr{E}(\mathbb{T})$ the space of smooth functions on $\mathbb{T}$ equipped with the family of seminorms

$$\|f\|_{(m)} = \sup_{0 \le t \le 2\pi} |\tilde{f}^{(m)}(t)|.$$

$\mathscr{E}(\mathbb{T})$ becomes a Fréchet space with the topology defined by this family of seminorms.

If $f \in \mathscr{E}(\mathbb{T})$ and $k \in \mathbb{Z}$, the Fourier coefficient of index k of f is

$$\hat{f}(k) = \frac{1}{2i\pi} \int_{|z|=1} f(z) z^{-k-1}\, dz = \frac{1}{2\pi} \int_0^{2\pi} \tilde{f}(t) e^{-ikt}\, dt.$$

A sequence $(c_k)_k$ is the sequence of Fourier coefficients of a smooth function if and only if it is rapidly decreasing; that is, for any positive s we have $\sup_{k\in\mathbb{Z}}(|k|^s|c_k|) < \infty$. Moreover the sequence $(\sum_{|k|\le n} \hat{f}(k)z^k)_n$ converges to f in the topology of $\mathscr{E}(\mathbb{T})$.

2.2. *Analytic Functions on the Circle*

Let $r > 1$ and $C_r = \{z \in \mathbb{C}; 1/r < |z| < r\}$. We denote by $\mathscr{O}_r$ the Banach space of bounded holomorphic functions in C_r endowed with the norm $\|f\|_r = \sup_{z\in C_r}|f(z)|$. The space of analytic functions on $\mathbb{T}$ is $\mathscr{A}(\mathbb{T}) = \operatorname{ind}\lim\{\mathscr{O}_r; r > 1\}$, the inductive limit of the spaces $\mathscr{O}_r$ as $r \to 1$. Let $f \in \mathscr{A}(\mathbb{T})$. Then f belongs to some $\mathscr{O}_r$. Let $1 < \rho < r$; f has a Laurent expansion $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ in C_r where $c_k = \frac{1}{2i\pi}\int_{|z|=r'} f(z)z^{-k-1}\,dz$, r' being any number such that $1/\rho \le r' \le \rho$. Taking $r' = \rho$ if $k \ge 0$ and $r' = 1/\rho$ otherwise, we obtain $|c_k| \le \|f\|_\rho \rho^{-|k|}$. It follows that for all $k \in \mathbb{Z}$, $|c_k| \le \|f\|_r r^{-|k|}$. If one takes $r' = 1$, it is easily seen that c_k is the corresponding Fourier coefficient of the associated analytic function $\tilde{f}$ defined on $\mathbb{R}$. So the c_k's are called the Fourier coefficients of f. We have the following characterization: a sequence $(a_k)_k$ is the sequence of Fourier coefficients of a $f \in \mathscr{A}(\mathbb{T})$ if and only if $\limsup_{|k|\to\infty}|a_k|^{1/|k|} < 1$.

THEOREM 2.1. *$\mathscr{A}(\mathbb{T})$ is continuously embedded in $\mathscr{E}(\mathbb{T})$ as a dense subspace.*

Proof. Clearly the considered map is one to one. Let $f(z) = \sum_{k=-\infty}^{\infty} \hat{f}(k)z^k$ be an element of $\mathscr{E}(\mathbb{T})$. The functions $g_n(z) = \sum_{|k|\le n} \hat{f}(k)z^k$ belong to $\mathscr{A}(\mathbb{T})$ and converge to f in the topology of $\mathscr{E}(\mathbb{T})$ showing the density of $\mathscr{A}(\mathbb{T})$ in $\mathscr{E}(\mathbb{T})$. We now consider $f(z) = \sum_{k=-\infty}^{\infty} c_k z^k$ in $\mathscr{O}_r$. Hence we have $\tilde{f}(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$. It follows that if $m \in \mathbb{N}\setminus\{0\}$, then $\tilde{f}^{(m)}(t) = \sum_{k=-\infty}^{\infty}(ik)^m c_k e^{ikt}$ and consequently $|\tilde{f}^{(m)}(t)| \le \|f\|_r \sum_{k=-\infty}^{\infty}|k|^m r^{-|k|}$. Now, using $\sup_{x>0} x^m r^{-x} = (1/\ln r)^m e^{-m} m^m$, we find $|\tilde{f}^{(m)}(t)| \le \|f\|_r \sum_{k=-\infty}^{\infty}(1/\sqrt{r})^{|k|}(2/\ln r)^m e^{-m} m^m$. Hence, we obtain

$$\|f\|_{(m)} \le C(am)^m \|f\|_r,$$

where a and C are constants depending only on r. This shows the continuity of the embedding on $\mathscr{O}_r$ and hence on $\mathscr{A}(\mathbb{T})$. ∎

2.3. *Distributions on the Circle*

The space of distributions on the circle is the topological dual $\mathscr{E}'(\mathbb{T})$ of $\mathscr{E}(\mathbb{T})$. This means that a sequence $(T_m)_m$ of distributions converges to a distribution T if and only if for each $f \in \mathscr{E}(\mathbb{T})$, $T_m(f)$ converges to $T(f)$ in $\mathbb{C}$. Let $k \in \mathbb{Z}$ and $T \in \mathscr{E}'(\mathbb{T})$. The function $z \mapsto z^k$ defined on $\mathbb{T}$ belongs to $\mathscr{E}(\mathbb{T})$. The Fourier coefficient of index k of T is the number $\hat{T}(k) = T(z \mapsto z^k)$. We have $T = \sum_{k=-\infty}^{\infty} \hat{T}(k)z^k$ in the topology of $\mathscr{E}'(\mathbb{T})$. The Fourier coefficients are characterized as follows: $(A_k)_k$ is the sequence of

Fourier coefficients of a distribution if and only if $(A_k)_k$ is slowly increasing; that is, there are positive constants C and s such that for all $k \in \mathbb{Z}$, $|A_k| \leq C(1 + |k|^s)$. Moreover, if $f \in \mathscr{E}(\mathbb{T})$, then $T(f) = \sum_{k=-\infty}^{\infty} \overline{\hat{T}(k)}\hat{f}(k)$.

2.4. *Hyperfunctions on the Circle*

The space $\mathscr{B}(\mathbb{T})$ of hyperfunctions on the circle is the topological dual $\mathscr{A}'(\mathbb{T})$ of $\mathscr{A}(\mathbb{T})$. Consequently, $H \in \mathscr{B}(\mathbb{T})$ if and only if H is continuous on each $\mathscr{O}_r$. Let $k \in \mathbb{Z}$ and $H \in \mathscr{B}(\mathbb{T})$. Since the function $z \mapsto z^k$ belongs to $\mathscr{A}(\mathbb{T})$, we define the Fourier coefficient of index k of H as being the number $\hat{H}(k) = \overline{H(z \mapsto z^k)}$. Moreover $H = \sum_{k=-\infty}^{\infty} \hat{H}(k) z^k$ in the topology of $\mathscr{B}(\mathbb{T})$. We have the following characterization: $(B_k)_k$ is the sequence of Fourier coefficients of a hyperfunction if and only if $(B_k)_k$ is of infraexponential type, that is, $\limsup_{|k| \to \infty} |B_k|^{1/|k|} \leq 1$. Moreover, if $g \in \mathscr{A}(\mathbb{T})$, then $H(f) = \sum_{k=-\infty}^{\infty} \overline{\hat{H}(k)}\hat{g}(k)$.

