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q-Functions and extreme topological measures

S. Butler

Department of Mathematics, University of Illinois at Urbana-Champaign, 273 Altgeld Hall, 1409 West Green Street, Urbana, IL 61801, USA

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Abstract

q-Functions provide a method for constructing topological measures. We give necessary and sufficient conditions for a composition of a q-function and a topological measure to be a topological measure. Regular and extreme step q-functions are characterized by certain regions in $\mathbb{R}^n$. Then extreme q-functions are used to study extreme topological measures. For example, we prove (under some assumptions on the underlying set) that given n, there are different types of extreme topological measures with values $0, 1/n, \ldots, 1$. In contrast, in the case of measures the only extreme points are $\{0, 1\}$-valued, i.e., point masses.
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1. Introduction

Topological measures (previously called quasi-measures) are set functions that generalize regular Borel measures. The corresponding functionals, called quasi-linear functionals, respectively generalize linear functionals in that they are only assumed to be linear on singly generated subalgebras of $C(X)$. This paper deals with the extreme points of the space of topological measures and some construction techniques.

E-mail address: svbutler@math.uiuc.edu.

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In the next section we give some basic results and examples. The third section deals with q-functions. They provide a way of constructing new topological measures and sometimes an alternative description of topological measures obtained by other techniques, such as by image transformations. We give necessary and sufficient conditions for a composition of a q-function and a topological measure to be a topological measure. Uniform and extreme step q-functions are completely characterized by certain regions in n-dimensional Euclidean space. In the last section we study extreme points of the space of all topological measures on a given set. In the case of measures, the only extreme points are $\{0, 1\}$-valued, i.e., point masses. In the space of topological measures for each natural n there are extreme topological measures that assume values $0, 1/n, \dots, 1$. More precisely, given n, one can find infinitely many finite sets of different cardinalities each of which defines an extreme topological measure with values $0, 1/n, \dots, 1$. For a certain class of underlying spaces (which includes all of our basic examples) we prove necessary and sufficient conditions for a composition of an extreme q-function and a topological measure to give an extreme topological measure.

2. Preliminaries

Let X be a compact Hausdorff space. Let $\mathcal{C}(X)$ (respectively $\mathcal{O}(X)$) denote the collection of closed (respectively open) subsets of X, and $\mathcal{A}(X) = \mathcal{C}(X) \cup \mathcal{O}(X)$. A *topological measure* on X is a function $\mu : \mathcal{A}(X) \to \mathbb{R}^+$ such that:

(i) $\mu(\bigsqcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$ ($\bigsqcup$ indicates disjoint union, and all A_i and $\bigsqcup_{i=1}^n A_i$ are assumed to be in $\mathcal{A}(X)$).
(ii) $\mu(U) = \sup\{\mu(C)\colon C \subseteq U, C \in \mathcal{C}(X)\}$ for all $U \in \mathcal{O}(X)$.

From (i) and (ii) it also follows that topological measures are monotone. Topological measures are also countably additive: if $A = \bigsqcup_{i=1}^\infty A_i$, where A, A_i $(i = 1, 2, \dots) \in \mathcal{A}(X)$, then $\mu(A) = \sum_{i=1}^\infty \mu(A_i)$. (See [8].) While topological measures resemble Borel measures, they need not be subadditive. Later we will give examples of topological measures that are not restrictions of Borel measures to the collection $\mathcal{A}(X)$. A positive topological measure has a (necessarily unique) extension to a regular Borel measure on X if and only if for any open sets U and V we have $\mu(U \cup V) \leqslant \mu(U) + \mu(V)$ (see [11]). The class of all normalized topological measures on X (i.e., topological measures satisfying the condition $\mu(X) = 1$) will be denoted by $TM(X)$.

The definition of a topological measure deals with closed and open sets. Often we may restrict our attention to solid sets. A set is *solid* if it and its complement are both connected. For example, let X be a square, D be a disk inside the square, and C be the boundary of D, a circle. Then D is a solid set, while C is not, since the set $X \setminus C$ is disconnected. We shall denote closed solid (respectively open solid) subsets of X by $\mathcal{C}_s(X)$ (respectively $\mathcal{O}_s(X)$), and $\mathcal{A}_s(X) = \mathcal{C}_s(X) \cup \mathcal{O}_s(X)$.

We are interested in spaces with genus 0, in which case we write $g(X) = 0$. For a precise definition of genus see [3]. Any simply connected space has genus 0. In fact, it was shown in [9] that for any space X with the cohomology module $H^1(X) = 0$ we have $g(X) = 0$.

In particular, the unit square in $\mathbb{R}^n$, $n \geqslant 2$, and the unit sphere in $\mathbb{R}^n$, $n \geqslant 3$, have genus 0. We call a compact, Hausdorff, connected, locally connected space with genus 0 a *q-space*. In this paper X is a q-space unless otherwise stated.

Definition 2.1. A solid set function on a q-space X is a function $\mu : \mathcal{A}_s(X) \to [0, 1]$ such that

(1) if $C_1 \sqcup \cdots \sqcup C_n \subseteq C$, $C, C_1, \ldots, C_n \in \mathcal{C}_s(X)$, then $\sum_{i=1}^n \mu(C_i) \leqslant \mu(C)$;
(2) $\mu(U) = \sup\{\mu(C)\colon C \subseteq U,\ C \in \mathcal{C}_s(X)\}$ for $U \in \mathcal{O}_s(X)$;
(3) $\mu(U) + \mu(X \setminus U) = 1$ for $U \in \mathcal{O}_s(X)$.

Theorem 2.2. *A solid set function on a q-space extends uniquely to a topological measure on X.*

The proof of this theorem is in [3].

Now we are ready to give some examples of topological measures.

Example 2.3. Let X be the unit square and B be the boundary of X. Fix a point p in $X \setminus B$. Define μ on solid sets as follows: $\mu(A) = 1$ if (1) $B \subset A$ or (2) $p \in A$ and $A \cap B \neq \emptyset$. Otherwise, we let $\mu(A) = 0$. Then μ is a solid set function and hence extends to a topological measure on X. To demonstrate that μ is not a measure, we shall show that μ is not subadditive. Let A_1 be a closed solid set consisting of two adjacent sides of B, A_2 be a closed solid set that is the other two adjacent sides of B, and $A_3 = X \setminus B$, an open solid subset of X. Then $X = A_1 \cup A_2 \cup A_3$, $\mu(X) = 1$, but $\mu(A_1) + \mu(A_2) + \mu(A_3) = 0$.

Example 2.4. Let X be a sphere. Fix three points p_1, p_2, p_3 in X. Define μ on solid sets as follows: $\mu(A) = 1$ if A contains the majority of the three points, otherwise, $\mu(A) = 0$. The resulting topological measure is non-subadditive, since $\mu(X) = 1$, and it is easy to represent X as a union of three overlapping solid sets each of which contains exactly one of the points p_1, p_2, p_3.

