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Castelnuovo–Mumford regularity of seminormal simplicial affine semigroup rings

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ABSTRACT

We show that the Eisenbud–Goto conjecture holds for (homogeneous) seminormal simplicial affine semigroup rings. Moreover, we prove an upper bound for the Castelnuovo–Mumford regularity in terms of the dimension, which is similar as in the normal case. Finally, we compute explicitly the regularity of full Veronese rings.

1. Introduction

Let K be a field, and let $R = K[x_1, \ldots, x_n]$ be a standard graded polynomial ring, that is, all variables x_i have degree 1. Let M be a finitely generated graded R-module. By $H^i_{R_+}(M)$ we denote the ith local cohomology module of M with respect to the homogeneous maximal ideal R_+ of R, and we set $a(H^i_{R_+}(M)) := \max\{r \mid H^i_{R_+}(M)_r \neq 0\}$ with the convention that $a(0) = -\infty$. The *Castelnuovo–Mumford regularity* (or *regularity* for short) $\operatorname{reg} M$ of M is defined by

$$\operatorname{reg} M := \max\{i + a(H^i_{R_+}(M)) \mid i \geqslant 0\}.$$

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The regularity $\operatorname{reg} M$ is an important invariant, for example, the ith syzygy module of M can be generated by elements of degree smaller or equal to $\operatorname{reg} M + i$. Moreover, one can use the regularity of a homogeneous ideal to bound the degrees in certain minimal Gröbner bases. For more information we refer to the paper of Eisenbud and Goto [1], and to Bayer and Stillman [2]. So it is natural to ask for bounds for the regularity of a homogeneous ideal I of R; note that $\operatorname{reg} I = \operatorname{reg} R/I + 1$. Denote by $\operatorname{codim} R/I := \dim_K [R/I]_1 - \dim R/I$ the codimension of R/I and by $\deg R/I$ its degree. An open conjecture is

Conjecture 1.1 *(Eisenbud–Goto [1]). If K is algebraically closed and I is a homogeneous prime ideal of R, then*

$$\operatorname{reg} R/I \leqslant \deg R/I - \operatorname{codim} R/I.$$

By a result of Gruson, Lazarsfeld, and Peskine [3] Conjecture 1.1 holds if $\dim R/I = 2$. The Cohen–Macaulay case was proven by Treger [4], and the Buchsbaum case by Stückrad and Vogel [5,6]. Conjecture 1.1 also holds if $\deg R/I \leqslant \operatorname{codim} R/I + 2$ by a result of Hoa, Stückrad, and Vogel [7], and in characteristic zero for smooth surfaces by Lazarsfeld [8] and for certain smooth threefolds by Ran [9]. Moreover, Giaimo [10] showed that the conjecture still holds for connected reduced curves.

Since the Eisenbud–Goto conjecture is widely open, it would be nice to prove it for more cases; in the following we will consider homogeneous simplicial affine semigroup rings. A semigroup is called *affine* if it is finitely generated and isomorphic to a submonoid of $(\mathbb{Z}^m, +)$ for some $m \in \mathbb{N}^+$. Let B be an affine semigroup. The *affine semigroup ring* $K[B]$ associated to B is defined as the K-vector space with basis $\{t^b \mid b \in B\}$ and multiplication given by the K-bilinear extension of $t^a \cdot t^b = t^{a+b}$. Since B is an affine semigroup we have $G(B) \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}$; where $G(B)$ denotes the group generated by B. Hence $G(B) \otimes_{\mathbb{Z}} \mathbb{R}$ is a finite dimensional $\mathbb{R}$-vector space with canonical embedding $G(B) \subseteq G(B) \otimes_{\mathbb{Z}} \mathbb{R}$ given by $x \mapsto x \otimes 1$. We say that B is *simplicial* if the corresponding cone $C(B)$ is generated by linearly independent elements, where $C(X) := \{\sum_{i=1}^{k} r_i x_i \mid k \in \mathbb{N}^+,\ r_i \in \mathbb{R}_{\geqslant 0},\ x_i \in X\}$ for $X \subseteq G(B) \otimes_{\mathbb{Z}} \mathbb{R}$. An element $x \in B$ is called a *unit* if $-x \in B$. We say that B is *positive* if 0 is its only unit. In this case, the *Hilbert basis* $\operatorname{Hilb}(B)$, that is, the set of irreducible elements of B, is a unique minimal generating set of B; an element $x \in B$ is called *irreducible* if it is not a unit and if for $x = y + z$ with $y, z \in B$ it follows that y or z is a unit. Moreover, we say that B is *homogeneous* if B is positive and there is a positive $\mathbb{Z}$-grading on $K[B]$ in which every t^b for $b \in \operatorname{Hilb}(B)$ has degree 1. See [11, Chapter 2]. In the following we will assume that B is homogeneous. We will always consider the above $\mathbb{Z}$-grading on $K[B]$, moreover, by $\operatorname{reg} K[B]$ we mean the regularity of $K[B]$ with respect to the canonical R-module structure which is induced by the homogeneous surjective K-algebra homomorphism

$$\pi : R = K[x_1, \ldots, x_n] \twoheadrightarrow K[B],$$

given by $x_i \mapsto t^{a_i}$; where $\operatorname{Hilb}(B) = \{a_1, \ldots, a_n\}$. Hence $R/\ker\pi \cong K[B]$, where $\ker\pi$ is a homogeneous prime ideal of R. In case that B is simplicial we will also call $K[B]$ a simplicial affine semigroup ring.

By extending the ground field if necessary, (the inequality in) Conjecture 1.1 holds for $K[B]$ in particular if $\dim K[B] = 2$, if $K[B]$ is Buchsbaum, and if $\deg K[B] \leqslant \operatorname{codim} K[B] + 2$. The conjecture also holds if $\operatorname{codim} K[B] = 2$ by Peeva and Sturmfels [12] and for simplicial affine semigroup rings with isolated singularity by Herzog and Hibi [13]. In [14, Theorem 3.2], Hoa and Stückrad presented a very good bound for the regularity of simplicial affine semigroup rings, moreover, they provided some cases where Conjecture 1.1 holds. However, the Eisenbud–Goto conjecture is still widely open even for simplicial affine semigroup rings. In the case that B is simplicial and seminormal (see Definition 3.1) we can confirm the Eisenbud–Goto conjecture for $K[B]$, we obtain the following:

Main result *(Theorem 3.12, Theorem 3.16). Let K be an arbitrary field and let B be a homogeneous affine semigroup. If B is simplicial and seminormal, then*

$$\operatorname{reg} K[B] \leqslant \min\bigl\{\dim K[B] - 1, \deg K[B] - \operatorname{codim} K[B]\bigr\}.$$

This result is more or less well known if B is normal (see Definition 3.1), since $K[B]$ is Cohen–Macaulay in this case; see [15, Theorem 1], [11, Theorem 6.10], and Remark 3.9. In fact, $K[B]$ is not necessary Buchsbaum if B is simplicial and seminormal, see Example 3.3. To prove Conjecture 1.1 in the seminormal simplicial case we will use an idea of Hoa and Stückrad, namely, one can decompose $K[B]$ into a direct sum of certain monomial ideals and compute $\operatorname{reg} K[B]$ in terms of the regularity of the ideals. This becomes even more powerful in the seminormal case, since seminormality can be characterized in terms of the decomposition by a result of Li [16].

