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Fixed Point Theory for Self Maps between Fréchet Spaces

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Using Krasnoselskii's fixed point theorem in a cone, we present a new fixed point theory for multivalued self maps between Fréchet spaces. Our analysis relies on a diagonal process and a result on hemicompact maps due to K. K. Tan and X. Z. Yuan (1994, *J. Math. Anal. Appl.* **185**, 378–390). An application is given to illustrate the theory. © 2001 Academic Press

1. INTRODUCTION

The Schauder–Tychonoff theorem states that if K is a closed, convex subset of a Fréchet space and $F: K \to K$ is continuous and completely continuous then F has a fixed point in K. This fixed point theorem has been widely used to establish the existence of solutions to various differential and integral inclusions (see [4, 6, 7] and the references there). One particular problem which arises "if one is not careful" when one applies the Schauder–Tychonoff theorem can be explained easily if one considers the nonlinear boundary value problem

$$\begin{aligned} y''' &= y^{\alpha} + y^{\beta} \quad \text{on } (0,1) \\ y(0) &= y(1) = y'(1) = 0, \end{aligned} \tag{1.1}$$

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with $0 < \alpha < 1 < \beta$. Note that $y \equiv 0$ is a solution to (1.1) and it would take a lot more work to show via Schauder's fixed point theorem that (1.1) has a nontrivial solution. Of course, it is well known [3, 5] that (1.1) has a nontrivial solution $y \in C[0,1]$ with $y(t) \geq 0$ for $t \in [0,1]$ and that there exists a $r > 0$ with $r \leq |y|_0 \leq 1$ (here $|y|_0 = \sup_{t \in [0,1]} |y(t)|$). As a result it would be of interest to present a fixed point theorem for a nonlinear self map F even if $F0$ is 0. Our starting point will be Krasnoselskii's fixed point theorem in a cone [6], which of course provides a fixed point theorem for a self map $F\colon K \to K$ where $K \subseteq E$; here E is a Banach space and K is an annulus.

Our paper has two main sections. In Section 2 this fixed point is extended to the Fréchet space setting for compact single valued maps. The existence of a nontrivial fixed point will be established by means of a diagonal process. Indeed, the ideas presented automatically provide a generalization of the Schauder–Tychonoff theorem. To illustrate the applicability of our theory we present a result which guarantees the existence of a nontrivial $C[0,\infty)$ solution to the nonlinear operator equation

$$y(t) = \int_0^\infty K(t,s) F(s, y(s))\, ds \qquad \text{for } t \in [0,\infty).$$

Section 3 extends the results in Section 2 to upper semicontinuous, k-set contractive $(0 \leq k < 1)$ maps. Our theory again relies on a diagonal process together with a result on hemicompact maps [8]. For the convenience of the reader we state the result from [8] which we will use in Section 3.

THEOREM 1.1 [8]. *Let (X, d) be a metric space, D a nonempty, complete subset of X, and $G\colon D \to 2^X$ a condensing map with $G(D)$ bounded. Then G is hemicompact (i.e., each sequence (x_n) in D has a convergent subsequence whenever $d(x_n, Gx_n) \to 0$ as $n \to \infty$).*

2. SINGLE VALUED MAPS

First, for the convenience of the reader we recall Krasnoselskii's fixed point theorem in a cone C of a Banach space $E = (E, \|\cdot\|)$. For notational purposes, for $\eta > 0$ let

$$U_\eta = \{y \in E\colon \|y\| < \eta\},\ \partial U_\eta = \{y \in E\colon \|y\| = \eta\},$$

and

$$\overline{U_\eta} = \{y \in E\colon \|y\| \leq \eta\}.$$

THEOREM 2.1 [6, p. 94]. *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone in E, $r_1 > 0$, $r_2 > 0$, $r_1 \neq r_2$ with $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$. Let $F\colon C \cap (\overline{U_R} \setminus U_r) \to C$ be a continuous, compact map with*

$$\|Fx\| \geq \|x\| \qquad \text{for all } x \in \partial U_{r_2} \cap C \tag{2.1}$$

and

$$\|Fx\| \leq \|x\| \qquad \text{for all } x \in \partial U_{r_1} \cap C \tag{2.2}$$

holding. Then F has at least one fixed point $y \in C$ with $r \leq \|y\| \leq R$.

Note that Theorem 2.1 immediately guarantees a fixed point theorem for self maps between Banach spaces.

THEOREM 2.2. *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone in E, $r > 0$, and $R > r$. Let $F\colon C \cap (\overline{U_R} \setminus U_r) \to C \cap (\overline{U_R} \setminus U_r)$ be a continuous, compact map. Then F has at least one fixed point $y \in C$ with $r \leq \|y\| \leq R$.*

Proof. We will use Theorem 2.1 with $r_1 = R$ and $r_2 = r$. Let $x \in \partial U_R \cap C$ (so $\|x\| = R$), and since $F\colon C \cap (\overline{U_R} \setminus U_r) \to C \cap (\overline{U_R} \setminus U_r)$ we have that

$$\|Fx\| \leq R = \|x\|,$$

i.e., (2.2) holds with $r_1 = R$.

