On a fourth order Stekloff eigenvalue problem*

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Abstract

We study a biharmonic Stekloff eigenvalue problem. We prove some new results and we collect and refine a number of known results. Moreover, we highlight the main open problems still to be solved.

1 Introduction and results

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^2$, let $d \in \mathbb{R}$ and consider the boundary eigenvalue problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Delta u = d u_\nu & \text{on } \partial\Omega \ , \end{cases} \tag{1.1}$$

where u_ν denotes the outer normal derivative of u on $\partial\Omega$. We are interested in studying the eigenvalues of (1.1), namely those values of d for which the problem admits nontrivial solutions, the corresponding eigenfunctions. The purpose of the present paper is to collect a number of known (and old) results, to prove some new results and to suggest some open questions.

Elliptic problems with eigenvalues in the boundary conditions are usually called Stekloff problems from their first appearance in [20]. By solution of (1.1) we mean a function $u \in H^2 \cap H^1_0(\Omega)$ such that

$$\int_\Omega \Delta u \Delta v \, dx = d \int_{\partial\Omega} u_\nu v_\nu \, dS \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega). \tag{1.2}$$

By taking $v = u$ in (1.2), it is clear that all the eigenvalues of (1.1) are strictly positive. Let $\mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega)$ and

$$d_1 = d_1(\Omega) := \min_{u \in \mathcal{H}(\Omega)} \frac{\displaystyle\int_\Omega |\Delta u|^2}{\displaystyle\int_{\partial\Omega} u_\nu^2} \ . \tag{1.3}$$

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The number d_1 represents the least positive eigenvalue and $d_1^{-1/2}$ is the norm of the compact linear operator $H^2 \cap H_0^1(\Omega) \to L^2(\partial\Omega)$, defined by $u \mapsto u_\nu|_{\partial\Omega}$. Moreover, as pointed out by Kuttler [10], d_1 is the sharp constant for a priori estimates for the Laplace equation

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

where $g \in L^2(\partial\Omega)$. Indeed, using Fichera's principle of duality (see [6] and also (1.11) below), for the solution v of (1.4) one has

$$d_1(\Omega) \cdot \|v\|^2_{L^2(\Omega)} \le \|g\|^2_{L^2(\partial\Omega)}$$

and $d_1(\Omega)$ is the largest possible constant for this inequality.

The first eigenvalue d_1 also plays a crucial role in the positivity preserving property for the biharmonic operator Δ^2 under the boundary conditions $u = \Delta u - du_\nu = 0$ on $\partial\Omega$, see [3, 7]. It is shown there that if $d \ge d_1$, then the positivity preserving property fails, whereas it holds when d is in a left neighborhood of d_1 (possibly $d \in (-\infty, d_1)$). We refer to [3, 7] for further details. We also refer to [11] for several inequalities between the eigenvalues of (1.1) and other eigenvalue problems.

The boundary condition in (1.1) has an interesting interpretation in theory of elasticity. Consider the model problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \Delta u - (1-\sigma)\kappa u_\nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain with smooth boundary, $\sigma \in (-1, 1/2)$ is the Poisson ratio and κ is the mean curvature of the boundary. Problem (1.5) describes the deformation u of the linear elastic supported plate Ω under the action of the transversal exterior force $f = f(x)$, $x \in \Omega$. The Poisson ratio σ of an elastic material is the negative transverse strain divided by the axial strain in the direction of the stretching force. In other words, this parameter measures the transverse expansion (resp. contraction) if $\sigma \ge 0$ (resp. $\sigma < 0$) when the material is compressed by an external force. We refer to [13, 21] for more details. The restriction on the Poisson ratio is due to thermodynamic considerations of strain energy in the theory of elasticity. As shown in [13], there exist materials for which the Poisson ratio is negative and the limit case $\sigma = -1$ corresponds to materials with an infinite flexural rigidity, see [21, p.456]. This limit value for σ is strictly related to the eigenvalue problem (1.1). Indeed, if we assume that Ω is the unit disk, then $d = (1-\sigma)\kappa = 1-\sigma$. Moreover, by Theorem 3 below, the first eigenvalue d of (1.1) is equal to 2, which implies $\sigma = -1$. Hence, the limit value $\sigma = -1$, which is not allowed from a physical point of view, also changes the structure of the stationary problem (1.5): when Ω is the unit disk and $d = (1-\sigma)\kappa = 2$, (1.5) either admits an infinite number of solutions or it admits no solutions at all, depending on f.