The topological measures in the last two examples (first presented in [1] and [3]) are not subadditive and hence are not Borel measures.

We conclude this section with some topological facts.

Lemma 2.5. *Let X be a q-space. Let $K \subseteq U$, where $K \in \mathcal{C}_s(X)$, and $U \in \mathcal{O}(X)$. Then there exists $V \in \mathcal{O}_s(X)$ such that $K \subseteq V \subseteq \bar{V} \subseteq U$ (here $\bar{V}$ denotes the closure of the set V).*

Remark 2.6.

(a) This lemma (given in [3]) immediately implies that whenever $K \subseteq U$, $K \in \mathcal{C}(X)$, $U \in \mathcal{O}_s(X)$, there exists $C \in \mathcal{C}_s(X)$ such that $K \subseteq C \subseteq U$.

(b) The closure of a solid set need not be solid. However, given $K \subseteq U$, where $K \in \mathcal{C}(X)$, $U \in \mathcal{O}_s(X)$ or $K \in \mathcal{C}_s(X)$, $U \in \mathcal{O}(X)$ it is not difficult to show that there exists $V \in \mathcal{O}_s(X)$ and $C \in \mathcal{C}_s(X)$ such that $K \subseteq V \subseteq C \subseteq U$.

We equip the space $TM(X)$ with topology given by a subbase $B = \{\hat{U}(\alpha)\colon\ U \in \mathcal{O}_s(X),\ \alpha \in [0,1]\}$, where $\hat{U}(\alpha) = \{\mu \in TM(X)\colon\ \mu(U) > \alpha\}$. Using results from [5] and [1], one can show that $TM(X)$ is a compact Hausdorff convex space.

3. q-Functions

A topological measure on a q-space can be given by a solid set function or by a convex combination of several other topological measures (since $TM(X)$ is a convex space). Another way to get a topological measure is to take the composition of a q-function with a solid set function.

Definition 3.1. A function $f:[0,1] \to [0,1]$ is called a q-function if

(1) f is continuous from the right;
(2) $f(0) = 0,\ f(1) = 1,\ f(x^-) + f(1-x) = 1$ for $0 < x \leqslant 1$;
(3) $\sum_{i=1}^n x_i < 1$ implies $\sum_{i=1}^n f(x_i) \leqslant f(\sum_{i=1}^n x_i)$.

Remark 3.2. From the definition (first given in [4]) we observe the following:

(a) f is nondecreasing on $[0,1]$, and it is enough to define f on $[0,1/2)$.
(b) The points of continuity (hence, discontinuity) of f come in pairs $(x, 1-x)$. Note that f is continuous at 0 and 1. If f is continuous at $1/2$, then $f(1/2) = 1/2$.
(c) The set of all q-functions is convex.

Remark 3.3. If $\sum_{i=1}^n x_i = 1$ and f is continuous at some $x_j \neq 0$, then we still have the inequality $\sum_{i=1}^n f(x_i) \leqslant 1 = f(\sum_{i=1}^n x_i)$.

Proof. If $x_j = 1$, the case is trivial. So suppose that $0 < x_j < 1$. Without loss of generality assume that $j = n$, i.e., $0 < x_n < 1$ and f is continuous at x_n. Then

$$\begin{aligned}\sum_{i=1}^n f(x_i) = \sum_{i=1}^{n-1} f(x_i) + f(x_n) &\leqslant f\left(\sum_{i=1}^{n-1} x_i\right) + f(x_n)\\ &= f(1-x_n) + f\left(x_n^-\right) = 1 = f(1) = f\left(\sum_{i=1}^n x_i\right). \qquad \square\end{aligned}$$

We will describe how to construct topological measures by q-functions.

Definition 3.4. The split spectrum of a topological measure ν is the set $\{\alpha \in (0,1)\colon$ there exist disjoint closed solid sets $C,\ C'$ with $\nu(C) = \alpha,\ \nu(C') = 1-\alpha\}$.

Theorem 3.5. *Let X be a q-space, ν a topological measure on X, f a q-function. Define function μ on solid subsets of X by letting $\mu(C) = f(\nu(C))$ for all $C \in \mathcal{C}_s(X)$ and $\mu(U) = 1 - \mu(X \backslash U)$ for all $U \in \mathcal{O}_s(X)$. Then μ is a solid set function on X (hence, extends uniquely to a topological measure on X) if and only if f is continuous on the split spectrum of ν.*

Proof. ($\Leftarrow$) We need to check the first two conditions in Definition 2.1.

(1) Suppose that $C_1, \dots, C_n, C \in \mathcal{C}_s(X)$ and $C_1 \sqcup \dots \sqcup C_n \subseteq C$. If $\sum_{i=1}^n \nu(C_i) < 1$, then

$$\sum_{i=1}^n \mu(C_i) = \sum_{i=1}^n f\big(\nu(C_i)\big) \leqslant f\left(\sum_{i=1}^n \nu(C_i)\right) \leqslant f\big(\nu(C)\big) = \mu(C).$$

If $\sum_{i=1}^n \nu(C_i) = 1$, then $\nu(C) = 1$, hence, $\mu(C) = 1$. Now we show that any $\nu(C_i) \in (0, 1)$ is in the split spectrum of ν. (Suppose that $0 < \nu(C_n) < 1$. We have: $\sum_{i=1}^{n-1} \nu(C_i) = 1 - \nu(C_n)$, i.e., $\nu(C_1 \sqcup \dots \sqcup C_{n-1}) = 1 - \nu(C_n) = \nu(X \setminus C_n)$. Using Remark 2.6, we can find a closed solid set C_n' such that $C_1 \sqcup \dots \sqcup C_{n-1} \subseteq C_n' \subseteq X \setminus C_n$ and $\nu(C_n') = 1 - \nu(C_n)$. Therefore, $\nu(C_n)$ is in the split spectrum of ν.) Thus f is continuous at all points $\nu(C_i)$ with $0 \leqslant \nu(C_i) \leqslant 1$ and by Remark 3.3,

$$\sum_{i=1}^n \mu(C_i) = \sum_{i=1}^n f\big(\nu(C_i)\big) \leqslant 1 = \mu(C).$$