On the other hand, if $x \in \partial U_r \cap C$ (so $\|x\| = r$) then once again since $F\colon C \cap (\overline{U_R} \setminus U_r) \to C \cap (\overline{U_R} \setminus U_r)$ we have

$$\|Fx\| \geq r = \|x\|,$$

i.e., (2.1) holds with $r_2 = r$.

The result follows from Theorem 2.1. ∎

We now extend Theorem 2.2 to an "applicable" fixed point theorem in the Fréchet space setting. Let $N_0 = \{1, 2, \dots\}$. E is a Fréchet space endowed with a family of seminorms $\{|\cdot|_n\colon n \in N_0\}$ with

$$|x|_1 \leq |x|_2 \leq \cdots \qquad \text{for all } x \in E.$$

Also assume that for each $n \in N_0$ that $(E_n, |\cdot|_n)$ is a Banach space, and suppose that

$$E_1 \supseteq E_2 \supseteq \cdots$$

with $E = \bigcap_{n=1}^{\infty} E_n$ and $|x|_n \leq |x|_{n+1}$ for all $x \in E_{n+1}$ (here $n \in N_0$). For each $n \in N_0$, let C_n be a cone in E_n and assume that

$$C_1 \supseteq C_2 \supseteq \cdots.$$

For $\rho > 0$ and $n \in N_0$, let

$$U_{n,\rho} = \{x \in E_n : |x|_n < \rho\} \qquad \text{and} \qquad \Omega_{n,\rho} = U_{n,\rho} \cap C_n.$$

Note that

$$\partial_{C_n}\Omega_{n,\rho} = \partial_{E_n}U_{n,\rho} \cap C_n \qquad \text{and} \qquad \overline{\Omega_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n$$

(the first closure is with respect to C_n, whereas the second is with respect to E_n). In addition, note that since $|x|_n \le |x|_{n+1}$ for all $x \in E_{n+1}$ that

$$\Omega_{1,\rho} \supseteq \Omega_{2,\rho} \supseteq \cdots$$

and

$$\overline{\Omega_{1,\rho}} \supseteq \overline{\Omega_{2,\rho}} \supseteq \cdots.$$

We first establish a result which guarantees that the inclusion

$$y = Fy \tag{2.3}$$

has a solution in E.

The main points needed to establish the existence of solutions to (2.3) are the following:

(i) the existence of continuous maps $F_n : C_n \cap (\overline{U_{n,R}} \setminus U_{n,r}) \to C_n$;

(ii) the sequence of maps $\{F_n\}$ has the property that a convergent sequence of fixed points $\{y_n\}$ of $\{F_n\}$ converges to a fixed point of F; and

(iii) the assumptions on F_n are such that Theorem 2.2 can be applied.

We note here that F_n need not be the restriction of F to E_n (see Theorem 2.4).

DEFINITION 2.1. Fix $k \in N_0$. If $x, y \in E_k$ then we say that $x = y$ in E_k if $|x - y|_k = 0$ (i.e., if $x - y = 0$; here 0 is the zero in E_k).

DEFINITION 2.2. If $x, y \in E$ then we say that $x = y$ in E if $x = y$ in E_k for each $k \in N_0$.

THEOREM 2.3. *Let $r > 0$, $R > 0$, $\gamma > 0$ be constants with $\gamma < r < R$. Suppose that the following conditions are satisfied*:

$$\textit{For each } n \in N_0,\ F_n : C_n \cap \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \to C_n \cap \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \textit{ is continuous}. \tag{2.4}$$

$$\textit{For each } n \in N_0, \textit{ the map } \mathcal{K}_n\colon \left(\overline{U_{n,R}} \setminus U_{n,\gamma}\right) \cap C_n \to 2^{C_n} \ (\textit{nonempty subsets of } C_n), \textit{ given by } \mathcal{K}_n y = \bigcup_{m=n}^{\infty} F_m y \ (\textit{see Remark 2.2}), \textit{ is compact}. \tag{2.5}$$

For every $k \in N_0$ *and any subsequence* $A \subseteq \{k, k+1, \dots,\}$,

$$\textit{if } x \in C_n,\ n \in A, \textit{ is such that } R \geq |x|_n \geq r, \textit{ then } |x|_k \geq \gamma. \tag{2.6}$$

If there exists a $w \in E$ *and a sequence* $\{y_n\}_{n \in N_0}$ *with* $y_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n$ *and* $y_n = F_n y_n$ *in* E_n *such that for every* $k \in N_0$ *there exists a subsequence* $S \subseteq \{k+1, k+2, \dots\}$ *of* N_0 *with* $y_n \to w$ *in* E_k *as* $n \to \infty$ *in* S, *then* $w = Fw$ *in* E. (2.7)

Then (2.3) *has a solution* $y_1 \in E$ *with* $y_1 \in \bigcap_{n=1}^{\infty}((\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n)$.

Remark 2.1. In Theorem 2.3 it is automatically assumed for each $n \in N_0$ that $F_n\colon (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n \to C_n$.