We are firstly interested in the description of the spectrum of (1.1). Throughout this paper we endow the Hilbert space $H^2 \cap H_0^1(\Omega)$ with the scalar product

$$(u, v) = \int_\Omega \Delta u \Delta v \, dx \; . \tag{1.6}$$

Consider the subspace

$$Z = \left\{ v \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0, \; u = 0 \text{ on } \partial\Omega \right\} \tag{1.7}$$

and denote by V the completion of Z with respect to the scalar product in (1.6). Then, we prove

Theorem 1. *Assume that* $\Omega \subset \mathbb{R}^n$ *(*$n \geq 2$*) is an open bounded domain with* C^2 *boundary. Then, problem (1.1) admits infinitely many (countable) eigenvalues. The only eigenfunction of one sign is the one corresponding to the first eigenvalue. The set of eigenfunctions forms a complete orthonormal system in* V*.*

The vector space V also has a different interesting characterization:

Theorem 2. *Assume that* $\Omega \subset \mathbb{R}^n$ *(*$n \geq 2$*) is an open bounded domain with* C^2 *boundary. Then, the space* $H^2 \cap H^1_0(\Omega)$ *admits the following orthogonal decomposition with respect to the scalar product (1.6)*

$$H^2 \cap H^1_0(\Omega) = V \oplus H^2_0(\Omega).$$

Moreover, if $v \in H^2 \cap H^1_0(\Omega)$ *and if* $v = v_1 + v_2$ *is the corresponding orthogonal decomposition, then* $v_1 \in V$ *and* $v_2 \in H^2_0(\Omega)$ *are weak solutions of*

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega \end{cases} \qquad \text{and} \qquad \begin{cases} \Delta^2 v_2 = \Delta^2 v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \\ (v_2)_\nu = 0 & \text{on } \partial\Omega\ . \end{cases} \tag{1.8}$$

When $\Omega = B$ (the unit ball) we may determine explicitly all the eigenvalues of (1.1). To this end, consider the spaces of harmonic homogeneous polynomials:

$$\mathcal{D}_k := \{P \in C^\infty(\mathbb{R}^n);\ \Delta P = 0 \text{ in } \mathbb{R}^n,\ P \text{ is an homogeneous polynomial of degree } k-1\}.$$

Also, denote by μ_k the dimension of $\mathcal{D}_k$. In particular, we have

$$\mathcal{D}_1 = \text{span}\{1\}\ , \qquad \mu_1 = 1\ ,$$

$$\mathcal{D}_2 = \text{span}\{x_i;\ (i = 1, ..., n)\}\ , \qquad \mu_2 = n\ ,$$

$$\mathcal{D}_3 = \text{span}\{x_i x_j;\ x_1^2 - x_h^2;\ (i, j = 1, ..., n,\ i \neq j,\ h = 2, ..., n)\}\ , \qquad \mu_3 = \frac{n^2 + n - 2}{2}\ .$$

Then, we prove

Theorem 3. *If* $n \geq 2$ *and* $\Omega = B$*, then for all* $k = 1, 2, 3, ...$*:*
(i) the eigenvalues of (1.1) are $d_k = n + 2(k-1)$*;*
(ii) the multiplicity of d_k *equals* μ_k*;*
(iii) for all $\psi_k \in \mathcal{D}_k$*, the function* $\varphi_k(x) := (1 - |x|^2)\psi_k(x)$ *is an eigenfunction corresponding to* d_k*.*

Remark 1. Theorems 1 and 3 become false if $n = 1$. It is shown in [3] that the one dimensional problem

$$u^{iv} = 0 \quad \text{in } (-1, 1)\ , \qquad u(\pm 1) = u''(-1) + du'(-1) = u''(1) - du'(1) = 0\ , \tag{1.9}$$

admits only *two* eigenvalues, $d_1 = 1$ and $d_2 = 3$, each one of multiplicity 1. The reason of this striking difference is that the "boundary space" of (1.9) has precisely dimension 2, one for each endpoint of the interval $(-1, 1)$. This result is consistent with Theorem 3 since $\mu_1 = \mu_2 = 1$ and $\mu_3 = 0$ whenever $n = 1$.