In Section 2 we will recall the decomposition of simplicial affine semigroup rings. Moreover, we will introduce sequences with $*$-property which will be useful to prove the main result in Section 3. Finally, we will compute explicitly the Castelnuovo–Mumford regularity of full Veronese rings in Section 4. We set $M_{d,\alpha} := \{(u_1, \dots, u_d) \in \mathbb{N}^d \mid \sum_{i=1}^{d} u_i = \alpha\}$ where $d, \alpha \in \mathbb{N}^+$, moreover, we define $B_{d,\alpha}$ to be the submonoid of $(\mathbb{N}^d, +)$ which is generated by $M_{d,\alpha}$. In Theorem 4.2 we will show that $\operatorname{reg} K[B_{d,\alpha}] = \lfloor d - \frac{d}{\alpha} \rfloor$. For a general consideration of seminormal rings we refer to [17,18], and for unspecified notation to [11,19].

2. Basics

In the following we will assume that the homogeneous affine semigroup B is simplicial, that is, we assume that there are linearly independent elements $e_1, \dots, e_d \in C(B)$ such that $C(B) = C(\{e_1, \dots, e_d\})$. Without loss of generality we may assume that $e_1, \dots, e_d \in \operatorname{Hilb}(B)$. Consider the $\mathbb{R}$-vector space isomorphism $\varphi : \operatorname{span}(\{e_1, \dots, e_d\}) \to \mathbb{R}^d$ where e_i is mapped to the element in $\mathbb{N}^d$ all of whose coordinates are zero except the ith coordinate which is equal to α for some $\alpha \in \mathbb{N}^+$, that is, $\varphi(e_i) = (0, \dots, 0, \alpha, 0, \dots, 0)$. By construction we have $\varphi(B) \subseteq \mathbb{R}^d_{\geqslant 0}$, since $C(B) = C(\{e_1, \dots, e_d\})$, hence $\varphi(B) \subseteq \mathbb{Q}^d_{\geqslant 0}$ by the Gaussian elimination. Thus, by choosing a suitable α we may assume that $\varphi(\operatorname{Hilb}(B)) \subset \mathbb{N}^d$, or equivalently, $\varphi(B) \subseteq \mathbb{N}^d$. The affine semigroup $\varphi(B)$ is again homogeneous, it follows that the coordinate sum of all elements of $\varphi(\operatorname{Hilb}(B))$ is equal to α, see [11, Proposition 2.20]. The isomorphism $B \cong \varphi(B)$ of semigroups induces an isomorphism of $\mathbb{Z}$-graded rings $K[B] \cong K[\varphi(B)]$. This enables us to identify a homogeneous simplicial affine semigroup B with its image $\varphi(B)$ in $\mathbb{N}^d$. Thus, we may assume that B is the submonoid of $(\mathbb{N}^d, +)$ which is generated by a set $\{e_1, \dots, e_d, a_1, \dots, a_c\} \subseteq M_{d,\alpha}$, where

$$e_1 := (\alpha, 0, \dots, 0), \quad e_2 := (0, \alpha, 0, \dots, 0), \quad \dots, \quad e_d := (0, \dots, 0, \alpha).$$

Let $a_i = (a_{i[1]}, \dots, a_{i[d]})$; since $\alpha \in \mathbb{N}^+$ can be chosen to be minimal, we may assume that the integers $a_{i[j]}$, $i = 1, \dots, c$, $j = 1, \dots, d$, are relatively prime. Moreover, we assume that $c \geqslant 1$, since the case $c = 0$ is not relevant in our context. Note that K is an arbitrary field, $\dim K[B] = d$, and $\operatorname{codim} K[B] = c$. Our notation tries to follow the notation in [14].

By $x_{[i]}$ we denote the ith component of x and $\deg x := (\sum_{j=1}^{d} x_{[j]})/\alpha$, for $x \in G(B)$. We define $A := \langle e_1, \dots, e_d \rangle$ to be the submonoid of B generated by $e_1, \dots, e_d$, and we set

$$B_A := \{x \in B \mid x - a \notin B \ \forall a \in A \setminus \{0\}\}.$$

Note that B_A is finite. Moreover, if $x \notin B_A$ then $x + y \notin B_A$ for all $x, y \in B$. We define $x \sim y$ if $x - y \in G(A) = \alpha\mathbb{Z}^d$, thus, $\sim$ is an equivalence relation on $G(B)$. Every element of $G(B)$ is equivalent to an element of $G(B) \cap D$, where $D := \{(x_{[1]}, \dots, x_{[d]}) \in \mathbb{Q}^d \mid 0 \leqslant x_{[i]} < \alpha \ \forall i\}$ and for all $x, y \in G(B) \cap D$ with $x \neq y$ we have $x \nsim y$. Hence the number of equivalence classes $f := \#(G(B) \cap D)$ on $G(B)$ is finite. Every element of B is by construction equivalent to an element of B_A. Moreover, for arbitrary $x, y \in B$ we have $x - y \sim x + (\alpha - 1)y \in B$, hence there are also f equivalence classes on B and on B_A. By $\Gamma_1, \dots, \Gamma_f$ we denote the equivalence classes on B_A. For $t = 1, \dots, f$ we define

$$h_t := \bigl(\min\{m_{[1]} \mid m \in \Gamma_t\}, \min\{m_{[2]} \mid m \in \Gamma_t\}, \dots, \min\{m_{[d]} \mid m \in \Gamma_t\}\bigr).$$

Note that $x - h_t \in A$ for all $x \in \Gamma_t$. This shows that $h_t \in G(B) \cap \mathbb{N}^d$. Let $T := K[y_1, \dots, y_d]$ be a standard graded polynomial ring, that is, all variables y_i have degree 1. We define $\tilde{\Gamma_t} := \{y^{(x-h_t)/\alpha} \mid x \in \Gamma_t\}$, where $u/\alpha := (u_{[1]}/\alpha, \dots, u_{[d]}/\alpha)$ and $y^u := y_1^{u_{[1]}} \cdot \ldots \cdot y_d^{u_{[d]}}$ for $u = (u_{[1]}, \dots, u_{[d]}) \in \mathbb{N}^d$. We obtain $\tilde{\Gamma_t} \subset T$, and therefore $I_t := \tilde{\Gamma_t} T$ is a monomial ideal in T for all $t = 1, \dots, f$. It follows that $\operatorname{ht} I_t \geqslant 2$ (height), since $\gcd \tilde{\Gamma_t} = 1$. See [14, Section 2]. By [14, Proposition 2.2(i)] we obtain

$$K[B] \cong \bigoplus_{t=1}^{f} I_t(-\deg h_t) \tag{2.1}$$

as $\mathbb{Z}$-graded T-modules (the T-module structure on $K[B]$ is induced by $T \cong K[A] \subseteq K[B]$, $y_i \mapsto t^{e_i}$). Hence $\deg K[B] = f$. Denote by $K[B]_+$ and T_+ the homogeneous maximal ideals of $K[B]$ and T. Using [20, Theorem 13.1.6] twice, we get $a(H^i_{R_+}(K[B])) = a(H^i_{K[B]_+}(K[B])) = a(H^i_{T_+}(K[B]))$. Thus,

$$\operatorname{reg} K[B] = \max\{\operatorname{reg} I_t + \deg h_t \mid t = 1, \dots, f\}, \tag{2.2}$$

where $\operatorname{reg} I_t$ denotes the regularity of I_t as a $\mathbb{Z}$-graded T-module; see also [14, Proposition 2.2(ii)]. This shows that the regularity of $K[B]$ is independent of K for $\dim K[B] \leqslant 5$ by [21, Corollary 1.4].

