



Orthosymmetric block reflectors[☆]

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Abstract

We develop a general theory of reflectors and block reflectors in a class of non-Euclidean scalar product spaces generated by orthosymmetric scalar product matrices J . These J -reflectors are generalizations of ordinary Householder transformations, and we show that they can always be expressed in a Householder-like representation. Reflection and mapping properties of J -reflectors are completely described.

Block J -reflectors can be used for block annihilation in QR-like factorizations, where Q is J -unitary and R is block upper triangular.

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1. Introduction

Let G be a “tall” rectangular matrix, $G \in \mathbb{F}^{m \times n}$, $m \geq n$, where \mathbb{F} is either the field of complex numbers \mathbb{C} or the field of real numbers \mathbb{R} . One of the most useful factorizations of G in practice is the QR factorization

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$$G = QR = [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1$$

with unitary Q and upper triangular R_1 . Note that unitary matrices preserve the ordinary Euclidean scalar product.

In some problems it is advisable to preserve a different, non-Euclidean structure, described by a given matrix $J \in \mathbb{F}^{m \times m}$, which generates the underlying “scalar product”. Then we seek a QR-like factorization of G , where Q is unitary with respect to the scalar product generated by J , i.e., $Q^* J Q = J$, and R_1 is block upper triangular.

Two classical examples, most frequently used in practice, are the hyperbolic (or indefinite) J , given by

$$J = \text{diag}(j_{11}, \dots, j_{mm}), \quad j_{ii} \in \{-1, 1\}, \quad (1.1)$$

and the symplectic J , given by

$$J = \text{diag}(J_0, \dots, J_0), \quad J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad (1.2)$$

with even m .

The so-called triangular case of the hyperbolic (or indefinite) QR factorization has been known for more than 20 years [4] and the detailed analysis is given in [17]. The general case has been described recently in [16].

The symplectic QR factorization has a similar history (see [5,18]). To avoid possible confusion in terminology, it should be noted that in [5] the permuted form of the symplectic QR is called the SR factorization, and the name symplectic is reserved for a factorization where Q is both unitary and J -unitary.

The ordinary QR factorization of G is usually computed by premultiplying G by a suitably chosen sequence of elementary unitary matrices, designed to annihilate certain elements or parts of columns of the working matrix, until the required form of R is reached. Two types of these elementary unitary transformations are in widespread use – Givens rotations and Householder reflectors.

Similarly, when $J \neq I$, we can use either J -rotations or J -reflectors to compute the corresponding QR factorization, if it exists. But, at least in the hyperbolic case, it is not hard to see that such basic transformations can be used to compute only the triangular form of the factorization. The general case requires the so-called block transformations which can annihilate larger portions of the working matrix.

Givens-like algorithms which use J -rotations and block J -rotations for computing the hyperbolic and the symplectic QR factorization are given in [16,18], respectively. Our goal is to show that the same can be accomplished by Householder-like algorithms, as well. To do this, we need to construct the required block J -reflectors, which are generalizations of basic J -reflectors and ordinary unitary block reflectors.

Indefinite (or hyperbolic) J -reflectors have a long history. Bunse-Gerstner [4] has introduced them in the early 1980s to obtain an indefinite analogue of the eigenvalue QR algorithm. Rader and Steinhard [14] have used the symmetrized form of J -reflectors (see later) to achieve efficient deletion of data in the least squares algorithms. Later, Cybenko and Berry [7] have used J -reflectors for efficient factorization of matrices with small displacement rank.

Ordinary block reflectors have been used in the mid 1970s by Brønlund and Lunde Johnsen [3] and Dietrich [8] for block annihilation. A complete theory of ordinary block reflectors has been given by Schreiber and Parlett in [15], and we will use it extensively in our generalizations.

Quite recently, a unified theory of the so-called \mathbb{G} -reflectors, i.e., reflectors in various scalar product spaces has been developed in [12,13]. The main difference between \mathbb{G} and J -reflectors lies in the fact that J -reflectors (both basic and block) are always reflectors (see Section 2), while \mathbb{G} -reflectors need not be, especially in the symplectic case. Also, the general non-triangular case of the hyperbolic QR factorization cannot be accomplished by basic reflectors, either \mathbb{G} or J .

Finally, Mackey et al. [12] discuss all three important cases of scalar products – generated by complex sesquilinear forms, complex bilinear forms, and real bilinear forms. After some thought, we have decided to limit our discussion here only to the cases of complex sesquilinear and real bilinear forms, which correspond to the usual Euclidean scalar product in \mathbb{C} and \mathbb{R} , respectively. For, in these two cases, we can use the common Moore–Penrose version of the generalized inverse.

The whole theory can also be done for complex bilinear forms, but requires a different (non-standard) type of the generalized inverse, and some additional preparatory results.

The rest of the paper is organized as follows. In the next section we introduce the required notation, define both basic and block J -reflectors, and discuss some elementary properties of these reflectors. Reflection properties are completely described in Section 3. It concludes with a Householder-like representation theorem for J -reflectors. Sections 4 and 5 deal with mapping properties of block J -reflectors. The first one contains straight forward generalizations of mapping results from [15], which provide only a partial solution to the mapping problem. A complete solution is given in Section 5.

Computational aspects and applications of orthosymmetric block reflectors for computing the corresponding QR factorization will be given in [19].

2. J -Reflectors and block J -reflectors

2.1. J -Scalar products

First, we introduce some notation which will be frequently used throughout the paper. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$ and let $J \in \mathbb{F}^{m \times m}$ be a given nonsingular (or invertible) matrix. A scalar product generated by J , or a J -scalar product on \mathbb{F}^m is defined by

$$[x, y] = \langle Jx, y \rangle = y^* Jx \quad \forall x, y \in \mathbb{F}^m, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual Euclidean scalar product. Here and henceforward A^* denotes the conjugate transpose of A , so $A^* = A^T$ if $\mathbb{F} = \mathbb{R}$.

If J is Hermitian, then $[\cdot, \cdot]$ becomes an indefinite scalar product (as defined in [9] or [10]). Otherwise, we may lose some of the properties of indefinite scalar products (namely, antisymmetry $[x, y] = \overline{[y, x]}$), but most of the other useful properties remain valid for any nonsingular J . The most important one is nondegeneracy: if $[x, y] = 0$ for all $y \in \mathbb{F}^m$, then $x = 0$.

Similarly, many definitions of objects in indefinite scalar product spaces do not depend on the structure of J , and remain valid in this more general setting. Moreover, many of these objects retain their properties. We list only a few of them, that will be used later (see [9,10] for details).

The right J -adjoint matrix $A^{[*]}$ of $A \in \mathbb{F}^{m \times m}$, is defined by

$$[Ax, y] = [x, A^{[*]}y] \quad \forall x, y \in \mathbb{F}^m.$$

In terms of ordinary (or I -) adjoints, it is easy to show that

$$A^{[*]} = (J^*)^{-1} A^* J^*. \tag{2.2}$$

Similarly, the left J -adjoint matrix ${}^{[*]}A$ of A , is defined by

$$[x, Ay] = [{}^{[*]}Ax, y] \quad \forall x, y \in \mathbb{F}^m,$$

which gives

$${}^{[*]}A = J^{-1}A^*J. \tag{2.3}$$

To avoid developing two theories (the right-handed one and the left-handed one), we are interested in spaces where these two J -adjoints of A are always equal, for all matrices $A \in \mathbb{F}^{m \times m}$. From (2.2) and (2.3), the requirement ${}^{[*]}A = A^{[*]}$ for all A , leads to

$$(J^{-1}J^*)A = A(J^{-1}J^*) \quad \text{for all } A \in \mathbb{F}^{m \times m},$$

i.e., $J^{-1}J^*$ commutes with all matrices A . This is possible if and only if $J^{-1}J^*$ is a scalar multiple of the identity, or

$$J^* = \tau J \quad \text{for some } \tau \in \mathbb{F}.$$

Since $(J^*)^* = J$, we have $J = \bar{\tau}J^*$, and multiplication by τ yields

$$J^* = \tau J = |\tau|^2 J^*,$$

so $|\tau| = 1$. From now on, we will assume that J satisfies

$$J^* = \tau J, \quad \tau \in \mathbb{F}, \quad |\tau| = 1. \tag{2.4}$$

This implies $[x, y] = \bar{\tau}[y, x]$ as a replacement for antisymmetry. J -scalar products which satisfy (2.4) are also called orthosymmetric scalar products (see Section 7 in [13]).

From (2.4), it is obvious that $\sqrt{\tau}J$ is Hermitian. So, an orthosymmetric matrix J is equal to a unimodularly scaled Hermitian matrix.

The (left and right) J -adjoint of A will be denoted by $A^{[*]}$. Note that (2.4) also gives $(A^{[*]})^{[*]} = A$.

A matrix U is J -unitary if it preserves the J -scalar product, $[Ux, Uy] = [x, y]$, for all $x, y \in \mathbb{F}^m$, or equivalently,

$$U^*JU = J. \tag{2.5}$$

Similarly, A is J -Hermitian (or J -selfadjoint) if $A^{[*]} = A$. From (2.3), this is equivalent to

$$A = J^{-1}A^*J. \tag{2.6}$$

We will use the same terminology J -adjoint, J -unitary, and J -Hermitian for both cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$, i.e., when $\mathbb{F} = \mathbb{R}$, the conjugate transpose simply becomes the ordinary transpose. This convention greatly simplifies the presentation, and also avoids any possible confusion that might arise by using the standard “real” terminology J -transpose, J -orthogonal, and J -symmetric, which has a different meaning when $\mathbb{F} = \mathbb{C}$, and a complex bilinear form is used to generate a scalar product.

The hyperbolic J from (1.1) is obviously nonsingular and Hermitian, so (2.4) holds with $\tau = 1$. By (2.1), it generates an indefinite scalar product which will be called a hyperbolic scalar product.

The symplectic J from (1.2) is again nonsingular, but skew-Hermitian (even skew-symmetric) with even m , and (2.4) holds with $\tau = -1$. The corresponding J -scalar product in (2.1) can be considered in both cases $\mathbb{F} = \mathbb{C}$ and $\mathbb{F} = \mathbb{R}$. When $\mathbb{F} = \mathbb{R}$, J generates a real skew-symmetric scalar product (as defined in [2]), which will be called a symplectic scalar product.

2.2. Nondegenerate subspaces and matrices

Let \mathcal{M} be an arbitrary subset of \mathbb{F}^m and let $[\cdot, \cdot]$ be a J -scalar product on \mathbb{F}^m that satisfies (2.4). The J -orthogonal companion of \mathcal{M} in \mathbb{F}^m is defined by

$$\mathcal{M}^{[\perp]} = \{x \in \mathbb{F}^m \mid [x, y] = 0 \text{ for all } y \in \mathcal{M}\}.$$

Obviously, $\mathcal{M}^{[\perp]}$ is always a subspace in \mathbb{F}^m . From now on we will assume that \mathcal{M} is also a subspace of \mathbb{F}^m . Then $\dim \mathcal{M} + \dim \mathcal{M}^{[\perp]} = m$ and $(\mathcal{M}^{[\perp]})^{[\perp]} = \mathcal{M}$ (see [10] for details).

Strictly speaking, $\mathcal{M}^{[\perp]}$ is the right J -orthogonal companion of \mathcal{M} . We can define the left J -orthogonal companion in a similar fashion, but there is no need for distinction, since J is orthosymmetric by assumption.

A subspace \mathcal{M} is said to be nondegenerate (with respect to the J -scalar product $[\cdot, \cdot]$) if $\mathcal{M} \cap \mathcal{M}^{[\perp]} = \{0\}$, i.e., if $x \in \mathcal{M}$ and $[x, y] = 0$ for all $y \in \mathcal{M}$, then $x = 0$. Otherwise, \mathcal{M} is degenerate.

The following well-known fact from [2] will be used later for reflection properties of block J -reflectors.

Proposition 2.1. *Let $[x, y] = \langle Jx, y \rangle$ be a J -scalar product on \mathbb{F}^m and let \mathcal{M} be a subspace of \mathbb{F}^m . The following statements are equivalent:*

- (i) \mathcal{M} is nondegenerate,
- (ii) $\mathcal{M}^{[\perp]}$ is nondegenerate,
- (iii) $\mathcal{M}^{[\perp]}$ is a direct complement to \mathcal{M} in \mathbb{F}^m , i.e., $\mathcal{M} \dot{+} \mathcal{M}^{[\perp]} = \mathbb{F}^m$.

Proof. The proof in [10] for indefinite scalar products works for any nonsingular J . □

Similarly, let $W \in \mathbb{F}^{m \times p}$ be an arbitrary matrix, and let $\mathcal{R}(W)$ and $\mathcal{N}(W)$ denote the range of W and the null-space of W , respectively. Then W is said to be nondegenerate (with respect to the J -scalar product $[\cdot, \cdot]$) if and only if $\mathcal{R}(W)$ is nondegenerate. Otherwise, W is degenerate.