By combining Theorems 1 and 3 we obtain

Corollary 1. *Assume that $n \geq 2$ and that $\Omega = B$. Assume moreover that for all $k \in \mathbb{N}$ the set $\{\psi_k^\ell : \ell = 1, ..., \mu_k\}$ is a basis of $\mathcal{D}_k$ chosen in such a way that the corresponding functions φ_k^ℓ are orthonormal with respect to the scalar product (1.6). Then, for any $u \in V$ there exists a sequence $\{\alpha_k^\ell\} \in \ell^2$ ($k \in \mathbb{N}$; $\ell = 1, ..., \mu_k$) such that*

$$u(x) = (1 - |x|^2) \sum_{k=1}^{\infty} \sum_{\ell=1}^{\mu_k} \alpha_k^\ell \psi_k^\ell(x) \qquad \textit{for a.e. } x \in B \ .$$

We now restrict our attention to the *first* eigenvalue $d_1(\Omega)$. To this end, we recall a consequence of Fichera's principle of duality [6]. Assume that $\partial\Omega \in C^2$ and let

$$C^2_H(\Omega) := \{v \in C^2(\overline{\Omega});\ \Delta v = 0 \text{ in } \Omega\} \ .$$

We consider the norm defined by $\|v\|_H := \|v\|_{L^2(\partial\Omega)}$ for all $v \in C^2_H(\overline{\Omega})$. Then, we consider

$$\mathbf{H} := \text{ the closure of } C^2_H(\overline{\Omega}) \text{ with respect to the norm } \|\cdot\|_H \ .$$

Finally, we define

$$\delta_1 = \delta_1(\Omega) := \min_{h \in \mathbf{H}\setminus\{0\}} \frac{\displaystyle\int_{\partial\Omega} h^2}{\displaystyle\int_{\Omega} h^2} \ . \tag{1.10}$$

The minimum in (1.10) is achieved. To see this, combine the continuous embedding (for weakly harmonic functions) $H^{-1/2}(\partial\Omega) \subset L^2(\Omega)$ (see Théorème 6.6 in Ch. 2 in [14]) with the compact embedding $L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$.

Fichera's principle [6] states that

$$\delta_1(\Omega) = d_1(\Omega) \qquad \text{for all } \Omega \text{ such that } \partial\Omega \in C^2 \ . \tag{1.11}$$

For the reader's convenience we quote a new proof of this statement in Section 5.

Problem 1. Does equality (1.11) hold also if $\partial\Omega \notin C^2$? This question is strictly related with the following: are the maps $\Omega \mapsto d_1(\Omega)$ and $\Omega \mapsto \delta_1(\Omega)$ continuous with respect to Hausdorff convergence of domains?

In view of the important applications explained in the introduction, one is interested in finding both lower and upper bounds for $d_1(\Omega)$. First, we extend to any space dimension $n \geq 2$ a lower bound for $d_1(\Omega)$ obtained for planar domains ($n = 2$) by Payne [15], see also Kuttler [9] for a different proof. This estimate is useful for *convex* domains since it involves the mean curvature of the boundary as stated in the following

Theorem 4. *Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) a bounded convex open domain with C^2 boundary. For all $x \in \partial\Omega$, let $K(x)$ denote the mean curvature at x and let*

$$\underline{K} := \min_{x \in \partial\Omega} K(x) \ .$$

Then $d_1(\Omega) \geq n\underline{K}$ and equality holds if and only if Ω is a ball.

