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Letter to the Editor

An inhomogeneous eigenvalue problem

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Abstract

In this paper it is shown that the general solution of an inhomogeneous boundary value problem may only consist of the general solution and that it is not necessary to superpose a partial solution. The eigenfrequencies of a membrane subject to inhomogeneous boundary conditions are calculated using the program Mathematica.
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1. Introduction

According to the usual definition, an eigenvalue problem is to determine values of a parameter k (eigenvalue) for which a homogeneous linear differential equation has nontrivial solutions (eigenfunctions) under the prescribed boundary conditions [2]. In [4], this definition demands that the boundary conditions should be homogeneous. For membrane oscillations described by the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

homogeneous boundary conditions imply that the membrane is clamped along the whole boundary. But how may the eigenfrequency of a membrane be calculated, if the membrane were not clamped all around? In this case, the boundary conditions are called inhomogeneous. A boundary value problem is called inhomogeneous if either the boundary conditions or the differential equation (or both) are inhomogeneous. A partial differential equation with inhomogeneous boundary conditions may be transformed into an inhomogeneous equation subject to homogeneous boundary conditions [2]. To

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show this, we consider an inhomogeneous linear equation

$$Du(x,y)=f(x,y), \tag{2}$$

and use the setup

$$u=w+v, \tag{3}$$

so that

$$Du=Dw+Dv=f(x,y), \tag{4}$$

defining

$$Dv=f(x,y), \quad Dw=0. \tag{5}$$

Demanding now a homogeneous boundary condition

$$u(\text{boundary})=0=v(\text{boundary})+w(\text{boundary}),$$

one obtains

$$v(\text{boundary})=-w(\text{boundary}) \tag{6}$$

(homogenization of an inhomogeneous boundary condition). Now the function $v(x,y)$ should satisfy the inhomogeneous conditions for $w(x,y)$ and must be such that (5) is satisfied.

It is quite difficult to construct such a function v [1]. However, it is well known that the general solution of an inhomogeneous equation of type (2) consists of the superposition of the general solution of the pertinent homogeneous equation and a partial solution of (2), see [3]. But in this paper, it will be shown that there exist general solutions of an inhomogeneous eigenvalue problem without the superposition of a partial solution of the inhomogeneous equation.

2. Eigenvalues of an inhomogeneous boundary value problem

Let us consider the Helmholtz equation (1) together with homogeneous and inhomogeneous boundary conditions on a rectangle of the dimensions $2a\times 2b$. We assume

$$u(x=\pm a,y)=0 \tag{7}$$

and

$$u(x,y=\pm b)=\cos\left(\frac{\pi}{2a}x\right) \quad [\text{or } g(x), g(\pm a)=0]. \tag{8}$$

This guarantees $u(\pm a,\pm b)=0$ and a continuous boundary value. Inserting the setup

$$u(x,y)=\cos\left(\frac{\pi}{2a}x\right)f(y) \tag{9}$$

into (1) one obtains

$$f''(y) + \left(k^2 - \left(\frac{\pi}{2a}\right)^2\right) f(y) = 0. \tag{10}$$

Solution (9) satisfies the homogeneous boundary condition (7). To be able to satisfy the inhomogeneous condition (8), we have to solve (10). Using the abbreviation

$$\beta = \sqrt{k^2 - \pi^2/4a^2}, \tag{11}$$

the solution of (10) is given by $f(y) = \cos\beta y$ and the solution of (1) reads

$$u(x,y) = \cos\left(\frac{\pi}{2a}\,x\right)\cos\left(\sqrt{k^2 - \pi^2/4a^2}\,y\right). \tag{12}$$

In order to satisfy the inhomogeneous condition (8), one must have

$$\cos\left(\sqrt{k^2 - \pi^2/4a^2}\,b\right) = 1$$

or

$$\sqrt{k^2 - \pi^2/4a^2}\,b = n\pi, \quad n = 1, 2 \ldots. \tag{13}$$

This is the equation determining the eigenvalue k and (12) is a solution of the inhomogeneous problem given by (1), (7) and (8).

