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ORIGINAL ARTICLE

On generalized ϕ -recurrent LP-Sasakian Manifolds



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Abstract The object of the present paper was to introduce the notion of *generalized ϕ -recurrent LP-Sasakian manifold* and study its various geometric properties with the existence by an interesting example. Among others it is shown that a generalized ϕ -recurrent LP-Sasakian manifold is an Einstein manifold. Also it is proved that there exists no generalized projectively ϕ -recurrent LP-Sasakian manifold.

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1. Introduction

On the analogy of Sasakian manifolds, in 1989 Matsumoto [1] introduced the notion of LP-Sasakian manifolds. Again the same notion is introduced by Mihai and Rosca [2] and obtained many interesting results. LP-Sasakian manifolds are also studied by Aqeel et al. [3], Bagewadi et al. [4], De et al. [5], Mihai et al. [6], Murathan et al. [7], Shaikh et al. [8–12], Venkatesha and Bagewadi [13] and others. The study of Riemann symmetric manifolds began with the work of Cartan [14]. The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [15] introduced the notion of local ϕ -symmetry on a Sasakian manifold. Generalizing the notion of local ϕ -symmetry of Takahashi [15], De et al. [16] introduced the notion of ϕ -recurrent Sasakian manifolds. Also De et al. [17] studied the ϕ -recurrent Kenmotsu manifolds. In this connection, it may be mentioned that Shaikh and Hui [18] studied the locally ϕ -symmetric β -Kenmotsu manifolds.

In the context of Lorentzian geometry, the notion of local ϕ -symmetry on an LP-Sasakian manifold is introduced and studied by Shaikh and Baishya [9] with several examples. Also Shaikh et al. [11] studied ϕ -recurrent LP-Sasakian manifolds and proved the existence of such notion by several non-trivial examples. Generalizing all these notions of local ϕ -symmetry, in the present paper we introduce *generalized ϕ -recurrent LP-Sasakian manifolds*.

The notion of generalized recurrent manifolds introduced by Dubey [19] and studied by De and Guha [20]. A Riemannian manifold (M^n, g) , $n > 2$, is called generalized recurrent [20] if its curvature tensor R of type $(1, 3)$ satisfies the condition

$$\nabla R = A \otimes R + B \otimes G, \quad (1.1)$$

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where A and B are non-vanishing 1-forms defined by $A(\cdot) = g(\cdot, \rho_1)$, $B(\cdot) = g(\cdot, \rho_2)$ and the tensor G of type (1,3) is given by

$$G(X, Y)Z = g(Y, Z)X - g(X, Z)Y \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$; $\chi(M)$ being the Lie algebra of smooth vector fields on M and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g . The 1-forms A and B are called the associated 1-forms of the manifold.

Specially, if the 1-form B vanishes, then (1.1) turns into the notion of recurrent manifold introduced by Walker [21].

A Riemannian manifold (M^n, g) , $n > 2$, is called a generalized Ricci-recurrent manifold [22] if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$\nabla S = A \otimes S + B \otimes g, \tag{1.3}$$

where A and B are non-vanishing 1-forms defined in (1.1).

In particular, if $B = 0$, then (1.3) reduces to the notion of Ricci-recurrent manifold introduced by Patterson [23].

A Riemannian manifold (M^n, g) , $n > 2$, is called a super generalized Ricci-recurrent manifold [24] if its Ricci tensor S of type (0,2) satisfies the condition

$$\nabla S = \alpha \otimes S + \beta \otimes g + \gamma \otimes \eta \otimes \eta, \tag{1.4}$$

where α , β and γ are non-vanishing unique 1-forms.

In particular, if $\beta = \gamma$, then (1.4) reduces to the notion of quasi-generalized Ricci-recurrent manifold introduced by Shaikh and Roy [25].