By rescaling, it is not difficult to see that the map $\Omega \mapsto d_1(\Omega)$ is homogeneous of degree -1, namely, for any Ω and $k > 0$ we have $d_1(\Omega) = k d_1(k\Omega)$. This suggests that d_1 should somehow be related to the "isoperimetric ratio" $|\partial\Omega|/|\Omega|$. As noticed by Kuttler [9], this is indeed the case: by taking $h \equiv 1$ in (1.10) and using (1.11) one readily gets

$$d_1(\Omega) \le \frac{|\partial\Omega|}{|\Omega|} \qquad \text{for all } \Omega \ . \tag{1.12}$$

In Section 7, we determine *extremal sets* for the isoperimetric inequality (1.12):

Theorem 5. *Let $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ be a bounded connected domain with C^2 boundary. Then*

$$d_1(\Omega) = \frac{|\partial\Omega|}{|\Omega|}$$

if and only if Ω is a ball.

Problem 2. The above mentioned homogeneity also suggests that the map $\Omega \mapsto d_1(\Omega)$ could be monotone decreasing with respect to domain inclusions. Is this true?

We now deal with a particular class of nonsmooth domains. It is clear that (planar) rectangles are slightly easier to handle than general domains. We consider the two families of rectangles

$$\forall a \in \left(0, \frac{\pi}{2}\right] \ , \quad R_a := (0, \pi - a) \times (0, a) \ , \qquad \forall \alpha \in (0, \sqrt{\pi}] \ , \quad Q_\alpha := \left(0, \frac{\pi}{\alpha}\right) \times (0, \alpha). \tag{1.13}$$

We note that the R_a's have the *same perimeter* 2π as the unit disk, whereas the Q_α's have the *same area* π as the unit disk. Moreover, $R_{\pi/2}$ and $Q_{\sqrt{\pi}}$ are squares.

Smith [18, 19] conjectured that for any domain Ω, one has $d_1(\Omega) \ge d_1(\Omega^*)$, where Ω^* is the ball having the same measure as Ω. In particular, for planar domains Ω of measure π (as the unit disk), this would mean that $d_1(\Omega) \ge 2$. This conjecture was disproved by Kuttler [9] which shows that

$$d_1(Q_{\sqrt{\pi}}) < 1.9889... \tag{1.14}$$

In fact, Kuttler's estimate [9, p.3] is given for the square $Q' := (0,1)^2$ for which $d_1(Q') < 3.5254...$, so that (1.14) is obtained by rescaling this inequality. In order to prove (1.14), Kuttler uses directly the characterization (1.3) of $d_1(Q_{\sqrt{\pi}})$ and finds a suitable linear combination of a fourth order polynomial with the stress function of $Q_{\sqrt{\pi}}$ to get an upper bound for $d_1(Q_{\sqrt{\pi}})$. In Section 8, by using Fichera's principle (1.11) and a much simpler trial function, we improve (1.14) with the following

$$d_1(Q_{\sqrt{\pi}}) < 1.96256. \tag{1.15}$$

We now consider rectangles with sizes of different length. In Section 9 we show that if they maintain the same perimeter and become "thin" then d_1 tends to infinity:

Theorem 6. *For all $0 < a \le \frac{\pi}{2}$ let R_a be as in (1.13). Then,*

$$\frac{\pi}{2} \le \liminf_{a\to 0} \Big[a \cdot d_1(R_a)\Big] \le \limsup_{a\to 0} \Big[a \cdot d_1(R_a)\Big] \le \pi d_1(R_{\pi/2}) < (2.2146)\pi \ .$$

Theorem 6 complements some *numerical approximations* of d_1 for rectangles obtained by Kuttler [10] using a posteriori / a priori inequalities, see also [12]. We recall here his results: for our convenience, we scale [10, Table 1] to the case of the rectangles R_a defined in (1.13) and we add the last column, according to Theorem 6.