(2) If $C \subseteq U$, where $C \in \mathcal{C}_s(X), U \in \mathcal{O}_s(X)$, then $C \sqcup (X \setminus U) \subseteq X$, and by the first part $\mu(C) + \mu(X \setminus U) \leqslant 1$, i.e., $\mu(C) \leqslant 1 - \mu(X \setminus U) = \mu(U)$. For any $U \in \mathcal{O}_s(X)$,

$$\begin{aligned}\mu(U) &= 1 - \mu(X \setminus U) = 1 - f\big(\nu(X \setminus U)\big) = 1 - f\big(1 - \nu(U)\big) = f\big(\nu(U)^-\big) \\ &= \sup\big\{f\big(\nu(C)\big)\colon C \subseteq U,\ C \in \mathcal{C}_s(X)\big\} = \sup\big\{\mu(C)\colon C \subseteq U,\ C \in \mathcal{C}_s(X)\big\}.\end{aligned}$$

($\Rightarrow$) Suppose the opposite. Then there exist disjoint closed solid sets C, C' such that $\nu(C) = \alpha > 0$, $\nu(C') = 1 - \alpha$ and f is discontinuous at α. Since f is a q-function, hence, non-decreasing and continuous from the right, we may assume that $f(\alpha^-) = a$, and $f(\alpha) = b > a$. Then we have

$$\begin{aligned}\mu(X) &\geqslant \mu(C) + \mu(C') = f\big(\nu(C)\big) + f\big(\nu(C')\big) = f(\alpha) + f(1 - \alpha) \\ &= f(\alpha) + \big(1 - f(\alpha^-)\big) = b + 1 - a > 1.\end{aligned}$$

This contradicts the normalization condition $\mu(X) = 1$. □

In the previous theorem it suffices to have continuity of f on the intersection of the split spectrum with the interval $(0, 1/2]$.

Example 3.6. The topological measure μ defined in Example 2.4 in the previous section can be viewed as $\mu = f \circ \nu$, where f is a step q-function defined as follows: $f(x) = 0$ on $[0, 1/2)$ and $f(x) = 1$ on $[1/2, 1]$; and $\nu = (\delta_1 + \delta_2 + \delta_3)/3$, where δ_i is the point mass at the point p_i.

A weaker form of Theorem 3.5 was first suggested by J.F. Aarnes and A.B. Rustad in [4], where they also have done some work on continuous q-functions. For example, $f(x) = 1/2(1 - \cos \pi x)$, $p(x) = 3x^2 - 2x^3$ are continuous q-functions. A result by D.J. Grubb [7] states that any polynomial q-function is a linear combination of polynomials of the form $r(x) = c^{-1} \int_0^x t^n (1-t)^n \, dt$ where $c = \int_0^1 t^n (1-t)^n \, dt$.

In this paper we are interested in discontinuous q-functions. Discontinuous and, in particular, step q-functions may provide an alternative description of topological measures which arise from other techniques, such as image transformations. For a simple example, see [10, p. 28]. The following theorem gives necessary and sufficient conditions for a step function to be a q-function.

Theorem 3.7. *Let $n \geqslant 1$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = 1$, $I_i = [\alpha_i, \alpha_{i+1})$ for $i = 0, \ldots, n-1$; $I_n = [\alpha_n, 1]$. Define a step function f with $n+1$ values by $f(I_i) = \beta_i$ where $0 = \beta_0 < \beta_1 < \cdots < \beta_{n-1} < \beta_n = 1$. Then f is a q-function if and only if*

(a) $\alpha_i = 1 - \alpha_{n+1-i}$, $i = 1, \ldots, n$;
(b) $f(\alpha_i + \alpha_j) \geqslant \beta_i + \beta_j$ *whenever* $i + j \leqslant n$; $i, j \geqslant 1$;
(c) $\beta_i = 1 - \beta_{n-i}$, $i = 0, \ldots, n$.

Proof. ($\Leftarrow$) Suppose f is a step function that satisfies (a)–(c). It is easy to see that $f(0) = 0$, $f(1) = 1$, $f : [0, 1] \to [0, 1]$ is continuous from the right and non-decreasing. If $x \in (\alpha_i, \alpha_{i+1}]$ for $i = 0, \ldots, n$, then by (a) $1 - x \in I_{n-i}$. Hence, $f(x^-) + f(1-x) = \beta_i + \beta_{n-i} = 1$. To check the last condition of a q-function it is enough to show that $x_1 + x_2 < 1$ implies $f(x_1 + x_2) \geqslant f(x_1) + f(x_2)$. Let $x_1 + x_2 < 1$, $x_1 \in I_i$, $x_2 \in I_j$. Then $i + j \leqslant n$ (for if $i + j \geqslant n + 1$ then $x_1 + x_2 \geqslant \alpha_i + \alpha_j \geqslant 1$). Since $f = 0$ on I_0 and f is non-decreasing, we may assume that $i, j \geqslant 1$. Then (b) gives: $f(x_1 + x_2) \geqslant f(\alpha_i + \alpha_j) \geqslant \beta_i + \beta_j = f(x_1) + f(x_2)$.

($\Rightarrow$) Suppose f is a q-function.

(a) The points of discontinuity of f are $\alpha_1, \ldots, \alpha_n$. By Remark 3.2, points of discontinuity of a q-function come in pairs $(x, 1-x)$ so each $1 - \alpha_i$ for $i = 1, \ldots, n$ is also a point of discontinuity of f, i.e., $1 - \alpha_i = \alpha_{i^*}$ for some $i^* \in \{1, \ldots, n\}$. Since $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = 1$, it is not difficult to show that $\alpha_i = 1 - \alpha_{n+1-i}$ for $i = 1, \ldots, n$.

(b) If $i + j \leqslant n$, then $\alpha_i + \alpha_j < 1$ (since inequality $\alpha_i + \alpha_j \geqslant 1$ would imply $\alpha_j \geqslant 1 - \alpha_i$, i.e., $j \geqslant n + 1 - i$ by part (a), while we must have $i + j \leqslant n$). f is a q-function, hence, $f(\alpha_i + \alpha_j) \geqslant f(\alpha_i) + f(\alpha_j) = \beta_i + \beta_j$.

(c) For all $i = 0, \ldots, n$ we have: $\beta_i + \beta_{n-i} = f(\alpha_{i+1}^-) + f(\alpha_{n-i}) = f(\alpha_{i+1}^-) + f(1 - \alpha_{i+1}) = 1$. $\square$

Remark 3.8. It is enough to have conditions (a) and (c) in the previous theorem for $i = 1, \ldots, [(n+1)/2]$ (where $[x]$ denotes the maximal integer that is less than or equal to x). When $n = 2k$ ($k \geqslant 1$) function f has $2k + 1$ values; for $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = 1$ we have: $\alpha_k < 1/2$, $\alpha_{k+1} > 1/2$, $\alpha_i = 1 - \alpha_{n+1-i}$ for $i = 1, \ldots, k$. When $n = 2k + 1 (k \geqslant 0)$ function f has $2k + 2$ values; $\alpha_{k+1} = 1/2$, $\alpha_i = 1 - \alpha_{n+1-i}$ for $i = 1, \ldots, k + 1$.