Remark 2.2. The definition of $\mathcal{K}_n$ in (2.5) is as follows. If $y \in (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n$ and $y \notin (\overline{U_{n+1,R}} \setminus U_{n+1,\gamma}) \cap C_{n+1}$ then $\mathcal{K}_n y = F_n y$, whereas if $y \in (\overline{U_{n+1,R}} \setminus U_{n+1,\gamma}) \cap C_{n+1}$ and $y \notin (\overline{U_{n+2,R}} \setminus U_{n+2,\gamma}) \cap C_{n+2}$ then $\mathcal{K}_n y = F_n y \cup F_{n+1} y$, and so on.

Remark 2.3. We note that although (2.5)–(2.7) seem technical, they are in fact easily checked in practice (see Theorem 2.4).

Proof. Fix $n \in N_0$. Theorem 2.2 guarantees that $y = F_n y$ has a solution $y_n \in (\overline{U_{n,R}} \setminus U_{n,r}) \cap C_n$. Let's look at $\{y_n\}_{n \in N_0}$. Note that $y_n \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$ for each $n \in N_0$. To see this note that $|y_n|_n \leq R$ and $|x|_1 \leq |x|_n$ for all $x \in E_n$ imply that $|y_n|_1 \leq R$. Thus $y_n \in \overline{U_{1,R}}$ for each $n_0 \in N_0$.

On the other hand, $|y_n|_n \geq r$, $y_n \in C_n$, together with (2.6) implies that $|y_n|_1 \geq \gamma$, so $y_n \in (\overline{U_{1,R}} \setminus U_{1,\gamma})$ for each $n \in N_0$. Now (2.5) guarantees that there exist a subsequence N_1^* of N_0 and a $z_1 \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in N_1^*. Note in particular that $\gamma \leq |z_1|_1 \leq R$. Let $N_1 = N_1^* \setminus \{1\}$. Look at $\{y_n\}_{n \in N_1}$. Note that $y_n \in (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2$ for each $n \in N_1$. Again (2.5) guarantees that there exists a subsequence N_2^* of N_1 and a $z_2 \in (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in N_2^*. Note in particular that $\gamma \leq |z_2|_2 \leq R$ and $z_2 = z_1$ in E_1. Let $N_2 = N_2^* \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \cdots, N_k^* \supseteq \{k, k+1, \dots\},$$

and $z_k \in (\overline{U_{k,R}} \setminus U_{k,\gamma}) \cap C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in N_k^*. Note that $z_{k+1} = z_k$ in E_k for $k = 1, 2, \dots$. Also, let $N_k = N_k^* \setminus \{k\}$. Fix $k \in N_0$.

Let $y = z_k$ in E_k. Note that y is well defined and $y \in E_k$ for each $k = 1, 2 \dots$. Also, $y_n = F_n y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k). This together with (2.7) implies that $y = Fy$ in E. Note also that $y \in \bigcap_{n=1}^{\infty}((\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n)$. ∎

Remark 2.4. If (2.6) is removed in Theorem 2.3 then the above analysis guarantees the existence of a $y \in E$ with $y = Fy$ in E and $y \in \bigcap_{n=1}^{\infty}(\overline{U_{n,R}} \cap C_n)$; we are of course assuming here that $\mathscr{K}_n\colon \overline{U_{n,R}} \cap C_n \to 2^{C_n}$ is compact for each $n \in N_0$ in (2.5) (so we are automatically assuming that $F_n\colon \overline{U_{n,R}} \cap C_n \to C_n$ for each $n \in N_0$). (The definition of $\mathscr{K}_n$ is as follows: If $y \in \overline{U_{n,R}} \cap C_n$ and $y \notin \overline{U_{n+1,R}} \cap C_{n+1}$ then $\mathscr{K}_n y = F_n y$, whereas if $y \in \overline{U_{n+1,R}} \cap C_{n+1}$ and $y \notin \overline{U_{n+2,R}} \cap C_{n+2}$ then $\mathscr{K}_n y = F_n y \cup F_{n+1} y$, and so on.) If in addition all mentions of R, r in (2.4), (2.5), and (2.7) are removed and we use Schauder's fixed point theorem instead of Theorem 2.2 in the proof of Theorem 2.3, we get a generalization of the Schauder–Tychonoff theorem. For completeness we state the result.

Let E be a Fréchet space endowed with a family of seminorms $\{|\cdot|_n\colon n \in N_0\}$. Also, assume for each $n \in N_0$ that $(E_n, |\cdot|_n)$ is a Banach space and suppose that

$$E_1 \supseteq E_2 \supseteq \cdots$$

with $E = \bigcap_{n=1}^{\infty} E_n$. For each $n \in N_0$, let C_n be a closed, convex set in E_n and assume that

$$C_1 \supseteq C_2 \supseteq \cdots .$$

Suppose the following conditions are satisfied:

For each $n \in N_0$, $F_n\colon C_n \to C_n$ is continuous.

For each $n \in N_0$, the map $\mathscr{K}_n\colon C_n \to 2^{C_n}$, given by $\mathscr{K}_n y = \bigcup_{m=n}^{\infty} F_m y$, is compact.

If there exist a $w \in E$ and a sequence $\{y_n\}_{n \in N_0}$, with $y_n \in C_n$ and $y_n = F_n y_n$ in E_n such that for every $k \in N_0$ there exists a subsequence $S \subseteq \{k+1, k+2, \dots\}$ of N_0 with $y_n \to w$ in E_k as $n \to \infty$ in S, then $w = Fw$ in E.