Table 1. Numerical estimates for $d_1(R_a)$ of rectangles R_a such that $|\partial R_a| = 2\pi$.

smallest side a	$\pi/2$	$5\pi/11$	$4\pi/9$	$3\pi/7$	$2\pi/5$	$\pi/3$	$2\pi/7$	$\to 0$
$d_1(R_a) >$	2.2118	2.2261	2.2331	2.2459	2.2846	2.4493	2.666	$\to \infty$
$d_1(R_a) <$	2.2133	2.2304	2.2359	2.2498	2.2878	2.4542	2.6839	

In view of these results, Kuttler suggests a new and weaker conjecture, which we state for any space dimension n:

Conjecture 1. [9]
Let $B \subset \mathbb{R}^n$ denote the unit ball. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain whose surface measure satisfies $|\partial\Omega| = |\partial B|$. Then, $n = d_1(B) \le d_1(\Omega)$.

In other words, the first eigenvalue d_1 is expected to be minimal on the ball, among all domains having the *same surface measure*. In connection with this conjecture, we also make the following

Remark 2. For the *second order* Stekloff problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u_\nu = \lambda u & \text{on } \partial\Omega \ , \end{cases} \tag{1.16}$$

it is known (see [5, Theorem 3]) that the first (nontrivial) eigenvalue $\lambda_1 = \lambda_1(\Omega)$ satisfies $\lambda_1(\Omega) \le \lambda_1(\Omega^*)$ where

$$\lambda_1 = \lambda_1(\Omega) := \inf_{u \in \mathbb{H}(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2}{\displaystyle\int_{\partial\Omega} u^2} \qquad \text{and} \qquad \mathbb{H}(\Omega) := \left\{ u \in H^1(\Omega) \setminus H^1_0(\Omega);\ \int_{\partial\Omega} u = 0 \right\}.$$

Therefore, (1.15) and the possible validity of Conjecture 1 would show that the fourth order problem (1.1) and the second order problem (1.16) are completely different.

Consider now the rectangles Q_α defined in (1.13) and which have fixed area. By Theorem 6 and by rescaling (it suffices to put $\alpha = \sqrt{a\pi/(\pi - a)}$) we obtain

Corollary 2. *For all* $0 < \alpha \le \sqrt{\pi}$ *let* Q_α *be as in (1.13). Then,*

$$\frac{\pi}{2} \le \liminf_{\alpha \to 0} \Big[\alpha \cdot d_1(Q_\alpha) \Big] \le \limsup_{\alpha \to 0} \Big[\alpha \cdot d_1(Q_\alpha) \Big] \le \pi d_1(R_{\pi/2}) < (2.2146)\pi \ .$$

Therefore, also for thinning rectangles of fixed area, the first eigenvalue d_1 tends to infinity, although at a lower rate. Then, we may also rescale Table 1 above and obtain for the rectangles Q_α in (1.13):

Table 2. Numerical estimates for $d_1(Q_\alpha)$ of rectangles Q_α such that $|Q_\alpha| = \pi$.

smallest side α	$\sqrt{\pi}$	$\sqrt{5\pi/6}$	$\sqrt{4\pi/5}$	$\sqrt{3\pi/4}$	$\sqrt{2\pi/3}$	$\sqrt{\pi/2}$	$\sqrt{2\pi/5}$	$\to 0$
$d_1(Q_\alpha) >$	1.9601	1.9647	1.9667	1.97	1.9838	2.0465	2.1347	$\to \infty$
$d_1(Q_\alpha) <$	1.9615	1.9685	1.9693	1.9734	1.9866	2.0506	2.149	

The numerical estimate in the first column of Table 2 seems to show that (1.15) is almost optimal.

Problem 3. From (1.15) we know that the ball (at least if $n = 2$) *is not* the minimizer for d_1 among domains having the same measure. Does there exist an optimal shape for this minimization problem? At least in the class of *convex* domains, we feel that the answer could be affirmative.