We are interested in the case when a q-function has steps of equal height.

Corollary 3.9. *Let* $n \geqslant 1$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \alpha_{n+1} = 1$, $I_i = [\alpha_i, \alpha_{i+1})$ *for* $i = 0, \ldots, n-1$; $I_n = [\alpha_n, 1]$. *Define a step function* $f(I_i) = i/n$ *for* $i = 0, \ldots, n$. *Then* f *is a* q*-function if and only if*

(a) $\alpha_i = 1 - \alpha_{n+1-i}$ *for* $i = 0, \ldots, n$;
(b*) $\alpha_i + \alpha_j \geqslant \alpha_{i+j}$ *whenever* $i + j \leqslant n$; $i, j \geqslant 1$.

Proof. Condition (c) in the previous theorem is trivial. We will show that (b) in the theorem and (b*) in the corollary are equivalent. Assume that $i, j \geqslant 1$, $i + j \leqslant n$. By (b), we have: $f(\alpha_i + \alpha_j) \geqslant \beta_i + \beta_j = (i + j)/n = \beta_{i+j}$. This means that $\alpha_i + \alpha_j \in I_k$ for some $k \geqslant i + j$. Then $\alpha_i + \alpha_j \geqslant \alpha_{i+j}$, so (b*) holds. By (b*), we have: $\alpha_i + \alpha_j \geqslant \alpha_{i+j}$. Since f is non-decreasing, $f(\alpha_i + \alpha_j) \geqslant f(\alpha_{i+j}) = \beta_{i+j} = \beta_i + \beta_j$, so (b) holds. $\square$

Notice that in the previous corollary the collection $S = \{(\alpha_1, \ldots, \alpha_n)\colon \alpha_1, \ldots, \alpha_n$ satisfy (a) and (b*)$\}$ is non-empty. For example, if $\alpha_i = i/(n+1)$, then $(\alpha_1 \ldots \alpha_n) \in S$.

Example 3.10. In this example we show how to find the region that characterizes a step q-function with the steps of equal height. Consider a step function f with 5 values and steps of equal height. Then we have

$$0 = \alpha_0 < \alpha_1 < \alpha_2 < 1/2 < \alpha_3 = 1 - \alpha_2 < \alpha_4 = 1 - \alpha_1 < \alpha_5 = 1,$$

$\beta_i = i/4$ for $i = 0, \ldots, 4$ and the function f is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \alpha_1), \\ 1/4, & \text{if } x \in [\alpha_1, \alpha_2), \\ 1/2, & \text{if } x \in [\alpha_2, \alpha_3), \\ 3/4, & \text{if } x \in [\alpha_3, \alpha_4), \\ 1, & \text{if } x \in [\alpha_4, 1]. \end{cases}$$

By Corollary 3.9, f is a q-function if and only if the following system of inequalities is satisfied:

$$\begin{cases} 0 < \alpha_1 < \alpha_2 < \frac{1}{2}, \\ 2\alpha_1 \geqslant \alpha_2, \\ \alpha_1 + 2\alpha_2 \geqslant 1. \end{cases}$$

The last system of inequalities defines a region $S = \mathrm{ABCD}$ in the square $[0, \frac{1}{2}) \times [0, \frac{1}{2})$. See Fig. 1. Any point (α_1, α_2) from S gives us a q-function and if f is a q-function, then $(\alpha_1, \alpha_2) \in S$.

Remark 3.11. The technique demonstrated in the previous example works in the general situation as well. If f is a step q-function with $m + 1$ values $\beta_i = \frac{i}{m}$ for $i = 0, \ldots, m$, then f is determined by $n = [\frac{m}{2}]$ parameters $\alpha_1, \ldots, \alpha_n$. Conditions (a) and (b*) of Corollary 3.9 define a region S in $\mathbb{R}^n$. The region S is non-empty, since it contains

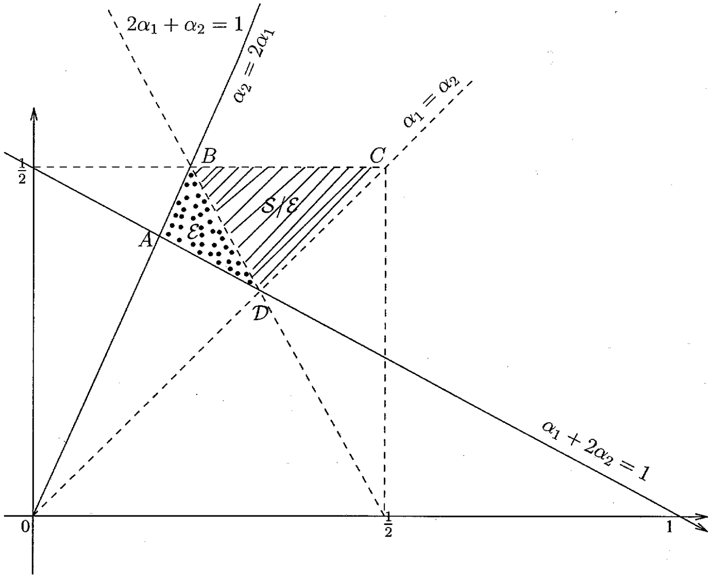

Fig. 1. Regions for q-functions and extreme q-functions.

point $(\frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1})$. The coordinates of any point from S (used as $\alpha_1, \dots, \alpha_n$) will yield a q-function with values $\frac{i}{m}$, $i = 0, \dots, m$. Conversely, if f is a q-function, then $(\alpha_1, \dots, \alpha_n) \in S$.

From Definition 3.1 it is clear that the set of all q-functions is convex, i.e., if f, g are q-functions, then so is $\alpha f + (1-\alpha)g$ for any $\alpha \in [0, 1]$. A q-function is *extreme* if $f = \frac{1}{2} f_1 + \frac{1}{2} f_2$ where f_1, f_2 are q-functions implies that $f = f_1 = f_2$.

Definition 3.12. The step q-function with $\alpha_i = i/(n+1)$ and $\beta_i = i/n$ for $i = 0, \dots, n$ we will call the uniform $(n+1)$-valued step q-function.

