Noncooperative Games

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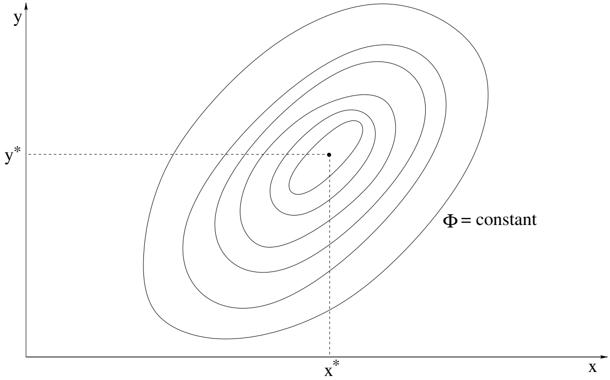
- introduction to non-cooperative games: solution concepts
- differential games in continuous time and economic models
- a game theoretical model of debt and bankruptcy

Optimal decision problem

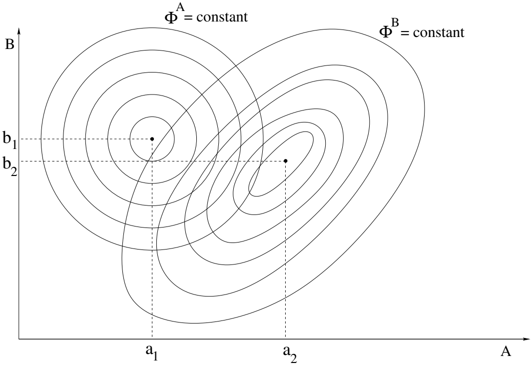
$$\text{maximize:} \quad \Phi(x, y)$$



The choice $(x^*, y^*) \in \mathbb{R}^2$ yields the maximum payoff

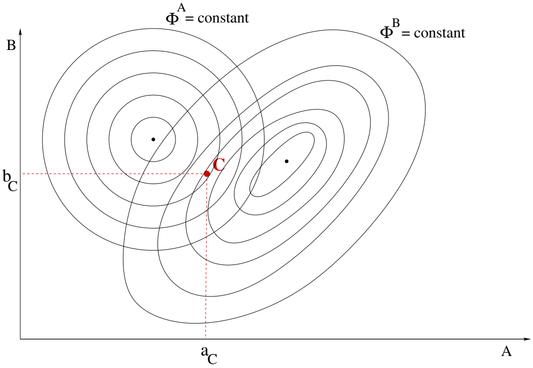
A game for two players

- Player A wishes to maximize his payoff $\Phi^A(a, b)$
- Player B wishes to maximize his payoff $\Phi^B(a, b)$



- Player A chooses the value of $a \in A$
- Player B chooses the value of $b \in B$

A cooperative solution



- maximize the sum of payoffs $\Phi^A(a, b) + \Phi^B(a, b)$
- split the total payoff fairly among the two players (how ???)

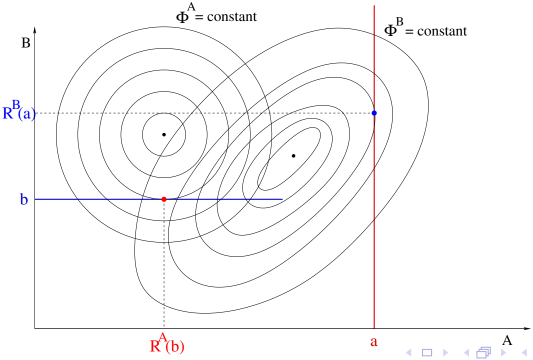
The best reply map

If Player A adopts the strategy a, the set of best replies for Player B is

$$R^B(a) \;=\; \left\{ b\,;\;\; \Phi^B(a,b) = \max_{s\in B} \Phi^B(a,s) \right\}$$