3. GENERALIZED HYPERFUNCTIONS ON THE CIRCLE

3.1. *The Algebra $\mathscr{X}_e(\mathbb{T})$ and the Ideal $\mathscr{N}_e(\mathbb{T})$*

Let $\mathscr{X}(\mathbb{T})$ denote the set of sequences of functions $(f_n)_n$ where $n \in \mathbb{N}$ and $f_n \in \mathscr{A}(\mathbb{T})$, and let $\mathscr{X}_e(\mathbb{T})$ denote the subset of $\mathscr{X}(\mathbb{T})$ whose elements $(f_n)_n$ are such that

$$\exists a > 1, \exists \eta \in \mathbb{N}, \exists r > 1 / \forall n > \eta, f_n \in \mathscr{O}_r, \|f_n\|_r \leq a^n.$$

We denote by $\mathscr{N}_e(\mathbb{T})$ the subset of $\mathscr{X}_e(\mathbb{T})$ whose elements $(f_n)_n$ satisfy the condition

$$\forall b \in]0,1[, \exists \eta \in \mathbb{N}, \exists r > 1 / \forall n > \eta, f_n \in \mathscr{O}_r, \|f_n\|_r \leq b^n.$$

Clearly $\mathscr{X}_e(\mathbb{T})$ is an algebra for usual termwise operations and $\mathscr{N}_e(\mathbb{T})$ is an ideal of $\mathscr{X}_e(\mathbb{T})$. We now define the following two spaces of sequences of complex numbers.

$\mathscr{X}_e(\mathbb{Z})$ is the set of sequences $(c_n)_n$ of sequences $c_n = (c_{n,k})_k$ such that

$$\exists a > 1, \exists \eta \in \mathbb{N}, \exists r > 1 / \forall n > \eta, \forall k \in \mathbb{Z}, |c_{n,k}| \leq a^n r^{-|k|}.$$

$\mathscr{N}_e(\mathbb{Z})$ is the set of sequences $(c_n)_n$ of sequences $c_n = (c_{n,k})_k$ such that

$$\forall b \in]0,1[, \exists \eta \in \mathbb{N}, \exists r > 1 / \forall n > \eta, \forall k \in \mathbb{Z}, |c_{n,k}| \leq b^n r^{-|k|}.$$

It is clear that $\mathscr{N}_e(\mathbb{Z})$ is a subset of $\mathscr{X}_e(\mathbb{Z})$. We have the following proposition:

PROPOSITION 3.1. *Let $(f_n)_n$ be a family of functions f_n which are holomorphic in some neighborhood of $\mathbb{T}$. Then we have*:

(i) $(f_n)_n \in \mathscr{X}_e(\mathbb{T})$ *if and only if* $(\hat{f}_n(k))_{n,k} \in \mathscr{X}_e(\mathbb{Z})$.

(ii) $(f_n)_n \in \mathscr{N}_e(\mathbb{T})$ *if and only if* $(\hat{f}_n(k))_{n,k} \in \mathscr{N}_e(\mathbb{Z})$.

Proof. Suppose that $(f_n)_n \in \mathscr{X}_e(\mathbb{T})$. Let $a > 1$, $r > 1$, and $\eta \in \mathbb{N}$ such that $f_n \in \mathscr{O}_r$ and $\|f_n\|_r \leq a^n$ for $n > \eta$. The Cauchy inequalities $|\hat{f}_n(k)| \leq \|f_n\|_r r^{-|k|}$ for $n > \eta$ and $k \in \mathbb{Z}$ show that $(\hat{f}_n(k))_{n,k} \in \mathscr{X}_e(\mathbb{Z})$. Conversely, suppose that $(\hat{f}_n(k))_{n,k} \in \mathscr{X}_e(\mathbb{Z})$. Choose $a > 1$, $r > 1 \eta \in \mathbb{N}$ such that $|\hat{f}_n(k)| \leq a^n r^{-|k|}$ for all $n > \eta$ and all $k \in \mathbb{Z}$. Let ρ be a number such that $0 < \rho < r$. We have $f_n(z) = \Sigma_{-\infty}^{+\infty} \hat{f}_n(k) z^k$ for $z \in C_\rho$ and $n > \eta$. It follows that $|f_n(z)| \leq a^n \Sigma_{-\infty}^{+\infty} (\rho/r)^{|k|} \leq C a^n$ for some constant $C > 0$. From that, we derive that $(f_n)_n \in \mathscr{X}_e(\mathbb{T})$. Part (ii) can be proved in the same way. ∎

Let $f \in \mathscr{A}(\mathbb{T})$. We have $f(z) = \Sigma_{-\infty}^{+\infty} \hat{f}(k) z^k$ in some C_r. We define

$$(\partial_\theta f)(z) = \sum_{-\infty}^{+\infty} (ik).\hat{f}(k) z^k.$$

If we consider the usual differentiation, we have

$$f'(z) = \sum_{-\infty}^{+\infty} k\hat{f}(k) z^{k-1} = \sum_{-\infty}^{+\infty} (k+1)\hat{f}(k+1) z^k.$$

It follows that

$$(\partial_\theta f)(z) = izf'(z)$$

$$\hat{f'}(k) = (k+1)\hat{f}(k+1)$$

$$\widehat{(\partial_\theta f)}(k) = ik\hat{f}(k).$$

Now, let $s \in \mathbb{Z}$ and $P \in \mathbb{C}[X]$. Define the operators $\breve{P}$ and t_s in the space of families of complex sequences of numbers $c = (c_n)$ where $c_n = (c_{n,k})_k$ with $k \in \mathbb{Z}$ by $\breve{P}.c = (P(k)c_{n,k})_{n,k}$ and $t_s.c = (c_{n,k+s})_{n,k}$. Then we have:

PROPOSITION 3.2. *Let $s \in \mathbb{Z}$ and $P \in \mathbb{C}[X]$. Then $(\breve{P} \circ t_s)\mathscr{X}_e(\mathbb{Z}) \subset \mathscr{X}_e(\mathbb{Z})$ and $(\breve{P} \circ t_s)\mathscr{N}_e(\mathbb{Z}) \subset \mathscr{N}_e(\mathbb{Z})$.*

Proof. Let $c = (c_{n,k})_{n,k} \in \mathscr{X}_e(\mathbb{Z})$. Let $a > 1$, $r > 1$, $\eta' \in \mathbb{N}$ such that $|c_{n,k}| \leq a^{n/2} r^{-2|k|}$ for all $k \in \mathbb{Z}$ and $n > \eta'$. Setting $d_n = (d_{n,k})_k$ with $d_{n,k} = P(k)c_{n,k+s}$, it follows that $|d_{n,k}| \leq |P(k)| a^{n/2} r^{-2|k+s|}$. According to $||k| - |k+s|| \leq |s|$ and $r > 1$, we obtain $r^{|k|-|k+s|} \leq r^{|s|}$ whence $|d_{n,k}| \leq$