Remark. This decomposition can be computed by using the Macaulay2 [22] package MonomialAlgebras [23], which has been developed by Janko Böhm, David Eisenbud, and the author. In this package we consider the case of affine semigroups $Q' \subseteq Q \subseteq \mathbb{N}^d$ such that $K[Q]$ is finite over $K[Q']$; the implemented algorithm decomposes the ring $K[Q]$ into a direct sum of monomial ideals in $K[Q']$. There is also an algorithm implemented computing $\operatorname{reg} K[Q]$ in the homogeneous case, moreover, there are functions available testing the Buchsbaum, Cohen–Macaulay, Gorenstein, normal, and the seminormal property in the simplicial case. Note that this decomposition works more general, for more information we refer to [24].

Definition 2.1. For an element $x \in B$ we say that a sequence $\lambda = (b_1, \dots, b_n)$ has $*$-*property* if $b_1, \dots, b_n \in \{e_1, \dots, e_d, a_1, \dots, a_c\}$ and $x - b_1 \in B$, $x - b_1 - b_2 \in B, \dots, x - (\sum_{j=1}^n b_j) \in B$; we say that the *length* of λ is n. Let $\lambda = (b_1, \dots, b_n)$ be a sequence with $*$-property of x; we define $x(\lambda, i) := x - (\sum_{j=1}^i b_j)$ for $i = 1, \dots, n$, and $x(\lambda, 0) := x$. By Λ_x we denote the set of all sequences with $*$-property of x with length $\deg x$, with the convention that $\Lambda_0 := \emptyset$.

By construction we have $\Lambda_x \neq \emptyset$ for all $x \in B \setminus \{0\}$. The definition of a sequence with $*$-property is motivated to control the degree of $K[B]$, the second assertion in Lemma 2.3 illustrates the usefulness of this construction. For elements $x, y \in G(B)$ we define $x \geqslant y$ if $x_{[k]} \geqslant y_{[k]}$ for all $k = 1, \dots, d$.

Remark 2.2. Let $\lambda = (b_1, \dots, b_n)$ be a sequence with $*$-property of x. We get $x(\lambda, i) \geqslant x(\lambda, j)$ for $0 \leqslant i \leqslant j \leqslant n$. Moreover, we have $\deg x(\lambda, i) = \deg x - i$ for $i = 0, \dots, n$. Hence for $\lambda \in \Lambda_x$ we get $x(\lambda, \deg x) = 0$.

Lemma 2.3. *Let $x \in B_A \setminus \{0\}$ and $\lambda = (b_1, \dots, b_n)$ be a sequence with $*$-property of x. Then*:

(1) *$x(\lambda, i) \in B_A$ for all $i = 0, \dots, n$.*
(2) *$x(\lambda, i) \nsim x(\lambda, j)$ for all $i, j \in \mathbb{N}$ with $0 \leqslant i < j \leqslant n$.*

Proof. (1) Follows from construction since if $y \notin B_A$ then $y + z \notin B_A$ for all $y, z \in B$.

(2) Suppose to the contrary that $x(\lambda, i) \sim x(\lambda, j)$ for some $i, j \in \mathbb{N}$ with $0 \leqslant i < j \leqslant n$. We have $x(\lambda, i) \geqslant x(\lambda, j)$, hence

$$x(\lambda, i) = x(\lambda, j) + \sum_{t=1}^{d} n_t e_t$$

for some $n_t \in \mathbb{N}$. Since $\deg x(\lambda, i) > \deg x(\lambda, j)$ we get that $n_t > 0$ for some $t \in \{1, \ldots, d\}$. Thus, $x(\lambda, i) - e_t \in B$ and therefore $x(\lambda, i) \notin B_A$ which contradicts claim (1). $\square$

Remark 2.4. Let $x \in B_A \setminus \{0\}$ and $\lambda = (b_1, \ldots, b_n)$ be a sequence with $*$-property of x. Suppose that $b_j \in \{e_1, \ldots, e_d\}$ for some $j \in \{1, \ldots, n\}$. Hence $x - b_j = x(\lambda, n) + \sum_{k=1, k \neq j}^{n} b_k \in B$ which contradicts $x \in B_A$. This shows that $b_1, \ldots, b_n \in \{a_1, \ldots, a_c\}$.

Lemma 2.3 implies that $\deg x \leqslant \deg K[B] - 1$ for all $x \in B_A$. This bound can be improved by using the following observation:

Remark 2.5. Consider the set $L = \{0, a_1, \ldots, a_c\}$, by construction $L \subseteq B_A$. Let $x \in L$ and $y \in B_A$ with $x \neq y$; suppose that $x \sim y$. Since $0 \leqslant x_{[i]} < \alpha$ for all $i = 1, \ldots, d$, we have $y \geqslant x$. By a similar argument as in Lemma 2.3(2) we get $y \notin B_A$. This shows that $x \nsim y$.

We define $\mathrm{r}(K[B]) := \max\{\deg x \mid x \in B_A\}$ and we will call $\mathrm{r}(K[B])$ the *reduction number* of $K[B]$, see [14, pp. 129, 135]. By using $\deg K[B] = f$, see Eq. (2.1), the latter remark, and Lemma 2.3 one can show that

$$\mathrm{r}\big(K[B]\big) \leqslant \deg K[B] - \operatorname{codim} K[B]. \tag{2.3}$$

We note that this equation was proved in [14, Theorem 1.1]. So whenever we have $\operatorname{reg} K[B] = \mathrm{r}(K[B])$ the Eisenbud–Goto conjecture holds for $K[B]$. It should be mentioned that this property does not hold in general. Even for a monomial curve in $\mathbb{P}^3$ the equality does not hold. For $B = \langle (40, 0), (0, 40), (35, 5), (11, 29)\rangle$ we get $\operatorname{reg} K[B] = 13 > 11 = \mathrm{r}(K[B])$. Note that we always have $\mathrm{r}(K[B]) \leqslant \operatorname{reg} K[B]$ by Eq. (2.2).

Example 2.6. Consider the monoid $B = \langle (4, 0), (0, 4), (3, 1), (1, 3)\rangle$. We have

$$B_A = \big\{(0, 0), (3, 1), (1, 3), (6, 2), (2, 6)\big\},$$

and therefore $\mathrm{r}(K[B]) = \max\{0, 1, 1, 2, 2\} = 2$. We get

$$\Gamma_1 = \big\{(0, 0)\big\}, \qquad \Gamma_2 = \big\{(3, 1)\big\}, \qquad \Gamma_3 = \big\{(1, 3)\big\}, \qquad \Gamma_4 = \big\{(6, 2), (2, 6)\big\},$$

and $h_1 = (0, 0)$, $h_2 = (3, 1)$, $h_3 = (1, 3)$, $h_4 = (2, 2)$. By this we obtain $I_1 = I_2 = I_3 = T$ and $I_4 = (y_1, y_2)T$, hence

$$\operatorname{reg} K[B] = \max\big\{\operatorname{reg} T + 0, \operatorname{reg} T + 1, \operatorname{reg} T + 1, \operatorname{reg}(y_1, y_2)T + 1\big\} = 2.$$

Lemma 2.7. *Let $x \in B_A$, $t \in \mathbb{N}^+$, $q \in \{1, \ldots, d\}$, and $x_{[q]} = t\alpha$. There exists a $\lambda \in \Lambda_x$ such that $(t-1)\alpha < x(\lambda, 1)_{[q]} < t\alpha$.*

Proof. Fix a $\nu = (b_1, \ldots, b_{\deg x}) \in \Lambda_x$. We have $x(\nu, \deg x) = 0$ by Remark 2.2, and therefore there is a $k \in \{1, \ldots, \deg x\}$ with $b_{k[q]} > 0$. Since $b_k \in \{a_1, \ldots, a_c\}$ by Remark 2.4 we get that $b_{k[q]} < \alpha$. The claim follows from the fact that $(b_{\sigma(1)}, \ldots, b_{\sigma(\deg x)}) \in \Lambda_x$ for every permutation σ of $\{1, \ldots, \deg x\}$, since $x = \sum_{j=1}^{\deg x} b_j$. $\square$

The next combinatorial lemma will be useful to prove the Eisenbud–Goto conjecture in the seminormal case in Theorem 3.16.