If Player B adopts the strategy b, the set of best replies for Player A is

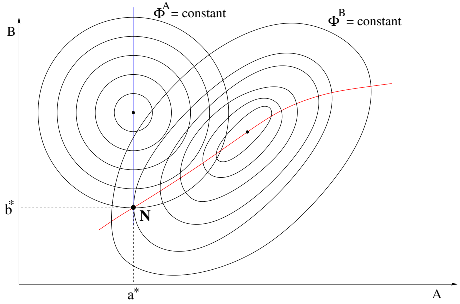
$$R^A(b) \;=\; \left\{ a\,;\;\; \Phi^A(a,b) = \max_{s\in A} \Phi^A(s,b) \right\}$$



Nash equilibrium solutions

A couple of strategies (a^*, b^*) is a **Nash equilibrium** if

$$a^* \in R^A(b^*) \qquad \text{and} \qquad b^* \in R^B(a^*)$$



Antoin Augustin Cournot (1838)
John Nash (1950)

Existence of Nash equilibria

Theorem. *Assume*

- *Sets of available strategies for the two players:* $A, B \subset \mathbb{R}^n$ *are compact and convex*
- *Payoff functions:* $\Phi^A, \Phi^B : A \times B \mapsto \mathbb{R}$ *are continuous*
- *For each* $a \in A$*, the set of best replies* $R^B(a) \subset B$ *is* **convex**
- *For each* $b \in B$*, the set of best replies* $R^A(b) \subset A$ *is* **convex**

Then the game admits at least one Nash equilibrium.

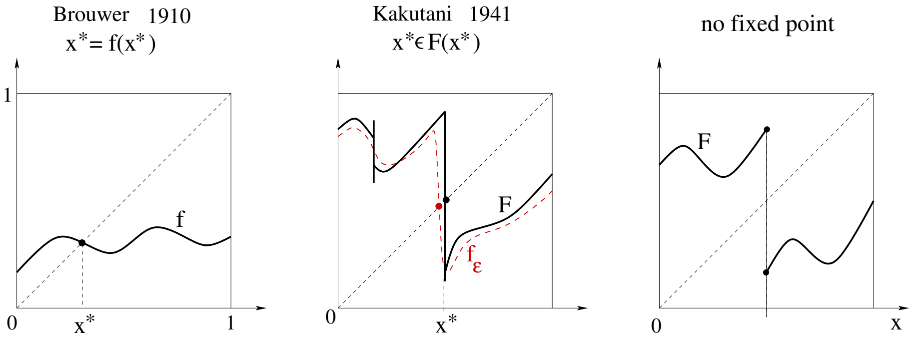
Proof. If the best reply maps are single valued, the map

$$(a, b) \ \mapsto \ (R^A(b),\ R^B(a))$$

is a continuous map from the compact convex set $A \times B$ into itself.
By Brouwer's fixed theorem, it has a fixed point (a^*, b^*).

If R^A, R^B are convex-valued, by Kakutani's fixed point theorem there exists $(a^*, b^*) \in (R^A(b^*),\ R^B(a^*))$

One-dimensional version of Brouwer's and Kakutani's theorems

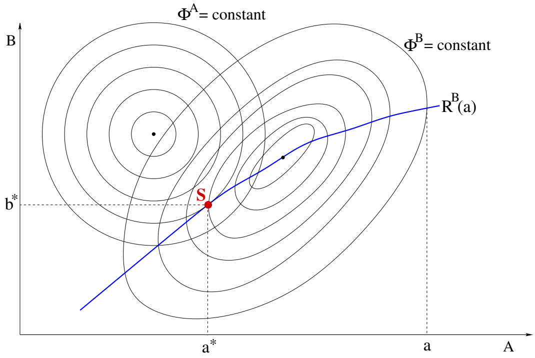


Luitzen Egbertus Jan Brouwer (1910)
Shizuo Kakutani (1941)
Arrigo Cellina (1969)

Stackelberg equilibrium

- Player A (the leader) announces his strategy $a \in A$ in advance
- Player B (the follower) adopts his best reply: $b \in R^B(a) \subseteq B$