2 Proof of Theorem 1

Let Z be as in (1.7) and define on Z the scalar product given by

$$(u,v)_W := \int_{\partial\Omega} u_\nu v_\nu \, dS \qquad \text{for all } u,v \in Z \tag{2.1}$$

and we denote by W the completion of Z with respect to this scalar product. Then, we prove:

Lemma 1. *The (Hilbert) space V is compactly embedded into the (Hilbert) space W.*

Proof. By definition of d_1 we have

$$\|u\|_W = \|u_\nu\|_{L^2(\partial\Omega)} \le d_1^{-1/2} \|\Delta u\|_{L^2(\Omega)} = d_1^{-1/2} \|u\|_V \qquad \forall u \in Z \tag{2.2}$$

and hence any Cauchy sequence in Z with respect to the norm of V is a Cauchy sequence with respect to the norm of W. Since V is the completion of Z with respect to (1.6), it follows immediately that $V \subset W$. The continuity of this inclusion can be obtained by density from (2.2).

It remains to prove that this embedding is compact. To this purpose, let $u_m \rightharpoonup u$ in V, so that also $u_m \rightharpoonup u$ in $H^2 \cap H^1_0(\Omega)$. Then by trace embedding and compact embedding $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ we obtain immediately $u_m \to u$ in W. □

Denote by $I_1 : V \to W$ the embedding $V \subset W$ and by $I_2 : W \to V'$ the continuous linear operator defined by

$$\langle I_2 u, v \rangle = (u,v)_W \qquad \forall u \in W,\ \forall v \in V.$$

Moreover, let $L : V \to V'$ be the linear operator given by

$$\langle Lu, v \rangle = \int_\Omega \Delta u \Delta v \, dx \qquad \forall u, v \in V.$$

Then L is an isomorphism and in view of Lemma 1, the linear operator $K = L^{-1} I_2 I_1 : V \to V$ is compact. Since for $n \ge 2$, V is an infinite dimensional Hilbert space and K is a compact self-adjoint operator with strictly positive eigenvalues then V admits an orthonormal base of eigenfunctions of K and the set of the eigenvalues of K can be ordered in a strictly decreasing sequence $\{\mu_i\}$ which converges to zero.

Therefore problem (1.2) admits an infinite set of eigenvalues given by $d_i = \frac{1}{\mu_i}$ and the eigenfunctions of (1.2) coincide with the eigenfunctions of K. The demonstration of Theorem 1 will be complete once we prove

Lemma 2. *If d_k is an eigenvalue of (1.1) corresponding to a positive eigenfunction φ_k then $d_k = d_1$.*

Proof. Since $\varphi_k > 0$ in Ω and $\varphi_k = 0$ on $\partial\Omega$, then $(\varphi_k)_\nu \le 0$ on $\partial\Omega$ and in turn $\Delta\varphi_k = d_k(\varphi_k)_\nu \le 0$ on $\partial\Omega$. Therefore by $\Delta^2\varphi_k = 0$ in Ω and the weak comparison principle, we infer $\Delta\varphi_k \le 0$ in Ω. Moreover, since $\varphi_k > 0$ in Ω and $\varphi_k = 0$ on $\partial\Omega$, the Hopf boundary lemma implies that $(\varphi_k)_\nu < 0$ on $\partial\Omega$. Let φ_1 be a positive eigenfunction corresponding to the first eigenvalue d_1 (see Theorem 1 in [3]). Then φ_1 also satisfies $(\varphi_1)_\nu < 0$ on $\partial\Omega$ and hence from

$$d_k \int_{\partial\Omega} (\varphi_k)_\nu(\varphi_1)_\nu \, dS = \int_\Omega \Delta\varphi_k \Delta\varphi_1 dx = d_1 \int_{\partial\Omega} (\varphi_k)_\nu(\varphi_1)_\nu \, dS > 0$$

we obtain $d_k = d_1$. □

3 Proof of Theorem 2

We start by proving that $Z^\perp = H^2_0(\Omega)$. Let $v \in Z$ and $w \in H^2 \cap H^1_0(\Omega)$. After two integrations by parts we obtain

$$\int_\Omega \Delta v \Delta w \, dx = \int_\Omega \Delta^2 v \, w \, dx + \int_{\partial\Omega} (w_\nu \Delta v - w(\Delta v)_\nu) \, dS = \int_{\partial\Omega} w_\nu \Delta v \, dS$$

for all $v \in Z$ and $w \in H^2 \cap H^1_0(\Omega)$. This proves that $w_\nu = 0$ on $\partial\Omega$ if and only if $w \in Z^\perp$ and hence $V^\perp = Z^\perp = H^2_0(\Omega)$.

