ENGS 199.12

GEOPHYSICAL FLUID DYNAMICS

Chapter 8 – Ekman Layer

Role of friction in GFD:
A horizontal boundary layer with a twist.

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Skip Section 8.1, except to note that in the presence of turbulence, the molecular viscosity of the fluid ν should be replaced by a much larger eddy viscosity ν_E.

On the importance of friction in a rotating fluid

When we treated the horizontal momentum equations, we noted that

1. Horizontal friction is negligible;
2. The relative importance of vertical friction compared to the Coriolis acceleration is measured by the Ekman number

$$Ek = \frac{\nu_E}{\Omega H^2}$$

in which ν_E is the turbulence-enhanced kinematic viscosity of the fluid (in m^2/s), Ω is the angular rotation rate (in 1/s), and H the height/depth scale (in m).

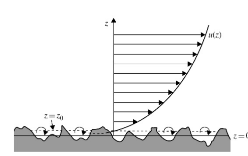


FIGURE 8.2 Velocity profile in the vicinity of a rough wall. The roughness height z_0 is smaller than the averaged height of the surface asperities. So, the velocity u falls to zero somewhere within the asperities, where local flow degenerates into small vortices between the peaks, and the negative values predicted by the logarithmic profile are not physically realized.

Even though the turbulence-enhanced viscosity ν_E (~10^{-2} m^2/s) is much larger than its molecular counterpart ν (~10^{-6} m^2/s), the Ekman number remains generally quite small. This implies that friction in GFD is most often negligible.

If friction is generally negligible in GFD flows, it nonetheless becomes important in thin boundary layers near bottom boundaries where the flow "scrubs against the floor."

How thin is this frictional boundary layer?

If we denote its thickness as d, then by definition d has the value that renders the Ekman number on the order of unity so as to make friction comparable to rotation:

$$Ek = \frac{\nu_E}{\Omega d^2} \sim 1 \quad \rightarrow \quad d^2 \sim \frac{\nu_E}{\Omega}, \quad d \sim \sqrt{\frac{\nu_E}{\Omega}}$$

For typical values $\nu_E \sim 10^{-2}$ m^2/s and $\Omega \sim 10^{-4}$/s, $d \sim 10$ m.

Note: As $\Omega \rightarrow 0$, d becomes very large and reaches infinity in the absence of rotation.
This is why basic (non-rotating) fluid mechanics does not possess a fixed boundary layer thickness.

Bottom Ekman Layer

Consider a horizontal geophysical flow that is
- Density homogeneous,
- Uniform in the horizontal,
- Above a horizontal boundary, and
- In the northern hemisphere.

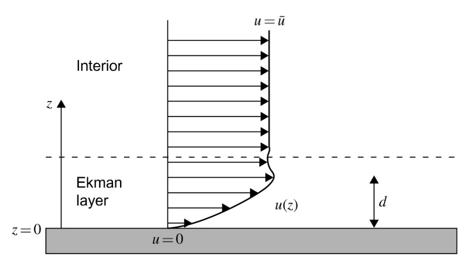


FIGURE 8.3 Frictional influence of a flat bottom on a uniform flow in a rotating framework.

By virtue of its horizontal uniformity, the flow has no horizontal gradients, thus:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial \upsilon}{\partial x} = \frac{\partial \upsilon}{\partial y} = 0$$

And, by virtue of its density homogeneity, its hydrostatic balance reduces to:

$$\frac{\partial p}{\partial z} = 0 \quad \rightarrow \quad p = \overline{p}(x, y)$$

The continuity equation turns out to yield:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad\rightarrow\quad \frac{\partial w}{\partial z}=0 \quad\rightarrow\quad w=0$$ because of impermeable flat bottom

The horizontal momentum equations are:

$$-f\upsilon=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu_E\frac{d^2u}{dz^2}$$

$$+fu=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}+\nu_E\frac{d^2\upsilon}{dz^2}$$

accompanied by the following boundary conditions:

Bottom $z=0: \quad u=\upsilon=0$

Interior $z\rightarrow\infty: \quad u\rightarrow\bar{u},\ \upsilon\rightarrow 0$

x-axis chosen in direction of the flow in the interior

Far above the boundary layer, in the interior ($z\rightarrow\infty$) where z-derivatives vanish, we have:

$$0=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$+f\bar{u}=-\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