What is the best strategy for the leader? $\max_{a\in A} \Phi^A(a, R^B(a))$



A couple of strategies (a^*, b^*) is a **Stackelberg equilibrium** if $b^* \in R^B(a^*)$ and

$$\Phi^A(a^*, b^*) \geq \Phi^A(a, b) \qquad \text{for all } \; a \in A, \quad b \in R^B(a)$$

Game theoretical models in Economics and Finance

- Sellers (choosing prices charged) vs. buyers (choosing quantities bought)
- Companies competing for market share (choosing production level, prices, amount spent on research & development or advertising)
- Auctions, bidding games
- Economic growth. Leading player: central bank (choosing prime rate) followers: private companies (choosing investment levels)
- Debt management. Lenders (choosing interest rate) vs. borrower (choosing repayment strategy)
- . . .

Differential games in finite time horizon

$$x(t) \in \mathbb{R}^n = \text{state of the system}$$

Dynamics: $\dot{x}(t) = f(x(t), u_1(t), u_2(t)), \qquad x(t_0) = x_0$

$u_1(\cdot),\ u_2(\cdot) =$ controls implemented by the two players

Goal of i-th player:

$$\text{maximize:} \qquad J_i \doteq \psi_i(x(T)) - \int_{t_0}^{T} L_i(x(t),\ u_1(t),\ u_2(t))\,dt$$

$$= \text{[terminal payoff] - [running cost]}$$

Differential games in infinite time horizon

Dynamics: $$\dot{x} = f(x, u_1, u_2), \qquad x(0) = x_0$$

Goal of i-th player:

$$\text{maximize:} \quad J_i \doteq \int_0^{+\infty} e^{-\gamma t}\, \Psi_i\big(x(t),\, u_1(t),\, u_2(t)\big)\, dt$$

(running payoff, exponentially discounted in time)

Example 1: an advertising game

- Two companies, competing for market share

state variable: $x(t) \in [0,1]$ = market share of company 1, at time t

$$\dot{x} = (1-x)\,u_1 - x\,u_2$$

controls: u_1, u_2 = advertising rates

$$\text{payoffs:} \quad J_i = N\,x_i(T)\,p_i - \int_0^T c_i u_i(t)\,dt \qquad i = 1,2$$

N = expected number of items purchased by consumers

p_i = profit made by player i on each sale

c_i = advertising cost

x_i = market share of player i $\qquad (x_1 = x, \quad x_2 = 1 - x)$

Example 2: harvesting of marine resources

$$x(t) \;=\; \text{amount of fish in a lake, at time } t$$

dynamics: $$\dot{x} \;=\; \alpha x(M - x) - xu_1 - xu_2$$

controls: $u_1, u_2 =$ harvesting efforts by the two players

payoffs: $$J_i \;=\; \int_0^{+\infty} e^{-\gamma t} \,(p\, x u_i - c_i u_i)\, dt$$

$p =$ selling price of fish

$c_i =$ harvesting cost

Example 3: a producer vs. consumer game

State variables: $\begin{cases} p & = & \text{price} \\ q & = & \text{size of the inventory} \end{cases}$

Controls: $\begin{cases} a(t) & = & \text{production rate} \\ b(t) & = & \text{consumption rate} \end{cases}$

The system evolves in time according to $\begin{cases} \dot{p} & = & p \ln(q_0/q) \\ \dot{q} & = & a - b \end{cases}$

Here q_0 is an "appropriate" inventory level

$$\text{Payoffs:} \qquad \begin{cases} J^{producer} & \doteq & \displaystyle\int_0^{+\infty} e^{-\gamma t} \Big[p(t) \cdot b(t) - c\big(a(t)\big) \Big]\, dt \\[2ex] J^{consumer} & \doteq & \displaystyle\int_0^{+\infty} e^{-\gamma t} \Big[\phi(b(t)) - p(t) b(t) \Big]\, dt \end{cases}$$