Then (2.3) has a solution $y_1 \in E$ with $y_1 \in \bigcap_{n=1}^{\infty} C_n$. (The definition of $\mathscr{K}_n$ is as follows: If $y \in C_n$ and $y \notin C_{n+1}$, then $\mathscr{K}_n y = F_n y$, whereas if $y \in C_{n+1}$ and $y \notin C_{n+2}$ then $\mathscr{K}_n y = F_n y \cup F_{n+1} y$, and so on.)

To illustrate the ideas involved in Theorem 2.3 we discuss the nonlinear integral equation

$$y(t) = \int_0^{\infty} K(t,s) F(s, y(s))\, ds \qquad \text{for } t \in [0, \infty). \tag{2.8}$$

THEOREM 2.4. *Let $1 \le p \le \infty$ be a constant and $1 \le q \le \infty$ the conjugate to p. Suppose the following conditions are satisfied*:

$$\text{For each } t \in [0,\infty), \text{ the map } s \mapsto K(t,s) \text{ is measurable.} \tag{2.9}$$

$$\sup_{t\in[0,\infty)} \left(\int_0^\infty |K(t,s)|^q \, ds \right)^{1/q} < \infty. \tag{2.10}$$

$$\int_0^\infty |K(t',s) - K(t,s)|^q \, ds \to 0 \text{ as } t \to t', \qquad \text{for each } t' \in [0,\infty). \tag{2.11}$$

F: $[0,\infty) \times \mathbf{R} \to \mathbf{R}$ is a L^p-Carathéodory function: By this we mean that

(a) *the map $x \mapsto F(x,u)$ is measurable for all $u \in \mathbf{R}$;*

(b) *for a.e. $x \in [0,\infty)$, the map $u \mapsto F(x,u)$ is continuous;*

(c) *for each $r > 0$ there exists $h_r \in L^p[0,\infty)$ with $|F(x,y)| \le h_r(x)$ for a.e. $x \in [0,\infty)$ and all $y \in \mathbf{R}$ with $|y| \le r$.* (2.12)

$$\text{For each } t \in [0,\infty),\ k(t,s) \ge 0 \text{ for a.e. } s \in [0,\infty). \tag{2.13}$$

$$F: [0,\infty) \times [0,\infty) \to [0,\infty). \tag{2.14}$$

$$\exists 0 < M < 1,\ \kappa \in L^p[0,\infty), \text{ and an interval } [a,b] \subseteq [0,\infty),\ a < b$$
$$\text{with } K(t,s) \ge M\kappa(s) \text{ for } t \in [a,b] \text{ and a.e. } s \in [0,\infty). \tag{2.15}$$

$$K(t,s) \le \kappa(s) \qquad \text{for } t \in [0,\infty) \text{ and a.e. } s \in [0,\infty). \tag{2.16}$$

There exist a function $\psi: [0,\infty) \to [0,\infty)$, continuous and nondecreasing, and a $\phi \in L^p[0,\infty)$ with $|F(s,y)| \le \phi(s)\psi(y)$ for all $y \in [0,\infty)$ and a.e. $s \in [0,\infty)$. (2.17)

$$\text{There exists a } \tau \in L^p[a,b] \text{ with } F(s,y) \ge \tau(s)\psi(y)$$
$$\text{for all } y \in [0,\infty) \text{ and a.e. } s \in [a,b]. \tag{2.18}$$

$$\exists r > 0, \text{ with } r \le K_1\psi(Mr) \text{ where } K_1 = \sup_{t\in[0,b]} \int_a^b K(t,s)\tau(s)\, ds. \tag{2.19}$$

$$\exists R > r, \text{ with } R \ge K_2\psi(R) \text{ where } K_2 = \sup_{t\in[0,\infty)} \int_0^\infty K(t,s)\phi(s)\, ds. \tag{2.20}$$

Choose $n_1 \in N_0$ with $n_1 \ge b$ and let $N_1 = \{n_1, n_1 + 1, \dots\}$. Then (2.8) *has a solution $y \in C[0,\infty)$ with $y \ge 0$ on $[0,\infty)$ and with $Mr \le |y|_n \le R$ for each $n \in N_1$ (here $|y|_n = \sup_{t\in[0,n]}|y(t)|$).*

Proof. Let $n \in N_1$ with

$$F_n y(t) = \int_0^n K(t,s) F(s, y(s))\, ds \qquad \text{for } t \in [0, n],$$

where $y \in C[0,n]$. Let $E_n = (C[0,n], |\cdot|_n)$ and

$$C_n = \left\{ y \in C[0,n] \colon y(t) \geq 0 \text{ for } t \in [0,n] \text{ and } \min_{t \in [a,b]} y(t) \geq M |y|_n \right\}.$$

For $\rho = r$ or R let

$$U_{n,\rho} = \{ x \in C[0,n] \colon |x|_n < \rho \}.$$

We seek to apply Theorem 2.3. Note that [7] guarantees that $F_n \colon C_n \cap (\overline{U_{n,R}} \setminus U_{n,r}) \to E_n$ is continuous for each $n \in N_1$. We *claim* that

$$F_n \colon C_n \cap \left(\overline{U_{n,R}} \setminus U_{n,r} \right) \to C_n \cap \left(\overline{U_{n,R}} \setminus U_{n,r} \right) \qquad \text{for each } n \in N_1. \tag{2.21}$$