Lemma 2.8. *Let $J \subseteq \{1,\dots,d\}$ with $\# J \geqslant 1$, and let $x \in B_A$ such that $x_{[q]} = \alpha$ for all $q \in J$. There exists a $\lambda \in \Lambda_x$ with the property: for all $p = 1,\dots,\# J$ there is a $q \in J$ such that $0 < x(\lambda, p)_{[q]} < \alpha$.*

Proof. Using induction on $k \in \mathbb{N}^+$ with $k \leqslant \# J$ as well as Lemma 2.7 we get a sequence $\lambda = (b_1,\dots,b_{\deg x}) \in \Lambda_x$ with the property: for all $p = 1,\dots,k$ there is a $q \in J$ such that $0 < x(\lambda, p)_{[q]} < \alpha$. In case that $x(\lambda,k)_{[q]} = \alpha$ for some $q \in J$ we can use Lemma 2.7 to get a sequence with $*$-property $(g_1,\dots,g_{\deg x(\lambda,k)}) \in \Lambda_{x(\lambda,k)}$ with $0 < (x(\lambda,k) - g_1)_{[q]} < \alpha$, since $x(\lambda,k) \in B_A$ by Lemma 2.3. By construction it follows that

$$\lambda' = (b_1,\dots,b_k, g_1,\dots,g_{\deg x(\lambda,k)}) \in \Lambda_x,$$

with the property: for all $p = 1,\dots,k+1$ there is a $q \in J$ such that $0 < x(\lambda', p)_{[q]} < \alpha$. Assume that $x(\lambda,k)_{[q]} < \alpha$ for all $q \in J$. Moreover, let us assume that $k+1 \leqslant \# J$. We clearly have $x(\lambda,k+1)_{[q]} < \alpha$ for all $q \in J$, that is, we need to show that $x(\lambda,k+1)_{[q]} > 0$ for some $q \in J$. Suppose to the contrary that $x(\lambda,k+1)_{[q]} = 0$ for all $q \in J$. Since $x(\lambda,k+1) \leqslant x$ and $\deg x(\lambda,k+1) = \deg x - (k+1) \geqslant \deg x - \# J$ (see Remark 2.2) we get that $x(\lambda,k+1) = x - (\sum_{q\in J} e_q)$. This contradicts $x \in B_A$, since $x(\lambda,k+1) \in B$. $\square$

3. The seminormal case

There are two closely related definitions:

Definition 3.1. Let U be an affine semigroup.

(1) We call U *normal* if $x \in G(U)$ and $tx \in U$ for some $t \in \mathbb{N}^+$ implies that $x \in U$.
(2) We call U *seminormal* if $x \in G(U)$ and $2x, 3x \in U$ implies that $x \in U$.

A domain S is called *seminormal* if for every element x in the quotient field $Q(S)$ of S such that $x^2, x^3 \in S$ it follows that $x \in S$. Note that the ring $K[U]$ is seminormal if and only if U is seminormal. This was first observed by Hochster and Roberts in [25, Proposition 5.32], provided that $U \subseteq \mathbb{N}^d$. For a proof in the general affine semigroup case we refer to [11, Theorem 4.76]. A similar result holds in the normal case, see [15, Proposition 1] and [11, Theorem 4.40]. To get new bounds for the regularity of $K[B]$, we need another characterization. We define the set $\mathrm{Box}(B)$ by

$$\mathrm{Box}(B) := \{x \in B \mid x_{[i]} \leqslant \alpha \ \forall i = 1,\dots,d\}.$$

Theorem 3.2. *(See [16, Theorem 4.1.1].) The simplicial affine semigroup B is seminormal if and only if B_A is contained in* $\mathrm{Box}(B)$.

In the following we will prove the Eisenbud–Goto conjecture for $K[B]$ if B is seminormal. As discussed in the introduction the conjecture holds in the Cohen–Macaulay and Buchsbaum case. Recall that $K[B]$ is Cohen–Macaulay if B is normal. Moreover, $K[B]$ is Cohen–Macaulay if B is seminormal and $\dim K[B] \leqslant 3$, see [26, Theorem 2.2]. This is not true for $\dim K[B] > 3$:

Example 3.3. Consider the monoid

$$B = \big\langle e_1,\dots,e_4, (1,1,0,0), (1,0,1,0), (0,0,1,1), (0,1,0,1)\big\rangle \subset \mathbb{N}^4,$$

with $\alpha = 2$. We have $B_A \subseteq \mathrm{Box}(B)$, thus, B is seminormal by Theorem 3.2. One can show that $(0,1,1,0)+e_1, (0,1,1,0)+e_4 \in B$, but $(0,1,1,0)+e_3 = (0,1,3,0) \notin B$. Hence $K[B]$ is not Buchsbaum by [27, Lemma 3]. Let U be a seminormal positive affine semigroup. Note that $K[U]$ is Cohen–Macaulay if $K[U]$ is Buchsbaum by [28, Proposition 4.15].

In case that Γ_t is contained in $\operatorname{Box}(B)$ for some $t \in \{1, \ldots, f\}$ we get $((x - h_t)/\alpha)_{[i]} \in \{0, 1\}$ for all $x \in \Gamma_t$ and for all $i = 1, \ldots, d$. Thus, I_t is a squarefree monomial ideal in T if $\Gamma_t \subseteq \operatorname{Box}(B)$. This shows that all ideals in the decomposition are squarefree in the seminormal case (see Theorem 3.2).

Lemma 3.4. *Let $\Gamma_t \subseteq \operatorname{Box}(B)$ for some $t \in \{1, \ldots, f\}$ with $\Gamma_t \neq \{0\}$. Let $x, y \in \Gamma_t$, and let $i \in \mathbb{N}$ with $1 \leqslant i \leqslant d$. We have*:

(1) *If $x_{[i]} \neq y_{[i]}$, then $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$.*
(2) *If $0 < x_{[i]} < \alpha$, then $x_{[i]} = y_{[i]}$.*
(3) *If $x_{[i]} \neq y_{[i]}$, then $x_{[i]} \in \{0, \alpha\}$ and $y_{[i]} = \alpha - x_{[i]}$.*
(4) *We have $0 < x_{[j]} < \alpha$ and $0 < x_{[k]} < \alpha$ for some $j, k \in \{1, \ldots, d\}$ with $j \neq k$.*
(5) *If $h_{t[i]} > 0$, then $h_{t[i]} = x_{[i]}$.*

Proof. (1) We have $x_{[i]} - y_{[i]} \in \alpha\mathbb{Z}$ and $x_{[i]} - y_{[i]} \in [-\alpha, \alpha]$, since $0 \leqslant x_{[i]} \leqslant \alpha$ and $0 \leqslant y_{[i]} \leqslant \alpha$. Hence $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$.