The paper is organized as follows. Section 2 is concerned with some preliminaries about LP-Sasakian manifolds. Section 3 deals with *generalized ϕ -recurrent LP-Sasakian manifolds* and obtained a necessary and sufficient condition for an LP-Sasakian manifold to be a generalized ϕ -recurrent LP-Sasakian manifold (see, Theorem 3.3). It is shown that a generalized ϕ -recurrent LP-Sasakian manifold is Einstein manifold and in such a manifold the 1-forms A and B are related by $A + B = 0$. We also study *generalized concircularly (resp., projectively) ϕ -recurrent LP-Sasakian manifolds*. It is proved that a generalized concircularly ϕ -recurrent LP-Sasakian manifold is super generalized Ricci-recurrent, but it can not be quasi-generalized Ricci-recurrent. Finally, the last section deals with an interesting example, which ensures the existence of generalized ϕ -recurrent LP-Sasakian manifold.

2. LP-Sasakian manifolds

An n -dimensional smooth manifold M is said to be an LP-Sasakian manifold [2,8] if it admits a (1, 1) tensor field ϕ , a unit timelike vector field ξ , an 1-form η and a Lorentzian metric g , which satisfy

$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \quad \phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \nabla_X \xi = \phi X, \tag{2.2}$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

for all $X, Y \in \chi(M)$, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g . It can be easily seen that in an LP-Sasakian manifold, the following relations hold:

$$\phi \xi = 0, \quad \eta(\phi X) = 0, \quad \text{rank } \phi = n - 1. \tag{2.4}$$

Again, if we put

$$\Omega(X, Y) = g(X, \phi Y)$$

for all $X, Y \in \chi(M)$, then the tensor field $\Omega(X, Y)$ is a symmetric (0,2) tensor field [1]. Also, since the 1-form η is closed in an LP-Sasakian manifold, we have [1,5]

$$(\nabla_X \eta)(Y) = \Omega(X, Y), \quad \Omega(X, \xi) = 0 \tag{2.5}$$

for all vector fields $X, Y \in \chi(M)$.

Let M be an n -dimensional LP-Sasakian manifold with structure (ϕ, ξ, η, g) . Then the following relations hold [1,8]:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.6}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{2.7}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \tag{2.8}$$

$$\begin{aligned} (\nabla_W R)(X, Y)\xi &= 2[\Omega(Y, W)X - \Omega(X, W)Y] - \phi R(X, Y)W \\ &\quad - g(Y, W)\phi X + g(X, W)\phi Y \\ &\quad - 2[\Omega(X, W)\eta(Y) - \Omega(Y, W)\eta(X)]\xi \\ &\quad - 2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(W). \end{aligned} \tag{2.9}$$

$$\begin{aligned} (\nabla_W R)(X, \xi)Z &= \Omega(W, Z)X - g(X, Z)\phi W \\ &\quad - R(X, \phi W)Z \end{aligned} \tag{2.10}$$

for all vector fields $X, Y, Z \in \chi(M)$, where R is the curvature tensor of the manifold.

The above results will be used in the later sections.

3. Generalized ϕ -recurrent LP-Sasakian manifolds

Definition 3.1. A tensor field K of type (1,3) on an LP-Sasakian manifold (M^n, g) , $n > 2$, is said to be *generalized K - ϕ -recurrent* if it satisfies the relation

$$\begin{aligned} \phi^2((\nabla_W K)(X, Y)Z) &= A(W)\phi^2(K(X, Y)Z) \\ &\quad + B(W)\phi^2(G(X, Y)Z) \end{aligned} \tag{3.1}$$

for all $X, Y, Z, W \in \chi(M)$, where A and B are non-vanishing 1-forms such that $A(X) = g(X, \rho_1)$, $B(X) = g(X, \rho_2)$. The 1-forms A and B are called the associated 1-forms of the manifold.

In particular, if $K = R$ (resp., \tilde{C} , P) then the LP-Sasakian manifold (M^n, g) , $n > 2$, is said to be generalized ϕ -recurrent (resp., generalized concircularly ϕ -recurrent, generalized projectively ϕ -recurrent), where \tilde{C} and P denote the concircular and projective curvature tensor of type (1,3) and are respectively given by