$|P(k)|r^{-|k+s|}r^{|k|-|k+s|}a^{n/2}r^{-|k|} \leq (|P(k)|r^{-|k+s|+|s|}a^{-n/2})a^n r^{-|k|}$. Now, there is $\eta > \eta'$ such that for all $n > \eta$ and $k \in \mathbb{Z}$, $|P(k)|r^{-|k+s|+|s|}a^{-n/2} \leq 1$. Hence for all $n > \eta$ and $k \in \mathbb{Z}$ we have $|d_{n,k}| \leq a^n r^{-|k|}$ showing that $(d_n)_n \in \mathscr{X}_e(\mathbb{Z})$. The second part of the proposition can be proved in the same way. ∎

On a family $(f_n)_n$, the differential operators are defined componentwise. Hence, the following corollary is a straightforward consequence of the above proposition:

COROLLARY 3.3. *$\frac{d}{dz}(\mathscr{X}_e(\mathbb{T})) \subset \mathscr{X}_e(\mathbb{T})$ and $\partial_\theta(\mathscr{X}_e(\mathbb{T})) \subset \mathscr{X}_e(\mathbb{T})$. The same relations hold for $\mathscr{N}_e(\mathbb{T})$.*

3.2. *The Algebra of Generalized Hyperfunctions on the Circle*

DEFINITION 3.1. The algebra of generalized hyperfunctions on $\mathbb{T}$ is the factor algebra

$$\mathscr{H}(\mathbb{T}) = \mathscr{X}_e(\mathbb{T})/\mathscr{N}_e(\mathbb{T}).$$

The class of $(f_n)_n$ in $\mathscr{H}(\mathbb{T})$ will be denoted by $\mathrm{cl}(f_n)$ or $[f_n]$.

According to Corollary 3.3, we endow $\mathscr{H}(\mathbb{T})$ with two differential structures defined by

$$\frac{d}{dz}[f_n] = \left[\frac{df_n}{dz}\right] \qquad \text{and} \qquad \partial_\theta[f_n] = [\partial_\theta f_n].$$

Let $f, g \in \mathscr{O}_r$. We define the convolution of f and g by

$$(f * g)(z) = \frac{1}{2i\pi}\int_{|w|=\rho} f(w)g(zw^{-1})w^{-1}\,dw,$$

where $z \in C_r$ and ρ verify $\sup(1/r, |z|/r) < \rho < \inf(r, r|z|)$. It follows that

$$(f * g)(z) = \sum_{-\infty}^{+\infty} \hat{f}(k)\hat{g}(k)z^k.$$

This above formula allows us to define $S * T$ when S and T are hyperfunctions:

$$(S * T)(z) = \sum_{-\infty}^{+\infty} \hat{S}(k)\hat{T}(k)z^k.$$

It follows that $S * f \in \mathscr{A}(\mathbb{T})$ if $S \in \mathscr{B}(\mathbb{T})$ and $f \in \mathscr{A}(\mathbb{T})$. In the same way $S * f \in \mathscr{E}(\mathbb{T})$ if $S \in \mathscr{E}'(\mathbb{T})$ and $f \in \mathscr{E}(\mathbb{T})$.

If $n \in \mathbb{N}$ we set $\varphi_n(z) = \sum_{|k| \leq n} z^k$. We have $\varphi_n * \varphi_n = \varphi_n$ and $\lim_{n \to \infty} \varphi_n = \delta$ in $\mathscr{E}'(\mathbb{T})$ where δ is the Dirac distribution. Let $\mathscr{Q}(\mathbb{T})$ denote the subset of $\mathscr{A}(\mathbb{T})$ which consists of functions f such that $\hat{f}(k) = 0$ for $|k|$ large enough. We set $\tilde{\mathscr{X}}_e(\mathbb{T}) = \mathscr{X}_e(\mathbb{T}) \cap \mathscr{Q}(\mathbb{T})^{\mathbb{N}}$ and $\tilde{\mathscr{N}}_e(\mathbb{T}) = \mathscr{N}_e(\mathbb{T}) \cap \mathscr{Q}(\mathbb{T})^{\mathbb{N}}$. Clearly $\tilde{\mathscr{X}}_e(\mathbb{T})$ is an algebra and $\tilde{\mathscr{N}}_e(\mathbb{T})$ is an ideal of $\tilde{\mathscr{X}}_e(\mathbb{T})$. If $H \in \mathscr{B}(\mathbb{T})$, then $(H * \varphi_n)(z) = \sum_{|k| \leq n} \hat{H}(k) z^k$ and $\lim_{n \to \infty} H * \varphi_n = H$ in $\mathscr{B}(\mathbb{T})$. Consider the maps $\mathbf{i} : \mathscr{B}(\mathbb{T}) \to \mathscr{X}(\mathbb{T})$ and $\mathbf{i}_0 : \mathscr{A}(\mathbb{T}) \to \mathscr{X}(\mathbb{T})$ defined by $\mathbf{i}(H) = (H * \varphi_n)_n$ and $\mathbf{i}_0(f) = (f_n)_n$, where $f_n = f$ for all n.

Proposition 3.4. *$\mathbf{i}$ is a linear embedding and $\mathbf{i}_0$ is a one to one morphism of algebras such that $\mathbf{i}(\mathscr{B}(\mathbb{T})) \subset \mathscr{X}_e(\mathbb{T})$ and $\mathbf{i}_0(\mathscr{A}(\mathbb{T})) \subset \mathscr{X}_e(\mathbb{T})$. More precisely, if $H \in \mathscr{B}(\mathbb{T})$ then for any $a > 1$ there exist $r > 1$, $\eta \in \mathbb{N}$ such that $\|H * \varphi_n\|_r < a^n$ for $n > \eta$. If $f \in \mathscr{A}(\mathbb{T})$, then there exist $b \in]0, 1[$, $r > 1$, and $\eta \in \mathbb{N}$ such that $\|f - f * \varphi_n\|_r < b^n$ for $n > \eta$.*

Proof. The part of the proposition corresponding to i_0 is obvious. Let $H \in \mathscr{B}(\mathbb{T})$ and choose $a > 1$. Take $r = a^{1/3}$. There is a constant $C > 0$ such that $|\hat{H}(k)| \leq Cr^{|k|}$ for any $k \in \mathbb{Z}$. For $|k| \leq n$ one has $|\hat{H}(k)| \leq Cr^{2n}r^{-|k|}$. Further there exists $\eta \in \mathbb{N}$ such that $C < r^n$ for $n > \eta$ whence we obtain $|\hat{H}(k)| \leq r^{3n}r^{-|k|} = a^n r^{-|k|}$ for $n > \eta$ and $k \in \mathbb{Z}$. It follows that $(\widehat{H * \varphi_n}(k))_n \in \mathscr{X}_e(\mathbb{Z})$. Hence from Proposition 3.1, one obtains $(H * \varphi_n)_n \in \mathscr{X}_e(\mathbb{T})$. Now let $f \in \mathscr{A}(\mathbb{T})$ and set $f_n = f - f * \varphi_n$. We have $f_n = \sum_{|k| > n} \hat{f}(k) z^k$. Let $r > 1$ and $C' > 0$ such that $|\hat{f}(k)| \leq C' r^{-|k|}$ for all $k \in \mathbb{Z}$ and set $\rho = \sqrt{r}$. Since $\rho^{-|k|} \leq \rho^{-n}$ for $|k| > n$, it follows that $|\hat{f}_n(k)| \leq C' \rho^{-n} \rho^{-|k|}$. Now there exists $\eta' \in \mathbb{N}$ such that $C' \rho^{-n/2} < 1$ for all $n > \eta'$. Hence $|\hat{f}_n(k)| \leq \rho^{-n/2} \rho^{-|k|}$ for any $n > \eta$ and any $|k| > n$. From this it is easily shown that the announced inequality holds. ∎