If (2.21) holds then of course (2.4) is true for $n \in N_1$. To see (2.21) fix $n \in N_1$ and take $y \in C_n \cap (\overline{U_{n,R}} \setminus U_{n,r})$. Then $y(t) \geq M|y|_n \geq Mr$ for $t \in [a,b]$. Also, from (2.16) we have

$$|F_n y|_n \leq \int_0^n \kappa(s) F(s, y(s))\, ds. \tag{2.22}$$

Next note that (2.15) together with (2.22) yields

$$\begin{aligned} \min_{t \in [a,b]} F_n y(t) &= \min_{t \in [a,b]} \int_0^n K(t,s) F(s, y(s))\, ds \\ &\geq M \int_0^n \kappa(s) F(s, y(s))\, ds \geq M |F_n y|_n, \end{aligned}$$

so $F_n y \in C_n$. Also, since $r \leq |y|_n \leq R$ we have for $t \in [0,n]$ that

$$\begin{aligned} |F_n y(t)| &\leq \int_0^n K(t,s) \phi(s) \psi(y(s))\, ds \\ &\leq \psi(|y|_n) \int_0^\infty K(t,s) \phi(s)\, ds \leq K_2 \psi(R). \end{aligned}$$

This together with (2.20) gives

$$|F_n y|_n \leq K_2 \psi(R) \leq R,$$

so $F_n y \in \overline{U_{n,R}}$. In addition, (2.18) and (2.19) imply that

$$\begin{aligned} |F_n y|_n &\ge \sup_{t\in[0,n]} \int_a^b K(t,s) F(s,y(s))\,ds \\ &\ge \sup_{t\in[0,n]} \int_a^b K(t,s)\tau(s)\psi(y(s))\,ds \\ &\ge \psi(Mr) \sup_{t\in[0,n]} \int_a^b K(t,s)\tau(s)\,ds \\ &\ge \psi(Mr) \sup_{t\in[0,b]} \int_a^b K(t,s)\tau(s)\,ds = \psi(Mr)K_1 \ge r, \end{aligned}$$

so $F_n y \notin U_{n,r}$. Thus (2.21) holds and so (2.4) is true for $n \in N_1$. To show (2.5) for $n \in N_1$, fix $n \in N_1$. Let $y \in (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n$. Without loss of generality assume that there exists $l \in \{0,1,2,\dots\}$ with $y \in (\overline{U_{n+l,R}} \setminus U_{n+l,\gamma}) \cap C_{n+l}$ and $y \notin (\overline{U_{n+l+1,R}} \setminus U_{n+l+1,\gamma}) \cap C_{n+l+1}$. Then, by definition (see Remark 2.2), $\mathscr{K}_n y = \bigcup_{m=n}^{n+l} F_m y$. Since $y \in \overline{U_{n+l,R}}$, (2.12) guarantees that there exists a $h_R \in L^p[0,\infty)$ with $|F(s,y(s))| \le h_R(s)$ for a.e. $s \in [0, n+l]$. Now for $j \in \{0,1,\dots,l\}$ we have for $t \in [0,n]$ that

$$\begin{aligned} \left|F_{n+j}y(t)\right| &\le \int_0^{n+j} h_R(s)|K(t,s)|\,ds \\ &\le \left(\int_0^{n+j} [h_R(s)]^p\,ds\right)^{1/p} \left(\int_0^{n+j} |K(t,s)|^q\,ds\right)^{1/q} \\ &\le \left(\int_0^{\infty} [h_R(s)]^p\,ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^{\infty} |K(t,s)|^q\,ds\right)^{1/q}. \end{aligned}$$

Thus for $j \in \{0,1,\dots,l\}$ we have

$$|F_{n+j}y|_n \le \left(\int_0^{\infty} [h_R(s)]^p\,ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^{\infty} |K(t,s)|^q\,ds\right)^{1/q}$$

and so

$$|\mathscr{K}_n y|_n \le \left(\int_0^{\infty} [h_R(s)]^p\,ds\right)^{1/p} \sup_{t\in[0,\infty)} \left(\int_0^{\infty} |K(t,s)|^q\,ds\right)^{1/q}. \quad (2.23)$$

In addition, for $t_1, t_2 \in [0, n]$ and $j \in \{0, 1, \dots, l\}$ we have

$$\begin{aligned}
&\left| F_{n+j} y(t_1) - F_{n+j} y(t_2) \right| \\
&\quad \le \int_0^{n+j} h_R(s) \left| K(t_1, s) - K(t_2, s) \right| ds \\
&\quad \le \left(\int_0^\infty [h_R(s)]^p \, ds \right)^{1/p} \left(\int_0^\infty \left| K(t_1, s) - K(t_2, s) \right|^q ds \right)^{1/q},
\end{aligned}$$

and so

$$\begin{aligned}
&\left| F_{n+j} y(t_1) - F_{n+j} y(t_2) \right| \\
&\quad \le \left(\int_0^\infty [h_R(s)]^p \, ds \right)^{1/p} \left(\int_0^\infty \left| K(t_1, s) - K(t_2, s) \right|^q ds \right)^{1/q}. \qquad (2.24)
\end{aligned}$$

