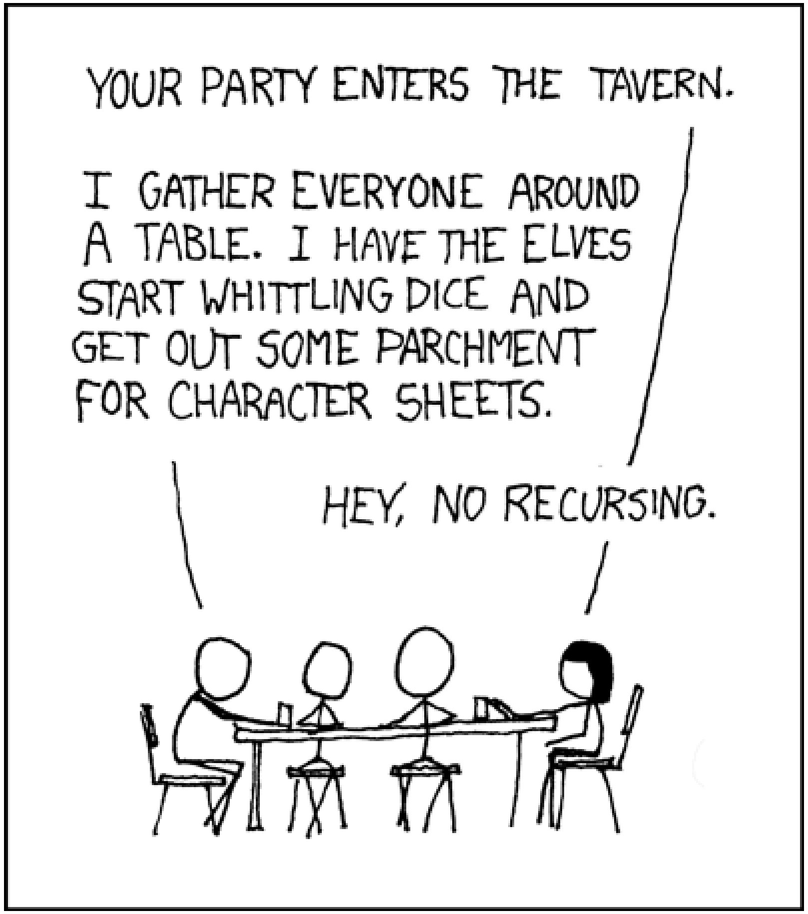
Fall 2015

Lecture 17: Strong induction & Recursive definitions



administrative

Midterm review session Sunday @ 1:00 pm (EEB 105)

MIDTERM MONDAY (IN THIS ROOM, USUAL TIME)

No office hours on Monday/Wednesday

Closed book.
One page (front and back) of notes allowed.

Exam includes induction!
Homework #5 is due on Friday, Nov 13th.

$$P(0)$$

$$\forall k\ \Big(\big(P(0) \land P(1) \land P(2) \land \cdots \land P(k)\big) \to P(k+1)\Big)$$

$$\therefore \forall n\ P(n)$$

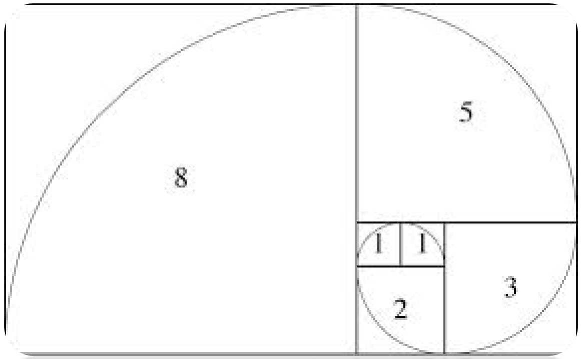
Follows from ordinary induction applied to

$$Q(n) = P(0) \land P(1) \land P(2) \land \cdots \land P(n)$$

1. By induction we will show that $P(n)$ is true for every $n \geq 0$
2. Base Case: Prove $P(0)$
3. Inductive Hypothesis:
   Assume that for some arbitrary integer $k \geq 0$, $P(j)$ is true for every j from 0 to k
4. Inductive Step:
   Prove that $P(k+1)$ is true using the Inductive Hypothesis (that $P(j)$ is true for all values $\leq k$)
5. Conclusion: Result follows by induction

review: Fibonacci numbers

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2$$



Theorem: $f_n < 2^n$ **for all** $n \geq 2$.