This proposition leads straightforwardly to the following

Corollary 3.5. *Let $\bar{\mathbf{i}} : \mathscr{B}(\mathbb{T}) \to \mathscr{H}(\mathbb{T})$ and $\bar{\mathbf{i}}_0 : \mathscr{A}(\mathbb{T}) \to \mathscr{H}(\mathbb{T})$. Then,*

$$H \mapsto [H_n], \qquad f \mapsto [f]$$

$\bar{\mathbf{i}}$ is a linear embedding and $\bar{\mathbf{i}}_0$ is a one to one morphism of algebras such that $\bar{\mathbf{i}} \mid_{\mathscr{Q}(\mathbb{T})} = \bar{\mathbf{i}}_0$. Moreover, for any $H \in \mathscr{B}(\mathbb{T})$ one has $\bar{\mathbf{i}}(\frac{dH}{dz}) = \frac{d}{dz}(\bar{\mathbf{i}}(H))$ and $\bar{\mathbf{i}}(\partial_\theta H) = \partial_\theta(\bar{\mathbf{i}}(H))$.

In the sequel, an element of $\bar{\mathbf{i}}(\mathscr{B}(\mathbb{T})$ will be referred to as a hyperfunction of $\mathscr{H}(\mathbb{T})$. By construction, the embedding of $\mathscr{B}(\mathbb{T})$ in $\mathscr{H}(\mathbb{T})$ shows that any hyperfunction in $\mathscr{H}(\mathbb{T})$ has a representative in $\tilde{\mathscr{X}}_e(\mathbb{T})$. In fact, this is valid for all generalized hyperfunctions.

Proposition 3.6. *Let $\sigma : \mathbb{N} \to \mathbb{N}$. Then $(f_n - f_n * \varphi_{\sigma(n)})_n \in \mathscr{N}_e(\mathbb{T})$ for all $(f_n)_n \in \mathscr{X}_e(\mathbb{T})$, if and only if $\lim_{n \to \infty} \sigma(n)/n = \infty$. Consequently, any generalized hyperfunction has infinitely many representatives in $\tilde{\mathscr{X}}_e(\mathbb{T})$.*

Proof. Suppose that $\lim_{n\to\infty}\sigma(n)/n=\infty$. Let $(f_n)_n\in\mathscr{X}_e(\mathbb{T})$. There are $a>1$, $r>1$, and $\eta\in\mathbb{N}$ such that $|\hat{f}_n(k)|\le a^n r^{-|k|}$ for $n>\eta$ and $k\in\mathbb{Z}$. Set $\rho=\sqrt{r}$. If $n>\eta$ and $|k|>\sigma(n)$, then $|\hat{f}_n(k)|\le a^n\rho^{-\sigma(n)}\rho^{-|k|}=(a\rho^{-\sigma(n)/n})^n\rho^{-|k|}$. Let $b\in]0,1[$. Since $\lim_{n\to\infty}\sigma(n)/n=\infty$, then there are $\eta'\in\mathbb{N}$, $\eta'\ge\eta$ such that $a\rho^{-\sigma(n)/n}\le b$ for all $n>\eta'$. Hence $|\hat{f}_n(k)|\le b^n\rho^{-|k|}$ for $n>\eta'$ and $|k|>\sigma(n)$, showing by use of Proposition 3.1 that $(f_n-f_n*\varphi_{\sigma(n)})_n\in\mathscr{N}_e(\mathbb{T})$. Conversely, suppose that $(f_n-f_n*\varphi_{\sigma(n)})_n\in\mathscr{N}_e(\mathbb{T})$ for all $(f_n)_n\in\mathscr{X}_e(\mathbb{T})$. Let $a>1$ and $(f_n)_n\in\mathscr{X}(\mathbb{T})$ such that $\hat{f}_n(k)=a^{n-|k|}$. From Proposition 3.1, $(f_n)_n\in\mathscr{X}_e(\mathbb{T})$. Let $s>0$ and $b=a^{-s}\in]0,1[$. Since $(f_n-f_n*f\varphi_{\sigma(n)})_n\in\mathscr{N}_e(\mathbb{T})$, there are $r>1$, $\eta\in\mathbb{N}$ such that $a^{n-|k|}<b^n r^{-|k|}$ for $n>\eta$ and $k>\sigma(n)$. Taking $k=\sigma(n)+1$, it follows that $\forall s>0, \exists\eta\in\mathbb{N}, \forall n>\eta, \sigma(n)/n>s$. That is, $\lim_{n\to\infty}\sigma(n)/n=\infty$. ∎

This proposition shows the surjectivity of the map from $\tilde{\mathscr{X}}_e(\mathbb{T})$ to $\mathscr{H}(\mathbb{T})$ which associates to each element its class. From the definition of $\tilde{\mathscr{N}}_e(\mathbb{T})$, this map is also injective, so we obtain:

COROLLARY 3.7. *We have* $\mathscr{H}(\mathbb{T})=\tilde{\mathscr{X}}_e(\mathbb{T})/\tilde{\mathscr{N}}_e(\mathbb{T})$.

Remark 3.1. It may be seen that $(\varphi_{\sigma(n)})_n\in\mathscr{X}_e(\mathbb{T})$ if and only if $\sigma(n)/n$ is bounded.

3.3. *Generalized Numbers of Exponential Type*

Let $\mathscr{S}$ be the set of complex valued sequences $(z_n)_{n\in\mathbb{N}}$. Such an element will be simply denoted $(z_n)_n$. Let $\mathscr{C}_e$ be set of $(z_n)_n\in\mathscr{S}$ such that

$$\exists a>1, \exists\eta\in\mathbb{N}/\forall n>\eta, |z_n|\le a^n.$$

We denote by $\mathscr{I}_e$ the set of elements $(z_n)_n\in\mathscr{C}_e$ such that

$$\forall b\in]0,1[, \exists\eta\in\mathbb{N}/\forall n>\eta, |z_n|\le b^n.$$

It may be seen that $\mathscr{C}_e$ is a subalgebra of $\mathscr{S}$ and that $\mathscr{I}_e$ is an ideal of $\mathscr{C}_e$.

DEFINITION 3.2. The algebra of complex generalized constants of *exponential type*, is the quotient algebra $\mathscr{C}=\mathscr{C}_e/\mathscr{I}_e$.

We shall refer to an element of $\mathscr{C}$ as a generalized number or a generalized constant. Every complex number z is identified with a generalized number in a natural way. They will be referred to as classical numbers. We denote by $\mathscr{T}$ the subset of $\mathscr{C}$ formed by the elements having a representative where each term belongs to $\mathbb{T}$.