From Proposition 2.1 it follows that W is nondegenerate if and only if $\mathcal{R}(W)^{[\perp]}$ is a direct complement to $\mathcal{R}(W)$ in \mathbb{F}^m . We also have the following characterization of nondegeneracy.

Proposition 2.2. *Let $J \in \mathbb{F}^{m \times m}$ be a nonsingular matrix which generates the J -scalar product in \mathbb{F}^m , and let $W \in \mathbb{F}^{m \times p}$. Then W is nondegenerate if and only if*

$$\mathcal{N}(W^* J W) = \mathcal{N}(W), \tag{2.7}$$

or, equivalently

$$\mathcal{R}(W^* J^* W) = \mathcal{R}(W^*). \tag{2.8}$$

Proof. First we use the facts that $\mathcal{R}(BC) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(BC) \supseteq \mathcal{N}(C)$ for matrices B and C . By taking $B = W^*$ and $C = J^* W$, we obtain $\mathcal{R}(W^* J^* W) \subseteq \mathcal{R}(W^*)$, and by taking $B = W^* J$ and $C = W$ we obtain $\mathcal{N}(W^* J W) \supseteq \mathcal{N}(W)$. It remains to prove that nondegeneracy of W is equivalent to the opposite inclusions.

Let W be nondegenerate and let $u \in \mathcal{N}(W^* J W)$. Then $W^* J W u = 0$ and $\langle W^* J W u, v \rangle = 0$, for all $v \in \mathbb{F}^p$. From the definition of the ordinary Euclidean adjoint, this is equivalent to

$$\langle J W u, W v \rangle = [W u, W v] = 0 \quad \text{for all } v \in \mathbb{F}^p.$$

By putting $x = Wu$ and $y = Wv$, we get $x, y \in \mathcal{R}(W)$ and

$$[x, y] = 0 \quad \text{for all } y \in \mathcal{R}(W).$$

Since $\mathcal{R}(W)$ is nondegenerate by assumption, we have $x = Wu = 0$, which shows $u \in \mathcal{N}(W)$.

Conversely, suppose $\mathcal{N}(W^*JW) \subseteq \mathcal{N}(W)$. Let $x \in \mathcal{R}(W)$ be such that

$$[x, y] = 0 \quad \text{for all } y \in \mathcal{R}(W).$$

Since $x = Wu$ and $y = Wv$, for some $u, v \in \mathbb{F}^P$, we get

$$[Wu, Wv] = \langle JWu, Wv \rangle = 0 \quad \text{for all } v \in \mathbb{F}^P.$$

This is equivalent to $\langle W^*JWu, v \rangle = 0$, for all $v \in \mathbb{F}^P$, and nondegeneracy of the Euclidean scalar product in \mathbb{F}^P implies $W^*JWu = 0$, or $u \in \mathcal{N}(W^*JW)$. Since $\mathcal{N}(W^*JW) \subseteq \mathcal{N}(W)$, we also have $u \in \mathcal{N}(W)$, or $x = Wu = 0$, which shows that $\mathcal{R}(W)$ is nondegenerate. This proves the first equivalence.

By using $\mathcal{N}(A)^\perp = \mathcal{R}(A^*)$, which is valid for any matrix A , it follows immediately that (2.7) is equivalent to (2.8). \square

Now, if J satisfies (2.4), i.e., J is an orthosymmetric scalar product matrix, we can freely interchange the roles of J and J^* in (2.7) and (2.8). More precisely, we will use the following characterizations of nondegeneracy.

Proposition 2.3. *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix, and let $W \in \mathbb{F}^{m \times P}$. Then W is nondegenerate if and only if*

$$\mathcal{R}(W^*JW) = \mathcal{R}(W^*). \tag{2.9}$$

In terms of rank, W is nondegenerate if and only if it satisfies the following “rank condition”:

$$\text{rank}(W^*JW) = \text{rank}(W). \tag{2.10}$$

Proof. As before, $\mathcal{R}(BC) \subseteq \mathcal{R}(B)$ implies $\mathcal{R}(W^*JW) \subseteq \mathcal{R}(W^*)$, for any J and W . By assumption, J satisfies (2.4). Thus $W^*J^*W = \tau W^*JW$ and

$$\mathcal{R}(W^*J^*W) = \mathcal{R}(W^*JW).$$

It follows immediately that (2.8) is equivalent to (2.9). By Proposition 2.2, this proves the first claim.

Since $\text{rank}(W) = \dim \mathcal{R}(W)$, and $\text{rank}(W) = \text{rank}(W^*)$, we always have $\text{rank}(W^*JW) \leq \text{rank}(W)$. From (2.9) it follows that W is nondegenerate if and only if (2.10) holds. \square

If $p = 1$, then $W \in \mathbb{F}^{m \times 1}$ is actually a vector, and will be denoted by w . If $w \neq 0$, then $\text{rank}(w) = 1$ and we see that w is degenerate if and only if $\text{rank}(w^*Jw) = 0$, i.e., $[w, w] = w^*Jw = 0$. Vectors w such that $[w, w] = 0$ are usually called J -neutral or isotropic vectors.

For example, if $\mathbb{F} = \mathbb{R}$ and $J^* = -J$, then it is easy to see that all vectors $w \in \mathbb{F}$ are isotropic. In particular, this is true for the symplectic J from (1.2).

2.3. Generalizations of Householder matrices

Let $w \in \mathbb{F}^m$, $w \neq 0$, be any vector. The well-known Householder matrix or a basic unitary reflector is defined by

$$H = H(w) = I - \frac{2}{w^*w} ww^*. \tag{2.11}$$

It reflects the one-dimensional subspace spanned by w with respect to the hyperplane perpendicular to w . Note that H has the following three properties:

$$H^*H = I, \quad H = H^*, \quad H^2 = I. \tag{2.12}$$

Moreover, any two of these properties imply the third one. In the view of this, we may say (for instance) that Hermiticity is a by-product of unitarity and the reflection property $H^2 = I$. Matrices A satisfying $A^2 = I$ are usually called involutory matrices, but we will call them reflectors, and the reasons for that will be made clear below.

Schreiber and Parlett [15] have used the following representation to define unitary block reflectors for any matrix $W \in \mathbb{F}^{m \times p}$

$$H = H(W) = I - 2W(W^*W)^+W^*, \tag{2.13}$$

where $(W^*W)^+$ denotes the Moore–Penrose (generalized) inverse of W^*W (see, for example, [1] or [6]). When $p = 1$ and $w \neq 0$, this definition obviously reduces to (2.11). These reflectors are usually referred to as block reflectors only when $p > 1$, and even though (2.13) is correct for any p , in practice we almost always have $p < m$. It is not hard to see that $H = H(W)$ reverses (or reflects) the range $\mathcal{R}(W)$ of W with respect to $\mathcal{R}(W)^\perp$, and still satisfies all three properties in (2.12). Finally, note that (2.13) gives $H(0) = I$.

Now let J be a given orthosymmetric scalar product matrix. We would like to generalize the concepts of reflectors and block reflectors to J -scalar product spaces. To this aim, we impose the J -scalar product analogues of properties (2.12), which are satisfied by unitary reflectors.

Definition 2.4. A matrix $H \in \mathbb{F}^{m \times m}$ will be called a J -reflector if it is J -unitary, J -Hermitian and a reflector, i.e.,

$$H^*JH = J, \quad JH = H^*J, \quad H^2 = I. \tag{2.14}$$

As before, it is sufficient to impose only two out of three conditions.

Proposition 2.5. Let $J \in \mathbb{F}^{m \times m}$ be a given nonsingular matrix. Any two equalities in (2.14) imply the third one.

Proof. First, suppose that H is J -unitary and J -Hermitian. Then

$$J = H^*JH = JH^2.$$

Since J is nonsingular, $H^2 = I$ follows.

Now, suppose that H is J -unitary and $H^2 = I$. Then H is nonsingular and $H^{-1} = H$, so

$$J = H^*JH = H^*JH^{-1}.$$

Multiplication by H from the right-hand side shows that H is J -Hermitian.

Finally, suppose that H is J -Hermitian and $H^2 = I$. Again, $H^{-1} = H$ and

$$H^*J = JH = JH^{-1}.$$

Multiplication by H from the right-hand side now shows that H is J -unitary. \square

The first requirement in (2.14) that H is J -unitary is somehow “natural” for practical purposes, as it preserves the structure of the problem, since J -unitary matrices always form a multiplicative

group. The last requirement $H^2 = I$ is crucial – it justifies the name reflector, as we will show below. It turns out that J -Hermicity is just another benefit caused by the other two requirements.

We have not yet shown that J -reflectors exist, but if they do, then $H^2 = I$ has many implications. First and foremost, H is diagonalizable, which immediately follows from the fact that the minimal polynomial of H is either

$$\mu(\lambda) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

or one of its factors. The eigenvalues of H belong to the set $\{-1, 1\}$. Furthermore, if $H \neq \pm I$, then H has two nontrivial reducing subspaces \mathcal{M}_- and \mathcal{M}_+ belonging to the eigenvalues -1 and 1 , respectively, with $\mathcal{M}_- \dot{+} \mathcal{M}_+ = \mathbb{F}^m$ and

$$\begin{aligned} Hx &= -x && \text{for all } x \in \mathcal{M}_-, \\ Hy &= y && \text{for all } y \in \mathcal{M}_+. \end{aligned}$$

In other words, H reflects \mathcal{M}_- with respect to \mathcal{M}_+ , which justifies the name reflector. Note that all these facts follow just from $H^2 = I$.

Finally, since H is also J -unitary, for all $x \in \mathcal{M}_-$ and $y \in \mathcal{M}_+$ we have

$$[x, y] = [Hx, Hy] = [-x, y] = -[x, y],$$

which gives $[x, y] = 0$. Hence, $\mathcal{M}_-^{[\perp]} = \mathcal{M}_+$ and vice versa, $\mathcal{M}_+^{[\perp]} = \mathcal{M}_-$, so both \mathcal{M}_- and \mathcal{M}_+ are nondegenerate. We see that H reflects a nondegenerate subspace with respect to its (nondegenerate) J -orthogonal complement. These facts can be summarized as follows.

Proposition 2.6. *Let $J \in \mathbb{F}^{m \times m}$ be a given orthosymmetric scalar product matrix.*

- (a) *A matrix $H \in \mathbb{F}^{m \times m}$ is a reflector, i.e., $H^2 = I$, if and only if there exists a pair of complementary subspaces \mathcal{M}_- and \mathcal{M}_+ in \mathbb{F}^m , meaning $\mathcal{M}_- \dot{+} \mathcal{M}_+ = \mathbb{F}^m$, such that*

$$\begin{aligned} Hx &= -x && \text{for all } x \in \mathcal{M}_-, \\ Hy &= y && \text{for all } y \in \mathcal{M}_+. \end{aligned} \tag{2.15}$$

- (b) *Let $H \in \mathbb{F}^{m \times m}$ be a reflector. Then H is a J -reflector if and only if the subspaces \mathcal{M}_- and \mathcal{M}_+ in (2.15) are mutually J -perpendicular, i.e., each one is a J -orthogonal complement of the other; $\mathcal{M}_-^{[\perp]} = \mathcal{M}_+$ and $\mathcal{M}_+^{[\perp]} = \mathcal{M}_-$. If so, both \mathcal{M}_- and \mathcal{M}_+ are nondegenerate.*

Proof. Let H be a reflector. If $H \neq \pm I$, the “if” part of both claims follows immediately from the above argument, since we can take \mathcal{M}_- and \mathcal{M}_+ to be the eigenspaces of H which correspond to the eigenvalues -1 and 1 , respectively.

The same remains valid when $H = \pm I$, if we allow one of these subspaces to be equal to $\{0\}$. This zero subspace then corresponds to the non-eigenvalue of H and remains nondegenerate, by definition. The other subspace is then equal to the whole space \mathbb{F}^m , and still contains all the eigenvectors of H . In particular, for $H = I$, we have $\mathcal{M}_- = \{0\}$ and $\mathcal{M}_+ = \mathbb{F}^m$ in (2.15), and for $H = -I$, we have $\mathcal{M}_- = \mathbb{F}^m$ and $\mathcal{M}_+ = \{0\}$ in (2.15). Hence, we always have a pair of nondegenerate subspaces with $\mathcal{M}_-^{[\perp]} = \mathcal{M}_+$ and $\mathcal{M}_+^{[\perp]} = \mathcal{M}_-$.

For the converse of both claims, just note that the action of H (or H^2) on the whole space \mathbb{F}^m is completely determined by its action on a pair of complementary subspaces \mathcal{M}_- and \mathcal{M}_+ in \mathbb{F}^m .

Then $H^2 = I$ follows immediately from (2.15). Similarly, from (2.15) we see that H preserves the J -scalar product on both subspaces \mathcal{M}_- and \mathcal{M}_+ . If $\mathcal{M}_-^{[\perp]} = \mathcal{M}_+$, then $[x, y] = [y, x] = 0$, for all $x \in \mathcal{M}_-$ and $y \in \mathcal{M}_+$. Finally,

$$[H(x + y), H(x + y)] = [x + y, x + y]$$

follows easily by direct calculation, which completes the proof. \square

In the view of Proposition 2.5, there is also a “ J -Hermitian flavour” of Proposition 2.6.