Proposition 3.13. *Suppose f is a step q-function with points of discontinuity at $\alpha_1, \dots, \alpha_n$. Suppose $f = \frac{1}{2} f_1 + \frac{1}{2} f_2$, where f_1, f_2 are q-functions. Then f_1, f_2 are also step q-functions. The sets of points of discontinuity of f_1 and f_2 are subsets of the set $\{\alpha_1, \dots, \alpha_n\}$ whose union is the whole set $\{\alpha_1, \dots, \alpha_n\}$.*

Proof. f is defined by: $f = \beta_i$ on I_i, where $0 = \beta_0 < \beta_1 < \dots < \beta_n = 1$, $0 = \alpha_0 < \alpha_1 < \dots < \alpha_n < \alpha_{n+1} = 1$, $I_i = [\alpha_i, \alpha_i + 1)$ for $i = 0, \dots, n-1$, and $I_n = [\alpha_n, 1]$. It is clear that $f_1 = f_2 = 0$ on I_0 and $f_1 = f_2 = 1$ on I_n. Since f_1, f_2 are q-functions, they are non-decreasing, hence, they must be constant on each interval I_i $(i = 0, \dots, n)$. At each of the points $\alpha_1, \dots, \alpha_n$ at least one of the functions f_1, f_2 must have a jump. The statement of the proposition follows. □

Proposition 3.14. *The uniform $(n+1)$-valued step q-function is extreme.*

Proof. Let f be the uniform $(n+1)$-valued step q-function, $n \geqslant 1$. Suppose that $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$. Notice first that by Proposition 3.13, f_1 and f_2 are also step functions defined on the same intervals I_i as f. (Recall: $I_i = \left[\frac{i}{n+1}, \frac{i+1}{n+1}\right)$ for $i = 0, \ldots, n-1$, and $I_n = \left[\frac{n}{n+1}, 1\right]$). To show that $f_1(I_i) = f_2(I_i) = f(I_i)$ for $i = 1, \ldots, n$, it is enough to check that $f_1\left(\frac{i}{n+1}\right) = f_2\left(\frac{i}{n+1}\right) = f\left(\frac{i}{n+1}\right) = \frac{i}{n}$ for $i = 1, \ldots, [\frac{n}{2}]$.

Since $\frac{n}{n+1} < 1$, from Definition 3.1 we have

$$nf_1\left(\frac{1}{n+1}\right) \leqslant f_1\left(\frac{n}{n+1}\right) \leqslant f(1) = 1,$$

that is, $f_1\left(\frac{1}{n+1}\right) \leqslant \frac{1}{n}$. The same thing holds for f_2. Since $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$, we have

$$f_1\left(\frac{1}{n+1}\right) = f_2\left(\frac{1}{n+1}\right) = \frac{1}{n} = f\left(\frac{1}{n+1}\right).$$

Now, $\frac{i}{n+1} + \frac{n-i}{n+1} < 1$, so for any i such that $1 < i \leqslant [\frac{n}{2}]$ we have

$$f_1\left(\frac{i}{n+1}\right) + (n-i)f_1\left(\frac{1}{n+1}\right) \leqslant f_1\left(\frac{i}{n+1} + \frac{n-1}{n+1}\right) \leqslant 1.$$

Then

$$f_1\left(\frac{i}{n+1}\right) \leqslant 1 - (n-i)f_1\left(\frac{1}{n+1}\right) = 1 - \frac{n-i}{n} = \frac{i}{n}.$$

Since also $f_2(\frac{i}{n+1}) \leqslant \frac{i}{n}$, we have

$$f_1\left(\frac{i}{n+1}\right) = f_2\left(\frac{i}{n+1}\right) = \frac{i}{n} = f\left(\frac{i}{n+1}\right). \qquad \square$$

Example 3.15. In this example we will find the region which characterizes extreme step q-functions (with the steps of equal height). Consider again a step function f with 5 values and steps of equal height from Example 3.10. There we found the region S whose points determine f as a q-function. To find the region $E \subset S$ that corresponds to extremal q-functions, we shall first find its complement in S.

Suppose that $f = \frac{1}{2}f_1 + \frac{1}{2}f_2$ where f, f_1, f_2 are q-functions. Since f_1, f_2 are q-functions, they must be $1/2$ on $[\alpha_2, \alpha_3)$. We may assume that $f_1 > f > f_2$ on $[\alpha_1, \alpha_2)$. If, say, $f_1 = 1/4 + \varepsilon,\ f_2 = 1/4 - \varepsilon$ on $[\alpha_1, \alpha_2)$, then on $[\alpha_3, \alpha_4)$ we have: $f_1 = 3/4 - \varepsilon,\ f_2 = 3/4 + \varepsilon$. Notice that we may assume that ε is as small as we wish using the convexity of the set of q-functions. Choose $\varepsilon < 1/12$. Let $\gamma_i = f_1(\alpha_i),\ \delta_i = f_2(\alpha_i)$ for $i = 1, 2, 3$. Then $\gamma_1 = 1/4 + \varepsilon,\ \gamma_2 = 1/2,\ \gamma_3 = 3/4 - \varepsilon,\ \delta_1 = 1/4 - \varepsilon,\ \delta_2 = 1/2,\ \delta_3 = 3/4 + \varepsilon$.

If f_1, f_2 are q-functions, according to (b) in Theorem 3.7, we must have: $f_1(\alpha_i + \alpha_j) \geqslant \gamma_i + \gamma_j$ and $f_2(\alpha_i + \alpha_j) \geqslant \delta_i + \delta_j$ whenever $1 < i + j \leqslant 4$. Hence, $f_1(2\alpha_1) \geqslant 2\gamma_1 = 1/2 + 2\varepsilon$. By our choice of ε, $1/2 + 2\varepsilon < 3/4 - \varepsilon = \gamma_3$. Then we must have: $f_1(2\alpha_1) \geqslant \gamma_3$, i.e., $2\alpha_1$ must be in I_i for $i \geqslant 3$. Then $2\alpha_1 \geqslant 1 - \alpha_2$, in other words, $2\alpha_1 + \alpha_2 \geqslant 1$. Proceeding similarly for other inequalities for f_1 and f_2 when $1 < i + j \leqslant 4$, we get the following system of inequalities:

$$\begin{cases} 0 < \alpha_1 < \alpha_2 < 1/2, \\ \alpha_2 \leqslant 2\alpha_1, \\ \alpha_1 + 2\alpha_2 \geqslant 1, \\ 2\alpha_1 + \alpha_2 \geqslant 1. \end{cases}$$

Notice that the first three inequalities define the region S in Example 3.10. All four inequalities define the subregion of S in which coordinates of its points give such parameters that all three functions f, f_1, f_2 are q-functions and $f = (1/2)f_1 + (1/2)f_2$. Clearly, f is not extreme in this case. The complement of this region in S yields a region $E = \Delta ABD$ that corresponds to extreme q-functions f (since for points in E f is still a q-function, but at least one of f_1, f_2 fails to be a q-function). See Fig. 1.

The technique demonstrated in the previous example may be used to find regions of extreme q-functions in the case of an arbitrary number of steps.

For a fixed topological measure ν, compositions of q-functions with ν define an affine map from the collection of all q-functions to $TM(X)$. If μ is in the image of this affine map and is extreme (i.e., $\mu = (1/2)\mu_1 + (1/2)\mu_2$ where $\mu_1, \mu_2 \in TM(X)$ implies that $\mu = \mu_1 = \mu_2$), it can be represented as the composition of an extreme q-function with ν. The converse is not true in general.