Let $f\in\mathscr{H}(\mathbb{T})$ with a representative $(f_n)_n$. Let $z\in\mathscr{T}$ with a representative $(z_n)_n$ such that $z_n\in\mathbb{T}$. There exist $a>1$, $r>1$, and $\eta\in\mathbb{N}$ such that for all $n>\eta$, $\|f_n\|_r\le a^n$. It follows that $|f_n(z_n)|\le a^n$ for $n>\eta$, showing that $(f_n(z_n))_n\in\mathscr{C}_e$. For $(v_n)_n\in\mathscr{X}_e(\mathbb{T})$ and $(y_n)_n\in\mathscr{C}_e$, we set $v_n(y_n)=0$ if

v_n is not defined at the point y_n. Now let $(\psi_n)_n \in \mathcal{N}_e$ and $(t_n)_n \in \mathcal{I}_e$. Let $b \in]0,1[$. Choose $a > 1$, $r(1) > 1$, and $\eta_1 \in \mathbb{N}$ such that $\|f_n'\|_{r(1)} \le a^n$ for $n > \eta_1$ and set $b_1 = b^2/a \in]0,1[$. There exist $r(2) > 1$ and $\eta_2 \in \mathbb{N}$ such that $|t_n| \le b_1^n$ and $\|\psi_n\|_{r(2)} \le b_1^n$ for $n > \eta_2$. Let $\eta \ge \max(\eta_1, \eta_2)$ such that $2b^n < 1$ for $n > \eta$ and choose r such that $1 < r < \min(r(1), r(2), b_1 + 1)$. From the condition $r < b_1 + 1$, it follows that $z_n + t_n \in C_r$ when $|t_n| \le b_1^n$. Taking $n > \eta$ and writing $(f_n + \psi_n)(z_n + t_n) - f_n(z_n) = f_n(z_n + t_n) - f_n(z_n) + \psi_n(z_n + t_n)$, we find $|f_n(z_n + t_n) - f_n(z_n)| \le |t_n| \, \|f_n'\|_r + \|\psi_n\|_r \le b_1^n(a^n + 1) \le 2b^{2n} < b^n$. It follows that the equality $[(f_n + \psi_n)(z_n + t_n)] = [f_n(z_n)]$ holds in $\mathcal{C}$. This enables us to give the following

Definition 3.3. Let $f \in \mathcal{H}(\mathbb{T})$ and $z \in \mathcal{T}$. The value of f at z is the generalized number $f(z) = [f_n(z_n)]$ where $(f_n)_n$ and $(z_n)_n$ are arbitrary representatives of f and z, respectively.

Let $f \in \mathcal{H}(\mathbb{T})$. Let $(f_n)_n$ and $(g_n)_n$ denote two representatives of f. We have

$$\left| \frac{1}{2i\pi} \int_{|z|=1} f_n(z) z^{-k-1}\, dz - \frac{1}{2i\pi} \int_{|z|=1} g_n(z) z^{-k-1}\, dz \right| \le \sup_{z \in \mathbb{T}} |f_n(z) - g_n(z)|.$$

From $(f_n - g_n)_n \in \mathcal{N}_e(\mathbb{T})$, it follows that $cl(\sup_{z\in\mathbb{T}} |f_n(z) - g_n(z)|) \in \mathcal{I}_e$, whence $cl(\frac{1}{2i\pi}\int_{|z|=1} f_n(z) z^{-k-1}\, dz) = cl(\frac{1}{2i\pi}\int_{|z|=1} g_n(z) z^{-k-1}\, dz)$. This leads to the following

Definition 3.4. The Fourier coefficient of rank k of the generalized hyperfunction f is the generalized number

$$\hat{f}(k) = cl\left(\frac{1}{2i\pi} \int_{|z|=1} f_n(z) z^{-k-1}\, dz \right),$$

where $(f_n)_n$ is an arbitrary representative of f.

Proposition 3.8. *Let $f \in \mathcal{H}(\mathbb{T})$.*

(i) *f admits a primitive in $\mathcal{H}(\mathbb{T})$ with respect to ∂_θ if and only if $\hat{f}(0) = 0$.*

(ii) *f admits a primitive in $\mathcal{H}(\mathbb{T})$ with respect to d/dz if and only if $\hat{f}(-1) = 0$.*

Proof. Since (i) and (ii) are equivalent, it is sufficient to prove (ii). Let $f \in \mathcal{H}(\mathbb{T})$. Suppose that $\hat{f}(-1) = 0$. Let $(f_n)_n$ be a representative of f. Define $(F_n)_n$ by $\widehat{F_n}(k) = \hat{f_n}(k)/(k+1)$ if $k \ne -1$ and $\widehat{F_n}(-1) = 0$. It is clear that $(F_n)_n \in \mathscr{X}_e(\mathbb{T})$; set $F = [F_n]$. Moreover, since $\widehat{F'_n}(k) = (k+1)\widehat{F_n}(k)$, the only nonzero Fourier coefficient of $f_n - F'_n$ is $\hat{f}_n(-1) - \widehat{F'_n}(-1) = \hat{f}_n(-1)$. Hence, from $\hat{f}(-1) = 0$ and Proposition 3.1, we find $F' = f$. Conversely, suppose that there is $F \in \mathscr{H}(\mathbb{T})$ such that $F' = f$. Set $F = [F_n]$ and $f = [f_n]$. Since $(F'_n - f_n)_n \in \mathscr{N}_e(\mathbb{T})$, for all $b \in]0,1[$ there are $r > 1$, $\eta \in \mathbb{N}$ such that $|(k+1)\widehat{F_n}(k) - \hat{f_n}(k)| < b^n r^{-|k|}$ for all $k \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Taking $k = -1$, we obtain $(\hat{f_n}(-1))_n \in \mathscr{I}_e$, that is, $\hat{f}(-1) = 0$. ∎

3.4. *Invertible Elements*

Let $\mathscr{C}^*$ denote the subset of invertible elements of $\mathscr{C}$. Then we have the following characterisation of $\mathscr{C}^*$:

THEOREM 3.9. *Let $x \in \mathscr{C}$. Then $x \in \mathscr{C}^*$ if and only if x admits a representative $(x_n)_n$ such that*

$$\exists b \in]0,1[, \exists \eta \in \mathbb{N} / \forall n > \eta, |x_n| > b^n. \qquad (*)$$