3. The general solution

We are now searching for a general solution of (1) making the setup

$$u(x,y) = \sum_{m=0}^{\infty} A_m \cos\left[\frac{\pi}{2a}(2m+1)x\right] f_n(y), \quad m = 0, 1, 2 \ldots . \tag{14}$$

This solution again satisfies the homogeneous boundary condition (7). For $f(y)$ one has

$$f_n(y)\left[k^2 - \left(\frac{\pi}{2a}(2m+1)\right)^2 x\right] + f_n''(y) = 0. \tag{15}$$

In the derivation of (15) we have used the fact that a sum vanishes if all terms vanish. The solution of (15) is given by

$$f_n(y) = B_n \cos\left[\sqrt{k_{nm}^2 - \left(\frac{\pi}{2a}(2m+1)\right)^2}\, y\right]. \tag{16}$$

Since k now depends on n and m, we used the designation k_{nm}. The inhomogeneous boundary condition contains the function $g(x)$, $g(\pm a) = 0$ and the general solution of (1) takes the form

$$u(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_m \cos\left[\frac{\pi}{2a}(2m+1)x\right] B_n \cos\left[\sqrt{k_{nm}^2 - \left(\frac{\pi}{2a}(2m+1)\right)^2}\, y\right]. \tag{17}$$

Now we have to satisfy the inhomogeneous boundary condition (8) and to find the eigenvalue k. Condition (8) yields

$$u(x, \pm b) = g(x) = \sum_{m,n}^{\infty} A_m B_n \cos\left[\frac{\pi}{2a}(2m+1)x\right] \cos\left[\sqrt{k_{nm}^2 - \left(\frac{\pi}{2a}(2m+1)\right)^2}\, b\right]. \tag{18}$$

With respect to (8) the second cos-term should be equal to 1. Thus

$$\sqrt{k_{nm}^2 - \left(\frac{\pi}{2a}(2m+1)\right)^2}\, b = 2n\pi, \quad n = 1, 2, \ldots \tag{19}$$

must be valid. This determines the eigenvalue k and

$$u(x, \pm b) = g(x) = \sum_m A_m \cos\left[\frac{\pi}{2a}(2m+1)x\right] \tag{20}$$

is a cos-Fourier series expansion of the given (even) function g, if all $B_n = 1$. Expansion (20) also satisfies $g(\pm a) = 0$.

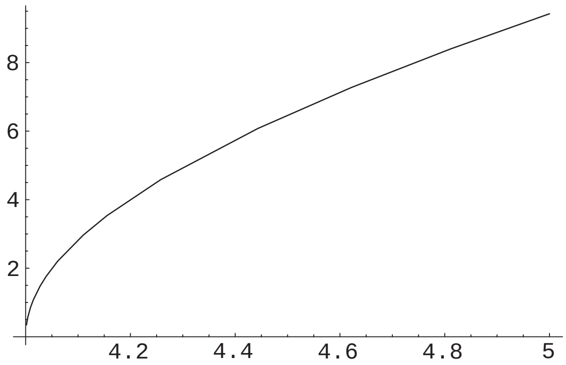
Solution (17) is a general solution and no partial solution of the pertinent inhomogeneous equation or a homogenization of the inhomogeneous boundary condition is necessary. Questions of convergence of the solution have not been investigated. This problem will depend on the boundary condition $g(x)$.

4. A numerical example with Mathematica

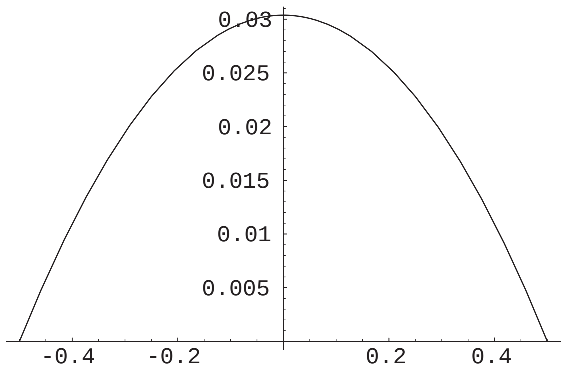
Using the program package Mathematica [1] we can give a numerical example.

```
In[1] := (* code 57 :
            Inhomogeneous eigenvalue problem. Solve the
            Helmholtz equation for a membrane,
            define a rectangle 2a × 2b as the domain,
            assume a homogeneous condition u(a,y) = 0
            and an inhomogeneous u(x,b) = g(x) * )
         Clear[g, a, b];
         g[x_] = Sqrt[a^2 + b^2 -  x^2] -  b;
         a = 0.5;
         (* Find a value of b
            so that the calculations converge.
            The simple eigenvalue π * Sqrt[b^2 – 4/a^2]
            must be real * )
         Plot[ π * Sqrt[b^2 – 4/a^2],
         {b,a,10.* a}]
Plot :: plnr : π Sqrt[b -  -- ] is not a machine-size real number
at b = 0.5000001875‘.

Plot :: plnr : π Sqrt[b -  -- ] is not a machine-size real number
at b = 0.682551462078121‘.

Plot :: plnr : π Sqrt[b -  -- ] is not a machine-size real number
at b = 0.8816395993671815‘.
```

```
General :: stop : Further output of Plot :: plnr will be
suppressed during this calculation.
```




```
Out[1]= -Graphics-
In[2]:=  b = 4.1;
         (* Check the satisfaction of the boundary conditions * )
         Plot[g[x],{x, − a,a}]
         g[a]
```