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}G(X, Y)Z \tag{3.2}$$

and

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y]. \tag{3.3}$$

We consider a generalized ϕ -recurrent LP-Sasakian manifold. Then by virtue of (2.1), we have from (3.1) that

$$\begin{aligned} (\nabla_W R)(X, Y)Z + \eta((\nabla_W R)(X, Y)Z)\xi \\ = A(W)[R(X, Y)Z + \eta(R(X, Y)Z)\xi] \\ + B(W)[G(X, Y)Z + \eta(G(X, Y)Z)\xi], \end{aligned} \tag{3.4}$$

from which it follows that

$$\begin{aligned} g((\nabla_W R)(X, Y)Z, U) + \eta((\nabla_W R)(X, Y)Z)\eta(U) \\ = A(W)[g(R(X, Y)Z, U) + \eta(R(X, Y)Z)\eta(U)] \\ + B(W)[g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)]. \end{aligned} \quad (3.5)$$

Taking an orthonormal frame field and then contracting (3.5) over X and U and then using (1.2), we get

$$\begin{aligned} (\nabla_W S)(Y, Z) = -g((\nabla_W R)(\xi, Y)Z, \xi) + A(W)S(Y, Z) + \{(n-2)B(W) \\ - A(W)\}g(Y, Z) - \{A(W) + B(W)\}\eta(Y)\eta(Z). \end{aligned} \quad (3.6)$$

Using (2.6), (2.9) and the relation $g((\nabla_W R)(X, Y)Z, U) = -g((\nabla_W R)(X, Y)U, Z)$, we have

$$g((\nabla_W R)(\xi, Y)Z, \xi) = 0. \quad (3.7)$$

By virtue of (3.7), it follows from (3.6) that

$$\begin{aligned} (\nabla_W S)(Y, Z) = A(W)S(Y, Z) + \{(n-2)B(W) - A(W)\}g(Y, Z) \\ - \{A(W) + B(W)\}\eta(Y)\eta(Z). \end{aligned} \quad (3.8)$$

Setting $Z = \xi$ in (3.8) and using (2.7), we obtain

$$(\nabla_W S)(Y, \xi) = (n-1)\{A(W) + B(W)\}\eta(Y). \quad (3.9)$$

Also we have

$$\begin{aligned} (\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi) \\ = (n-1)g(Y, \phi W) - S(Y, \phi W), \text{ using (2.5) and (2.7).} \end{aligned} \quad (3.10)$$

From (3.9) and (3.10), we obtain

$$\begin{aligned} (n-1)g(Y, \phi W) - S(Y, \phi W) \\ = (n-1)\{A(W) + B(W)\}\eta(Y). \end{aligned} \quad (3.11)$$

Again plugging $Y = \xi$ in (3.11), we get

$$A(W) + B(W) = 0 \text{ for all } W. \quad (3.12)$$

By virtue of (3.12), it follows from (3.11) that

$$S(Y, \phi W) = (n-1)g(Y, \phi W). \quad (3.13)$$

Substituting Y by ϕY in (3.13) and then using (2.2) and (2.8), we get

$$S(Y, W) = (n-1)g(Y, W).$$

This leads to the following:

Theorem 3.1. *A generalized ϕ -recurrent LP-Sasakian manifold is an Einstein manifold and moreover the associated 1-forms A and B are related by $A + B = 0$.*

Corollary 3.1. [8] *An LP-Sasakian manifold is Ricci-semisymmetric if and only if it is Einstein.*

Thus from Theorem 3.1 and Corollary 3.1, we can state the following:

Theorem 3.2. *A generalized ϕ -recurrent LP-Sasakian manifold is Ricci-semisymmetric.*

Using (2.9) and the relation $g((\nabla_W R)(X, Y)Z, U) = -g((\nabla_W R)(X, Y)U, Z)$ in (3.4), we have

$$\begin{aligned} (\nabla_W R)(X, Y)Z \\ = [g(X, W)g(\phi Y, Z) - g(Y, W)g(\phi X, Z) + 2\{g(Y, \phi W)g(X, Z) \\ - g(X, \phi W)g(Y, Z)\} - 2\{g(X, \phi W)\eta(Y) - g(Y, \phi W)\eta(X)\}\eta(Z) \\ - 2\{\eta(Y)g(\phi X, Z) - \eta(X)g(\phi Y, Z)\}\eta(W) - g(\phi R(X, Y)W, Z)]\xi \\ + A(W)[R(X, Y)Z + \eta(R(X, Y)Z)\xi] + B(W)[g(Y, Z)X \\ + g(Y, Z)\eta(X)\xi - g(X, Z)Y - g(X, Z)\eta(Y)\xi]. \end{aligned} \quad (3.14)$$