Now (2.11), (2.23), and (2.24) guarantee that $\{\mathscr{K}_n y \colon y \in (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n\}$ is uniformly bounded and equicontinuous on $[0, n]$. The Arzela–Ascoli theorem implies that $\mathscr{K}_n \colon (\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n \to 2^{C_n}$ is compact, so (2.5) holds for $n \in N_1$. Next we show that (2.6) is true for $n \in N_1$ with $\gamma = Mr$. To see this fix $k \in N_1$ and take any subsequence $A \subseteq \{k, k+1, \dots\}$. Now if $x \in C_n$, $n \in A$, is such that $R \ge |x|_n \ge r$, then $x(t) \ge M|x|_n \ge Mr = \gamma$ for $t \in [a, b]$ and so $\min_{t \in [a,b]} x(t) \ge \gamma$. Thus $|x|_k = \sup_{t \in [0,k]} |x(t)| \ge \gamma$, so (2.6) holds for $n \in N_1$. It remains to show that (2.7) is satisfied for $n \in N_1$. Suppose there exist $w \in C[0, \infty)$ and a sequence $\{y_n\}_{n \in N_1}$ with $y_n \in (\overline{U_{n,R}} \setminus U_{n,r}) \cap C_n$ and $y_n(t) = F_n y_n(t)$, $t \in [0, n]$, such that for every $k \in N_1$ there exists a subsequence $S \subseteq \{k+1, k+2, \dots\}$ of N_1 with $y_n \to w$ in $C[0, k]$ as $n \to \infty$ in S. If we show that

$$w(t) = \int_0^\infty K(t, s) F(s, w(s)) \, ds \qquad \text{for } t \in [0, \infty),$$

then (2.7) is true for $n \in N_1$. Fix $t \in [0, \infty)$. Consider $k \ge t$ and $n \in S$ (as described above). Then $y_n(s) = F_n y_n(s)$, $t \in [0, n]$, for $n \in S$ and so

$$y_n(t) - \int_0^k K(t, s) F(s, y_n(s)) \, ds = \int_k^n K(t, s) F(s, y_n(s)) \, ds.$$

Now (2.12) guarantees that there exists a $h_R \in L^p[0, \infty)$ with $|F(s, y_n(s))| \le h_R(s)$ for a.e. $s \in [0, n]$ and so

$$\begin{aligned}
\left| y_n(t) - \int_0^k K(t, s) \, (s, y_n(s)) \, ds \right| &\le \int_k^n K(t, s) h_R(s) \, ds \\
&\le \int_k^\infty K(t, s) h_R(s) \, ds. \qquad (2.25)
\end{aligned}$$

Let $n \to \infty$ through S in (2.25) and use the Lebesgue dominated convergence theorem to obtain

$$\left| w(t) - \int_0^k K(t,s)F(s,w(s))\,ds \right| \le \int_k^\infty h_R(s)|K(t,s)|\,ds$$

since $y_n \to w$ in $C[0,k]$. Finally, we let $k \to \infty$ to conclude that

$$w(t) - \int_0^\infty K(t,s)F(s,w(s))\,ds = 0.$$

Thus (2.7) holds (with N_0 replaced by N_1). To deduce the result we apply Theorem 2.3 (with N_0 replaced by N_1). ∎

Remark 2.5. Note from the above proof that (2.19) could be replaced in Theorem 2.4 with the following: $\exists r > 0$ with $r \le \psi(Mr)\sup_{t \in [0,n_1]} \int_a^b K(t,s)\tau(s)\,ds$.

Remark 2.6. Note that if $\psi(x) = x^\alpha$, $0 \le \alpha < 1$, then clearly (2.19) and (2.20) hold since

$$\lim_{x \to 0} \frac{x}{\psi(Mx)} = M^{-\alpha} \lim_{x \to 0} x^{1-\alpha} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{x}{\psi(x)} = \lim_{x \to \infty} x^{1-\alpha} = \infty.$$

3. MULTIVALUED MAPS

First we recall [1] the Petryshyn–Krasnoselskii Fixed Point Theorem for multivalued maps between Banach spaces. Let $E = (E, \|\cdot\|)$ be a Banach space, and for notational purposes for $\eta > 0$ let

$$U_\eta = \{y \in E : \|y\| < \eta\},\ \partial U_\eta = \{y \in E : \|y\| = \eta\},$$

and

$$\overline{U_\eta} = \{y \in E : \|y\| \le \eta\}.$$

THEOREM 3.1 [1]. *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone in E, $r_1 > 0$, $r_2 > 0$, and $r_1 \ne r_2$ with $R = \max\{r_1, r_2\}$ and $r = \min\{r_1, r_2\}$. Assume that $\|\cdot\|$ is increasing with respect to C and $F: C \cap \overline{U_R} \to CK(C)$ (here $CK(C)$ denotes the family of nonempty compact, convex subsets of C) is an upper semicontinuous, k-set contractive (here $0 \le k < 1$) map with*

$$\|y\| \ge \|x\| \quad \text{for all } y \in Fx \quad \text{and} \quad x \in \partial U_{r_2} \cap C \tag{3.1}$$

and

$$\|y\| \leq \|x\| \qquad \textit{for all } y \in Fx \qquad \textit{and} \qquad x \in \partial U_{r_1} \cap C \tag{3.2}$$

holding. Then F has at least one fixed point $y \in C$ with $r \leq \|y\| \leq R$.