(2) We have $x_{[i]} - y_{[i]} \notin \{-\alpha, \alpha\}$ and therefore $x_{[i]} = y_{[i]}$ by claim (1).

(3) By claims (1) and (2) we have $x_{[i]} - y_{[i]} \in \{-\alpha, \alpha\}$ and $x_{[i]} \in \{0, \alpha\}$. Hence $y_{[i]} = \alpha - x_{[i]}$.

(4) Suppose to the contrary that $0 < x_{[j]} < \alpha$ for exactly one $j \in \{1, \ldots, d\}$, that is, $x_{[l]} \in \{0, \alpha\}$ for all $l \in \{1, \ldots, d\} \setminus \{j\}$. Hence $\sum_{l=1}^{d} x_{[l]} \notin \alpha\mathbb{N}$ which contradicts $x \in B$. If $x_{[l]} \in \{0, \alpha\}$ for all $l = 1, \ldots, d$ we have $x \sim 0$. Hence $0 \in \Gamma_t$, that is, $\Gamma_t = \{0\}$ which contradicts our assumption.

(5) We have $0 < h_{t[i]} \leqslant x_{[i]} \leqslant \alpha$, hence $h_{t[i]} = x_{[i]}$, since $h_{t[i]} - x_{[i]} \in \alpha\mathbb{Z}$. $\square$

Remark 3.5. Consider an element $x \in \operatorname{Box}(B) \cap B_A$. Since $x_{[i]} \leqslant \alpha$ for all $i = 1, \ldots, d$ we have $\deg x \leqslant d$. On the other hand there is only one element in $\operatorname{Box}(B)$ with degree d, that is, $(\alpha, \ldots, \alpha)$, but $(\alpha, \ldots, \alpha) \notin B_A$. This shows that $\deg x \leqslant d - 1$. By Theorem 3.2 we get $\operatorname{r}(K[B]) \leqslant d - 1$ if B is seminormal. In Theorem 3.12 we obtain a similar bound for the regularity of $K[B]$ in the seminormal case.

Definition 3.6. For a monomial $m = y_1^{c_1} \cdot \ldots \cdot y_d^{c_d}$ in T we define $\deg m = \sum_{j=1}^{d} c_j$. Let I be a monomial ideal in T with minimal set of monomial generators $\{m_1, \ldots, m_s\}$. Let $F = y_1^{b_1} \cdot \ldots \cdot y_d^{b_d}$ be the least common multiple of $\{m_1, \ldots, m_s\}$. We define $\operatorname{var}(I) := \deg F$, moreover, we define the set $\operatorname{supp}(I) \subseteq \{1, \ldots, d\}$ by $i \in \operatorname{supp}(I)$ if $b_i \neq 0$.

Remark 3.7. Let $t \in \{1, \ldots, f\}$; we note that $\tilde{\Gamma}_t$ is always a minimal set of monomial generators of I_t. Moreover, every monomial ideal in T has a unique minimal set of monomial generators. By construction we get that I_t is a proper ideal in T if and only if $\#\Gamma_t \geqslant 2$. Since $\operatorname{ht} I_t \geqslant 2$ we have $\operatorname{var}(I_t) \neq 1$. Hence I_t is a proper ideal if and only if $\operatorname{var}(I_t) \geqslant 2$. Moreover, if I_t is a proper ideal, then $\deg h_t \geqslant 1$, since $h_t \in G(B) \cap \mathbb{N}^d$ and $h_t \neq 0$.

Consider the squarefree monomial ideal $I = (y_1 y_2, y_2 y_5 y_6)T$ in $T = K[y_1, \ldots, y_6]$. We have $\operatorname{var}(I) = 4$ and $\operatorname{supp}(I) = \{1, 2, 5, 6\}$. So $\operatorname{supp}(I)$ is the set of indices of the variables which occur in the minimal generators of a monomial ideal I in T. Note that we always have $\operatorname{var}(I) = \#\operatorname{supp}(I)$ in the case that I is a squarefree monomial ideal. Hence $\operatorname{var}(I_t) = \#\operatorname{supp}(I_t)$ if $\Gamma_t \subseteq \operatorname{Box}(B)$ for some $t \in \{1, \ldots, f\}$.

Lemma 3.8. *Let $\Gamma_t \subseteq \operatorname{Box}(B)$ for some $t \in \{1, \ldots, f\}$. Then*

$$\operatorname{var}(I_t) \leqslant d - 1 - \deg h_t.$$

Proof. If $\#\Gamma_t = 1$, then we get $\operatorname{var}(I_t) = 0$ and $\deg h_t \leqslant d - 1$ by Remark 3.5. So we may assume that $\#\Gamma_t \geqslant 2$. Let $x \in \Gamma_t$; by Lemma 3.4(4) there are some $j, k \in \{1, \ldots, d\}$ with $j \neq k$ such that $0 < x_{[j]}, x_{[k]} < \alpha$. Hence $0 < h_{t[j]}, h_{t[k]} < \alpha$, since $x - h_t \in A$. By Lemma 3.4(5) we get that $h_{t[q]} = 0$ for all

$q \in \operatorname{supp}(I_t)$. We have $\#\operatorname{supp}(I_t) = \operatorname{var}(I_t)$, since I_t is squarefree. Let $J := \{1, \ldots, d\} \setminus \operatorname{supp}(I_t)$; we get $j, k \in J$ and $h_{t[q]} \leqslant \alpha$ for all $q \in J$, and it follows that

$$\deg h_t = \frac{1}{\alpha} \sum\nolimits_{q \in J} h_{t[q]} < d - \#\operatorname{supp}(I_t) = d - \operatorname{var}(I_t). \qquad \square$$

Remark 3.9. Consider a normal homogeneous affine semigroup U. One can show that $\operatorname{reg} K[U] \leqslant \dim K[U] - 1$. This can be deduced from the proof of [29, Corollary 4.7] and [29, Corollary 3.8], and the fact that $K[U]$ is Cohen–Macaulay by [15, Theorem 1] or [11, Theorem 6.10]. The next theorem obtains a similar bound for seminormal simplicial affine semigroup rings.

Definition 3.10. We define the set $\Gamma(B) \subseteq \{\Gamma_1, \ldots, \Gamma_f\}$ by $\Gamma_t \in \Gamma(B)$ for $t \in \{1, \ldots, f\}$ if $\operatorname{reg} K[B] = \operatorname{reg} I_t + \deg h_t$.

By Eq. (2.2) we obtain $\Gamma(B) \neq \emptyset$. Note that the ideals and shifts corresponding to the elements of $\Gamma(B)$ are computed by the function `regularityMA` in [23].