Proof. First, the condition $(*)$ doesn't depend on the chosen representative $(x_n)_n$. Let $(y_n)_n$ denote another representative of x. From $(*)$, there are b_1, $\eta_1 \in \mathbb{N}$ such that $|x_n| > b_1^n$ for $n > \eta_1$. Let $b \in]0, b_1[$. Since $(x_n - y_n)_n \in \mathscr{I}_e$, there is $\eta_2 \in \mathbb{N}$ such that $|x_n - y_n| < b^n$ for $n > \eta_2$. Let $\eta \geq \max(\eta_1, \eta_2)$. If $n > \eta$, then $x_n \neq 0$, and $|1 - y_n/x_n| = |x_n - y_n|\,|x_n|^{-1} < (bb_1)^n$. Since $bb_1 < 1$, we may suppose that η is large enough for $|y_n/x_n| > 1/2$ when $n > \eta$. It follows that if $n > \eta$, then $|y_n| > |x_n|/2 \geq (b_1/2)^n$, showing that $(y_n)_n$ fulfills $(*)$. Now, let $z_n = 0$ if $x_n = 0$ and $z_n = 1/x_n$ if $x_n \neq 0$. Clearly $(z_n)_n \in \mathscr{C}_e$. Set $z = [z_n]$. Since $x_n z_n = 1$ for n large enough, it follows that $xz = 1$, proving that $x \in \mathscr{C}^*$. Conversely, suppose that $x \in \mathscr{C}^*$. Let $z = x^{-1}$ and set $x = [x_n]$, $y = [y_n]$. Since $x_n z_n \to 1$ as $n \to \infty$ and $(z_n)_n \in \mathscr{C}_e$, there are $a > 1$ and $\eta \in \mathbb{N}$ such that $|x_n z_n| > 1/2$ and $|z_n| < a^n$ for $n > \eta$. It follows that $|x_n| > (a^{-1})^n/2 \geq (a^{-1}/2)^n$. Hence $(x_n)_n$ fulfills $(*)$. ∎

We now characterize invertible elements of $\mathscr{H}(\mathbb{T})$.

THEOREM 3.10. *An element $f \in \mathscr{H}(\mathbb{T})$ is invertible if and only if it admits a representative $(f_n)_n$ such that*

$$\exists b \in]0,1[, \exists r > 1, \exists \eta \in \mathbb{N} / \forall n > \eta, f_n \in \mathscr{O}_r, \inf_{z \in C_r} |f_n(z)| > b^n. \qquad (**)$$

Proof. First we show that the condition $(**)$ is independent on the choice of the representative $(f_n)_n$. Let $(h_n)_n$ denote another representative of f. Let $b_1 \in]0,1[$, $r_1 > 1$, $\eta_1 \in \mathbb{N}$ such that $\inf_{z \in C_r} |f_n(z)| > b_1^n$ for $n > \eta_1$. Set $b = b_1/2$. Since $(f_n - h_n)_n \in \mathscr{N}_e(\mathbb{T})$, there are $r > 1$ and

$\eta \in \mathbb{N}$ such that $r \le r_1$, $\eta \ge \eta_1$, and $\|f_n - h_n\|_r < b^n$ for $n > \eta$. It follows that for any $z \in C_r$ and $n > \eta$, we have $|f_n(z) - h_n(z)| < b^n$ whence $b_1^n < \inf_{z \in C_r}|f_n(z)| \le \inf_{z \in C_r}|h_n(z)| + b^n$ if $n > \eta$. Since $b_1^n > 2b^n$ for $n > \eta$, we have $\inf_{z \in C_r}|h_n(z)| > b^n$ for $n > \eta$ showing that $(h_n)_n$ fulfills $(**)$. Now suppose that f satisfies $(**)$ and let $(f_n)_n$ denote a representative of f. Let $b \in]0,1[$, $r > 1$, $\eta \in \mathbb{N}$ such that $\inf_{z \in C_r}|f_n(z)| > b^n$ for $n > \eta$. Set $g_n = f_n^{-1}$ if $n > \eta$ and $g_n = 1$ otherwise. We have $\|g_n\|_r = (\inf_{z \in C_r}|f_n(z)|)^{-1} < (1/b)^n$ for $n > \eta$ proving that $(g_n)_n) \in \mathscr{X}_e(\mathbb{T})$. Hence f is invertible with $f^{-1} = g$. Conversely, suppose that f is invertible and set $f = [f_n]$, $g = f^{-1} = [g_n]$. Let $a > 1$, $r(1) > 1$, and $\eta_1 \in \mathbb{N}$ such that $g_n \in O_{r(1)}$, $\|g_n\|_{r(1)} < a^n$ for $n > \eta_1$. From $fg = 1$, for any $b \in]0,1[$ there are $r > 1$, $\eta \in \mathbb{N}$ with $r < r(1)$, $\eta > \eta_1$ such that $f_n g_n \in \mathscr{O}_r$ and $\|f_n g_n - 1\|_r < b^n$ for all $n > \eta$. Hence we have $|f_n(z)|\,|g_n(z)| > 1 - b^n$ for $z \in C_r$ and $n > \eta$ whence $\|g_n\|_r \inf_{z \in C_r}|f_n(z)| \ge 1 - b^n$ if $n > \eta$. It follows that $\inf_{z \in C_r}|f_n(z)| \ge (1 - b^n)(1/a)^n > (1/2a)^n$ for n large enough. ∎

We obtain straightforwardly the following

COROLLARY 3.11. *$f \in \mathscr{H}^*(\mathbb{T})$ if and only if f admits a representative $(f_n)_n$ for which there is $r > 1$ such that $f_n \in \mathscr{O}_r$ for n large enough and such that $cl(\inf_{z \in C_r}|f_n(z)|) \in \mathscr{C}^*$.*

In the sequel we denote by $\mathscr{H}^*(\mathbb{T})$ the set of invertible elements of $\mathscr{H}(\mathbb{T})$.

3.5. *A Vanishing Theorem*

Like in many algebras of Colombeau type [2, 3], there are generalized functions which vanish at each classical point without being the zero function. In our case, let $\alpha > 1$ and set $u_n(z) = ((z^2 - 1)/2z)\exp[n^\alpha((z^2 - 1)/2z)^2]$. We first show that $(u_n)_n \in \mathscr{X}_e(\mathbb{T})$. Let $1 < \rho < \sqrt{3/2}$ and $1/\rho < |z| < \rho$. We have $|u_n(z)| \le ((\rho^2 + 1)/2\rho)\exp[n^\alpha \Re((z^2 - 1)/2z)^2]$ and we find that $\Re((z^2 - 1)/2z)^2 = (z^2 + \bar{z}^2)(|z|^4 + 1) - 4|z|^4/8|z|^4$. Set $z^2 = x + iy$. Then $(z^2 + \bar{z}^2)(|z|^4 + 1) - 4|z|^4 = 2x(|z|^4 + 1) - 4|z|^4 \le 2x - |z|^4 = -(x - y)^2 \le 0$. It follows that $\|u_n\|_\rho \le (\rho^2 + 1)/2\rho$ showing that $(u_n)_n \in \mathscr{X}_e(\mathbb{T})$. Let $u = [u_n]$. A simple calculation shows that if $\theta \in \mathbb{R}$, then $u_n(e^{i\theta}) = i \sin\theta \exp(-n^\alpha \sin^2\theta)$. Since $\alpha > 1$, then, if $\sin\theta \ne 0$, for all $b \in]0,1[$ there exists $\eta \in \mathbb{N}$ such that $\exp(-n^\alpha \sin^2\theta) < b^n$ for $n > \eta$. One deduces that $u(z) = 0$ for all $z \in \mathbb{T}$. It may also be seen that $\sup_{\theta \in \mathbb{R}}|u_n(e^{i\theta})| = (2en^\alpha)^{-1/2}$. Then, for all $r > 1$ and all $n \ne 0$ we have $u_n \in \mathscr{O}_r$ and $\|u_n\|_r \ge (2en^\alpha)^{-1/2}$, proving that $u \ne 0$.