Proposition 3.16. *Let X be a q-space. Let $\nu = (\nu_1 + \cdots + \nu_n)/n$, where $\nu_1, \ldots, \nu_n$ are different $\{0,1\}$-valued topological measures. Let f be the uniform $(n+1)$-valued step q-function (hence, extreme). Then $\mu = f \circ \nu$ is not an extreme topological measure.*

Proof. Note first that by Theorem 3.5, $\mu = f \circ \nu$ is a topological measure on X. Let $\sharp(A) = |\{i:\ \nu_i(A) = 1\}|$. It is not difficult to see that $\mu(A) = \nu(A)$ when A is a solid set with $\sharp(A) = 0, 1, \ldots, n$. Hence, $\mu = \nu$, and μ is not an extreme topological measure, while f is an extreme q-function. □

As we see, the composition of an extreme q-function with a topological measure is a necessary but not sufficient condition for producing an extreme topological measure. However, in the next section we prove that under certain conditions the composition of an extreme q-function with a topological measure will result in an extreme topological measure.

4. Finitely defined and extreme topological measures

From the examples above it is clear that some topological measures can be determined by only a finite set of points, while others can not. Compare, for instance, Examples 2.4 and 2.3. In this section we are interested in topological measures of the first kind.

Definition 4.1. A topological measure μ on X is finitely defined if there is a finite subset $F \subset X$ such that $\sum_{i=1}^{n} \mu(A_i) \leqslant \mu(A)$ whenever $\bigsqcup_{i=1}^{n}(A_i \cap F) \subseteq A \cap F$, where $A, A_1, \ldots, A_n \in \mathcal{A}_s(X)$.

Remark 4.2. The set F that defines a topological measure μ on X is not unique. If F satisfies the condition in Definition 4.1, then so does any set $F' \supseteq F$.

It is easy to see that the split spectrum of a finitely defined topological measure ν is the set $\{\nu(C)\colon\ C \in \mathcal{C}_s(X)\} \cap (0,1)$. The split spectrum of a finitely defined topological measure is finite.

Example 4.3. Let X be a q-space. An example is the unit square. Pick points $p_1, \dots, p_m$ inside the square (avoiding the boundary). Let $F = \{p_1, \dots, p_m\}$, $\nu = (\delta_1 + \dots + \delta_m)/m$, where δ_i is a point mass at p_i. If f is any uniform $(n+1)$-valued step q-function such that

$$\left\{\frac{1}{n+1}, \frac{2}{n+1}, \dots, \frac{n}{n+1}\right\} \cap \left\{\frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\right\} = \emptyset,$$

then by Theorem 3.5, $\mu = f \circ \nu$ defines a topological measure on X. It is clear that μ is finitely defined by the set F.

Example 4.4. If $m = 2n + 1$ with $n \geqslant 1$ and q is the uniform 2-valued step function (i.e., $q = 0$ on $[0, 1/2)$ and $q = 1$ on $[1/2, 1]$), then the above procedure gives a $\{0, 1\}$-valued finitely defined topological measure μ. For example, if $m = 5$, then $F = \{p_1, \dots, p_5\}$ and the resulting $\{0, 1\}$-valued topological measure μ is defined on solid set as follows: $\mu(A) = 1$ if A contains three or more points, otherwise $\mu(A) = 0$. This μ is a composition of the uniform 2-valued step q-function with a measure $(\delta_1 + \dots + \delta_5)/5$.

Definition 4.5. A topological measure that can be represented as the composition of the uniform $(n + 1)$-valued step function with a topological measure is called a uniform topological measure.

Theorem 4.6. *Let $\mu_1, \dots, \mu_m$, where $m \geqslant 1$, be $\{0, 1\}$-valued topological measures, $\mu = (\mu_1 + \dots + \mu_m)/m$, and q be the uniform $(n + 1)$-valued step q-function. If m and $n + 1$ are relatively prime, then $\nu = q \circ \mu$ is a uniform topological measure on X. If $\mu_1, \dots, \mu_m$ are finitely defined, then so is ν.*

Proof. By Theorem 3.5 and Remark 4.2, to show that ν is a topological measure it is enough to check that the sets $\{\frac{i}{n+1}\colon\ i = 1, \dots, n\}$ and $\{\frac{j}{m}\colon\ j = 1, \dots, m - 1\}$ are disjoint. Suppose the opposite. Then $j/m = i/(n+1)$ for some $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m - 1\}$. Then m divides $j(n + 1)$. Since m and $(n + 1)$ are relatively prime, m must divide j, which gives a contradiction. Therefore, ν is a topological measure. It is clear that ν is uniform. If $\mu_1, \dots, \mu_m$ are finitely defined by sets $F_1, \dots, F_m$ then ν is finitely defined by the set $F = F_1 \cup \dots \cup F_m$. $\square$

Remark 4.7. Since a topological measure is uniquely determined by its values on solid sets (see Theorem 2.2), to check that a measure μ is extreme it is enough show that for all $A \in \mathcal{A}_s(X)$ we have $\mu(A) = \mu_1(A)$. In fact, it is enough to check this for open solid sets.

Notice that any topological measure that assumes only values 0 and 1 is extreme. In particular, topological measures in Examples 2.3 and 4.4 are extreme. In Theorem 4.12 below we will present a rich collection of extreme topological measures which is not limited to topological measures with values 0 and 1.

Definition 4.8. A finite set F is freely imbedded in X if for any open connected set U and any finite set $P \subseteq F \cap U$ there exists a closed solid set C such that $C \cap F = P$.

Example 4.9. Any finite subset of the unit sphere S^k, $k \geqslant 2$, is freely imbedded. For $k \geqslant 3$ any finite set is freely imbedded in the unit ball B^k. A finite set F that does not intersect the boundary is freely imbedded in the unit ball B^k for $k = 1, 2$.

Notice that if X is the unit square in $\mathbb{R}^2$ and F is a finite subset of X that has points on the boundary of the square, then F may not be freely imbedded in X. Suppose $F = \{p_1, p_2, p_3, p_4\}$ where each segment of the boundary contains exactly one of these points, and points p_1, p_2 are on the opposite segments. Then F is not freely imbedded in X for there is no closed solid set C such that $C \cap F = \{p_1, p_2\}$.

Definition 4.10. The family of sets $\{P_1, \ldots, P_k\}$ is called an n-disjoint k-chain if the sets $P_i, \ldots, P_{i+n-1}$ are disjoint for $i = 1, \ldots, k$ (indices are mod k).

Before we prove the main theorem, we will need the following lemma [6].