The equations for the flow inside the boundary layer are:

$$-f\upsilon=\nu_E\frac{d^2u}{dz^2}$$

$$+f(u-\bar{u})=\nu_E\frac{d^2\upsilon}{dz^2}$$

The solution to this set of linear equations is, after some tedious algebra:

$$u=\bar{u}+e^{-z/d}\left(A\cos\frac{z}{d}+B\sin\frac{z}{d}\right)$$

$$\upsilon=\quad e^{-z/d}\left(B\cos\frac{z}{d}-A\sin\frac{z}{d}\right)$$

in which d is now precisely defined as $d=\sqrt{\frac{2\nu_E}{f}}\quad (f>0)$

called *Ekman depth*

Application of the bottom boundary conditions yields:

$$u=\bar{u}\left(1-e^{-z/d}\cos\frac{z}{d}\right)$$

$$\upsilon=\bar{u}\,e^{-z/d}\sin\frac{z}{d}$$

Transverse transport:

$$V=\int_0^\infty \upsilon\,dz=\bar{u}\,\frac{d}{2}$$

to the left of the main current above

This curve is called the *Ekman spiral*.

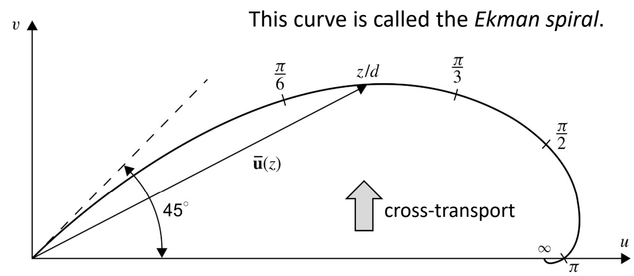


FIGURE 8.4 The velocity spiral in the bottom Ekman layer. The figure is drawn for the northern hemisphere ($f>0$), and the deflection is to the left of the current above the layer. The reverse holds for the southern hemisphere.

Generalization to Non-uniform Currents

Let us now suppose that the flow in the interior (= sufficiently away from the bottom, $z >> d$) is two-dimensional, with variations in both directions:

$$-f\bar{\upsilon} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial x} \qquad +f\bar{u} = -\frac{1}{\rho_0}\frac{\partial \bar{p}}{\partial y}$$

with $\bar{p} = \bar{p}(x, y)$ $\Longrightarrow$ $\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{\upsilon}}{\partial y} = 0$ with the assumption of f constant

Within the boundary layer, the equations now are (with the variables x and y acting as mere parameters):

$$-f(\upsilon - \bar{\upsilon}) = \nu_E \frac{d^2 u}{dz^2}$$

$$+f(u - \bar{u}) = \nu_E \frac{d^2 \upsilon}{dz^2}$$

The solution is a slightly augmented version of the earlier solution:

$$u = \bar{u}\left(1 - e^{-z/d}\cos\frac{z}{d}\right) - \bar{\upsilon}\, e^{-z/d}\sin\frac{z}{d}$$

$$\upsilon = \bar{u}\, e^{-z/d}\sin\frac{z}{d} + \bar{\upsilon}\left(1 - e^{-z/d}\cos\frac{z}{d}\right)$$

Ekman pumping

The volumetric transport due to the twisting of the boundary layer now has two components

$$U = \int_0^\infty (u - \bar{u})\, dz = -\frac{d}{2}(\bar{u} + \bar{\upsilon})$$

$$V = \int_0^\infty (\upsilon - \bar{\upsilon})\, dz = +\frac{d}{2}(\bar{u} - \bar{\upsilon})$$

The divergence of this volumetric flow is non-zero:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{d}{2}\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{\upsilon}}{\partial x}\right) + \frac{d}{2}\left(\frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{\upsilon}}{\partial y}\right)$$

$$= -\frac{d}{2}\underbrace{\left(\frac{\partial \bar{\upsilon}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right)}_{\text{vorticity of the interior flow}} = -\frac{d}{2\rho_0 f}\nabla^2 \bar{p}$$

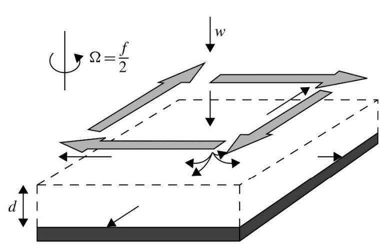


FIGURE 8.5 Divergence in the bottom Ekman layer and compensating downwelling in the interior. Such a situation arises in the presence of an anticyclonic gyre in the interior, as depicted by the large horizontal arrows. Similarly, interior cyclonic motion causes convergence in the Ekman layer and upwelling in the interior.

There is thus a non-zero vertical velocity in the interior:

$$\bar{w} = -\int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y}\right) dz = \frac{d}{2}\left(\frac{\partial \bar{\upsilon}}{\partial x} - \frac{\partial \bar{u}}{\partial y}\right) = \frac{d}{2\rho_0 f}\nabla^2 \bar{p}$$

This is called *Ekman pumping*.