$c(a) =$ production cost, $\quad \phi(b) =$ utility to the consumer

Solution concepts

- No outcome can be optimal simultaneously for all players

Different outcomes may arise, depending on

- information available to the players
- their ability and willingness to cooperate

Seek: feedback strategies: $u_1^*(x)$, $u_2^*(x)$ with the following properties

- Given the strategy $u_2 = u_2^*(x)$ adopted by the second player, for every initial data $x(0) = y$, the assignment $u_1 = u_1^*(x)$ provides a solution to the **optimal control problem for the first player:**

$$\max_{u_1(\cdot)} \int_0^\infty e^{-\gamma t}\, \Psi_1(x, u_1, u_2^*(x))\, dt$$

subject to

$$\dot{x} = f(x, u_1, u_2^*(x)), \qquad x(0) = y$$

- Similarly, given the strategy $u_1 = u_1^*(x)$ adopted by the first player, the feedback control $u_2 = u_2^*(x)$ provides a solution to the optimal control problem for the second player.

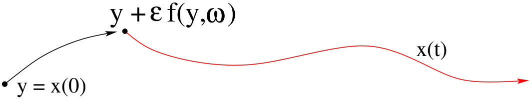
Solving an optimal control problem by PDE methods

$$V(y) \; = \; \inf_{u(\cdot)} \int_0^{+\infty} e^{-\gamma t}\, L(x(t), u(t))\, dt$$

subject to:

$$\dot{x}(t) \; = \; f(x(t), u(t)) \qquad\qquad x(0) = y \qquad\qquad u(t) \in U$$

$V(y) =$ minimum cost, if the system is initially at y



If we use the constant control $u(t) = \omega$ for $t \in [0, \varepsilon]$
then we play optimally for $t \in [\varepsilon, \infty[$, the total cost is

$$\begin{aligned} J^{\varepsilon,\omega} &= \left(\int_0^{\varepsilon} + \int_{\varepsilon}^{+\infty}\right) e^{-\gamma t} L(x(t), u(t))\, dt \\ &= \varepsilon\, L(y,\omega) + e^{-\gamma\varepsilon}\, V\Big(y + \varepsilon f(y,\omega)\Big) + o(\varepsilon) \\ &= \varepsilon\, L(y,\omega) + (1-\gamma\varepsilon)V(y) + \nabla V(y)\cdot \varepsilon f(y,\omega) + o(\varepsilon) \\ &\geq V(y) \end{aligned}$$

Minimize w.r.t. ω:

$$V(y) = V(y) - \gamma\varepsilon V(y) + \varepsilon \cdot \min_{\omega\in U} \Big\{ L(y,\omega) + \nabla V(y)\cdot f(y,\omega) \Big\} + o(\varepsilon)$$

The Hamilton-Jacobi PDE for the value function

$$V(y) \;=\; V(y) - \gamma\varepsilon V(y) + \varepsilon \cdot \min_{\omega\in U} \Big\{ L(y,\omega) + \nabla V(y)\cdot f(y,\omega) \Big\} + o(\varepsilon)$$

Letting $\varepsilon \to 0$ we obtain

$$\gamma V(y) \;=\; \min_{\omega\in U} \Big\{ L(y,\omega) + \nabla V(y)\cdot f(y,\omega) \Big\} \;\doteq\; H(y, \nabla V(y))$$

If $V(\cdot)$ is known, the optimal feedback control can be recovered by

$$u^*(y) \;=\; \operatorname*{argmin}_{\omega\in U} \Big\{ L(y,\omega) + \nabla V(y)\cdot f(y,\omega) \Big\}$$

An example

minimize: $$\int_0^\infty e^{-\gamma t} \left(\phi(x(t)) + \frac{u^2(t)}{2} \right) dt$$