Lemma 4.11. *Suppose F is freely imbedded in X. Suppose $P_1, \ldots, P_k$ are disjoint subsets of F, and $C_i \in \mathcal{C}_s(X)$ are such that $C_i \cap F = P_i$ for $i = 1, \ldots, m$, $m \leqslant k$. Then there exist closed solid sets $C_1, \ldots, C_k$ such that $C_i \cap F = P_i$ for $i = 1, \ldots, k$.*

Proof. By induction it is enough to prove the lemma for $k = m + 1$. Let $U = X \setminus \bigcup_{i=1}^m C_i$. Then U is open, connected and $P_{m+1} \subseteq U$. By Definition 4.8, find a closed solid set $C_{m+1} \subseteq U$ such that $C_{m+1} \cap F = P_{m+1}$. $\square$

Theorem 4.12. *Let $\mu_1, \ldots, \mu_{(n+1)l-1}$ (where n, l are natural numbers) be $\{0, 1\}$-valued topological measures on X that are finitely defined by disjoint sets $F_1, \ldots, F_{(n+1)l-1}$. Let $F = F_1 \sqcup \cdots \sqcup F_{(n+1)l-1}$, $\mu = \frac{\mu_1 + \cdots + \mu_{(n+1)l-1}}{(n+1)l-1}$, and q be the uniform $(n+1)$-valued step q-function. If F is freely imbedded in X, then $\nu = q \circ \mu$ is a topological measure on X that is extreme if and only if $n = 1$ or $l \geqslant 2$.*

Proof. By Theorem 4.6, ν is a uniform finitely defined topological measure. If $n \geqslant 2$, $l = 1$, then ν is not extreme by Proposition 3.16. For the converse, notice first that when $n = 1$ the topological measure ν is $\{0, 1\}$-valued, hence, extreme. We may assume now that $n, l \geqslant 2$. For a solid set A in X let $\sharp(A) = |\{i\colon\ \mu_i(A) = 1\}|$. Then

$$\nu(A) = (q \circ \mu)(A) = \begin{cases} 0, & \text{if } \sharp A = 0, \ldots, l-1, \\ 1/n, & \text{if } \sharp A = l, \ldots, 2l-1, \\ m/n, & \text{if } \sharp A = ml, \ldots, (m+1)l-1, \\ 1, & \text{if } \sharp A = nl, \ldots, (n+1)l-1. \end{cases}$$

Suppose $\nu = (1/2)\nu_1 + (1/2)\nu_2$, where ν_1, ν_2 are topological measures on X. To show that $\nu = \nu_1 = \nu_2$ it is enough to check that $\nu(U) = \nu_1(U)$ for open solid sets U. Let $k = (n+1)l - 1$.

(1) Notice first that for any open or closed set A with $\nu(A) = 0$ (or 1) we also have $\nu_1(A) = 0$ (or 1). Then clearly $\nu(U) = \nu_1(U)$ for open solid sets U with $\sharp U = 0, \dots, l-1$ or $\sharp U = nl, \dots, k$.

(2) Let $U \in \mathcal{O}_s(X)$ be such that $\sharp U = l$, so $\nu(U) = 1/n$. We may assume that $\mu_i(U) = 1$ for $i = 1, \dots, l$ and $\mu_i(U) = 0$ for $i = l+1, \dots, k$. We may also assume that $F_i \subseteq U$ for $1 \leqslant i \leqslant l$ and $F_i \cap U = \emptyset$ for $1 \leqslant l+1 \leqslant k$. (If this is not the case, replace the sets F_i by the sets K_i defined as follows: $K_i = U \cap F_i$ for $i = 1, \dots, l$ and $K_i = U^c \cap F_i$ for $i = l+1, \dots, k$. Now let A be any solid set which contains F_j for some index j and does not intersect F_i for all $i \neq j$. Then clearly $\sharp A = 1$. The same holds, i.e., $\sharp A = 1$ if A is such that A contains K_j but does not intersect K_i for all $i \neq j$. Indeed, if $1 \leqslant j \leqslant l$, then $U \cap F_j = K_j \subseteq A \cap F_j$ and the fact that $\mu_j(U) = 1$ then implies that $\mu_j(A) = 1$. If $l+1 \leqslant j \leqslant k$, then $U^c \cap F_j = K_j \subseteq A \cap F_j$, and since $\mu_j(U^c) = 1$, we have $\mu_j(A) = 1$. Similarly one shows that $\mu_i(A) = 0$ for all $i \neq j$. Therefore, $\sharp A = 1$.)

Form k finite subsets $P_1, \dots, P_k$ of F as

$$P_i = F_{(i-1)l+1}, F_{(i-1)l+2}, \dots, F_{il},$$

where indices are mod k. Then $F = \bigcup_{i=1}^k P_i$ and $\{P_1, \dots, P_k\}$ is an n-disjoint k-chain.

Find a closed solid set C_1 such that $P_1 \subseteq C_1 \subseteq U$. The existence of C_1 follows from Remark 2.6. Now apply Lemma 4.11 to find disjoint closed solid sets $C_1, \dots, C_n$ such that $F \cap C_i = P_i$ for $i = 1, \dots, n$. Applying Lemma 4.11 again, we find C_{n+1} such that the sets $C_2, \dots, C_{n+1}$ are disjoint and $F \cap C_i = P_i$ for $i = 2, \dots, n+1$. Continuing in this fashion, we get a family of closed solid sets $\{C_1, \dots, C_k\}$ which is an n-disjoint k-chain with the property $F \cap C_i = P_i$ for $i = 1, \dots, k$. Since $C_i \cap F = P_i$, it is clear that $\sharp C_i = l$ and hence $\nu(C_i) = \frac{1}{n}$ for $i = 1, \dots, k$. Since the sets $C_1, \dots, C_n$ are disjoint, we have

$$\begin{aligned}\nu(C_1 \sqcup C_2 \sqcup \dots \sqcup C_n) &= \nu(C_1) + \nu(C_2) + \dots + \nu(C_n) \\ &= 1/n + 1/n + \dots + 1/n = 1.\end{aligned}$$

Then by part (1) we also have

$$\begin{aligned}\nu_1(C_1) + \nu_1(C_2) + \dots + \nu_1(C_n) &= \nu_1(C_1 \sqcup C_2 \sqcup \dots \sqcup C_n) \\ &= \nu(C_1 \sqcup C_2 \sqcup \dots \sqcup C_n) = 1.\end{aligned}$$

Let $x_i = \nu_1(C_i)$. Then we get $x_1 + x_2 + \dots + x_n = 1$. Since $C_1, \dots, C_k$ is an n-disjoint k-chain, repeating the argument for disjoint sets $C_i, \dots, C_{i+n-1}$, we obtain $x_i + \dots + x_{i+n-1} = 1$ for all $i = 1, 2, \dots$ (mod k). From the equations $x_i + \dots + x_{i+n-1} = 1$ and $x_{i+1} + \dots + x_{i+n} = 1$ we get $x_i = x_{i+n}$, and hence $x_i = x_{i+pn}$ for $p = 0, 1, 2, \dots$. Since $(n, k) = 1$, n is a generator for the cyclic group $\mathbb{Z}_k$. Hence, $x_1 = x_2 = \dots = x_k$, which implies that $x_i = \frac{1}{n}$ for $i = 1, \dots, k$. In particular, $\nu_1(C_1) = x_1 = \frac{1}{n}$. Then $\nu_1(U) \geqslant \nu_1(C_1) = \frac{1}{n}$. Similarly we get: $\nu_2(U) \geqslant \frac{1}{n}$. Since $\nu(U) = \frac{1}{2}\nu_1(U) + \frac{1}{2}\nu_2(U)$, we must have: $\nu_1(U) = \nu_2(U) = \nu(U) = \frac{1}{n}$.