Applying ϕ^2 on both sides, we get the relation (3.1). Hence we can state the following:

Theorem 3.3. *An LP-Sasakian manifold is generalized ϕ -recurrent if and only if the relation (3.14) holds.*

Changing W, X, Y cyclically in (3.5) and adding them, we get by virtue of Bianchi identity and (3.12) that

$$\begin{aligned} A(W)[g(R(X, Y)Z, U) - g(G(X, Y)Z, U) \\ + \{\eta(R(X, Y)Z) - \eta(G(X, Y)Z)\}\eta(U)] \\ + A(X)[g(R(Y, W)Z, U) - g(G(Y, W)Z, U) \\ + \{\eta(R(Y, W)Z) - \eta(G(Y, W)Z)\}\eta(U)] \\ + A(Y)[g(R(W, X)Z, U) - g(G(W, X)Z, U) \\ + \{\eta(R(W, X)Z) - \eta(G(W, X)Z)\}\eta(U)] = 0. \end{aligned}$$

Contracting the above relation over Y and Z , we get

$$\begin{aligned} A(W)[S(X, U) - (n-1)g(X, U)] - A(X)[S(W, U) - (n-1)g(W, U)] \\ + A(R(W, X)U) - A(R(W, X)\xi)\eta(U) - A(X)g(W, U) + A(W)g(X, U) \\ - \{A(X)\eta(W) - A(W)\eta(X)\}\eta(U) = 0. \end{aligned} \quad (3.15)$$

Again contracting (3.15) over X and U and using (2.6), we get

$$S(W, \rho_1) = \frac{r - (n-1)(n-2)}{2}g(Y, \rho_1). \quad (3.16)$$

This leads to the following:

Theorem 3.4. *In a generalized ϕ -recurrent LP-Sasakian manifold, $\frac{r-(n-1)(n-2)}{2}$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector ρ_1 .*

We now consider a generalized concircularly ϕ -recurrent LP-Sasakian manifold (M^n, g) , $n > 2$. Then from (3.1) we have

$$\begin{aligned} \phi^2 \left((\nabla_W \tilde{C})(X, Y)Z \right) = A(W)\phi^2 \left(\tilde{C}(X, Y)Z \right) \\ + B(W)\phi^2(G(X, Y)Z), \end{aligned} \quad (3.17)$$

where A and B are defined as in (3.1).

Let us consider a generalized concircularly ϕ -recurrent LP-Sasakian manifold. Then by virtue of (2.1), it follows from (3.17) that

$$\begin{aligned} (\nabla_W \tilde{C})(X, Y)Z = -\eta \left((\nabla_W \tilde{C})(X, Y)Z \right) \xi \\ + A(W) \left[\tilde{C}(X, Y)Z + \eta \left(\tilde{C}(X, Y)Z \right) \xi \right] \\ + B(W) [G(X, Y)Z + \eta(G(X, Y)Z)\xi], \end{aligned} \quad (3.18)$$

from which it follows that

$$\begin{aligned} g \left((\nabla_W \tilde{C})(X, Y)Z, U \right) = -\eta \left((\nabla_W \tilde{C})(X, Y)Z \right) \eta(U) \\ + A(W) \left[g \left(\tilde{C}(X, Y)Z, U \right) + \eta \left(\tilde{C}(X, Y)Z \right) \eta(U) \right] \\ + B(W) [g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)]. \end{aligned} \quad (3.19)$$