We now give sufficient conditions, based on pointvalues, for a generalized hyperfunction to be zero. We start by the following

DEFINITION 3.5. Let $f \in \mathscr{H}(\mathbb{T})$. Then f is said to *vanish analytically* at a point $\zeta \in \mathbb{T}$, if f admits a representative $(f_n)_n \in \mathscr{X}_e(\mathbb{T})$ such that for any

$b \in]0,1[$ there are $\lambda > 0$, $\mu > 0$, and $\eta \in \mathbb{N}$ such that $\forall m \in \mathbb{N}$, $\forall n > \eta$, $|f_n^{(m)}(\zeta)| \le \lambda \mu^m m! b^n$.

This definition does not depend on the chosen representative $(f_n)_n$. If h is a holomorphic function in a neighborhood of some C_r and $m \in \mathbb{N}$, $\xi \in C_r$, then we have from the Cauchy formula

$$h^{(m)}(\xi) = \frac{m!}{2i\pi}\left[\int_{|z|=r} \frac{h(z)\,dz}{(z-\zeta)^{m+1}} - \int_{|z|=1/r} \frac{h(z)\,dz}{(z-\zeta)^{m+1}}\right].$$

It follows that

$$|h^{(m)}(\xi)| \le \left(\frac{r+|\xi|}{r-|\xi|}\right)\left(\frac{r|\xi|}{r-|\xi|}\right)^m m!\|h\|_r.$$

Consequently if $g_n = f_n + \psi_n$ with $(\psi_n)_n \in \mathcal{N}_e(\mathbb{T})$, we find for $\psi_n \in \mathcal{O}_r$ and $m \in \mathbb{N}$,

$$|g_n^{(m)}(\zeta)| \le |f_n^{(m)}(\zeta)| + \left(\frac{r+1}{r-1}\right)\left(\frac{r}{r-1}\right)^m m!\|h\|_r.$$

Let $b \in]0,1[$. There are $r > 1$, $\eta_1 \in \mathbb{N}$ such that $\psi_n \in \mathcal{O}_r$ and $\|\psi_n\|_r < (b/2)^n$ for $n > \eta_1$. From the definition, there are $\lambda > 0$, $\mu > 0$, $\eta_2 \in \mathbb{N}$ such that $|f_n^{(m)}(\zeta)| \le \lambda\mu^m m!(b/2)^n$ for $n > \eta_2$. Taking $\alpha = \max(\lambda, (r+1)/(r-1))$, $\beta = \max(\mu, r/(r-1))$, $\eta = \max(\eta_1, \eta_2)$, we find $|g_n^{(m)}(\zeta)| \le \alpha\beta^m m! b^n$ for $n > \eta$.

Let $f \in \mathcal{H}(\mathbb{T})$ vanish analytically at $\zeta \in \mathbb{T}$. We denote by $\mu(f,\zeta)$ the lower bound of the set of constants $\mu > 0$ for which there exists a representative $(f_n)_n$ of f satisfying the conditions of Definition 3.5.

THEOREM 3.12. *Let $f \in \mathcal{H}(\mathbb{T})$ vanish analytically at $\zeta \in \mathbb{T}$. If $\mu(f,\zeta) < \frac{1}{\pi(e-1)}$, then $f = 0$.*

Proof. Suppose that $\mu(f,\zeta) < \frac{1}{\pi(e-1)}$ and let $(f_n)_n$ denote a representative of f satisfying the conditions of Definition 3.5. Keeping the notations of Definition 3.5, we may suppose that there exists $r > 1$ such that $f_n \in \mathcal{O}_r$ for all $n > \eta$. Let $\rho_1 < (r-1)/r$. Since $\overline{D}(\zeta, \rho_1) \subset C_r$, we have for $z \in \overline{D}(\zeta, \rho_1)$ and $n > \eta$,

$$f_n(z) = \sum_{m=0}^{\infty} \frac{f_n^{(m)}(\zeta)}{m!}(z-a)^m.$$

It follows that if $k \in \mathbb{N}$, $z \in \overline{D}(\zeta, \rho_1)$ and $n > \eta$, then

$$f_n^{(k)}(z) = \sum_{m=0}^{\infty} \frac{f_n^{(m+k)}(\zeta)}{m!}(z - \zeta)^m.$$

Hence, we have for $z \in \overline{D}(\zeta, \rho_1)$ and $n > \eta$,

$$|f_n^{(k)}(z)| \leq \lambda b^n \sum_{m=0}^{\infty} \frac{\mu^{m+k}(m+k)!}{m!}\rho_1^m,$$

$$|f_n^{(k)}(z)| \leq \lambda \mu^k k! b^n \sum_{m=0}^{\infty} \frac{(m+k)!}{m!k!}(\mu\rho_1)^m.$$

For $m \neq 0$ and $k \neq 0$, we have from Stirling's formula,

$$\frac{(m+k)!}{m!k!} \leq C.\frac{(m+k)^{m+k+1/2}}{m^{m+1/2}k^{k+1/2}},$$

for some constant C. Writing $(m+k)^{m+k+1/2}/m^{m+1/2}k^{k+1/2} = (1+\frac{k}{m})^m(1+\frac{m}{k})^k(\frac{m+k}{mk})^{1/2}$, it follows that

$$\frac{(m+k)^{m+k+1/2}}{m^{m+1/2}k^{k+1/2}} \leq \frac{\sqrt{2}}{(mk)^{1/4}}e^{m+k}.$$

Consequently there is a constant $C' > 0$ such that for all $k \in \mathbb{N}$, one has

$$|f_n^{(k)}(z)| \leq C'(e\mu)^k k! b^n \sum_{m=0}^{\infty} (e\mu\rho_1)^m.$$

Hence, if we take $\rho_1 < \min((e\mu)^{-1}, (r-1)/r)$, then there is $C_1 > 0$ such that

$$\forall z \in \overline{D}(\zeta, \rho_1), \forall n > \eta, \forall m \in \mathbb{N}, |f_n^{(k)}(z)| \leq C_1(e\mu)^m m! b^n.$$

Let $z \in \mathbb{T} \cap \overline{D}(\zeta, \rho_1)$ and set $z = e^{i\theta}$, $\zeta = e^{i\alpha}$ with θ, $\alpha \in [0, 2\pi[$. Since $z - \zeta = 2i\sin(\frac{\theta-\alpha}{2})e^{i(\theta+\alpha)/2}$ and $|z - \zeta| = \rho_1$, it follows that $\theta - \alpha = \pm 2\arcsin(\rho_1/2)$. Let $z_1 = e^{i[\alpha + 2\arcsin(\rho_1/2)]}$ and $z_2 = e^{i[\alpha - 2\arcsin(\rho_1/2)]}$. From the above inequalities we find that

$$|f_n^{(k)}(z_i)| \leq C_1(e\mu)^m m! b^n, \qquad i = 1, 2; m \in \mathbb{N}, n > \eta.$$