Ekman layer over a sloping bottom

See Section 8.5 of textbook on pages 250-251, if interested.

Bottom line is

$$\overline{w} = \left(u \frac{\partial b}{\partial x} + \upsilon \frac{\partial b}{\partial y} \right) + \frac{d}{2} \left(\frac{\partial \overline{\upsilon}}{\partial x} - \frac{\partial \overline{u}}{\partial y} \right)$$

No surprise. Vertical velocity in interior = that necessary to follow the bottom + Ekman pumping

Surface Ekman layer

Now, we flip the situation upside down
and change the boundary condition
from zero velocity to a prescribed surface stress.

At the surface ($z = 0$), we impose:

$$\rho_0 \nu_E \frac{\partial u}{\partial z} = \tau^x$$

$$\rho_0 \nu_E \frac{\partial \upsilon}{\partial z} = \tau^y$$

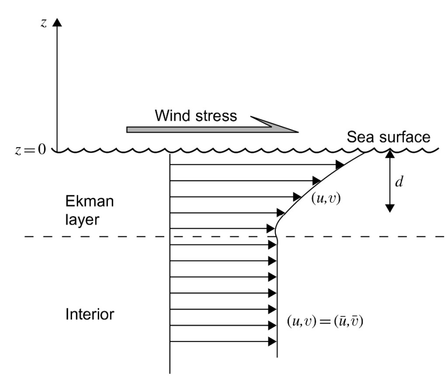


FIGURE 8.6 The surface Ekman layer generated by a wind stress on the ocean.

Toward the interior ($z \to -\infty$), the velocity becomes the interior velocity:

$$u \to \overline{u}$$

$$\upsilon \to \overline{\upsilon}$$

Solution is:

$$u = \bar{u} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) - \tau^y \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]$$

$$\upsilon = \bar{\upsilon} + \frac{\sqrt{2}}{\rho_0 f d} e^{z/d} \left[\tau^x \sin\left(\frac{z}{d} - \frac{\pi}{4}\right) + \tau^y \cos\left(\frac{z}{d} - \frac{\pi}{4}\right) \right]$$

Properties of this solution:

- The surface speed is at 45º from the wind,
- The vertically integrated velocity is at 90º from the wind,

to the right in Northern Hemisphere,
to the left in the Southern Hemisphere.

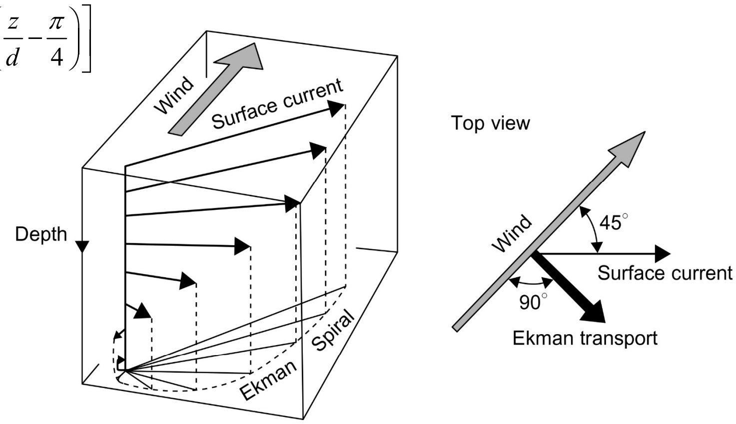


FIGURE 8.7 Structure of the surface Ekman layer. The figure is drawn for the northern hemisphere ($f > 0$), and the deflection is to the right of the surface stress. The reverse holds for the southern hemisphere.

Ekman pumping below surface

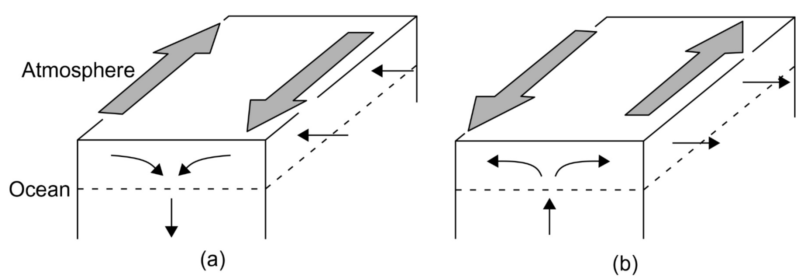


FIGURE 8.8 Ekman pumping in an ocean subject to sheared winds (case of northern hemisphere).