Reasoning which is essentially the same as that in Theorem 2.2 (except here we use Theorem 3.1) establishes the following fixed point theorem for self maps between Banach spaces.

THEOREM 3.2. *Let $E = (E, \|\cdot\|)$ be a Banach space, $C \subseteq E$ a cone in E, $r > 0$, $R > r$, and $\|\cdot\|$ increasing with respect to C. Let $F: C \cap \overline{U_R} \to CK(C)$ be an upper semicontinuous, k-set contractive $(0 \leq k < 1)$ map with $F: C \cap (\overline{U_R} \setminus U_r) \to CK(C \cap (\overline{U_R} \setminus U_r))$. Then F has at least one fixed point $y \in C$ with $r \leq \|y\| \leq R$.*

We now extend Theorem 3.2 to the Fréchet space setting. Let $N_0 = \{1, 2, \dots\}$. E is a Fréchet space endowed with a family of seminorms $\{|\cdot|_n: n \in N_0\}$ with

$$|x|_1 \leq |x_2| \leq \cdots \qquad \text{for all } x \in E.$$

Also assume for each $n \in N_0$ that $(E_n, |\cdot|_n)$ is a Banach space and suppose that

$$E_1 \supseteq E_2 \supseteq \cdots$$

with $E = \bigcap_{n=1}^{\infty} E_n$ and $|x|_n \leq |x|_{n+1}$ for all $x \in E_{n+1}$ (here $n \in N_0$). For each $n \in N_0$, let C_n be a cone in E_n and assume that $|\cdot|_n$ is increasing with respect to C_n. Also, assume that

$$C_1 \supseteq C_2 \supseteq \cdots .$$

For $\rho > 0$ and $n \in N_0$, let

$$U_{n,\rho} = \{x \in E_n: |x|_n < \rho\} \qquad \text{and} \qquad \Omega_{n,\rho} = U_{n,\rho} \cap C_n.$$

Note that

$$\partial_{C_n} \Omega_{n,\rho} = \partial_{E_n} U_{n,\rho} \cap C_n \qquad \text{and} \qquad \overline{\Omega_{n,\rho}} = \overline{U_{n,\rho}} \cap C_n$$

(the first closure is with respect to C_n, whereass the second is with respect to E_n). In addition, note that since $|x|_n \leq |x|_{n+1}$ for all $x \in E_{n+1}$ that

$$\Omega_{1,\rho} \supseteq \Omega_{2,\rho} \supseteq \cdots$$

and

$$\overline{\Omega_{1,\rho}} \supseteq \overline{\Omega_{2,\rho}} \supseteq \cdots .$$

We first establish a result which guarantees that the inclusion

$$y \in Fy \tag{3.3}$$

has a solution in E.

DEFINITION 3.1. Fix $k \in N_0$. We say that $x \in Fy$ in E_k if there exists $w \in Fy$ with $x = w$ in E_k.

THEOREM 3.3. *Let $r > 0$, $R > 0$, $\gamma > 0$ be constants with $\gamma < r < R$. Suppose that the following conditions are satisfied*:

For each $n \in N_0$, $F_n\colon C_n \cap \overline{U_{n,R}} \to CK(C_n)$ *is an upper semicontinuous map.* (3.4)

$$\text{For each } n \in N_0,\ F_n\colon C_n \cap \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \to CK\left(C_n \cap \left(\overline{U_{n,R}} \setminus U_{n,r}\right)\right). \tag{3.5}$$

For each $n \in N_0$, *the map* $\mathscr{K}_n\colon \overline{U_{n,R}} \cap C_n \to 2^{C_n}$, *given by* $\mathscr{K}_n y = \bigcup_{m=n}^{\infty} F_m y$ (*see Remark* 3.1), *is k-set contractive* $(0 \le k < 1)$. (3.6)

For every $k \in N_0$ *and any subsequence* $A \subseteq \{k, k+1, \dots\}$, *if* $x \in C_n$, $n \in A$, *is such that* $R \ge |x|_n \ge r$ *then* $|x|_k \ge \gamma$. (3.7)

If there exist a $w \in E$ *and a sequence* $\{y_n\}_{n \in N_0}$ *with* $y_n \in \left(\overline{U_{n,R}} \setminus U_{n,r}\right) \cap C_n$ *and* $y_n \in F_n y_n$ *in* E_n *such that for every* $k \in N_0$ *there exists a subsequence* $S \subseteq \{k+1, k+2, \dots\}$ *of* N_0 *with* $y_n \to w$ *in* E_k *as* $n \to \infty$ *in* S, *then* $w \in Fw$ *in* E. (3.8)

Then (3.3) *has a solution* $y_1 \in E$ *with* $y_1 \in \bigcap_{n=1}^{\infty}((\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n)$.