bounding the Fibonacci numbers

Theorem: $2^{\frac{n}{2}-1} \leq f_n < 2^n$ for all $n \geq 2$

$f_0 = 0$; $f_1 = 1$; $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$

Theorem: $2^{n/2-1} \leq f_n < 2^n$ **for all** $n \geq 2$

Proof:

1. Let P(n) be "$2^{n/2-1} \leq f_n < 2^n$. By (strong) induction we prove P(n) for all n ≥ 2.
2. **Base Case:** P(2) is true: $f_2=1$, $2^{2/2-1}=2^0=1 \leq f_2$, $2^2=4>f_2$
3. **Ind.Hyp:** Assume $2^{j/2-1} \leq f_j < 2^j$ for all integers j with $2 \leq j \leq k$ for for some arbitrary integer k ≥ 2.
4. **Ind. Step:** $\boxed{\text{Goal: Show } 2^{(k+1)/2-1} \leq f_{k+1} < 2^{k+1}}$

   <u>Case k=2</u>: P(3) is true: $f_3=f_2+f_1=1+1=2$, $2^{3/2-1}=2^{1/2} \leq 2 = f_3$, $2^3=8 > f_3$

   <u>Case k≥3:</u>

   $$f_{k+1} = f_k + f_{k-1} \geq 2^{k/2-1} + 2^{(k-1)/2-1} \quad \text{by I.H. since k-1} \geq 2$$

   $$> 2^{(k-1)/2-1} + 2^{(k-1)/2-1} = 2 \cdot 2^{(k-1)/2-1} = 2^{(k+1)/2-1}$$

   $$f_{k+1} = f_k + f_{k-1} < 2^k + 2^{(k-1)} \quad \text{by I.H. since k-1} \geq 2$$

   $$< 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$$

The divisibility theorem

Theorem: For any positive integers n, d, there are integers q, r such that $n = dq + r$ and $0 \leq r \leq d - 1$.

running time of Euclid’s algorithm

running time of Euclid's algorithm

Theorem: Suppose that Euclid's algorithm takes n steps for $\gcd(a, b)$ with $a > b$, then $a \geq f_{n+1}$.

Proof:

Set $r_{n+1} = a, r_n = b$ then Euclid's algorithm computes

$$
\begin{aligned}
r_{n+1} &= q_n r_n + r_{n-1} \\
r_n &= q_{n-1} r_{n-1} + r_{n-2} \\
&\vdots \\
r_3 &= q_2 r_2 + r_1 \\
r_2 &= q_1 r_1
\end{aligned}
$$

each quotient $q_i \geq 1$

$r_1 \geq 1$

Recursive definition

- Basis step: $0 \in S$
- Recursive step: if $x \in S$, then $x + 2 \in S$
- Exclusion rule: Every element in S follows from basis steps and a finite number of recursive steps

recursive definition of sets

Basis: $6 \in S$; $15 \in S$;
Recursive: if $x, y \in S$, then $x + y \in S$;

Basis: $[1, 1, 0] \in S, [0, 1, 1] \in S$;
Recursive:
if $[x, y, z] \in S$, $\alpha \in \mathbb{R}$, then $[\alpha x, \alpha y, \alpha z] \in S$
if $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$
then $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$

Powers of 3:

recursive definitions of sets: general form

Recursive definition

- *Basis step:* Some specific elements are in S
- *Recursive step:* Given some existing named elements in S some new objects constructed from these named elements are also in S.
- *Exclusion rule*: Every element in S follows from basis steps and a finite number of recursive steps

- An *alphabet* Σ is any finite set of characters.

- The set Σ^* of *strings* over the alphabet Σ is defined by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Palindromes are strings that are the same backwards and forwards.

Basis:

ε is a palindrome and any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome then apa is a palindrome for every $a \in \Sigma$.

all binary strings with no 1's before 0's

function definitions on recursively defined sets

Length:

len $(\varepsilon) = 0$;

len $(wa) = 1 +$ len(w); for $w \in \Sigma^*, a \in \Sigma$

Reversal:

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*, a \in \Sigma$

Concatenation:

function definitions on recursively defined sets

Length:

len $(\varepsilon) = 0$;

len $(wa) = 1 +$ len(w); for $w \in \Sigma^*, a \in \Sigma$

Reversal:

$\varepsilon^{\mathrm{R}} = \varepsilon$

$(wa)^{\mathrm{R}} = aw^{\mathrm{R}}$ for $w \in \Sigma^*, a \in \Sigma$

Concatenation:

$x \bullet \varepsilon = x$ for $x \in \Sigma^*$

$x \bullet wa = (x \bullet w)a$ for $x, w \in \Sigma^*, a \in \Sigma$