Contracting (3.19) over X and U and using (2.10) and (3.2), we get

$$\begin{aligned} (\nabla_W S)(Y, Z) = A(W)S(Y, Z) \\ + (n-2) \left[B(W) + \frac{dr(W)}{n(n-1)} - \left\{ \frac{r}{n(n-1)} + \frac{1}{n-2} \right\} A(W) \right] \\ g(Y, Z) - \left[\frac{dr(W)}{n(n-1)} + \left\{ 1 - \frac{r}{n(n-1)} \right\} A(W) + B(W) \right] \eta(Y)\eta(Z), \end{aligned} \quad (3.20)$$

which can be written as

$$\nabla S = A \otimes S + \psi \otimes g + H \otimes \eta \otimes \eta, \tag{3.21}$$

where

$$\psi(W) = (n-2) \left[B(W) + \frac{dr(W)}{n(n-1)} - \left\{ \frac{r}{n(n-1)} + \frac{1}{n-2} \right\} A(W) \right]$$

and

$$H(W) = - \left[\frac{dr(W)}{n(n-1)} + \left\{ 1 - \frac{r}{n(n-1)} \right\} A(W) + B(W) \right].$$

Hence we can state the following:

Theorem 3.5. *A generalized concircularly ϕ -recurrent LP-Sasakian manifold is a super generalized Ricci-recurrent manifold.*

Setting $Y = Z = \xi$ in (3.20) and using (2.7), we get

$$dr(W) = \{r - n(n-1)\}A(W) - n(n-1)B(W). \tag{3.22}$$

This leads to the following:

Theorem 3.6. *In a generalized concircularly ϕ -recurrent LP-Sasakian manifold, the 1-forms A and B are related by the equation (3.22).*

Corollary 3.2. *In a generalized concircularly ϕ -recurrent LP-Sasakian manifold with non-zero constant scalar curvature, the associated 1-forms A and B are related by*

$$\{r - n(n-1)\}A - n(n-1)B = 0.$$

We now consider a generalized concircularly ϕ -recurrent LP-Sasakian manifold which is quasi-generalized Ricci-recurrent [25]. Then $\psi(W) = H(W)$, where ψ and H are 1-forms given in (3.21). The equality of ψ and H gives us

$$dr(W) = rA(W) - n(n-1)B(W). \tag{3.23}$$

From (3.22) and (3.23), we obtain $A = 0$, which is inadmissible. Hence we can state the following:

Theorem 3.7. *A generalized concircularly ϕ -recurrent LP-Sasakian manifold can not be a quasi-generalized Ricci-recurrent manifold.*

We now consider a generalized projectively ϕ -recurrent LP-Sasakian manifold (M^n, g) , $n > 2$. Then from (3.1) we have

$$\phi^2((\nabla_w P)(X, Y)Z) = A(W)\phi^2(P(X, Y)Z) + B(W)\phi^2(G(X, Y)Z), \tag{3.24}$$

where A and B are defined as in (3.1).

Let us consider a generalized projectively ϕ -recurrent LP-Sasakian manifold. Then by virtue of (2.1), it follows from (3.24) that

$$\begin{aligned} (\nabla_w P)(X, Y)Z &= -\eta((\nabla_w P)(X, Y)Z)\xi \\ &\quad + A(W)[P(X, Y)Z + \eta(P(X, Y)Z)\xi] \\ &\quad + B(W)[G(X, Y)Z + \eta(G(X, Y)Z)\xi], \end{aligned} \tag{3.25}$$

from which it follows that

$$\begin{aligned} g((\nabla_w P)(X, Y)Z, U) &= -\eta((\nabla_w P)(X, Y)Z)\eta(U) \\ &\quad + A(W)[g(P(X, Y)Z, U) + \eta(P(X, Y)Z)\eta(U)] \\ &\quad + B(W)[g(G(X, Y)Z, U) + \eta(G(X, Y)Z)\eta(U)]. \end{aligned} \tag{3.26}$$

Contracting (3.26) over X and U and using (2.9), (3.10) and (3.3), we get

$$\begin{aligned} (\nabla_w S)(Y, Z) &= A(W)S(Y, Z) + (n-1)[(n-2)B(W) - A(W)]g(Y, Z) \\ &\quad + [S(Z, \phi W) - (n-1)B(W)\eta(Z)]\eta(Y) \\ &\quad + (n-1)\eta(R(Y, \phi W)Z). \end{aligned} \tag{3.27}$$

Setting $Y = Z = \xi$ in (3.27), we obtain $B = 0$, which is not possible. Hence we can state the theorem:

Theorem 3.8. *There exists no generalized projectively ϕ -recurrent LP-Sasakian manifold (M^n, g) , $n > 3$.*