(3) Let $U \in \mathcal{O}_s(X)$ be such that $l < \sharp(A) < nl - 1$, i.e., $\sharp(U) = ml + t$ for some $m \in \{1, \dots, n-1\}$, $t \in \{0, \dots, l-1\}$. All cases are similar, so we will show the proof for the case $t = l-1$, i.e., $\sharp(U) = (m+1)l - 1$. Say, $\mu_i(U) = 1$ for $i = 1, \dots, (m+1)l - 1$ and $\mu_i(A) = 0$ for $i = (m+1)l, \dots, k = (n+1)l - 1$. Then $\nu(U) = m/n$. As before, we may assume that $F_i \subseteq U$ for $i = 1, \dots, (m+1)l - 1$ and $F_i \cap U = \emptyset$ for $i = (m+1)l, \dots, k$.

Let $P_0 = F_1 \sqcup \cdots \sqcup F_{(m+1)l-1}$. By Remark 2.6, there exists a closed solid set C_0 such that $P_0 \subseteq C_0 \subseteq U$. Split the remaining $n-m$ sets $F_{(m+1)l}, \ldots, F_{(n+1)l-1}$ into $n-m$ disjoint finite sets $P_1, \ldots, P_{n-m}$ of equal size. Thus each P_i consists of exactly l sets F_j, $j \in \{(m+1)l, \ldots, (n+1)l-1\}$. Since the sets $P_0, P_1, \ldots, P_{n-m}$ are disjoint subsets of F, we may use Lemma 4.11 to find closed solid sets $C_1, \ldots, C_{n-m}$ such that $C_0, C_1, \ldots, C_{n-m}$ are all disjoint and $F \cap C_i = P_i$ for $i = 0, 1, \ldots, n-m$. Note that for $i = 1, \ldots, n-m$ the fact that $F \cap C_i = P_i$ means that $\sharp C_i = l$, and then $\nu(C_i) = \nu_1(C_i) = 1/n$ by part (2). Considering a closed set $C_0 \sqcup C_1 \sqcup \cdots \sqcup C_{n-m}$, we obtain

$$\begin{aligned} \nu(C_0 \sqcup C_1 \sqcup \cdots \sqcup C_{n-m}) &= \nu(C_0) + \nu(C_1) + \cdots + \nu(C_{n-m}) \\ &= m/n + 1/n + \cdots + 1/n = 1. \end{aligned}$$

Hence, we also have:

$$\nu_1(C_0 \sqcup C_1 \sqcup \cdots \sqcup C_{n-m}) = 1, \quad \text{i.e., } \nu_1(C_0) + \nu_1(C_1) + \cdots + \nu_1(C_{n-m}) = 1.$$

Then

$$\nu_1(C_0) = 1 - \nu_1(C_1) - \cdots - \nu_1(C_{n-m}) = 1 - \frac{1}{n} - \cdots - \frac{1}{n} = \frac{m}{n}.$$

Thus, $\nu_1(U) \geqslant \nu_1(C_0) = \frac{m}{n}$. Similarly, $\nu_2(U) \geqslant \frac{m}{n}$. Since $\nu(U) = \frac{1}{2}\nu_1(U) + \frac{1}{2}\nu_2(U)$, we have: $\nu_1(U) = \nu_2(U) = \nu(U) = \frac{k}{n}$. The proof is complete now. □

Remark 4.13. Theorem 4.12 was first proved using chained families of sets on solidly chainable spaces. These ideas were developed beautifully by J.F. Aarnes into less technical and at the same time more powerful and elegant concepts of n-disjoint k-chains and freely imbedded sets. The author is grateful to J.F. Aarnes for sharing his results [6] which allowed for a shorter, more transparent proof of Theorem 4.12.

Example 4.14. Let X be the unit square. Let $F = \{p_1, p_2, \ldots, p_{2n+1}\}$ be a subset of X that does not intersect the boundary. Let μ_i be the point mass at the point p_i. Define a topological measure ν on X by its value on solid sets by: $\nu(A) = (q \circ \mu)(A) = m/n$, if $\sharp A = 2m$ or $2m+1$ for $m = 0, \ldots, n$. As before, $\sharp(A) = |\{i \in \{1, \ldots, 2n+1\}\colon\ \mu_i(A) = 1\}| = |\{i \in \{1, \ldots, 2n+1\}\colon\ p_i \in A\}|$. Then we can say that $\nu = q \circ \mu$, where q is the uniform $(n+1)$-valued step q-function, and $\mu = \frac{\mu_1 + \mu_2 + \cdots + \mu_{2n+1}}{2n+1}$. By Theorem 4.12, ν is an extreme topological measure.

With $F = \{p_1, p_2, p_3, p_4, p_5\}$ and the uniform 3-valued step q-function one gets the extreme topological measure from [2].

Example 4.15. Let X be the unit square. Let $F = \{p_1, p_2, \ldots, p_5\}$ be a subset of X that does not intersect the boundary, and μ_i be the point mass at the point p_i. Define a topological measure $\nu = f \circ \mu$, where f is the uniform 4-valued step q-function, and $\mu = \frac{\mu_1 + \cdots + \mu_5}{5}$. In other words,

$$\nu(A) = (f \circ \mu)(A) = \begin{cases} 0, & \text{if } \sharp A = 0, 1, \\ 1/3, & \text{if } \sharp A = 2, \\ 2/3, & \text{if } \sharp A = 3, \\ 1, & \text{if } \sharp A = 4, 5. \end{cases}$$

Let f_1 be the uniform 3-valued step q-function, f_2 be the uniform 2-valued step q-function, $\nu_1 = f_1 \circ \mu$, $\nu_2 = f_2 \circ \mu$. Then it is not difficult to check that $\nu = \frac{2}{3}\nu_1 + \frac{1}{3}\nu_2$, hence, ν is not extreme.

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