Proposition 2.7. *Let $J \in \mathbb{F}^{m \times m}$ be a given orthosymmetric scalar product matrix. Also, let $H \in \mathbb{F}^{m \times m}$ be any matrix and let $P \in \mathbb{F}^{m \times m}$ be defined by*

$$P = \frac{1}{2}(I - H). \tag{2.16}$$

(a) *Then $H \in \mathbb{F}^{m \times m}$ is a reflector if and only if there exists a pair of complementary subspaces \mathcal{M}_- and \mathcal{M}_+ in \mathbb{F}^m , such that P is the projector of \mathbb{F}^m onto \mathcal{M}_- along \mathcal{M}_+ , i.e.,*

$$\begin{aligned} Px &= x && \text{for all } x \in \mathcal{M}_-, \\ Py &= 0 && \text{for all } y \in \mathcal{M}_+, \end{aligned} \tag{2.17}$$

which can also be written as $\mathcal{R}(P) = \mathcal{M}_-$ and $\mathcal{N}(P) = \mathcal{M}_+$.

(b) *Let $H \in \mathbb{F}^{m \times m}$ be a reflector. Then H is a J -reflector if and only if P is J -Hermitian.*

Proof. From the definition (2.16) of P , it is trivial to see that (2.15) for H is equivalent to (2.17) for P . So, the first claim follows directly from Proposition 2.6(a).

Let H be a reflector, i.e., $H^2 = I$. By Definition 2.4 and Proposition 2.5, H is a J -reflector if and only if H is J -Hermitian. Finally, from (2.16) it is obvious that H is J -Hermitian if and only if P is J -Hermitian. \square

Projectors P such that $\mathcal{R}(P)^{[\perp]} = \mathcal{N}(P)$ (and vice versa) are usually called J -orthogonal projectors. The statements of Proposition 2.7 can then be rephrased as: H is a (J -)reflector if and only if P , defined by (2.16), is a (J -orthogonal) projector.

We also have the following “short” characterization of J -reflectors.

Corollary 2.8. *Let $J \in \mathbb{F}^{m \times m}$ be a given orthosymmetric scalar product matrix. A matrix $H \in \mathbb{F}^{m \times m}$ is a J -reflector if and only if there exists a nondegenerate subspace \mathcal{M}_- in \mathbb{F}^m , such that*

$$\begin{aligned} Hx &= -x && \text{for all } x \in \mathcal{M}_-, \\ Hy &= y && \text{for all } y \in \mathcal{M}_-^{[\perp]}. \end{aligned} \tag{2.18}$$

Moreover, H is uniquely determined by \mathcal{M}_- and vice versa.

Proof. The first part of the claim follows directly from Proposition 2.6, since $\mathcal{M}_+ = \mathcal{M}_-^{[\perp]}$ and $\mathcal{M}_- \dot{+} \mathcal{M}_+ = \mathbb{F}^m$. The uniqueness follows from Proposition 2.7, as P in (2.16) is uniquely determined by its range \mathcal{M}_- and null-space $\mathcal{M}_-^{[\perp]}$, and $H = I - 2P$. \square

This will be used later to obtain a Householder-like representation of J -reflectors.

2.4. Basic J -reflectors

To our knowledge, the first nonunitary basic reflectors have been introduced by Bunse-Gerstner [4] for the hyperbolic scalar product matrix J from (1.1) in the following form:

$$H = H(w) = I - 2ww^*J, \quad w \in \mathbb{F}^m, \quad w^*Jw = 1.$$

This immediately implies nondegeneracy of w , since w is nonisotropic. A comparison with (2.13) readily gives the following generalization, which, as we will see, is the correct one for any orthosymmetric J .

Definition 2.9 (*Basic J -reflector*). Let $J \in \mathbb{F}^{m \times m}$ be a given orthosymmetric scalar product matrix. For a given vector $w \in \mathbb{F}^m$, a matrix $H \in \mathbb{F}^{m \times m}$ defined by

$$H = H(w) = I - 2w(w^*Jw)^+w^*J \tag{2.19}$$

will be called a basic J -reflector (generated by w).

These basic J -reflectors are J -reflectors in the sense of Definition 2.4, which also shows not only the existence of such reflectors, but an abundance of them, for any orthosymmetric J .

Proposition 2.10. *Let H be a basic J -reflector from (2.19). Then H is J -unitary, J -Hermitian and $H^2 = I$.*

Proof. The matrix J is orthosymmetric by assumption, so $J^* = \tau J$, with $\tau \in \mathbb{F}$, $|\tau| = 1$. Since $(A^*)^+ = (A^+)^*$ for any A (see [6]), from (2.19) we have

$$\begin{aligned} H^* &= I - 2J^*w(w^*J^*w)^+w^* = I - 2\tau Jw(w^*(\tau J)w)^+w^* \\ &= I - 2\tau Jw(\tau^{-1}(w^*Jw)^+)w^* = I - 2Jw(w^*Jw)^+w^*. \end{aligned}$$

By using the ‘‘cancellation’’ law $A^+AA^+ = A^+$ for the generalized inverse (see [6]), with scalar $A = w^*Jw$, we conclude that

$$H^*JH = (I - 2Jw(w^*Jw)^+w^*)J(I - 2w(w^*Jw)^+w^*J) = J,$$

which, by (2.5), proves that H is a J -unitary matrix. From

$$J^{-1}H^*J = J^{-1}(I - 2Jw(w^*Jw)^+w^*)J = I - 2w(w^*Jw)^+w^*J = H,$$

we see that H satisfies (2.6), so H is J -Hermitian. Finally, by Proposition 2.5, these two relations imply $H^2 = I$. \square

If J is not definite, there exist nonzero isotropic vectors $w \in \mathbb{F}^m$. Then $w^*Jw = 0$, so $(w^*Jw)^+ = 0$ and $H(w) = I$. For such vectors w , $H(w)$ cannot reflect vectors, and cannot be used for mapping and annihilation purposes. This phenomenon of degeneracy will become even more evident later, when we discuss block J -reflectors.

All other properties of basic J -reflectors, like mapping and reflection properties, will be described in a broader context of block J -reflectors.

Remark 2.11. Some other generalizations of ordinary basic reflectors have also been proposed, mostly for the hyperbolic scalar product matrix J . For example, a matrix $\tilde{H} \in \mathbb{F}^{m \times m}$ defined by

$$\tilde{H} = \tilde{H}(w) = J - 2w(w^*Jw)^+w^* \tag{2.20}$$

is called a “hyperbolic Householder” matrix by Rader and Steinhard [14]. For $J = J^* = J^{-1}$, it is easy to see that \tilde{H} is J -unitary, Hermitian and satisfies $\tilde{H}^{-1} = J\tilde{H}J$, so it is not a reflector. Even if the first J in (2.20) is replaced by J^{-1} , the corresponding \tilde{H} is J -unitary if and only if J is unitary, i.e., $J^{-1} = J^*$. Then $\tilde{H} = HJ^{-1}$ and all properties of \tilde{H} can be deduced from this relation.

Mackey et al. [13] have used a slightly different approach for generalization of unitary reflectors. They define the so-called \mathbb{G} -reflectors as elementary matrices of the form $G = I + uv^*$ for some $u, v \in \mathbb{F}^m$, which are also J -unitary (in our terminology). These \mathbb{G} -reflectors can be expressed as

$$G = I + \beta uu^*J,$$

where $\beta \in \mathbb{F}$ satisfies some additional constraints (see Theorem 4.2 in [13]). When compared with (2.19), we see that basic J -reflectors are \mathbb{G} -reflectors, with $u = w$ and $\beta = -2(w^*Jw)^+$, since it is easy to verify that this choice of β always satisfies those additional constraints on β .

The only difference is that \mathbb{G} -reflectors are elementary matrices, while basic J -reflectors are always reflectors, i.e., $H^2 = I$. We prefer to impose the stronger reflection requirement $H^2 = I$, not only because it ensures that H is always diagonalizable, but also because it is easily generalized to obtain block reflectors.

2.5. Block J -reflectors

Block J -reflectors are a natural, but nontrivial generalization of unitary block reflectors and basic J -reflectors. They are defined by the following generalization of (2.13) and (2.19).

Definition 2.12 (*Block J -reflector*). Let $J \in \mathbb{F}^{m \times m}$ be a given orthosymmetric scalar product matrix. For any given $W \in \mathbb{F}^{m \times p}$, a matrix $H \in \mathbb{F}^{m \times m}$ defined by

$$H = H(W) = I - 2W(W^*JW)^+W^*J \tag{2.21}$$

will be called a block J -reflector (generated by W).

Strictly speaking, H should be called a block J -reflector only if $p \geq 2$. For otherwise, it is already a basic J -reflector. But, to simplify the presentation, we will consider basic J -reflectors as a special case of block J -reflectors for $p = 1$, just as (2.19) is a special case of (2.21).

It is easy to see that Proposition 2.10 remains valid in the block case.

Proposition 2.13. *Let H be any block J -reflector defined by (2.21). Then H is J -unitary, J -Hermitian and $H^2 = I$, i.e., H is a J -reflector.*

Proof. The scalar product matrix J is orthosymmetric by assumption, so $J^* = \tau J$, with $\tau \in \mathbb{F}$, $|\tau| = 1$. From (2.21) we have

$$\begin{aligned} H^* &= I - 2J^*W(W^*J^*W)^+W^* = I - 2\tau JW(W^*(\tau J)W)^+W^* \\ &= I - 2\tau JW(\tau^{-1}(W^*JW)^+)W^* = I - 2JW(W^*JW)^+W^*. \end{aligned}$$

Then

$$\begin{aligned} H^* J H &= (I - 2JW(W^* J W)^+ W^*) J (I - 2W(W^* J W)^+ W^* J) \\ &= J - 2JW(W^* J W)^+ W^* J - 2JW(W^* J W)^+ W^* J \\ &\quad + 4JW(W^* J W)^+ W^* J W(W^* J W)^+ W^* J. \end{aligned}$$

The “cancellation” law $A^+ A A^+ = A^+$ for $A = W^* J W$ gives

$$H^* J H = J - 4JW(W^* J W)^+ W^* J + 4JW(W^* J W)^+ W^* J = J,$$

which shows that H is J -unitary. From

$$\begin{aligned} J^{-1} H^* J &= J^{-1} (I - 2JW(W^* J W)^+ W^*) J \\ &= I - 2W(W^* J W)^+ W^* J = H, \end{aligned}$$

we see that H is J -Hermitian. Finally, by Proposition 2.5, these two relations imply $H^2 = I$, so H is a J -reflector. \square

At the end of the next section we will show that the converse of this result is also valid – every J -reflector (in the sense of Definition 2.4) can be represented as a block J -reflector, with appropriate choice of W .

3. Reflection properties of block J -reflectors

Now we would like to analyze the reflection properties of any given block J -reflector $H = H(W)$. Since H is also a J -reflector, by Corollary 2.8, we know that

$$\begin{aligned} Hx &= -x \quad \text{for all } x \in \mathcal{M}_-, \\ Hy &= y \quad \text{for all } y \in \mathcal{M}_-^{[\perp]}, \end{aligned}$$

where \mathcal{M}_- is a nondegenerate subspace in \mathbb{F}^m .

To completely describe the reflection properties of $H = H(W)$, we need explicit expressions for \mathcal{M}_- and $\mathcal{M}_-^{[\perp]}$ in terms of W .

First we will consider the case when W is nondegenerate, and then we will see what happens when W is degenerate. We begin by the following general result which describes $\mathcal{R}(W)^{[\perp]}$ for any W .

Proposition 3.1. *Let $J \in \mathbb{F}^{m \times m}$ be a nonsingular matrix which generates the J -scalar product in \mathbb{F}^m , and let $W \in \mathbb{F}^{m \times p}$. Then*

$$\mathcal{R}(W)^{[\perp]} = \mathcal{R}(J^* W)^\perp = \mathcal{N}(W^* J). \tag{3.1}$$

In addition, if J is orthosymmetric, then $\mathcal{R}(J^ W) = \mathcal{R}(J W)$ and $\mathcal{N}(W^* J) = \mathcal{N}(W^* J^*)$.*

Proof. Let $x \in \mathcal{R}(W)^{[\perp]}$. Then

$$\langle x, y \rangle = \langle Jx, y \rangle = \langle x, J^* y \rangle = 0 \quad \text{for all } y \in \mathcal{R}(W).$$

Put $y' = J^* y$. Since $y = Wv$ for some $v \in \mathbb{F}^p$, we get $y' = J^* Wv \in \mathcal{R}(J^* W)$. Also, any $y' \in \mathcal{R}(J^* W)$ can be written as $y' = J^* y$ for some $y \in \mathcal{R}(W)$. Thus

$$\langle x, y' \rangle = 0 \quad \text{for all } y' \in \mathcal{R}(J^* W),$$

which shows $x \in \mathcal{R}(J^* W)^\perp$.