As for the bottom Ekman layer below a geostrophic flow that has vorticity,
a shear in the wind stress (wind stress curl $\neq 0$) causes convergence or divergence of the Ekman flow,
and there is a vertical velocity emanating from the Ekman layer penetrating into the interior below:

$$\bar{w} = +\int_{-\infty}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial \upsilon}{\partial y} \right) dz = \frac{1}{\rho_0} \left[\frac{\partial}{\partial x}\left(\frac{\tau^y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau^x}{f}\right) \right]$$

Ekman spiral observed below the ice

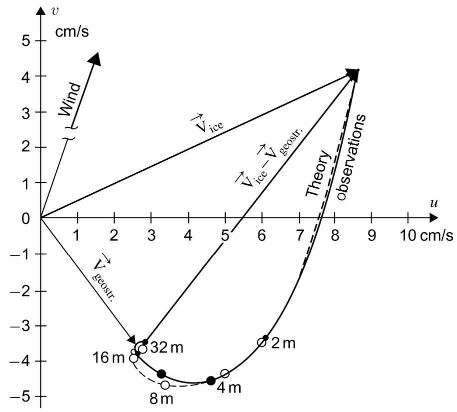


FIGURE 8.9 Comparison between observed currents below a drifting ice floe at 84.3°N and theoretical predictions based on an eddy viscosity $\nu_E = 2.4 \times 10^{-3}$ m^2/s. (*Reprinted from Deep-Sea Research, 13, Kenneth Hunkins, Ekman drift currents in the Arctic Ocean, p. 614, ©1966, with kind permission from Pergamon Press Ltd, Headington Hill Hall, Oxford 0X3 0BW, UK*)

Observations show that
- The ice drifts at a velocity to the right of the wind;
- The Ekman spiral ties nicely the water flow from under the ice down to the geostrophic interior;
- The transverse Ekman transport is the right of the velocity of the ice relative to the ocean interior velocity;
- The theory matches well with the observations (after fitting the value for the eddy viscosity).

Ekman spiral in turbulent flows

The Ekman spiral theory assumes that turbulence behaves as enhanced molecular diffusion.

This is the exception rather than the rule, and the theory needs to be revised if the turbulence behaves differently.

Generally, under shear-induced turbulence, as it is in the water with a free surface exposed to the wind, one notes the following:

- The angle of the surface flow with respect to the surface wind stress is significantly less than 45°, and in the range between 5° and 20°.
- The depth of the Ekman spiral is best given by

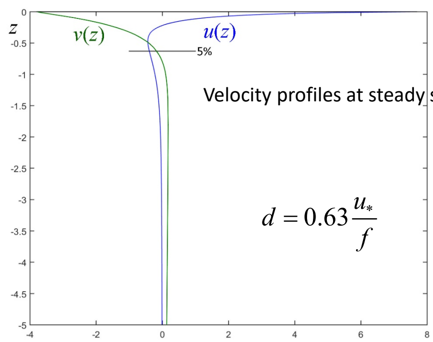
$$d \simeq 0.40 \frac{u_*}{f} \qquad \text{with} \quad u_* = \sqrt{\frac{\tau_{wind}}{\rho_0}} \quad \text{(in m/s)}$$

Benoit's recent model for shear turbulence:

$$\frac{\partial u}{\partial t} - f\upsilon = \frac{u_*}{\pi^2}\int_{-\infty}^{0}\left[u(z',t)-u(z,t)\right]\left[\frac{1}{(z'-z)^2}+\frac{1}{(z'+z)^2}\right]dz'$$

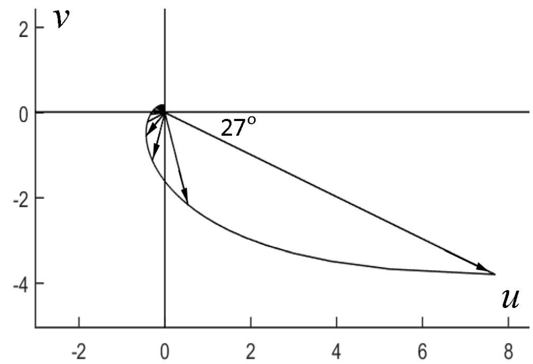
$$\frac{\partial \upsilon}{\partial t} + fu = \frac{u_*}{\pi^2}\int_{-\infty}^{0}\left[\upsilon(z',t)-\upsilon(z,t)\right]\left[\frac{1}{(z'-z)^2}+\frac{1}{(z'+z)^2}\right]dz'$$

integrated over time until steady state is reached.



Velocity profiles at steady state.

$$d = 0.63\frac{u_*}{f}$$



Above figure:
Ekman velocity spiral at steady state.