Proposition 3.11. *Let $\Gamma_t \in \Gamma(B)$ for some $t \in \{1, \ldots, f\}$. If $\Gamma_t \subseteq \operatorname{Box}(B)$, then*

$$\operatorname{reg} K[B] \leqslant \dim K[B] - 1.$$

Proof. We need to show that $\operatorname{reg} I_t + \deg h_t \leqslant d - 1$. In case that $\#\Gamma_t = 1$ this follows from Remark 3.5. Assume that $\#\Gamma_t \geqslant 2$; by Lemma 3.8 we get

$$\operatorname{var}(I_t) - \operatorname{ht} I_t + 1 \leqslant d - 1 - \deg h_t - 2 + 1 = d - 2 - \deg h_t, \tag{3.1}$$

since $\operatorname{ht} I_t \geqslant 2$. Hence $\operatorname{reg} I_t + \deg h_t \leqslant d - 2$ by [30, Theorem 3.1] and Eq. (3.1). $\square$

By Theorem 3.2 and Proposition 3.11 we get the following theorem:

Theorem 3.12. *If B is seminormal, then*

$$\operatorname{reg} K[B] \leqslant \dim K[B] - 1.$$

Note that the bound established in Theorem 3.12 is sharp. Assume $\alpha \geqslant d$ in Theorem 4.2; we get $\operatorname{reg} K[B_{d,\alpha}] = d - 1$ and of course $B_{d,\alpha}$ is seminormal. Consider $B = \langle (3,0,0), (0,3,0), (0,0,3), (2,1,0), (1,0,2), (0,2,1), (1,1,1) \rangle$. One can show that $\Gamma_t = \{(2,2,2)\}$ for some t and therefore $\Gamma_t \subseteq \operatorname{Box}(B)$. Using Macaulay2 [22] we get $\operatorname{reg} K[B] = 2$, hence $\Gamma_t \in \Gamma(B)$. Moreover, since $(4,2,0) \in B_A$ it follows that $K[B]$ is not seminormal by Theorem 3.2. Thus, the condition in Proposition 3.11 is not equivalent to B being seminormal.

Proposition 3.13. *Let $\Gamma_t \in \Gamma(B)$ for some $t \in \{1, \ldots, f\}$. If $\Gamma_t \subseteq \operatorname{Box}(B)$ and $\dim K[B] \leqslant 5$, then*

$$\operatorname{reg} K[B] = \operatorname{r}\bigl(K[B]\bigr).$$

Proof. We have $\operatorname{r}(K[B]) \leqslant \operatorname{reg} K[B]$ by Eq. (2.2). We show that $\operatorname{reg} I_t$ is equal to the maximal degree of a generator of I_t. By this we get

$$\operatorname{reg} K[B] = \operatorname{reg} I_t + \deg h_t = \max\{\deg x \mid x \in \Gamma_t\},$$

and hence $\mathrm{r}(K[B]) = \operatorname{reg} K[B]$. Keep in mind that I_t is squarefree. The case $\#\Gamma_t = 1$ follows from construction. We therefore may assume that $\#\Gamma_t \geqslant 2$, or equivalently, $\operatorname{var}(I_t) \geqslant 2$; note that $\deg h_t \geqslant 1$, see Remark 3.7. Let $d \leqslant 3$; by Lemma 3.8 we get $\operatorname{var}(I_t) \leqslant 1$ which contradicts $\#\Gamma_t \geqslant 2$. Let $d = 5$; by Lemma 3.8 we have to consider the cases $\operatorname{var}(I_t) \in \{2, 3\}$. Suppose that $\operatorname{var}(I_t) = 2$; the ideal I_t is of the form $I_t = (y_k, y_l)T$ for some $k, l \in \{1, \ldots, 5\}$ with $k \neq l$, since $\operatorname{ht} I_t \geqslant 2$. It follows that $\operatorname{reg} I_t = 1$. By a similar argument we get the assertion for $d = 4$ and $\operatorname{var}(I_t) = 2$. Let $d = 5$ and $\operatorname{var}(I_t) = 3$. Since $\operatorname{ht} I_t \geqslant 2$ the only ideals possible are

$$I_{t_1} = (y_k, y_l, y_m)T, \qquad I_{t_2} = (y_k y_l, y_m)T, \qquad I_{t_3} = (y_k y_l, y_k y_m, y_l y_m)T$$

for some $k, l, m \in \{1, \ldots, 5\}$ which are pairwise not equal. Using [30, Theorem 3.1] we obtain $\operatorname{reg} I_{t_1} = 1$ and $\operatorname{reg} I_{t_2} = \operatorname{reg} I_{t_3} = 2$ and we are done. $\square$

By Theorem 3.2 and Proposition 3.13 it follows that $\operatorname{reg} K[B] = \mathrm{r}(K[B])$ if B is seminormal and $\dim K[B] \leqslant 5$. Thus, the Eisenbud–Goto conjecture holds in this case by Eq. (2.3). Theorem 3.16 will confirm the conjecture in any dimension in the seminormal case. Note that Proposition 3.13 could fail for $d \geqslant 6$. Let us consider the squarefree monomial ideal $I = (y_1 y_2, y_3 y_4)T$ with $\operatorname{var}(I) = 4$. So $\operatorname{reg} I = 3$ is bigger than the maximal degree of a generator of I which is 2.

Lemma 3.14. *Let $\Gamma_t \subseteq \operatorname{Box}(B)$ for some $t \in \{1, \ldots, f\}$. Let $n \in \Gamma_t$ and $m \in \tilde{\Gamma}_t$ such that $m = y^{(n-h_t)/\alpha}$. Then:*

(1) *$n_{[q]} = 0$ for all $q \in \operatorname{supp}(I_t) \setminus \operatorname{supp}(mT)$.*
(2) *$n_{[q]} = \alpha$ for all $q \in \operatorname{supp}(mT)$.*

Proof. (1) Suppose that there is a $q \in (\operatorname{supp}(I_t) \setminus \operatorname{supp}(mT)) \neq \emptyset$ such that $n_{[q]} > 0$. Since $q \in \operatorname{supp}(I_t)$ we have $h_{t[q]} = 0$ by Lemma 3.4(5), and therefore $n_{[q]} = \alpha$, because $h_{t[q]} - n_{[q]} \in \alpha\mathbb{Z}$ and $n_{[q]} \leqslant \alpha$. This implies $q \in \operatorname{supp}(mT)$ which is a contradiction.

(2) Let $q \in \operatorname{supp}(mT)$; we have $n_{[q]} \geqslant \alpha$. Moreover, we get $n_{[q]} \leqslant \alpha$, since $\Gamma_t \subseteq \operatorname{Box}(B)$. $\square$

The above lemma is false in general. For the monoid B in Example 2.6 we have $\Gamma_4 = \{(6, 2), (2, 6)\}$, that is, $h_4 = (2, 2)$, and $\tilde{\Gamma}_4 = \{y_1, y_2\}$. For $n \in \Gamma_4$ we get that $n_{[i]} > 0$ for $i = 1, 2$. But $\operatorname{supp}(I_4) = \{1, 2\}$ and $\# \operatorname{supp}(y_1 T) = \# \operatorname{supp}(y_2 T) = 1$. As a consequence of the next proposition the Eisenbud–Goto conjecture holds if B is seminormal.

Proposition 3.15. *Let $\Gamma_t \in \Gamma(B)$ for some $t \in \{1, \ldots, f\}$. If $\Gamma_t \subseteq \operatorname{Box}(B)$, then*

$$\operatorname{reg} K[B] \leqslant \deg K[B] - \operatorname{codim} K[B].$$

Proof. By construction we need to show that $\operatorname{reg} I_t + \deg h_t \leqslant \deg K[B] - c$. If $\#\Gamma_t = 1$ the assertion follows from Eq. (2.3). Let $\#\Gamma_t \geqslant 2$, equivalently, I_t is a proper ideal, see Remark 3.7. We have $\Gamma_t = \{n_1, \ldots, n_{\#\Gamma_t}\}$ and $\tilde{\Gamma}_t = \{m_1, \ldots, m_{\#\Gamma_t}\}$; we may assume that $m_i = y^{(n_i - h_t)/\alpha}$. We set $J_k := (m_1, \ldots, m_k)T$ and $g(k) := \operatorname{var}(J_k) - \operatorname{ht} J_k + 1 + \deg h_t$ for $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \#\Gamma_t$. Note that $J_{\#\Gamma_t} = I_t$, moreover, J_k is a (proper) squarefree monomial ideal in T, since $\Gamma_t \subseteq \operatorname{Box}(B)$, hence $\operatorname{var}(J_k) = \# \operatorname{supp}(J_k)$. We show by induction on $k \in \mathbb{N}$ with $1 \leqslant k \leqslant \#\Gamma_t$ that there is a set L_k with the following properties:

(i) $L_k \subseteq B_A$.
(ii) $\#L_k \geqslant g(k) - 1$.
(iii) $x \nsim y$ for all $x, y \in L_k$ with $x \neq y$.
(iv) $\deg x \geqslant 2$ for all $x \in L_k$.
(v) $x_{[q]} = 0$ for all $x \in L_k$ and for all $q \in \operatorname{supp}(I_t) \setminus \operatorname{supp}(J_k)$.