Now, let $x \in \mathcal{R}(J^*W)^\perp$. Then $\langle x, y' \rangle = 0$ for all $y' \in \mathcal{R}(J^*W)$. Since $y' = J^*y$ for some $y \in \mathcal{R}(W)$, and $J^*y \in \mathcal{R}(J^*W)$ for all $y \in \mathcal{R}(W)$, we get

$$\langle x, J^*y \rangle = \langle Jx, y \rangle = [x, y] = 0 \quad \text{for all } y \in \mathcal{R}(W).$$

Hence $x \in \mathcal{R}(W)^{[\perp]}$. This proves the first equality in (3.1).

The second one follows from $\mathcal{R}(A)^\perp = \mathcal{N}(A^*)$ with $A = J^*W$.

Finally, if J is orthosymmetric, from (2.4) we immediately get $\mathcal{R}(J^*W) = \mathcal{R}(JW)$ and $\mathcal{N}(W^*J) = \mathcal{N}(W^*J^*)$. \square

The identity $\mathcal{R}(W)^{[\perp]} = J^{-1}\mathcal{R}(W)^\perp$ is also valid (see (2.2.3) in [10]), but we will not need it here.

3.1. Reflection for nondegenerate W

In [15], the following ‘‘cancellation’’ law

$$W(W^*W)^+W^*W = W$$

is essential to obtain reflection properties of unitary block reflectors. This, of course, holds for any W , without restrictions. The nondegeneracy of W ensures a similar ‘‘cancellation’’ law that will be used for block J -reflectors.

Theorem 3.2. *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix. For any given $W \in \mathbb{F}^{m \times p}$, the ‘‘cancellation’’ law*

$$W(W^*JW)^+W^*JW = W \tag{3.2}$$

holds if and only if W is nondegenerate.

Proof. According to Moore’s definition of the generalized inverse A^+ of A (see [6]), A^+A is the orthogonal projector onto $\mathcal{R}(A^+)$, i.e., $A^+A = P_{\mathcal{R}(A^+)}$. Furthermore, $\mathcal{R}(A^+) = \mathcal{R}(A^*)$. If we use these facts for $A = W^*JW$, since J is orthosymmetric, we obtain

$$(W^*JW)^+(W^*JW) = P_{\mathcal{R}((W^*JW)^+)} = P_{\mathcal{R}((W^*JW)^*)} = P_{\mathcal{R}(W^*JW)}. \tag{3.3}$$

By Proposition 2.3, W is nondegenerate if and only if $\mathcal{R}(W^*JW) = \mathcal{R}(W^*)$. From this and (3.3), we see that $(W^*JW)^+(W^*JW) = P_{\mathcal{R}(W^*)}$ if and only if W is nondegenerate. Finally, since $P_{\mathcal{R}(W^*)}$ is Hermitian, we have

$$WP_{\mathcal{R}(W^*)} = (P_{\mathcal{R}(W^*)}W^*)^* = (W^*)^* = W,$$

so the ‘‘cancellation’’ law (3.2) is equivalent to the nondegeneracy of W . \square

Now, it is easy to show that Lemma 1 from [15], which describes the reflection properties of unitary block reflectors, can be generalized to block J -reflectors that are defined by (2.21) with nondegenerate matrices W .

Theorem 3.3. *Let $H = H(W)$ be a block J -reflector defined by (2.21). If W is nondegenerate, then $\mathcal{M}_- = \mathcal{R}(W)$, and*

$$\begin{aligned} Hx &= -x \quad \text{for all } x \in \mathcal{R}(W), \\ Hy &= y \quad \text{for all } y \in \mathcal{R}(W)^{[\perp]}, \end{aligned}$$

i.e., H reflects (or reverses) $\mathcal{R}(W)$ with respect to $\mathcal{R}(W)^{[\perp]}$.

Proof. If $x \in \mathcal{R}(W)$, then $x = Wu$ for some $u \in \mathbb{F}^p$. The definition (2.21) of H and Theorem 3.2 yield

$$\begin{aligned} Hx &= x - 2W(W^*JW)^+W^*Jx = x - 2W(W^*JW)^+W^*JWu \\ &= x - 2Wu = x - 2x = -x. \end{aligned}$$

Let $y \in \mathcal{R}(W)^{[\perp]}$. From (3.1) we know that $\mathcal{R}(W)^{[\perp]} = \mathcal{N}(W^*J)$, so $W^*Jy = 0$. This gives

$$Hy = y - 2W(W^*JW)^+W^*Jy = y,$$

which completes the proof. \square

This result deserves a few comments. In the “vector” case $p = 1$, it is easy to see that $w \neq 0$ is nondegenerate if and only if w is nonisotropic, i.e., $w^*Jw \neq 0$. Otherwise, $H(w) = I$ and it reflects nothing but the trivial zero subspace.

On the other hand, in the “block” case $p \geq 2$, we require not only $W^*JW \neq 0$, but that W is nondegenerate, which is now a much stronger condition. But, it can be viewed in an entirely different light. In this case, $\mathcal{R}(W)$ may contain some nonzero isotropic vectors. As long as none of these vectors are J -perpendicular to the whole subspace, W is nondegenerate and $H(W)$ reflects the whole range of W . In other words, for $p \geq 2$, nondegeneracy becomes less restrictive than nonisotropy. This fact will be heavily used in Section 5, and also in [19] (with $p = 2$) to obtain both the hyperbolic and the symplectic QR factorization.

Finally, note that Proposition 2.3 provides a practical way to check the nondegeneracy of a given W . We only have to verify the range equality (2.9), or the rank condition (2.10).

3.2. Reflection for degenerate W

Now we consider what happens when W is allowed to be degenerate in (2.21). We will show that $H(W)$ can always be generated by a nondegenerate matrix which is simply related to W .

For any matrix $W \in \mathbb{F}^{m \times p}$, let $Z \in \mathbb{F}^{m \times p}$ be defined by

$$Z = Z(W) = W(W^*JW)^+. \tag{3.4}$$

Observe that $H(W)$ from (2.21) can now be written as $H(W) = I - 2ZW^*J$. This matrix Z plays a key role in general description of reflection properties of $H(W)$.

Theorem 3.4. *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix. Let $W \in \mathbb{F}^{m \times p}$ be any matrix, and let Z be defined by (3.4). Then Z is nondegenerate and*

$$H(Z) = H(W(W^*JW)^+) = H(W). \tag{3.5}$$

Moreover, $\mathcal{R}(Z) \subseteq \mathcal{R}(W)$, and $\mathcal{R}(Z) = \mathcal{R}(W)$ holds if and only if W is nondegenerate.

Proof. First we establish the validity of (3.5). Since J is orthosymmetric, from (2.4) and (3.4) we obtain

$$Z^* = (W^*J^*W)^+W^* = \tau^{-1}(W^*JW)^+W^*.$$

By using $A^+AA^+ = A^+$ with $A = W^*JW$, we also have

$$\begin{aligned} Z^*JZ &= \tau^{-1}(W^*JW)^+W^*JW(W^*JW)^+ \\ &= \tau^{-1}(W^*JW)^+ = (W^*J^*W)^+. \end{aligned}$$

From these two relations and the definition (2.21) of $H(Z)$, we get

$$\begin{aligned} H(Z) &= I - 2Z(Z^*JZ)^+Z^*J \\ &= I - 2W(W^*JW)^+[\tau^{-1}(W^*JW)^+]^+\tau^{-1}(W^*JW)^+W^*J. \end{aligned}$$

Finally, by using $(A^+)^+ = A$ and $A^+AA^+ = A^+$ again, it follows that

$$\begin{aligned} H(Z) &= I - 2W(W^*JW)^+ \tau[(W^*JW)^+]^+ \tau^{-1}(W^*JW)^+W^*J \\ &= I - 2W(W^*JW)^+(W^*JW)(W^*JW)^+W^*J \\ &= I - 2W(W^*JW)^+W^*J = H(W). \end{aligned}$$

To show that Z is nondegenerate, we will verify that it satisfies the range equality (2.9). Clearly, $\mathcal{R}(Z^*JZ) \subseteq \mathcal{R}(Z^*)$. Since $Z^*JZ = (W^*J^*W)^+$,

$$\mathcal{R}(Z^*) = \mathcal{R}((W^*J^*W)^+W^*) \subseteq \mathcal{R}((W^*J^*W)^+) = \mathcal{R}(Z^*JZ),$$

so $\mathcal{R}(Z^*JZ) = \mathcal{R}(Z^*)$ and Z is nondegenerate.

This range equality can be simplified by using $\mathcal{R}(A^+) = \mathcal{R}(A^*)$. We get

$$\begin{aligned} \mathcal{R}(Z^*) &= \mathcal{R}(Z^*JZ) = \mathcal{R}((W^*J^*W)^+) = \mathcal{R}((W^*J^*W)^*) \\ &= \mathcal{R}(W^*JW) \subseteq \mathcal{R}(W^*). \end{aligned}$$

From (2.9) we conclude that $\mathcal{R}(Z^*) = \mathcal{R}(W^*)$ if and only if W is nondegenerate.

Note that $Z = W(W^*JW)^+$ immediately implies $\mathcal{R}(Z) \subseteq \mathcal{R}(W)$, so $\text{rank}(Z) \leq \text{rank}(W)$. Since $\text{rank}(B) = \text{rank}(B^*)$ for any B , we conclude that $\mathcal{R}(Z) = \mathcal{R}(W)$ if and only if $\mathcal{R}(Z^*) = \mathcal{R}(W^*)$, and this holds if and only if W is nondegenerate. \square

Now it is easy to give a complete characterization of reflection properties of block J -reflectors.

Theorem 3.5. *Let $W \in \mathbb{F}^{m \times p}$ be any matrix, and let $H = H(W)$ be a block J -reflector defined by (2.21). Then $\mathcal{M}_- = \mathcal{R}(Z)$, where $Z = W(W^*JW)^+$, and*

$$\begin{aligned} Hx &= -x \quad \text{for all } x \in \mathcal{R}(Z), \\ Hy &= y \quad \text{for all } y \in \mathcal{R}(Z)^{[\perp]}, \end{aligned}$$

i.e., H reflects (or reverses) $\mathcal{R}(Z)$ with respect to $\mathcal{R}(Z)^{[\perp]}$.

Proof. By Theorem 3.4, Z is nondegenerate and $H(Z) = H(W)$. The claim follows immediately from Theorem 3.3 for Z . \square

This result has the following interpretation. We have already shown that any block J -reflector H always reflects a nondegenerate subspace \mathcal{M}_- in \mathbb{F}^m . If W is degenerate, $\mathcal{R}(W)$ is degenerate (by definition) and cannot be reflected by any H . In this case, Theorem 3.4 gives that $\mathcal{R}(Z)$ is a proper nondegenerate subspace of $\mathcal{R}(W)$. By Theorem 3.5, $H(W)$ then reflects this “smaller” nondegenerate subspace $\mathcal{R}(Z)$, instead of the whole $\mathcal{R}(W)$.

However, it is not clear whether $\mathcal{R}(Z)$ is the “largest” nondegenerate subspace in $\mathcal{R}(W)$, in the sense that it is not a proper subspace of another nondegenerate subspace in $\mathcal{R}(W)$. We now show that this is indeed the case.

Proposition 3.6. *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix. Let $W \in \mathbb{F}^{m \times p}$ be a degenerate matrix, and let Z be defined by (3.4). If \mathcal{M} is a subspace in \mathbb{F}^m such that*

$$\mathcal{R}(Z) \subset \mathcal{M} \subseteq \mathcal{R}(W), \tag{3.6}$$

then \mathcal{M} is degenerate, just like $\mathcal{R}(W)$.

Proof. To prove the claim, by Proposition 2.1, it is sufficient to show that $\mathcal{M} \cap \mathcal{M}^{[\perp]}$ is nontrivial.

Since $Z = W(W^*JW)^+$ is nondegenerate, from Proposition 2.1(iii) it follows that $\mathcal{R}(Z)^{[\perp]}$ is a direct complement to $\mathcal{R}(Z)$ in \mathbb{F}^m .

By assumption, $\mathcal{R}(Z)$ is a proper subspace of \mathcal{M} , so $\mathcal{R}(Z)^{[\perp]} \cap \mathcal{M}$ is a nontrivial subspace in \mathbb{F}^m . Therefore, there exists a vector $x \neq 0$ such that $x \in \mathcal{R}(Z)^{[\perp]} \cap \mathcal{M}$.

First, we analyze the implications of $x \in \mathcal{R}(Z)^{[\perp]}$ and then we will add the fact that $x \in \mathcal{M}$, as well. From (3.1) it follows that $x \in \mathcal{N}(Z^*J)$, or

$$Z^*Jx = (W^*J^*W)^+W^*Jx = 0.$$

Now we use $\mathcal{N}(B^+) = \mathcal{N}(B^*)$ for $B = W^*J^*W$, to obtain

$$(W^*J^*W)^*W^*Jx = 0.$$

By (3.6), $x \in \mathcal{M}$ also implies $x \in \mathcal{R}(W)$, so $x = Wu$ for some $u \in \mathbb{F}^p$. Thus

$$(W^*J^*W)^*W^*JWu = 0.$$

Since J is orthosymmetric, $J^* = \tau J$, this can be written as

$$\bar{\tau}(W^*JW)^*(W^*JW)u = 0.$$

From $\mathcal{N}(A^*A) = \mathcal{N}(A)$ for $A = W^*JW$, we get

$$W^*JWu = W^*Jx = 0,$$

which shows $x \in \mathcal{N}(W^*J) = \mathcal{R}(W)^{[\perp]}$.