4. An Example of generalized ϕ -recurrent LP-Sasakian manifold

Example 4.1. We consider a 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where (x, y, z) are the standard coordinates in \mathbb{R}^3 . Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame on M given by [10]

$$E_1 = e^z \frac{\partial}{\partial x}, \quad E_2 = e^{z-ax} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z},$$

where a is a non-zero constant such that $a \neq 1$. Let g be the Lorentzian metric defined by $g(E_1, E_3) = g(E_2, E_3) = g(E_1, E_2) = 0$, $g(E_1, E_1) = g(E_2, E_2) = 1$, $g(E_3, E_3) = -1$. Let η be the 1-form defined by $\eta(U) = g(U, E_3)$ for any $U \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi E_1 = -E_1$, $\phi E_2 = -E_2$ and $\phi E_3 = 0$. Then using the linearity of ϕ and g we have $\eta(E_3) = -1$, $\phi^2 U = U + \eta(U)E_3$ and $g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)$ for any $U, W \in \chi(M)$. Thus for $E_3 = \xi$, (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M .

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g and R be the curvature tensor of g . Then we have

$$[E_1, E_2] = -ae^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2.$$

Using Koszul formula for the Lorentzian metric g , we can easily calculate

$$\begin{aligned} \nabla_{E_1} E_1 &= -E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -E_1, \\ \nabla_{E_2} E_1 &= ae^z E_2, \quad \nabla_{E_2} E_2 = -ae^z E_1 - E_3, \quad \nabla_{E_2} E_3 = -E_2, \\ \nabla_{E_3} E_1 &= 0, \quad \nabla_{E_3} E_2 = 0, \quad \nabla_{E_3} E_3 = 0. \end{aligned}$$

From the above it can be easily seen that for $E_3 = \xi$, (ϕ, ξ, η, g) is an LP-Sasakian structure on M . Consequently $M^3(\phi, \xi, \eta, g)$ is an LP-Sasakian manifold. Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R as follows:

$$\begin{aligned} R(E_1, E_2)E_1 &= -(1 - a^2 e^{2z})E_2, \quad R(E_1, E_2)E_2 = (1 - a^2 e^{2z})E_1, \\ R(E_1, E_3)E_1 &= -E_3, \quad R(E_1, E_3)E_3 = -E_1, \\ R(E_2, E_3)E_2 &= -E_3, \quad R(E_2, E_3)E_3 = -E_2 \end{aligned}$$

and the components which can be obtained from these by the symmetry properties.

Since $\{E_1, E_2, E_3\}$ forms a basis of a 3-dimensional LP-Sasakian manifold, any vector field $X, Y, Z \in \chi(M)$ can be written as

$$\begin{aligned} X &= a_1E_1 + b_1E_2 + c_1E_3, \\ Y &= a_2E_1 + b_2E_2 + c_2E_3, \\ Z &= a_3E_1 + b_3E_2 + c_3E_3, \end{aligned}$$

where $a_i, b_i, c_i \in \mathbb{R}^+$ (the set of all positive real numbers), $i = 1, 2, 3$ such that a, b, c are not proportional. Then

$$\begin{aligned} R(X, Y)Z &= \{b_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) - c_3(a_1c_2 - a_2c_1)\}E_1 \\ &\quad - \{a_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) + c_3(b_1c_2 - b_2c_1)\}E_2 \\ &\quad - \{b_3(b_1c_2 - b_2c_1) + a_3(a_1c_2 - a_2c_1)\}E_3 \end{aligned} \quad (4.1)$$

and

$$\begin{aligned} G(X, Y)Z &= (a_2a_3 + b_2b_3 - c_2c_3)(a_1E_1 + b_1E_2 + c_1E_3) \\ &\quad - (a_1a_3 + b_1b_3 - c_1c_3)(a_2E_1 + b_2E_2 + c_2E_3). \end{aligned} \quad (4.2)$$