Let $k = 1$. We have $\operatorname{ht} J_1 = 1$ and $\operatorname{var}(J_1) + \deg h_t = \deg n_1$, that is, $g(1) = \deg n_1$. Fix a $\lambda \in \Lambda_{n_1}$ and set

$$L_1 := \{n_1(\lambda, 0), \dots, n_1(\lambda, \deg n_1 - 2)\},$$

clearly $\# L_1 = \deg n_1 - 1 = g(1) - 1$, hence (ii) is satisfied and by construction we get property (iv). By Lemma 2.3(1) $L_1 \subseteq B_A$ which shows (i), and by Lemma 2.3(2) property (iii) holds. By Lemma 3.14(1) we get $n_1(\lambda, 0)_{[q]} = 0$ for all $q \in \operatorname{supp}(I_t) \setminus \operatorname{supp}(J_1)$, hence (v) holds by construction of L_1.

Using induction on $k \leqslant \#\Gamma_t - 1$ the properties (i)–(v) hold for L_k. We define the set $J := \operatorname{supp}(m_{k+1}T) \setminus \operatorname{supp}(J_k)$. By Lemma 3.14(2) we get $n_{k+1[q]} = \alpha$ for all $q \in \operatorname{supp}(m_{k+1}T)$. Since $n_{k+1} \in B_A$ it follows that $\deg n_{k+1} \geqslant \# \operatorname{supp}(m_{k+1}T) + 1$. Moreover, since $n_{k+1[q]} = \alpha$ for all $q \in J$ we can fix, by Lemma 2.8, a $\lambda \in \Lambda_{n_{k+1}}$ with the property: for all $p = 1, \dots, \# J$ there is a $q \in J$ with $0 < n_{k+1}(\lambda, p)_{[q]} < \alpha$. There could be two cases:

Case 1. $\operatorname{supp}(J_k) \cap \operatorname{supp}(m_{k+1}T) \neq \emptyset$ (e.g., $J_2 = (y_1, y_2 y_3)T$ and $m_3 = y_3 y_4$).

Set

$$L_{k+1} := L_k \cup \{n_{k+1}(\lambda, 1), \dots, n_{k+1}(\lambda, \# J)\}.$$

In case that $J = \emptyset$ we set $L_{k+1} := L_k$.

(iii) By induction we get $x \nsim y$ for all $x, y \in L_k$ with $x \neq y$, moreover, $n_{k+1}(\lambda, i) \nsim n_{k+1}(\lambda, j)$ for all $i, j \in \mathbb{N}$ with $0 \leqslant i < j \leqslant \deg n_{k+1}$ by Lemma 2.3(2). Fix an $x \in L_k$ and let $p \in \{1, \dots, \# J\}$. By (v) $x_{[q]} = 0$ for all $q \in J$, moreover, there is a $q \in J$ such that $0 < n_{k+1}(\lambda, p)_{[q]} < \alpha$, hence $x \nsim n_{k+1}(\lambda, p)$. Thus, property (iii) is satisfied. This also shows that $\# L_{k+1} = \# L_k + \# J$.

(i) By Lemma 2.3(1) $n_{k+1}(\lambda, 1), \dots, n_{k+1}(\lambda, \# J) \in B_A$, since $n_{k+1} \in B_A$.

(iv) Since $\# \operatorname{supp}(m_{k+1}T) \geqslant \# J + 1$ we obtain $\deg n_{k+1} \geqslant \# J + 2$. Hence (iv) holds by construction.

(v) By induction $x_{[q]} = 0$ for all $x \in L_k$ and for all $q \in (\operatorname{supp}(I_t) \setminus \operatorname{supp}(J_k)) \supseteq (\operatorname{supp}(I_t) \setminus \operatorname{supp}(J_{k+1}))$. By Lemma 3.14(1) we have $n_{k+1[q]} = 0$ for all $q \in (\operatorname{supp}(I_t) \setminus \operatorname{supp}(m_{k+1}T)) \supseteq (\operatorname{supp}(I_t) \setminus \operatorname{supp}(J_{k+1}))$, hence property (v) holds by construction.

(ii) Since $\operatorname{supp}(J_{k+1}) = \operatorname{supp}(J_k) \cup \operatorname{supp}(m_{k+1}T)$ we get that $\operatorname{var}(J_{k+1}) = \operatorname{var}(J_k) + \# J$. We have $\operatorname{ht} J_{k+1} \geqslant \operatorname{ht} J_k$ and therefore

$$\begin{aligned} g(k+1) - 1 &\leqslant \# J + \operatorname{var}(J_k) - \operatorname{ht} J_k + 1 + \deg h_t - 1 = \# J + g(k) - 1 \\ &\leqslant \# J + \# L_k = \# L_{k+1}. \end{aligned}$$

Case 2. $\operatorname{supp}(J_k) \cap \operatorname{supp}(m_{k+1}T) = \emptyset$ (e.g., $J_2 = (y_1, y_2 y_3)T$ and $m_3 = y_4 y_5$).

Note that $J = \operatorname{supp}(m_{k+1}T)$, in particular, $\# J \geqslant 1$. Set

$$L_{k+1} := L_k \cup \{n_{k+1}(\lambda, 1), \dots, n_{k+1}(\lambda, \# J - 1)\}.$$

In case that $\# J = 1$ we set $L_{k+1} := L_k$.

(iii), (i), (iv), (v) Analogously, replace $\# J$ by $\# J - 1$ in the corresponding proofs in the first case. Moreover, $\# L_{k+1} = \# L_k + \# J - 1$ by construction.

(ii) We also have $\operatorname{var}(J_{k+1}) = \operatorname{var}(J_k) + \# J$. Since $\operatorname{supp}(J_k) \cap \operatorname{supp}(m_{k+1}T) = \emptyset$ we get that $m_{k+1} + J_k$ is a non-zero divisor of T/J_k. Hence $\operatorname{ht} J_{k+1} = \operatorname{ht} J_k + 1$ by Krull's Principal Ideal Theorem (see, for example, [19, Theorem 10.1]), and therefore

$$g(k+1)-1=\#J+\operatorname{var}(J_k)-\operatorname{ht}J_k-1+1+\deg h_t-1=\#J+g(k)-2$$
$$\leqslant \#J+\#L_k-1=\#L_{k+1}.$$

By this we obtain a set $L_{\#\Gamma_t}$ with the above properties, in particular

$$\#L_{\#\Gamma_t}\overset{\text{(ii)}}{\geqslant} g(\#\Gamma_t)-1=\operatorname{var}(I_t)-\operatorname{ht}I_t+1+\deg h_t-1. \tag{3.2}$$