Finally, (3.6) implies $\mathcal{R}(W)^{[\perp]} \subseteq \mathcal{M}^{[\perp]}$, so $x \in \mathcal{M}^{[\perp]}$, as well. We now have $x \neq 0$ and $x \in \mathcal{M} \cap \mathcal{M}^{[\perp]}$. Hence, \mathcal{M} must be degenerate, and $\mathcal{R}(Z)$ is a maximal nondegenerate subspace in $\mathcal{R}(W)$. \square

In other words, Theorem 3.5 is the best possible result for any block J -reflector $H(W)$ generated by $W \in \mathbb{F}^{m \times p}$.

For all practical purposes we should avoid trivial reflectors $H(W) = \pm I$. This means that both subspaces \mathcal{M}_- and $\mathcal{M}_-^{[\perp]}$ have to be nontrivial.

From Theorem 3.5 it is obvious that $\mathcal{M}_- = \mathcal{R}(Z)$ is trivial if and only if $Z = W(W^*JW)^+ = 0$. But, this can be simplified by the following argument. If $Z = 0$, then

$$0 = (W^*J)Z(W^*JW) = (W^*JW)(W^*JW)^+(W^*JW) = W^*JW$$

and vice versa. Therefore, $H(W) = I$ if and only if $W^*JW = 0$. To put it more practically, $W^*JW \neq 0$ ensures $H(W) \neq I$.

When $p = 1$, this boils down to nonisotropy, which is then equivalent to nondegeneracy for $w \neq 0$. As soon as $H(w) \neq I$, $H(w)$ reflects the whole $\mathcal{R}(w)$. If $p > 1$, the situation gets more complicated, for $H(W)$ may reflect a smaller subspace than $\mathcal{R}(W)$, if W is degenerate.

As for the other subspace $\mathcal{M}_-^{[\perp]}$, in practice we always have $p < m$, which immediately gives $H(W) \neq -I$.

3.3. Householder-like representation of J -reflectors

To round-up the basic theory of block J -reflectors, we will prove that every J -reflector (as defined by Definition 2.4) can be expressed as a block J -reflector that is generated by a certain nondegenerate matrix W (in the sense of Definition 2.12). In other words, there are no other J -reflectors.

The proof is based on Proposition 2.7, and we will find an explicit expression for the projector P from (2.16). For this, we need an even more general type of matrix inverse than the Moore–Penrose inverse (see, for example [1]).

Definition 3.7. For any matrix $A \in \mathbb{F}^{m \times p}$, let $A\{1, 2\}$ denote the set of all matrices $X \in \mathbb{F}^{p \times m}$ that satisfy only the first two of the following four equations (usually called the Penrose equations)

$$\begin{aligned} AXA &= A, \\ XAX &= X, \\ (AX)^* &= AX, \\ (XA)^* &= XA. \end{aligned}$$

Any matrix $X \in A\{1, 2\}$ is called a $\{1, 2\}$ -inverse of A , and will be denoted by A^I (as in [11]).

Note that the Moore–Penrose inverse A^+ of A satisfies all four Penrose equations, so the set $A\{1, 2\}$ is certainly non-empty.

We also need the following generalization of Proposition 12.7.5. from [11, pp. 431–432] to represent a projector in terms of $\{1, 2\}$ -inverses.

Proposition 3.8. Let $A, B \in \mathbb{F}^{m \times p}$ be two matrices such that

$$\mathcal{R}(A) \dot{+} \mathcal{N}(B^*) = \mathbb{F}^m \tag{3.7}$$

and let $(B^*A)^I$ be any $\{1, 2\}$ -inverse of B^*A . Then

$$P = A(B^*A)^I B^* \tag{3.8}$$

is the projector of \mathbb{F}^m onto $\mathcal{R}(A)$ along $\mathcal{N}(B^*)$.

Proof. By Definition 3.7, $(B^*A)^I$ satisfies the first two Penrose equations

$$(B^*A)(B^*A)^I(B^*A) = B^*A, \tag{3.9}$$

$$(B^*A)^I(B^*A)(B^*A)^I = (B^*A)^I. \tag{3.10}$$

First we have to show that $P = A(B^*A)^I B^*$ is indeed a projector, i.e., satisfies $P^2 = P$. This follows easily by using (3.10)

$$P^2 = A(B^*A)^I B^* A(B^*A)^I B^* = A(B^*A)^I B^* = P.$$

Since a projector is uniquely determined by its range and null-space, it remains to prove that $\mathcal{R}(P) = \mathcal{R}(A)$ and $\mathcal{N}(P) = \mathcal{N}(B^*)$.

From the definition of P , by using the “product rule” for the range and the null-space, we get

$$\mathcal{R}(P) = \mathcal{R}(A(B^*A)^I B^*) \subseteq \mathcal{R}(A), \quad \mathcal{N}(P) = \mathcal{N}(A(B^*A)^I B^*) \supseteq \mathcal{N}(B^*).$$

Now we only have to prove the opposite inclusions.

To prove that $\mathcal{N}(P) \subseteq \mathcal{N}(B^*)$, let $x \in \mathcal{N}(P)$. Then

$$Px = A(B^*A)^I B^*x = 0. \tag{3.11}$$

Since $x \in \mathbb{F}^m$, as well, from (3.7) it follows that x can be uniquely written as

$$x = y + z, \quad y \in \mathcal{R}(A), \quad z \in \mathcal{N}(B^*).$$

Then $y = Au$, for some $u \in \mathbb{F}^p$, and $B^*z = 0$. From (3.11), we now have

$$\begin{aligned} 0 &= A(B^*A)^I B^*x = A(B^*A)^I B^*(Au + z) \\ &= A(B^*A)^I B^*Au + A(B^*A)^I B^*z = A(B^*A)^I B^*Au. \end{aligned}$$

Multiplication by B^* from the left hand side, together with (3.9), yields

$$0 = B^*A(B^*A)^I B^*Au = B^*Au = B^*y.$$

Finally, by adding $B^*z = 0$, we get

$$0 = B^*y + B^*z = B^*(y + z) = B^*x,$$

which shows that $x \in \mathcal{N}(B^*)$. This completes the proof of $\mathcal{N}(P) = \mathcal{N}(B^*)$.

The second equality $\mathcal{R}(P) = \mathcal{R}(A)$ follows directly from this one, by considering dimensions. Since P is a projector, we have $\mathcal{R}(P) \dot{+} \mathcal{N}(P) = \mathbb{F}^m$. From this and (3.7) we get

$$\dim \mathcal{R}(P) + \dim \mathcal{N}(P) = \dim \mathcal{R}(A) + \dim \mathcal{N}(B^*) = m.$$

Finally, $\dim \mathcal{N}(P) = \dim \mathcal{N}(B^*)$ implies $\dim \mathcal{R}(P) = \dim \mathcal{R}(A)$. As we already know that $\mathcal{R}(P) \subseteq \mathcal{R}(A)$, we must have $\mathcal{R}(P) = \mathcal{R}(A)$. \square

The original version of this claim in [11] is stated for $B = A$, and (3.7) is immediately satisfied in this case.

Remark 3.9. The projector P in Proposition 3.8 is uniquely determined by its range $\mathcal{R}(A)$ and its null-space $\mathcal{N}(B^*)$. On the other hand, the representation (3.8) of P is not unique, for we can use any $\{1, 2\}$ -inverse $(B^*A)^I$ of B^*A in (3.8) to get the same projector P . To make this representation unique, as well, we can choose the Moore–Penrose inverse $(B^*A)^+$ of B^*A ,

$$P = A(B^*A)^+ B^*, \tag{3.12}$$

or any other $\{1, 2\}$ -inverse that is suitable for algebraic manipulation.

Here we will use the Moore–Penrose inverse, as it commutes with the conjugate transpose $*$. This makes it suitable for J -scalar products that are generated by complex sesquilinear and real bilinear forms.

In the case of complex bilinear forms, a different type of inverse is more suitable. Also, a slightly different version of Proposition 3.8 is required.

Theorem 3.10 (Householder-like representation). *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix, and let $H \in \mathbb{F}^{m \times m}$ be a J -reflector (as defined by Definition 2.4). Then there exists a nondegenerate matrix $W \in \mathbb{F}^{m \times p}$, for some p , such that*

$$H = H(W) = I - 2W(W^*JW)^+ W^*J, \tag{3.13}$$

i.e., H is a block J -reflector generated by W (as defined by Definition 2.12).

Proof. Since H is a J -reflector, by Corollary 2.8 and Proposition 2.7, there exists a nondegenerate subspace \mathcal{M}_- in \mathbb{F}^m , such that

$$P = \frac{1}{2}(I - H) \tag{3.14}$$

is the projector of \mathbb{F}^m onto \mathcal{M}_- along \mathcal{M}_-^{\perp} .

Let $\{w_1, \dots, w_p\} \subset \mathcal{M}_-$ be any finite set of vectors that spans \mathcal{M}_- ,

$$\text{span}(\{w_1, \dots, w_p\}) = \mathcal{M}_-. \tag{3.15}$$

Obviously, $p \geq \dim \mathcal{M}_-$. If we choose a basis in \mathcal{M}_- , then $p = \dim \mathcal{M}_- \leq m$.

Furthermore, let $W \in \mathbb{F}^{m \times p}$ be the matrix with columns w_1, \dots, w_p , i.e., $W = [w_1, \dots, w_p]$. Note that (3.15) implies $\mathcal{R}(W) = \mathcal{M}_-$. Since \mathcal{M}_- is nondegenerate, the matrix W is nondegenerate, by definition.

Moreover, $\mathcal{M}_-^{\perp} = \mathcal{R}(W)^{\perp}$, and (3.1) gives $\mathcal{M}_-^{\perp} = \mathcal{N}(W^*J)$. Also, from $\mathcal{M}_- \dot{+} \mathcal{M}_-^{\perp} = \mathbb{F}^m$, we have $\mathcal{R}(W) \dot{+} \mathcal{N}(W^*J) = \mathbb{F}^m$.

If we denote $A = W$ and $B^* = W^*J$, this establishes the validity of (3.7), and Proposition 3.8 provides a representation for the projector P in (3.14). By using (3.12), we get

$$P = W(W^*JW)^+W^*J.$$

Finally, from (3.14), we have $H = I - 2P$, and (3.13) follows. \square

Instead of (3.12), we can also use the general representation (3.8) of P , but there is no need to do so.

The projector P in (3.14) is uniquely determined by its range \mathcal{M}_- and its null-space \mathcal{M}_-^{\perp} . Since (3.15) is equivalent to $\mathcal{R}(W) = \mathcal{M}_-$, if we take any other matrix $W' \in \mathbb{F}^{m \times p'}$ such that $\mathcal{R}(W') = \mathcal{M}_-$, we would still get the same J -reflector, i.e., $H(W') = H(W) = H$.

4. Mapping by block J -reflectors

4.1. Mapping by unitary reflectors

Usefulness of the ordinary Householder matrices arises from the well known fact that, given any two vectors with the same Euclidean norm, then there exists a Householder matrix (or a basic unitary reflector) that maps one vector into the other, up to a factor of unit modulus in complex spaces.

More precisely, for any two vectors $g, f \in \mathbb{F}^m$ such that $g^*g = f^*f$, there exists a reflector $H = H(w)$ such that

$$Hg = \sigma f. \tag{4.1}$$

In the complex case $\mathbb{F} = \mathbb{C}$, the factor σ is given by

$$\sigma = \pm \frac{f^*g}{|f^*g|}, \quad \text{if } f^*g \neq 0 \tag{4.2}$$

and we can take any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, if $f^*g = 0$.

In the real case $\mathbb{F} = \mathbb{R}$, we have $\sigma = \pm 1$, or $Hg = \pm f$ in (4.1), i.e., a reflector can be used to map g into $\pm f$ without any further restrictions.

Furthermore, H in (4.1) is given by (2.11), with $w = \sigma f - g$, except for $f = g = 0$, when $w = 0$. Then we can take $H = H(0) = I$, but any other H will obviously do, as well.

The significance of condition (4.2) on σ will become more apparent below in the context of block reflectors. For now, just note that this choice of σ gives $(\sigma f)^* g = g^*(\sigma f)$, and $|\sigma| = 1$ implies $(\sigma f)^*(\sigma f) = f^* f = g^* g$.

A similar result is also valid for basic J -reflectors defined by (2.19). It will be stated later, as a consequence of a more general mapping result for block J -reflectors.