By virtue of (4.1), we have the following:

$$\begin{aligned} (\nabla_{E_1}R)(X, Y)Z &= a^2e^{2z}[(a_1b_2 - a_2b_1)(b_3E_3 + c_3E_2) \\ &\quad + (b_1c_2 - b_2c_1)(b_3E_1 - a_3E_2)], \end{aligned} \quad (4.3)$$

$$\begin{aligned} (\nabla_{E_2}R)(X, Y)Z &= (a_1b_2 - a_2b_1)\{a^2e^{2z}(c_3E_1 - a_3E_3) \\ &\quad - (b_3 + c_3)E_1\} - (a_1c_2 - a_2c_1)a^2e^{2z}(b_3E_1 - a_3E_2) \\ &\quad + (b_1c_2 - b_2c_1)(c_3 - b_3)ae^zE_1, \end{aligned} \quad (4.4)$$

$$(\nabla_{E_3}R)(X, Y)Z = 0. \quad (4.5)$$

From (4.1) and (4.2), we get

$$\begin{aligned} \phi^2(R(X, Y)Z) &= u_1E_1 + u_2E_2 \text{ and } \phi^2(G(X, Y)Z) \\ &= v_1E_1 + v_2E_2, \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} u_1 &= b_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) - c_3(a_1c_2 - a_2c_1), \\ u_2 &= -\{a_3(a_1b_2 - a_2b_1)(1 - a^2e^{2z}) + c_3(b_1c_2 - b_2c_1)\}, \\ v_1 &= a_1(b_2b_3 - c_2c_3) - a_2(b_1b_3 - c_1c_3), \\ v_2 &= b_1(a_2a_3 - c_2c_3) - b_2(a_1a_3 - c_1c_3). \end{aligned}$$

Also from (4.3)–(4.5), we obtain

$$\phi^2((\nabla_{E_i}R)(X, Y)Z) = p_iE_1 + q_iE_2, \text{ for } i = 1, 2, 3, \quad (4.7)$$

where

$$\begin{aligned} p_1 &= b_3(b_1c_2 - b_2c_1)a^2e^{2z}, \\ q_1 &= \{c_3(a_1b_2 - a_2b_1) - a_3(b_1c_2 - b_2c_1)\}a^2e^{2z}, \\ p_2 &= (a_1b_2 - a_2b_1)\{c_3(a^2e^{2z} - 1) - b_3\} - b_3(a_1c_2 - a_2c_1)a^2e^{2z} \\ &\quad + (c_3 - b_3)(b_1c_2 - b_2c_1)ae^z, q_2 = a_3(a_1c_2 - a_2c_1)a^2e^{2z}, p_3 = 0, q_3 = 0. \end{aligned}$$

Let us now consider the 1-forms as

$$\begin{aligned} A(E_1) &= \frac{v_2p_1 - v_1q_1}{u_1v_2 - u_2v_1}, \quad B(E_1) = \frac{u_1q_1 - u_2p_1}{u_1v_2 - u_2v_1}, \\ A(E_2) &= \frac{v_2p_2 - v_1q_2}{u_1v_2 - u_2v_1}, \quad B(E_2) = \frac{u_1q_2 - u_2p_2}{u_1v_2 - u_2v_1}, \\ A(E_3) &= 0, \quad B(E_3) = 0, \end{aligned} \quad (4.8)$$

where $v_2p_1 - v_1q_1 \neq 0, u_1q_1 - u_2p_1 \neq 0, v_2p_2 - v_1q_2 \neq 0, u_1q_2 - u_2p_2 \neq 0, u_1v_2 - u_2v_1 \neq 0$. From (3.4), we have

$$\begin{aligned} \phi^2((\nabla_{E_i}R)(X, Y)Z) &= A(E_i)\phi^2(R(X, Y)Z) \\ &\quad + B(E_i)\phi^2(G(X, Y)Z), \quad i = 1, 2, 3. \end{aligned} \quad (4.9)$$

By virtue of (4.6)–(4.8), it can be easily shown that the manifold satisfies the relation (4.9). Hence the manifold under consideration is a 3-dimensional generalized ϕ -recurrent LP-Sasakian manifold, which is not ϕ -recurrent. This leads to the following:

Theorem 4.1. *There exists a 3-dimensional generalized ϕ -recurrent LP-Sasakian manifold, which is neither ϕ -symmetric nor ϕ -recurrent.*

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