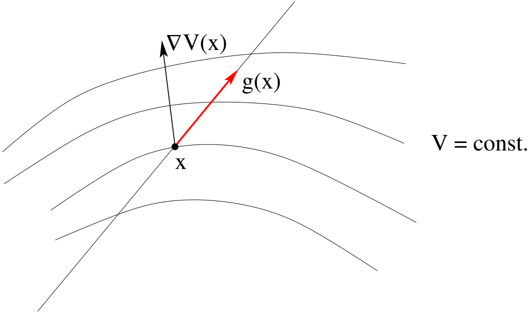
subject to: $\dot{x} \;=\; f(x) + g(x)\,u, \qquad x(0) = y, \qquad u(t) \in \mathbb{R}$

The value function $V(y) =$ minimum cost starting at y satisfies the PDE

$$\gamma V(x) \;=\; \phi(x) + \nabla V(x) \cdot f(x) - \frac{1}{2} \left(\nabla V(x) \cdot g(x) \right)^2$$

Finding the optimal feedback control

$$\dot{x} \;=\; f(x) + g(x)\,u$$



If $V(\cdot)$ is known, the optimal control can be recovered by

$$u^*(x) \;=\; \operatorname*{argmin}_{u\in\mathbb{R}} \left\{ \nabla V(x)\cdot g(x)\,u + \frac{u^2}{2} \right\} \;=\; -\nabla V(x)\cdot g(x)$$

Solving a differential game by PDE methods

$$\text{Dynamics:} \quad \dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$$

Player i seeks to minimize: $\quad J_i = \int_0^\infty e^{-\gamma t}\left(\phi_i(x(t)) + \frac{u_i^2(t)}{2}\right) dt$

The value functions V_1, V_2 for the two players satisfy the system of H-J equations

$$\begin{cases} \gamma V_1 = (f \cdot \nabla V_1) - \frac{1}{2}(g_1 \cdot \nabla V_1)^2 - (g_2 \cdot \nabla V_1)(g_2 \cdot \nabla V_2) + \phi_1 \\ \gamma V_2 = (f \cdot \nabla V_2) - \frac{1}{2}(g_2 \cdot \nabla V_2)^2 - (g_1 \cdot \nabla V_1)(g_1 \cdot \nabla V_2) + \phi_2 \end{cases}$$

Optimal feedback controls: $\quad u_i^*(x) = -\nabla V_i(x) \cdot g_i(x) \qquad i = 1, 2$

highly nonlinear, implicit !

Linear - Quadratic games

Assume that the dynamics is linear:

$$\dot{x} = (Ax + \mathbf{b}_0) + \mathbf{b}_1 u_1 + \mathbf{b}_2 u_2 \,, \qquad x(0) = y$$

and the cost functions are quadratic:

$$J_i = \int_0^{+\infty} e^{-\gamma t} \left(\mathbf{a}_i \cdot x + x^T P_i x + \frac{u_i^2}{2} \right) dt$$

Then the system of PDEs has a special solution of the form

$$V_i(x) = \alpha_i + \beta_i \cdot x + x^T \Gamma_i x \qquad i = 1, 2 \qquad (*)$$

$$\text{optimal controls:} \quad u_i^*(x) = -(\beta_i + 2x^T \Gamma_i) \cdot \mathbf{b}_i$$

To find this solution, it suffices to
determine the coefficients $\alpha_i, \beta_i, \Gamma_i$ by solving a system of algebraic equations

Validity of linear-quadratic approximations ?

Assume the dynamics is almost linear

$$\dot{x} \;=\; f_0(x)+g_1(x)u_1+g_2(x)u_2 \;\approx\; (Ax+\mathbf{b}_0)+\mathbf{b}_1 u_1+\mathbf{b}_2 u_2\,, \qquad\qquad x(0)=y$$

and the cost functions are almost quadratic

$$J_i \;=\; \int_0^{+\infty} e^{-\gamma t}\Big(\phi_i(x)+\frac{u_i^2}{2}\Big)\,dt \;\approx\; \int_0^{+\infty} e^{-\gamma t}\Big(\mathbf{a}_i\cdot x+x^T P_i x+\frac{u_i^2}{2}\Big)\,dt$$

Is it true that the nonlinear game has a feedback solution close to the linear-quadratic game?

References

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