Now consider the block case. For given matrices $G, F \in \mathbb{F}^{m \times q}$, $q \geq 1$, we seek a block reflector $H = H(W)$ such that

$$HG = F. \tag{4.3}$$

Necessary and sufficient conditions for the existence of such a reflector H in the unitary case are given by Schreiber and Parlett [15, Theorem 1].

Theorem 4.1 (Unitary reflector mapping theorem). *For two matrices $G, F \in \mathbb{F}^{m \times q}$, there exists a unitary block reflector H such that $HG = F$, if and only if G and F satisfy the following two properties:*

(i) *isometry property:*

$$F^* F = G^* G, \tag{4.4}$$

(ii) *symmetry property:*

$$G^* F = F^* G, \tag{4.5}$$

or, equivalently, $G^ F$ is Hermitian.*

When $q = 1$, it is easy to see that (4.2) is a direct consequence of (4.5) for g and σf , if $f^* g \neq 0$.

4.2. Mapping by block J -reflectors

The situation becomes far more complicated if \mathbb{F}^m is equipped with a general orthosymmetric J -scalar product. We will show that a straight forward generalization of Theorem 4.1 gives only necessary conditions for the existence of a block J -reflector H that maps G into F . In this section we will also give some sufficient conditions, but they are not necessary, as we will demonstrate. Nevertheless, these results are very useful for block annihilation purposes. A complete characterization will be given in the next section.

The construction is very similar to the one in [15]. First, we need orthosymmetric J -scalar product analogues of properties (4.4) and (4.5). These conditions on G and F are now

(i) J -isometry property:

$$F^* J F = G^* J G, \tag{4.6}$$

(ii) J -symmetry property (or symmetry with respect to J):

$$G^* J F = F^* J G. \tag{4.7}$$

It should be said that J -symmetry has nothing to do with J -symmetric matrices. Since we have already decided not to use the term “ J -symmetric”, no confusion should arise if we continue to use the shorter name “ J -symmetry property” for (4.7). Also note that (4.7) does not imply that $G^* J F$ is Hermitian, if $\tau \neq 1$ in (2.4).

Obviously, if F satisfies these conditions, so does $-F$. This is useful if we have some freedom of choice for F in (4.3), which is the case when we try to annihilate a certain portion or block of G by this transformation.

Let H in (4.3) be a block J -reflector defined by (2.21). From Proposition 2.13 it is easy to see that G and F satisfy both conditions (4.6) and (4.7). Since H is J -unitary, from (4.3) we have

$$F^* J F = G^* H^* J H G = G^* J G.$$

Similarly, H is J -Hermitian, so

$$G^* J F = G^* J H G = G^* H^* J G = F^* J G.$$

It follows that (4.6) and (4.7) are necessary for the existence of H , but may not be sufficient, as we will now demonstrate.

Since H is nonsingular, (4.3) also implies $\text{rank}(G) = \text{rank}(F)$. The following example shows that there exist matrices G and F that satisfy (4.6) and (4.7) for a particular choice of J , but have different ranks, so (4.3) is not possible.

Example 4.1. Let $J = \text{diag}(1, -1, 1, -1)$ be the hyperbolic scalar product in \mathbb{F}^4 , and let $G, F \in \mathbb{F}^{4 \times 2}$ be given by

$$G = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Obviously, $\text{rank}(G) = 2, \text{rank}(F) = 1$, and it is easy to verify that

$$F^* J F = G^* J G = G^* J F = F^* J G = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

so both (4.6) and (4.7) are satisfied by G and F . \square

Let D (for “difference”) and S (for “sum”) be the matrices defined by

$$D = F - G, \quad S = F + G. \tag{4.8}$$

This is equivalent to

$$G = \frac{1}{2}(S - D), \quad F = \frac{1}{2}(S + D). \tag{4.9}$$

Then we have the following J -analogue of Lemma 2 from [15].

Lemma 4.2. Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix, and let $G, F \in \mathbb{F}^{m \times q}$ be any two matrices. Then G and F satisfy (4.6) and (4.7), if and only if D and S , defined by (4.8), satisfy

$$D^* J S = 0.$$

Proof. First, suppose that G and F satisfy (4.6) and (4.7). From (4.6)–(4.8), we have

$$\begin{aligned} D^* J S &= (F^* - G^*) J (F + G) = F^* J F - G^* J F + F^* J G - G^* J G \\ &= (F^* J F - G^* J G) + (F^* J G - G^* J F) = 0. \end{aligned}$$

Now suppose that $D^*JS = 0$. Since J is orthosymmetric, this also implies $S^*JD = 0$. Then

$$\begin{aligned} \left(\frac{1}{2}(S \pm D)\right)^* J \left(\frac{1}{2}(S \pm D)\right) &= \frac{1}{4}(S^*JS \pm D^*JS \pm S^*JD + D^*JD) \\ &= \frac{1}{4}(S^*JS + D^*JD), \end{aligned}$$

and from (4.9) we get $F^*JF = G^*JG$. Similarly,

$$\begin{aligned} \left(\frac{1}{2}(S \mp D)\right)^* J \left(\frac{1}{2}(S \pm D)\right) &= \frac{1}{4}(S^*JS \mp D^*JS \pm S^*JD - D^*JD) \\ &= \frac{1}{4}(S^*JS - D^*JD), \end{aligned}$$

which proves $G^*JF = F^*JG$. \square

This result actually says that the two subspaces $\mathcal{R}(D)$ and $\mathcal{R}(S)$ are mutually J -orthogonal, i.e., $\mathcal{R}(D) \subseteq \mathcal{R}(S)^{\perp\perp}$, and vice versa, $\mathcal{R}(S) \subseteq \mathcal{R}(D)^{\perp\perp}$. If at least one of these subspaces is nondegenerate, the corresponding matrix D , or S , can be used to generate a reflector H that maps G into F , or $-F$. More precisely, we have the following generalization of Theorem 4.1 for orthosymmetric scalar products.

Theorem 4.3 (*J*-reflector partial mapping theorem). *Let G and F be two matrices in $\mathbb{F}^{m \times q}$ that satisfy (4.6) and (4.7), and let D, S be defined by (4.8). Then $H(D)G = F$ if and only if D is nondegenerate, i.e., D satisfies the rank condition*

$$\text{rank}(D^*JD) = \text{rank}(D). \tag{4.10}$$

Furthermore, $H(S)G = -F$ if and only if S is nondegenerate, i.e., S satisfies the rank condition

$$\text{rank}(S^*JS) = \text{rank}(S). \tag{4.11}$$

Proof. By putting $W = D$ into (2.21), together with (4.9), we obtain

$$\begin{aligned} H(D)G &= \frac{1}{2}(H(D)S - H(D)D) \\ &= \frac{1}{2}((S - 2D(D^*JD)^+D^*JS) - (D - 2D(D^*JD)^+D^*JD)) \\ &= \frac{1}{2}(S + D) - D(D^*JD)^+D^*JS + D(D^*JD)^+D^*JD - D. \end{aligned}$$

Lemma 4.2 gives $D^*JS = 0$, so

$$H(D)G = F + D(D^*JD)^+D^*JD - D.$$

We see that $H(D)G = F$ if and only if

$$D(D^*JD)^+D^*JD = D.$$

From Theorem 3.2 it follows that this ‘‘cancellation’’ law holds if and only if D is nondegenerate, which proves the first claim.

The second claim for S follows similarly, by calculating $H(S)G$.

Finally, by Proposition 2.3, rank conditions (4.10) and (4.11) are just an easy way to check the nondegeneracy of D and S , respectively. \square

Note that (4.10) implies the existence of a block J -reflector H in (4.3), and we can take $H = H(D)$. Likewise, (4.11) implies a similar mapping $HG = -F$ with $H = H(S)$. In both cases we have $p = q$ for the generating matrix W in (2.21). Additionally, if J, G and F are real matrices, then $H(D)$ and $H(S)$ are also real, even if $\mathbb{F} = \mathbb{C}$, which simplifies the computation.

In the unitary case $J = I$, both D and S are always nondegenerate. We can take $W = D$ to get $H(D)G = F$, and $W = S$ to get $H(S)G = -F$. This immediately proves Theorem 4.1, so (4.6) and (4.7) are necessary and sufficient for the existence of H in the unitary case.

Generally, when $J \neq I$, Theorem 4.3 gives only sufficient conditions for the existence of a mapping reflector H . If both D and S are nondegenerate, we have a choice of sign on the right hand side in (4.3). But, it can happen that at least one of them is degenerate. Moreover, as the next example shows, both of them can be degenerate, and there still exists a reflector H that maps G into F . Clearly, H is not generated by D or S in such a case.

Example 4.2. Let, as before, $J = \text{diag}(1, -1, 1, -1)$ be the hyperbolic scalar product in \mathbb{F}^4 , and let $G, F \in \mathbb{F}^{4 \times 2}$ be given by

$$G = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $\text{rank}(G) = \text{rank}(F) = 1$, and it is easy to verify that

$$F^* J F = G^* J G = G^* J F = F^* J G = 0,$$

so both (4.6) and (4.7) are satisfied by G and F . We also have

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 2 & 2 \\ 2 & 2 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

with $\text{rank}(D) = \text{rank}(S) = 1$, and again $D^* J D = S^* J S = 0$. Thus, both D and S are degenerate. We have $H(D) = H(S) = I$, but $G \neq F$.

On the other hand, if we take

$$W = \begin{bmatrix} 0 & 1/4 \\ 0 & -3/4 \\ -1 & -2 \\ -1 & 0 \end{bmatrix},$$

then W is nondegenerate, $\text{rank}(W) = \text{rank}(W^* J W) = 2$, with

$$(W^* J W)^{-1} = \begin{bmatrix} -7/8 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad H(W) = \begin{bmatrix} 1 & 0 & 1/4 & -1/4 \\ 0 & 1 & -3/4 & 3/4 \\ 1/4 & 3/4 & -5/4 & 1/4 \\ 1/4 & 3/4 & -1/4 & -3/4 \end{bmatrix}$$

and it is easy to check that $H(W)G = F$. \square

It is important that such examples exist only for $p \geq 2$ in (2.21), when H in (4.3) is a ‘‘true’’ block J -reflector that reflects a multidimensional subspace. If g and f are vectors ($q = 1$), and H in (4.3) is restricted to be a basic J -reflector ($p = 1$), this cannot happen.

4.3. Mapping by basic J -reflectors

In the case $p = q = 1$, we can easily establish a “full” mapping theorem, which gives necessary and sufficient conditions for the existence of a mapping basic J -reflector H in (4.3), analogous to the unitary case.

It should be noted that the results in this subsection are not new, since they follow from Theorem 8.2 in [13] (see the remark at the end of this subsection). However, we state and prove them here in the context of the whole theory of J -reflectors.

Theorem 4.4 (Basic J -reflector mapping theorem). *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix, and let $g, f \in \mathbb{F}^m$ be two distinct vectors. There exists a basic J -reflector $H = H(w)$ such that $H(w)g = f$ if and only if*

- (a) g and f satisfy the J -isometry property (4.6) and the J -symmetry property (4.7), and
- (b) $d = f - g \neq 0$ is nondegenerate, i.e., $d^*Jd \neq 0$.

Furthermore, whenever H exists, it is unique. More precisely, H can be generated by any vector $w \in \mathbb{F}^m$ such that

$$w = \lambda d, \quad \lambda \in \mathbb{F}, \quad \lambda \neq 0.$$

Then w is also nondegenerate and $H(w) = H(d)$.

Finally, the same remains valid if we replace f by $-f$, and d by $s = f + g$.

Proof. First note that $g \neq f$ is equivalent to $d \neq 0$, so d is nondegenerate if and only if $d^*Jd \neq 0$. With this in mind, Theorem 4.3 immediately implies that (a) and (b) are sufficient for the existence of H , with $H = H(d)$, which is a basic J -reflector.

We already know that (4.6) and (4.7) are necessary for the existence of H . Thus, it remains to prove that (b) is also necessary.

Suppose that $H(w)g = f$ for some $w \in \mathbb{F}^m$. Since $g \neq f$, we must have $H(w) \neq I$, which implies $w^*Jw \neq 0$, so $w \neq 0$ and w is nondegenerate.

From Proposition 2.10 it follows that $(H(w))^2 = I$, which also gives $H(w)f = g$. Hence, by subtraction, $H(w)d = -d$. Then (2.19) gives

$$H(w)d = (I - 2w(w^*Jw)^+w^*J)d = d - 2w(w^*Jw)^+w^*Jd = -d.$$

Since $w^*Jw \neq 0$, we have $(w^*Jw)^+ = 1/(w^*Jw)$, and the above relation reduces to

$$w \frac{w^*Jd}{w^*Jw} = d.$$

But, $d \neq 0$ also implies $w^*Jd \neq 0$, so

$$w = \frac{w^*Jw}{w^*Jd} d,$$

or $w = \lambda d$, with $\lambda \in \mathbb{F} \setminus \{0\}$. From $w^*Jw = |\lambda|^2(d^*Jd) \neq 0$ it follows that $d^*Jd \neq 0$, so d is nondegenerate. This completes the proof of (b).