Moreover, any point of a neighborhood of the arc of $\mathbb{T}$ delimited by z_1 and z_2 verifies the above inequality, and the length of such an arc is $4\arcsin(\rho_1/2)$. In the same way, if $\rho_2 < \min((e^2\mu)^{-1}, (r-1)/r)$, then there is $C_2' > 0$ such that

$$\forall z \in \overline{D}(z_i, \rho_2), i = 1, 2; \forall m \in \mathbb{N}, \forall n > \eta, |f_n^{(k)}(z_i)| \leq C_2(e^2\mu)^m m! b^n.$$

It follows that the above inequality holds on a neighborhood of an arc of $\mathbb{T}$ of length $4[\arcsin(\rho_1/2) + \arcsin(\rho_2/2)]$, where the constant C_2' is replaced by $C_2 = \max(C_1, C_2')$. Repeating this process j times, we obtain $C_j > 0$ such that for all z in a neighborhood of an arc of $\mathbb{T}$ of length $l_j = 4[\arcsin(\rho_1/2) + \cdots + \arcsin(\rho_j/2)]$, $0 < \rho_i < (e^i\mu)^{-1}$, we have

$$\forall m \in \mathbb{N}, \forall n > \eta, |f_n^{(k)}(z)| \leq C_j(e^j\mu)^m m! b^n.$$

In order that the above inequality holds for all the points of a neighborhood of $\mathbb{T}$, it suffices that there exists $j \geq 1$ such that $l_j \geq 2\pi$. Hence it is sufficient to have $4[(e\mu)^{-1} + \cdots + (e^j\mu)^{-1}] > 2\pi$ for some $j \geq 1$; that is, $\mu < (1 - e^{-j})/\pi(e-1)$. Since $\mu(f, \zeta) < \frac{1}{\pi(e-1)}$, we can choose $\mu < \frac{1}{\pi(e-1)}$. Therefore such an j exists proving the theorem. ∎

3.6. *A Differential Problem*

We are concerned with the following Cauchy problem,

$$(I) \quad u' + fu + gu^2 = 0; \qquad u(z_0) = \mu,$$

where $f, g \in \mathscr{H}(\mathbb{T})$, $z_0 \in \mathbb{T}$, and $\mu \in \mathscr{C}^*$. We seek a solution $u \in \mathscr{H}^*(\mathbb{T})$. We start with the following two lemmas:

LEMMA 3.13. *Let $(v_n)_n \in \mathscr{X}_e(\mathbb{T})$. Then $(\exp(v_n))_n \in \mathscr{X}_e(\mathbb{T})$ if and only if*:

$$(P) \quad \exists \alpha > 0, \exists r > 1, \exists \eta \in \mathbb{N}, \forall n > \eta, \sup_{\zeta \in C_r} (\Re v_n(\zeta)) \leq \alpha n.$$

If $(v_n)_n$ fulfills the condition (P), then the same holds for $(v_n + \psi_n)_n$ for all $(\psi_n)_n \in \mathscr{N}_e(\mathbb{T})$. Moreover $(\exp(v_n + \psi_n) - \exp(v_n))_n \in \mathscr{N}_e(\mathbb{T})$. Therefore if $v = [v_n]$, one may define e^v as $e^v = [\exp(v_n)]$.

Proof. Let $(v_n)_n \in \mathscr{X}_e(\mathbb{T})$. Let $r > 1$ such that $v_n \in \mathscr{O}_r$. Then the first part of the lemma follows from the equality $\|\exp(v_n)\|_r = \exp(\sup_{\zeta \in C_r}(\Re v_n(\zeta)))$. If $v_n, \psi_n \in \mathscr{O}_r$, then $\sup_{\zeta \in C_r}(\Re(v_n(\zeta) + \psi_n(\zeta))) \leq \sup_{\zeta \in C_r}(\Re v_n(\zeta)) + \|\psi_n\|_r$, and since $(\|\psi_n\|_r)_n \in \mathscr{N}_e(\mathbb{T})$, it follows that $(v_n + \psi_n)_n \in \mathscr{X}_e(\mathbb{T})$. From $|\exp(\psi_n) - 1| \leq |\psi_n| \exp(|\psi_n|)$, we derive that $(\exp(\psi_n) - 1)_n \in \mathscr{N}_e(\mathbb{T})$. Writing $\exp(v_n + \psi_n) - \exp(v_n) = \exp(v_n)(\exp(\psi_n) - 1)$ and according to $(\exp(v_n))_n \in \mathscr{X}_e(\mathbb{T})$, one obtains $(\exp(v_n + \psi_n) - \exp(v_n))_n \in \mathscr{N}_e(\mathbb{T})$. ∎

LEMMA 3.14. *For any $v \in \mathscr{H}(\mathbb{T})$, there exists $\lambda \in \mathscr{C}^*$ such that $v + \lambda \in \mathscr{H}^*(\mathbb{T})$.*

Proof. Let $(v_n)_n$ denote a representative of v. Then, there exist $a > 1$, $r > 1$, and $\eta \in \mathbb{N}$ such that $\|v_n\|_r \leq a^n$ for $n > \eta$. Let $\alpha \in \mathscr{C}^*$ with a representative $(\alpha_n)_n$ such that $\alpha_n > 0$ for $n > \eta$ and set $\lambda_n = \|v_n\|_r + \alpha_n$. Clearly $(\lambda_n)_n \in \mathscr{C}_e$ and $\inf_{\zeta \in C_r} |v_n(\zeta) + \lambda_n| \geq |\lambda_n| - \|v_n\|_r = \alpha_n$ for $n > \eta$. Since $\alpha \in \mathscr{C}^*$, it follows that $v + \lambda \in \mathscr{H}^*(\mathbb{T})$, where $\lambda = [\lambda_n]$. ∎

THEOREM 3.15. *Suppose that*

(H_1) *f admits a primitive F with respect to d/dz having a representative* $(F_n)_n$ *such that*

$$\exists \alpha > 0, \exists r > 1, \exists \eta \in \mathbb{N} / \forall n > \eta, \Re F_n(C_r) \subset [-\alpha n, \alpha n],$$

(H_2) ge^{-F} *admits a primitive G with respect to d/dz*,

(H_3) $G + \mu^{-1} e^{-F(z_0)} - G(z_0) \in \mathscr{H}^*(\mathbb{T})$.

Then, the problem (I) *admits a solution* $u \in \mathscr{H}^*(\mathbb{T})$.

Proof. According to Lemma 3.13, it follows from (H_1) that we can define e^{-F} and e^F. Then, ge^{-F} is a well defined generalized hyperfunction and (H_2) means that g is of the form $g = he^F$ where $\hat{h}(-1) = 0$. Now, set $g = [g_n]$, $G = [G_n]$, and $\lambda = \mu^{-1} e^{-F(z_0)} - G(z_0)$. The differential equation of the problem (I) being of Bernoulli type, we set $u = (G + \lambda)^{-1} e^{-F}$. According to Lemma 3.14, the condition (H_3) makes sense and u is then well defined. Moreover $u \in \mathscr{H}^*(\mathbb{T})$. Let us verify that u is a solution to (I). We have $u' = -g(G + \lambda)^{-2} e^{-2F} - f(G + \lambda)^{-1} e^{-F}$, that is, $u' + fu + gu^2 = 0$. Further, taking in account the definition of λ, we find $u(z_0) = \mu$. Hence u is a solution to (I) in $\mathscr{H}^*(\mathbb{T})$. ∎

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