Hence

$$\#L_{\#\Gamma_t}\geqslant \operatorname{reg}I_t+\deg h_t-1, \tag{3.3}$$

by [30, Theorem 3.1] and Eq. (3.2). We get a set

$$L:=L_{\#\Gamma_t}\cup\{0,a_1,\ldots,a_c\},$$

with $L\subseteq B_A$ such that $x\nsim y$ for all $x,y\in L$ with $x\neq y$ by (i), (iii), and Remark 2.5. Since $\deg K[B]=f$, see Eq. (2.1), we have

$$\deg K[B]\geqslant \#L\overset{\text{(iv)}}{=}\#L_{\#\Gamma_t}+c+1\overset{(3.3)}{\geqslant}\operatorname{reg}I_t+\deg h_t+c=\operatorname{reg}K[B]+c. \qquad \square$$

We obtain from Theorem 3.2 and Proposition 3.15 the following main result:

Theorem 3.16. *If B is seminormal, then*

$$\operatorname{reg}K[B]\leqslant \deg K[B]-\operatorname{codim}K[B].$$

Note that the bound of Theorem 3.16 is again sharp. For $d=2$ and $\alpha\geqslant 2$ we get that $\operatorname{reg}K[B_{2,\alpha}]=\lfloor 2-\frac{2}{\alpha}\rfloor=1$ and $\deg K[B_{2,\alpha}]-\operatorname{codim}K[B_{2,\alpha}]=\alpha-(\alpha+1)+2=1$, see Section 4.

4. Regularity of full Veronese rings

For $X,Y\subseteq\mathbb{N}^d$ we define $X+Y:=\{x+y\mid x\in X,\ y\in Y\}$, $nX:=X+\cdots+X$ (n times), and $0X:=0$. Recall that $B_{d,\alpha}$ denotes the submonoid of $(\mathbb{N}^d,+)$ which is generated by $M_{d,\alpha}=\{(u_1,\ldots,u_d)\in\mathbb{N}^d\mid \sum_{i=1}^d u_i=\alpha\}$. For example $B_{2,2}=\langle(2,0),(0,2),(1,1)\rangle$. We have

$$nM_{d,\alpha}=\left\{(u_1,\ldots,u_d)\in\mathbb{N}^d\;\middle|\;\sum_{i=1}^d u_i=n\alpha\right\}, \tag{4.1}$$

hence there is an isomorphism of K-vector spaces: $K[B_{d,1}]_{n\alpha}\cong K[B_{d,\alpha}]_n$. It is a well-known fact that $h_{K[B_{d,1}]}(n)=\binom{n+d-1}{d-1}$, where h_M denotes the Hilbert polynomial. This shows that $h_{K[B_{d,\alpha}]}(n)=h_{K[B_{d,1}]}(n\alpha)=\binom{n\alpha+d-1}{d-1}$ and therefore $\deg K[B_{d,\alpha}]=\alpha^{d-1}$. Moreover, we get $\operatorname{codim}K[B_{d,\alpha}]=\binom{\alpha+d-1}{d-1}-d$, since $\#M_{d,\alpha}=\binom{\alpha+d-1}{d-1}$. The semigroups $B_{d,\alpha}$ are normal, hence the ring $K[B_{d,\alpha}]$ is Cohen–Macaulay by [15, Theorem 1] and therefore $\#\Gamma_t=1$ for all $t=1,\ldots,f$, see [31, Theorem 6.4]. It follows that

$$\operatorname{reg}K[B_{d,\alpha}]=\operatorname{r}\bigl(K[B_{d,\alpha}]\bigr), \tag{4.2}$$

by Eq. (2.2). In the following we will compute the reduction number $\mathrm{r}(K[B_{d,\alpha}])$ which can also be computed by

$$\mathrm{r}\big(K[B_{d,\alpha}]\big) = \min\big\{r \in \mathbb{N} \bigm| rM_{d,\alpha} + \{e_1, \dots, e_d\} = (r+1)M_{d,\alpha}\big\},$$

see [14, pp. 129, 135].

Lemma 4.1. *Let $r \in \mathbb{N}$. The following assertions are equivalent*:

(1) $rM_{d,\alpha} + \{e_1, \dots, e_d\} = (r+1)M_{d,\alpha}$.
(2) $(r+1)\alpha > d(\alpha - 1)$.

Proof. (1) $\Rightarrow$ (2) Assume that $0 \leqslant (r+1)\alpha \leqslant d(\alpha-1)$. There is an element $x \in \mathbb{N}^d$ with $x_{[j]} \leqslant \alpha - 1$ for all $j = 1, \dots, d$ and $\sum_{j=1}^{d} x_{[j]} = (r+1)\alpha$. We have $x \in (r+1)M_{d,\alpha}$ by Eq. (4.1). Suppose that $x \in rM_{d,\alpha} + \{e_1, \dots, e_d\}$; we get $x = x' + e_j$ for some $x' \in \mathbb{N}^d$ and some $j \in \{1, \dots, d\}$ which contradicts $x_{[j]} \leqslant \alpha - 1$. Hence $x \notin rM_{d,\alpha} + \{e_1, \dots, e_d\}$.

(2) $\Rightarrow$ (1) Let $x \in (r+1)M_{d,\alpha}$. Suppose that $x_{[j]} \leqslant \alpha - 1$ for all $j = 1, \dots, d$. We get $(r+1)\alpha = \sum_{j=1}^{d} x_{[j]} \leqslant d(\alpha - 1)$. Thus, $x_{[j]} \geqslant \alpha$ for some $j \in \{1, \dots, d\}$ and therefore $x - e_j \in rM_{d,\alpha}$ by Eq. (4.1). Hence $(r+1)M_{d,\alpha} \subseteq rM_{d,\alpha} + \{e_1, \dots, e_d\}$, that is, $(r+1)M_{d,\alpha} = rM_{d,\alpha} + \{e_1, \dots, e_d\}$ and we are done. $\square$

Theorem 4.2. *We have*

$$\operatorname{reg} K[B_{d,\alpha}] = \left\lfloor d - \frac{d}{\alpha} \right\rfloor.$$

Proof. By Eq. (4.2) we need to show that $\mathrm{r}(K[B_{d,\alpha}]) = \lfloor d - \frac{d}{\alpha} \rfloor$. We get

$$\left(\left\lfloor d - \frac{d}{\alpha} \right\rfloor + 1\right)\alpha > \left(d - \frac{d}{\alpha}\right)\alpha = d(\alpha - 1),$$

and therefore $\mathrm{r}(K[B_{d,\alpha}]) \leqslant \lfloor d - \frac{d}{\alpha} \rfloor$ by Lemma 4.1. We may assume that $\lfloor d - \frac{d}{\alpha} \rfloor \geqslant 1$. We have

$$\left(\left\lfloor d - \frac{d}{\alpha} \right\rfloor - 1 + 1\right)\alpha \leqslant \left(d - \frac{d}{\alpha}\right)\alpha = d(\alpha - 1),$$

hence $\mathrm{r}(K[B_{d,\alpha}]) > \lfloor d - \frac{d}{\alpha} \rfloor - 1$ by Lemma 4.1 and we are done. $\square$

Example 4.3. By Theorem 4.2 we are able to compute the regularity of full Veronese rings. For $B_{20,2}$ we get $\operatorname{reg} K[B_{20,2}] = \lfloor 20 - \frac{20}{2} \rfloor = 10$. Moreover, we have $\deg K[B_{20,2}] - \operatorname{codim} K[B_{20,2}] = 2^{19} - \binom{2+19}{19} + 20 = 524\,098$.

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