For any $w \in \mathbb{F}^m$ such that $w = \lambda d$, with $\lambda \in \mathbb{F} \setminus \{0\}$, from (2.19) it is easy to verify that $H(w) = H(d)$. Therefore, H is unique, if it exists.

Finally, if we write $-f$ instead of f , then d becomes $(-f) - g = -s$, and the proof follows by taking $\lambda = -1$. \square

Due to its strength, it is worth to put this result in a broader perspective of earlier results from Section 2. First of all, since $H = H(w)$ is a basic J -reflector, it reflects at most a one-dimensional subspace. Hence, the situation $\mathcal{R}(z) \subsetneq \mathcal{R}(w)$ in Theorem 3.5 can occur only trivially, with $z(w) = 0$ in (3.4), which is equivalent to $H(w) = I$.

The assumption $g \neq f$ is used solely to establish the uniqueness of H (if it exists). In terms of Theorem 3.3, if H exists, then $d \in \mathcal{R}(w)$, and $d \neq 0$ uniquely determines this one-dimensional subspace $\mathcal{R}(w)$ that is reversed by $H(w)$.

If $g = f$, or $d = 0$, then d is nondegenerate by definition, and we can certainly use $H(d) = I$ to map g into itself. But, this H is not necessarily unique. When $m > 1$, we can use any other $w \in \mathbb{F}^m$ that is J -perpendicular to g , since $w^*Jg = 0$ immediately gives

$$H(w)g = (I - 2w(w^*Jw)^+w^*J)g = g - 2w(w^*Jw)^+w^*Jg = g.$$

If we choose a nondegenerate w , then $H(w) \neq I = H(d)$.

For practical purposes, when g is given and we can choose f in (4.3), it is useful to combine the mapping results of Theorem 4.4 for f and $-f$. We know that if f satisfies the assumption (a) of Theorem 4.4, i.e., (4.6) and (4.7), so does $-f$. Therefore, these necessary conditions for mapping provide a common starting point.

Suppose now that $g, f \in \mathbb{F}^m$ satisfy both (4.6) and (4.7), and let us denote the respective values in these relations by

$$\mu_1 = f^*Jf = g^*Jg, \quad \mu_2 = g^*Jf = f^*Jg. \tag{4.12}$$

In terms of $d = f - g$ and $s = f + g$, by using $d^*Js = 0$ from Lemma 4.2, it follows immediately that

$$\mu_1 = \frac{1}{4}(s^*Js + d^*Jd), \quad \mu_2 = \frac{1}{4}(s^*Js - d^*Jd), \tag{4.13}$$

and also

$$\begin{aligned} d^*Jd &= -2g^*Jd = -2d^*Jg = 2f^*Jd = 2d^*Jf, \\ s^*Js &= 2g^*Js = 2s^*Jg = 2f^*Js = 2s^*Jf. \end{aligned} \tag{4.14}$$

We distinguish the following two cases for the values of μ_1 and μ_2 in (4.12).

If $\mu_1 \neq 0$ or $\mu_2 \neq 0$, from (4.13) we must have $d^*Jd \neq 0$ or $s^*Js \neq 0$, so d or s is nondegenerate. By Theorem 4.3, there exists a basic J -reflector H that maps g into f , or $-f$.

On the other hand, if $\mu_1 = \mu_2 = 0$, then (4.13) implies $d^*Jd = s^*Js = 0$, as well. The mapping by basic J -reflectors is now possible only in trivial cases $d = 0$ or $s = 0$, i.e., $g = f$ or $g = -f$. Then we can use $H = I$, and possibly some other basic J -reflectors.

Finally, if $d \neq 0$ and $s \neq 0$, both d and s are degenerate, and Theorem 4.4 implies that there is no basic J -reflector that maps g into either f , or $-f$. In this case, $g \neq \pm f$, and $\mu_1 = \mu_2 = 0$ means that both g and f are isotropic and mutually J -perpendicular.

This argument has the following simple, but very useful consequence for many applications of mapping, including annihilation.

Let $g \in \mathbb{F}^m$ be a given nonisotropic vector, $g^*Jg \neq 0$. We only have to find an appropriate vector $f \in \mathbb{F}^m$ that satisfies the necessary mapping conditions (4.6) and (4.7). Then $\mu_1 \neq 0$, and there certainly exists a basic J -reflector H that maps g into f (or $-f$). Moreover, this H is generated by d (or s).

Actually, it is sufficient to find $f \in \mathbb{F}^m$ that satisfies only the J -isometry requirement (4.6). The J -symmetry condition (4.7) can then be satisfied easily by scaling f , similarly to (4.1) and (4.2) in the unitary case.

Proposition 4.5. Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix, i.e., $J^* = \tau J$, where $\tau \in \mathbb{F}$ and $|\tau| = 1$. Let $g, f \in \mathbb{F}^m$ be two nonisotropic vectors such that $g^* J g = f^* J f \neq 0$. There exists a basic J -reflector $H = H(w)$ such that

$$Hg = \sigma f. \tag{4.15}$$

In the complex case $\mathbb{F} = \mathbb{C}$, the factor σ is given by

$$\sigma = -\text{sign}(\sqrt{\tau}(g^* J g)) \cdot \sqrt{\tau} \cdot \frac{f^* J g}{|f^* J g|}, \quad \text{if } f^* J g \neq 0, \tag{4.16}$$

where $\sqrt{\tau}$ is any square root of τ , and we can take any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, if $f^* J g = 0$.

In the real case $\mathbb{F} = \mathbb{R}$, if we want $\sigma \in \mathbb{R}$ in (4.15), then $\tau = 1$. If $f^* J g = 0$, we can take $\sigma = \pm 1$ in (4.15). Otherwise, σ is given by (4.16).

Proof. We seek a basic J -reflector $H(w)$ that maps g into $f' = \sigma f$, for some $\sigma \in \mathbb{F}$. By Theorem 4.4, g and $f' = \sigma f$ must satisfy the necessary conditions (4.6) and (4.7). From (4.6) we get

$$\mu_1 = g^* J g = (\sigma f)^* J (\sigma f) = |\sigma|^2 f^* J f.$$

Since $g^* J g = f^* J f \neq 0$ by assumption, $\mu_1 \neq 0$, and we must have $|\sigma| = 1$. The J -symmetry property (4.7) for g and σf is

$$\mu_2 = g^* J (\sigma f) = (\sigma f)^* J g.$$

Since J is orthosymmetric, $g^* J f = \overline{(f^* J^* g)} = \overline{\tau(f^* J g)}$, and we obtain

$$\mu_2 = \sigma \overline{\tau(f^* J g)} = \overline{\sigma} (f^* J g). \tag{4.17}$$

If $f^* J g = 0$, i.e., g and f are mutually J -perpendicular, then (4.17) is satisfied for any $\sigma \in \mathbb{F}$. Note that we still need $|\sigma| = 1$, which follows from (4.6).

If $f^* J g \neq 0$, from (4.17) we get

$$\sigma^2 = \tau \cdot \frac{(f^* J g)^2}{|f^* J g|^2}.$$

This gives two solutions for σ

$$\sigma = \varrho \sqrt{\tau} \cdot \frac{f^* J g}{|f^* J g|}, \quad \varrho \in \{-1, 1\}. \tag{4.18}$$

Let $w = \sigma f - g$ (this is d for g and σf). Now, we only have to ensure that w is nondegenerate, i.e., $w^* J w \neq 0$, which will determine the choice of sign ϱ in (4.18). By using $g^* J g = f^* J f$ and (4.18), it is easy to show that

$$w^* J w = 2(g^* J g) - 2\varrho \sqrt{\tau} |f^* J g|,$$

which holds even when $f^* J g = 0$. Multiplication by $\sqrt{\tau}$ yields

$$\sqrt{\tau}(w^* J w) = 2\sqrt{\tau}(g^* J g) - 2\varrho |f^* J g|. \tag{4.19}$$

Note that $\sqrt{\tau} J$ is a Hermitian matrix, so $\sqrt{\tau}(a^* J a) \in \mathbb{R}$, for any vector $a \in \mathbb{F}^m$. Therefore, all terms in (4.19) are real. Now we choose the sign ϱ to get the opposite signs of terms that subtract on the right hand side in (4.19),

$$\varrho = -\text{sign}(\sqrt{\tau}(g^* J g)).$$

This choice of sign in (4.18) gives σ as in (4.16). Moreover, we get

$$\sqrt{\tau}(w^*Jw) = 2(\sqrt{\tau}(g^*Jg) + \text{sign}(\sqrt{\tau}(g^*Jg)) |f^*Jg|) > 0.$$

Again, this is true if $f^*Jg = 0$. We also have

$$|w^*Jw| = 2(|g^*Jg| + |f^*Jg|) = 2(|\mu_1| + |\mu_2|) > 0,$$

regardless of $f^*Jg \neq 0$ or not. This proves that w is nonisotropic, and by Theorem 4.4, $H(w)g = \sigma f$. \square

If $f^*Jg \neq 0$, it may happen that the other choice of sign ϱ in (4.18) also gives a nonisotropic vector w . In fact, from (4.19) we see that $w^*Jw = 0$ if and only if

$$\frac{\sqrt{\tau}(g^*Jg)}{|f^*Jg|} \in \{-1, 1\}.$$

If this happens, then we choose the opposite sign in (4.18). Otherwise, both choices are good.

In the real case $\mathbb{F} = \mathbb{R}$, orthosymmetry of J implies $\tau = \pm 1$. By Proposition 4.5, nontrivial mapping by basic J -reflectors is possible only if $\tau = 1$, i.e., when $J^* = J$. If $J^* = -J$, all vectors are isotropic, so $H(w) = I$, for all $w \in \mathbb{F}^m$, which shows that nontrivial mapping is not possible by basic J -reflectors.

Remark 4.6. We have already said that basic J -reflectors belong to the class of \mathbb{G} -reflectors from [13]. In fact, Theorem 4.4 is a special case of the \mathbb{G} -reflector mapping theorem [13, Theorem 8.2], and it is easy to see that these two theorems are equivalent for basic J -reflectors. This follows from the first equality in (4.14), which shows that the condition $[d, g] \neq 0$ from [13] is equivalent to the nondegeneracy of d .

However, since \mathbb{G} -reflectors form a wider class, we can find examples where there exists a \mathbb{G} -reflector G that maps g into f , while there is no basic J -reflector that does the same. In such a case, G is certainly not diagonalizable, and $G^2 \neq I$, so G cannot map f back into g . Neither of these is a drawback for practical purposes.

Most notable such examples occur in the symplectic case.

Note that restricting H to be a basic J -reflector in the above results is crucial for the validity of conclusions, as the next example shows.

Example 4.3. Again, let $J = \text{diag}(1, -1, 1, -1)$ be the hyperbolic scalar product in \mathbb{F}^4 , and let $g, f \in \mathbb{F}^4$ be the first columns of matrices G, F from Example 4.2, respectively,

$$g = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad f = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then $g \neq \pm f$, and $d, s \neq 0$. Also

$$f^*Jf = g^*Jg = g^*Jf = f^*Jg = 0,$$

so $d^*Jd = s^*Js = 0$. It follows that there is no basic J -reflector $H = H(w)$, with $w \in \mathbb{F}^4$, that maps g into either f , or $-f$.

On the other hand, if we allow H to be a block J -reflector, generated by some $W \in \mathbb{F}^{4 \times p}$ with $p > 1$, we can take $W \in \mathbb{F}^{4 \times 2}$ from Example 4.2 to get $H(W)g = f$. \square

The results of this section provide only a partial solution to the mapping problem, by giving sufficient conditions for the existence of H in (4.3). Nevertheless, they are very important for practical purposes. In most applications we use either basic J -reflectors, or block J -reflectors generated by D or S .

A complete solution to the “full” mapping problem — to find necessary and sufficient conditions for the existence of a mapping reflector H in (4.3), will be given in the next section.

5. Full mapping theorem

First, we state the following technical result about J -orthogonal complements which will be used throughout this section.

Proposition 5.1. *Let $J \in \mathbb{F}^{m \times m}$ be an orthosymmetric scalar product matrix, and let \mathcal{M}_1 and \mathcal{M}_2 be arbitrary subspaces in \mathbb{F}^m . Then*

- (a) $\mathcal{M}_1 \subseteq \mathcal{M}_2 \implies \mathcal{M}_1^{[\perp]} \supseteq \mathcal{M}_2^{[\perp]}$,
- (b) $(\mathcal{M}_1 + \mathcal{M}_2)^{[\perp]} = \mathcal{M}_1^{[\perp]} \cap \mathcal{M}_2^{[\perp]}$,
- (c) $(\mathcal{M}_1 \cap \mathcal{M}_2)^{[\perp]} = \mathcal{M}_1^{[\perp]} + \mathcal{M}_2^{[\perp]}$.