Remark 3.1. The definition of $\mathscr{K}_n$ in (3.6) is as follows. If $y \in \overline{U_{n,R}} \cap C_n$ and $y \notin \overline{U_{n+1,R}} \cap C_{n+1}$ then $\mathscr{K}_n y = F_n y$, whereas if $y \in \overline{U_{n+1,R}} \cap C_{n+1}$ and $y \notin \overline{U_{n+2,R}} \cap C_{n+2}$ then $\mathscr{K}_n y = F_n y \cup F_{n+1} y$, and so on.

Proof. Fix $n \in N_0$. Theorem 3.2 guarantees that $y \in F_n y$ has a solution $y_n \in (\overline{U_{n,R}} \setminus U_{n,r}) \cap C_n$. Let's look at $\{y_n\}_{n \in N_0}$. As in Theorem 2.3 it is easy to see that $y_n \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$ for each $n \in N_0$. Now Theorem 1.1 (with $X = E_1$, $G = \mathscr{K}_1$, and $D = (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$, and noting that $d_1(u_n, \mathscr{K}_1 u_n) = 0$ for each $n \in N_0$ since $|x|_1 \le |x|_n$ for all $x \in E_n$ and $y_n \in F_n y_n$ in E_n; here $d_1(x, S) = \inf_{y \in S} |x - y|_1$ if S is a nonempty subset of X) guarantees that there exist a subsequence N_1^* of N_0 and a $z_1 \in (\overline{U_{1,R}} \setminus U_{1,\gamma}) \cap C_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in N_1^*. Note in particular that $\gamma \le |z_1|_1 \le R$. Let $N_1 = N_1^* \setminus \{1\}$. Look at $\{y_n\}_{n \in N_1}$. Note that $y_n \in (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2$ for each $n \in N_1$. Now Theorem 1.1 (with $X = E_2$,

$G = \mathscr{K}_2$, $D = (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2$, and noting that $d_2(u_n, \mathscr{K}_2 u_n) = 0$ for each $n \in N_1$; here $d_2(x, S) = \inf_{y \in S} |x - y|_2$ if S is a nonempty subset of X) guarantees that there exist a subsequence N_2^* of N_1 and a $z_2 \in (\overline{U_{2,R}} \setminus U_{2,\gamma}) \cap C_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in N_2^*. Note in particular that $\gamma \le |z_2|_2 \le R$ and $z_2 = z_1$ in E_1. Let $N_2 = N_2^* \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^* \supseteq N_2^* \supseteq \cdots, N_k^* \subseteq \{k, k+1, \dots\},$$

and $z_k \in (\overline{U_{k,R}} \setminus U_{k,\gamma}) \cap C_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in N_k^*. Note that $z_{k+1} = z_k$ in E_k for $k = 1, 2, \dots$. Also, let $N_k = N_k^* \setminus \{k\}$. Fix $k \in N_0$. Let $y = z_k$ in E_k. Note that y is well defined and $y \in E_k$ for each $k = 1, 2 \dots$. Also, $y_n \in F_n y_n$ in E_n for $n \in N_k$ and $y_n \to y$ in E_k as $n \to \infty$ in N_k (since $y = z_k$ in E_k). This together with (3.8) implies that $y \in Fy$ in E. Note also that $y \in \bigcap_{n=1}^{\infty}((\overline{U_{n,R}} \setminus U_{n,\gamma}) \cap C_n)$. ∎

Remark 3.2. The idea in Theorem 3.3 together with the observation in Remark 2.4 immediately guarantees a generalization of Fan's fixed point theorem. For completeness we state the result: Let E be a Fréchet space endowed with a family of seminorms $\{|\cdot|_n : n \in N_0\}$. Also, assume that for each $n \in N_0$ that $(E_n, |\cdot|_n)$ is a Banach space and suppose that

$$E_1 \supseteq E_2 \supseteq \dots$$

with E $= \bigcap_{n=1}^{\infty} E_n$. For each $n \in N_0$, let C_n be a closed, convex set in E_n and assume that

$$C_1 \supseteq C_2 \supseteq \cdots .$$

Suppose the following conditions are satisfied:

For each $n \in N_0$, $F_n : C_n \to CK(C_n)$ is upper semicontinuous.

For each $n \in N_0$, the map $\mathscr{K}_n : C_n \to 2^{C_n}$, given by $\mathscr{K}_n y = \bigcup_{m=n}^{\infty} F_m y$, is k-set contractive $(0 \le k < 1)$.

If there exist a $w \in E$ and a sequence $\{y_n\}_{n \in N_0}$ with $y_n \in C_n$ and $y_n \in F_n y_n$ in E_n such that for every $k \in N_0$ there exists a subsequence $S \subseteq \{k+1, k+2, \dots\}$ of N_0 with $y_n \to w$ in E_k as $n \to \infty$ in S, then $w \in Fw$ in E.

Then (3.3) has a solution $y_1 \in E$ with $y_1 \in \bigcap_{n=1}^{\infty} C_n$. (The definition of $\mathscr{K}_n$ is as follows: If $y \in C_n$ and $y \notin C_{n+1}$ then $\mathscr{K}_n y = F_n y$, whereas if $y \in C_{n+1}$ and $y \notin C_{n+2}$ then $\mathscr{K}_n y = F_n y \cup F_{n+1} y$, and so on.)

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