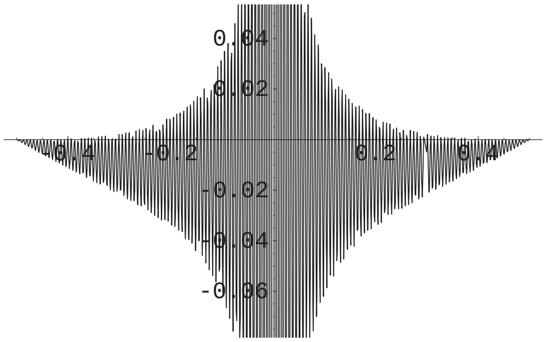



```
Out[2]= -Graphics-
Out[2]= 0.
In[3]:=  (* Is the simple eigenvalue k real ? * )
          k = π * Sqrt[b^2 − 4/a^2]
Out[3]= 2.82743
In[4]:=  (* Expand g(x) into a Fourier series * )
          < < Calculus‘FourierTransform’
          FourierTrigSeries[g[x], x, 2]
Out[4]= 0.0202651 + 2 Cos[2πx]
```

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(-4.1+\sqrt{17.06 - x^2}\right) \text{Cos}[2\pi x]\,dx+$$

$$2\,\mathrm{Cos}[4\pi x]\int_{-\frac{1}{2}}^{\frac{1}{2}}\left(-4.1+\sqrt{17.06-x^2}\right)\mathrm{Cos}[4\pi x]\,dx$$

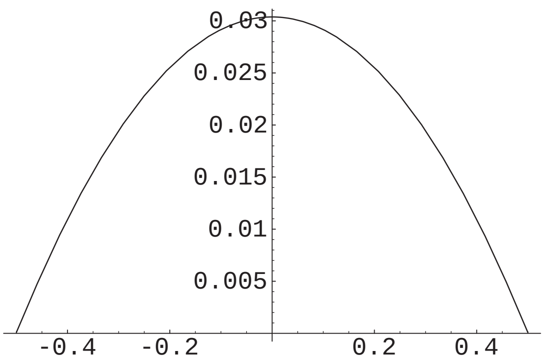
In[5]:=
```
(* Investigate the integrand and
    make a choice of how many terms *)
p = 150;
G[x_] = Sum[g[x]* Cos[2* n* πx],{n,1,p}];
    Plot[G[x],{x, − a,a}]
```



Out[5]= -Graphics-

In[6]:=
```
(* This is an oscillatory integrand.
    You may use option Method Oscillatory *)
Clear[J];
(* Calculate the expansion coefficients J[n] *)
Off[NIntegrate :: ncvb];
Off[NIntegrate :: slwcon];
Table[J[n] = NIntegrate[g[x] * Cos[2 * n * π x],
{x, − a,a}
(* , Method → Oscillatory * )],
Clear[GR];
GR[x_] = 0.0202651 + Sum[Cos[2 * n * π* x] * 2 * J[n], {n, 1, p}];
```

In[7]:=
```
Plot[GR[x],{x, − a,a}]
```



Out[7]= `-Graphics-`
In[8]:= `(* Calculate the error *)`
`g[0] - GR[0]`
Out[8] = 3.0249×10^{-7}
In[9]:= `g[a] − GR[a]`
Out[9]= `−0.0000820712`

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