Proof. Everything follows by straight forward arguments. Note that the proof of (c) can be simplified by using (b) and $(\mathcal{M}^{[\perp]})^{[\perp]} = \mathcal{M}$. \square

5.1. Necessary conditions revisited

Let $G, F \in \mathbb{F}^{m \times q}$, with $q \geq 1$, be given matrices. As in the previous section, we seek a J -reflector $H \in \mathbb{F}^{m \times m}$ that maps G into F

$$HG = F. \tag{5.1}$$

Now suppose that such a mapping J -reflector exists. Since $H^2 = I$, from (5.1) we also get $HF = G$. Let $D = F - G$, and $S = F + G$, as in (4.8). Then, it is easy to see that

$$\left. \begin{matrix} HG = F, \\ HF = G, \end{matrix} \right\} \text{ if and only if } \left\{ \begin{matrix} HD = -D, \\ HS = S. \end{matrix} \right.$$

The second pair of equations is equivalent to

$$\begin{matrix} Hx = -x & \text{for all } x \in \mathcal{R}(D), \\ Hy = y & \text{for all } y \in \mathcal{R}(S). \end{matrix} \tag{5.2}$$

On the other hand, from Corollary 2.8, we know that H is a J -reflector if and only if there exists a nondegenerate subspace $\mathcal{M} = \mathcal{M}_-$ in \mathbb{F}^m , such that (2.18) holds, i.e.,

$$\begin{matrix} Hx = -x & \text{for all } x \in \mathcal{M}, \\ Hy = y & \text{for all } y \in \mathcal{M}^{[\perp]}. \end{matrix} \tag{5.3}$$

By comparing (5.2) and (5.3), we get the following result.

Proposition 5.2. *A mapping J -reflector H in (5.1) exists if and only if there exists a nondegenerate subspace \mathcal{M} in \mathbb{F}^m , such that*

$$\mathcal{R}(D) \subseteq \mathcal{M} \quad \text{and} \quad \mathcal{R}(S) \subseteq \mathcal{M}^{[\perp]}. \tag{5.4}$$

Proof. If H in (5.1) exists, then (5.2) holds. Also, by Corollary 2.8, it uniquely determines a nondegenerate subspace \mathcal{M} such that (5.3) holds. Then, (5.4) follows easily by comparison.

On the other hand, again by Corollary 2.8, any nondegenerate subspace \mathcal{M} in \mathbb{F}^m uniquely determines a J -reflector H in (5.3). Now, (5.4) obviously gives (5.2), and this implies $HG = F$. \square

This version of the mapping theorem is not very useful in practical terms, as we still do not know how to construct this subspace \mathcal{M} . However, it provides additional insight into necessary conditions for the existence of H in (5.1).

From (5.4), by using Proposition 5.1(a), we get $\mathcal{R}(D)^{\perp} \supseteq \mathcal{M}^{\perp}$ and $\mathcal{R}(S)^{\perp} \supseteq \mathcal{M}$. Hence

$$\mathcal{R}(S) \subseteq \mathcal{R}(D)^{\perp}, \quad \mathcal{R}(D) \subseteq \mathcal{R}(S)^{\perp}. \tag{5.5}$$

These conditions can also be written as $D^*JS = 0$, and $S^*JD = 0$. Only one of them is significant, since the other one follows directly from orthosymmetry of J . By Lemma 4.2, it follows that (5.5) is equivalent to the well known necessary conditions — J -isometry (4.6) and J -symmetry (4.7). So, this is nothing new.

But, Proposition 5.2 also requires that \mathcal{M} is nondegenerate in (5.4), i.e., $\mathcal{M} \cap \mathcal{M}^{\perp} = \{0\}$, which yields

$$\mathcal{R}(D) \cap \mathcal{R}(S) = \{0\}$$

as another necessary condition. And this one is crucial – for it is also sufficient for the existence of H in (5.1), as we are about to show.

5.2. Full mapping theorem for block J -reflectors

Theorem 5.3 (Block J -reflector full mapping theorem). *Let G and F be two matrices in $\mathbb{F}^{m \times q}$, and let $D = F - G$, $S = F + G$. Also, let $J \in \mathbb{F}^{m \times m}$ be a given orthosymmetric scalar product matrix.*

Then there exists a J -reflector $H \in \mathbb{F}^{m \times m}$ that maps G into F , i.e., $HG = F$, if and only if D and S satisfy

$$\mathcal{R}(S) \subseteq \mathcal{R}(D)^{\perp} \tag{5.6}$$

and

$$\mathcal{R}(D) \cap \mathcal{R}(S) = \{0\}. \tag{5.7}$$

Moreover, H is generated by some nondegenerate matrix $W \in \mathbb{F}^{m \times p}$, where, in general, $p \neq q$.

The same conditions are also necessary and sufficient for the existence of a J -reflector $H' \in \mathbb{F}^{m \times m}$ that maps G into $-F$.

Proof. The argument above shows that (5.6) and (5.7) are necessary.

It remains to prove the sufficiency of these two conditions. By Proposition 5.2, we have to construct a nondegenerate subspace \mathcal{M} in \mathbb{F}^m , such that

$$\mathcal{R}(D) \subseteq \mathcal{M} \quad \text{and} \quad \mathcal{R}(S) \subseteq \mathcal{M}^{\perp}. \tag{5.8}$$

First, suppose that D is nondegenerate. In this case, (5.6) alone is sufficient, since $\mathcal{R}(D) \cap \mathcal{R}(D)^{\perp} = \{0\}$ and (5.6) implies (5.7). Then $\mathcal{M} = \mathcal{R}(D)$ satisfies (5.8), so it can be used to construct a mapping reflector in (5.1). (This is just another proof of Theorem 4.3.)

Therefore, from now on, we can assume that D is degenerate. Let

$$\mathcal{M}_1 = \mathcal{R}(D) + \mathcal{R}(D)^{\perp\perp}. \tag{5.9}$$

Since D is degenerate, this sum is certainly not a direct one. On the other hand, (5.7) implies that $\mathcal{R}(D) \dot{+} \mathcal{R}(S)$ is a direct sum, and (5.6) gives

$$\mathcal{R}(S) \subseteq \mathcal{R}(D) \dot{+} \mathcal{R}(S) \subseteq \mathcal{M}_1. \tag{5.10}$$

From (5.9), by Proposition 5.1(b), we also have

$$\mathcal{M}_1^{\perp\perp} = \mathcal{R}(D) \cap \mathcal{R}(D)^{\perp\perp}. \tag{5.11}$$

Since D is degenerate, this subspace is not trivial. Let $\ell = \dim \mathcal{M}_1^{\perp\perp} > 0$. Then $\dim \mathcal{M}_1 = m - \ell < m$.

The subspace $\mathcal{M}_1^{\perp\perp}$ in (5.11) is the “bad” part of $\mathcal{R}(D)$, and provides a starting point for the construction of \mathcal{M} .

Let $\mathcal{M}_{1,2}$ be any direct complement of $\mathcal{M}_1^{\perp\perp}$ in \mathbb{F}^m . Then,

$$\mathcal{M}_1^{\perp\perp} \dot{+} \mathcal{M}_{1,2} = \mathbb{F}^m \tag{5.12}$$

with $\dim \mathcal{M}_{1,2} = m - \ell$, and $\mathcal{M}_1^{\perp\perp} \cap \mathcal{M}_{1,2} = \{0\}$. Now, we define

$$\mathcal{M}_2 = \mathcal{M}_{1,2}^{\perp\perp}, \tag{5.13}$$

where $\dim \mathcal{M}_2 = m - \dim \mathcal{M}_{1,2} = \ell$.

As we will see, any such subspace \mathcal{M}_2 is suitable for the extension of $\mathcal{R}(D)$ to a nondegenerate subspace.

First, we will show that

$$\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}. \tag{5.14}$$

From (5.12), we have $(\mathcal{M}_1^{\perp\perp} \dot{+} \mathcal{M}_{1,2})^{\perp\perp} = (\mathbb{F}^m)^{\perp\perp} = \{0\}$. On the other hand, by Proposition 5.1(b) and (5.13), we get

$$(\mathcal{M}_1^{\perp\perp} \dot{+} \mathcal{M}_{1,2})^{\perp\perp} = (\mathcal{M}_1^{\perp\perp})^{\perp\perp} \cap \mathcal{M}_{1,2}^{\perp\perp} = \mathcal{M}_1 \cap \mathcal{M}_2,$$

which proves (5.14). Moreover, from $\dim \mathcal{M}_1 = m - \ell$ and $\dim \mathcal{M}_2 = \ell$, we see that $\mathcal{M}_1 \dot{+} \mathcal{M}_2 = \mathbb{F}^m$.

Finally, we define

$$\mathcal{M} = \mathcal{R}(D) \dot{+} \mathcal{M}_2. \tag{5.15}$$

This sum is obviously direct, as (5.9) gives $\mathcal{R}(D) \subseteq \mathcal{M}_1$, and $\mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$, by (5.14). Note that the added subspace \mathcal{M}_2 has the same dimension ℓ as the “bad” part $\mathcal{M}_1^{\perp\perp}$ of $\mathcal{R}(D)$.

We claim that $\mathcal{M} = \mathcal{R}(D) \dot{+} \mathcal{M}_2$ is a nondegenerate subspace. This is true if and only if $\mathcal{M} \cap \mathcal{M}^{\perp\perp} = \{0\}$.

To begin with, we need an expression for $\mathcal{M}^{\perp\perp}$. This follows from (5.15), by using Proposition 5.1(b) and (5.13). We get

$$\mathcal{M}^{\perp\perp} = (\mathcal{R}(D) \dot{+} \mathcal{M}_2)^{\perp\perp} = \mathcal{R}(D)^{\perp\perp} \cap \mathcal{M}_2^{\perp\perp} = \mathcal{R}(D)^{\perp\perp} \cap \mathcal{M}_{1,2}. \tag{5.16}$$

Suppose now that $x \in \mathcal{M} \cap \mathcal{M}^{\perp\perp}$. Since $x \in \mathcal{M}$, from (5.15) it follows that x can be uniquely represented as

$$x = x_1 + x_2, \quad x_1 \in \mathcal{R}(D), \quad x_2 \in \mathcal{M}_2. \tag{5.17}$$

From (5.16) and (5.9), we get the following sequence of inclusions

$$\mathcal{M}^{\perp\perp} = \mathcal{R}(D)^{\perp\perp} \cap \mathcal{M}_{1,2} \subseteq \mathcal{R}(D)^{\perp\perp} \subseteq \mathcal{R}(D) + \mathcal{R}(D)^{\perp\perp} = \mathcal{M}_1.$$

Therefore, $x \in \mathcal{M}^{\perp\perp}$ implies $x = x_1 + x_2 \in \mathcal{M}_1$. Since $x_1 \in \mathcal{R}(D) \subseteq \mathcal{M}_1$, and \mathcal{M}_1 is a subspace, we also get

$$x_2 = x - x_1 \in \mathcal{M}_1.$$

On the other hand, $x_2 \in \mathcal{M}_2$ in (5.17), which yields $x_2 \in \mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}$, by (5.14), so $x_2 = 0$.

This means that $x = x_1 \in \mathcal{R}(D)$ in (5.17). Since $x \in \mathcal{M}^{\perp\perp}$, by assumption, we get $x \in \mathcal{R}(D) \cap \mathcal{M}^{\perp\perp}$. This intersection can be simplified by using (5.16) and (5.11). We obtain

$$\mathcal{R}(D) \cap \mathcal{M}^{\perp\perp} = \mathcal{R}(D) \cap \mathcal{R}(D)^{\perp\perp} \cap \mathcal{M}_{1,2} = \mathcal{M}_1^{\perp\perp} \cap \mathcal{M}_{1,2}.$$

Finally, $\mathcal{M}_{1,2}$ was chosen as direct complement of $\mathcal{M}_1^{\perp\perp}$ in (5.12). We conclude that $x = 0$, which proves that \mathcal{M} is nondegenerate.

The first requirement $\mathcal{R}(D) \subseteq \mathcal{M}$ in (5.8) follows directly from the definition (5.15) of \mathcal{M} .

From (5.10), we have $\mathcal{R}(S) \subseteq \mathcal{M}_1$, and (5.14) implies $\mathcal{R}(S) \cap \mathcal{M}_2 = \{0\}$. Together with (5.7), this gives $\mathcal{R}(S) \cap \mathcal{M} = \{0\}$ in (5.15). In the end, since \mathcal{M} is nondegenerate, we get $\mathcal{R}(S) \subseteq \mathcal{M}^{\perp\perp}$, which is the second requirement in (5.8).

Finally, note that (5.7) is symmetric with respect to D and S , and (5.6) implies the “symmetric” inclusion $\mathcal{R}(D) \subseteq \mathcal{R}(S)^{\perp\perp}$ from (5.5). The proof for G and $-F$ then follows simply by interchanging the roles of D and S . To be precise, D becomes $-S$, but this is irrelevant. \square

If D is degenerate, then we have some freedom of choice for the complementary subspace $\mathcal{M}_{1,2}$ in (5.12). This immediately implies that H in (5.1) need not be unique (if it exists).

This result completes the basic general theory of J -reflectors. Computational aspects of this theory, with applications of mapping results for (block) annihilation, will be given in [19].

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