Let $v \in H^2 \cap H^1_0(\Omega)$ and consider the first Dirichlet problem in (1.8):

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Since $v_\nu \in H^{1/2}(\partial\Omega)$, by Lax-Milgram Theorem and Théorème 8.3 in Ch. 1 in [14], we deduce that (3.1) admits a unique solution $v_1 \in H^2 \cap H^1_0(\Omega)$ such that

$$\|\Delta v_1\|_{L^2(\Omega)} \le C \|v_\nu\|_{H^{1/2}(\partial\Omega)} \tag{3.2}$$

This proves that $v_1 \in V$. Let $v_2 = v - v_1$, then $(v_2)_\nu = 0$ on $\partial\Omega$ and, in turn, $v_2 \in H^2_0(\Omega)$. Moreover, by (3.1) we infer

$$\int_\Omega \Delta v_2 \Delta w \; dx = \int_\Omega \Delta v \Delta w \; dx - \int_\Omega \Delta v_1 \Delta w \; dx = \int_\Omega \Delta v \Delta w \; dx \qquad \forall w \in H^2_0(\Omega) \tag{3.3}$$

which proves that v_2 is a weak solution of the second problem in (1.8).

4 Proof of Theorem 3

We start with the following technical result:

Lemma 3. *Let $k \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with boundary $\partial\Omega$ of class C^{k+1}. Let $u \in C^{k+1}(\overline{\Omega})$ be such that $u \equiv 0$ on $\partial\Omega$. Consider the function $\varphi : \Omega \to \mathbb{R}$ defined by*

$$\varphi(x) := \frac{u(x)}{\text{dist}(x, \partial\Omega)} \; .$$

Then, there exists $\delta > 0$ such that $\varphi \in C^k(\Omega_\delta)$, with $\Omega_\delta = \{x \in \overline{\Omega};\ \text{dist}(x, \partial\Omega) < \delta\}$.

Proof. It is well-known (see e.g. [2]) that there exists $\delta > 0$ such that $x \mapsto \text{dist}(x, \partial\Omega)$ is of class $C^{k+1}(\Omega_\delta)$; this will give the "size" of δ. Therefore, by local C^{k+1}-charts, we may restrict our attention to the case where

$$\Omega = \mathbb{R}^n_+ \; , \qquad \partial\Omega = \{x = (x_1, ..., x_n) \in \mathbb{R}^n;\ x_1 = 0\} \; , \qquad \text{dist}(x, \partial\Omega) = x_1 \; , \qquad \varphi(x) = \frac{u(x)}{x_1} \; ,$$

with $u \in C^{k+1}(\{x_1 \geq 0\})$ and $u(x) = 0$ whenever $x \in \partial\Omega$. We denote $x = (x_1, x')$; then, by the mean value Theorem, we have

$$\varphi(x) = \frac{u(x) - u(0, x')}{x_1} = \frac{1}{x_1} \int_0^1 \frac{\partial u}{\partial x_1}(tx_1, x') x_1 \, dt = \int_0^1 \frac{\partial u}{\partial x_1}(tx_1, x') \, dt \; .$$

Since $\frac{\partial u}{\partial x_1} \in C^k(\{x_1 \geq 0\})$, we conclude that also $\varphi \in C^k(\{x_1 \geq 0\})$. □

Consider now an eigenfunction u of (1.1). Then $u \in C^\infty(\overline{B})$ and by Lemma 3 we can write

$$u(x) = (1 - |x|^2)\varphi(x) \tag{4.1}$$

with $\varphi \in C^\infty(\overline{B})$. We have

$$u_{x_i} = -2x_i\varphi + (1 - |x|^2)\varphi_{x_i},$$

and on ∂B,

$$u_\nu = x \cdot \nabla u = x \cdot (-2x\varphi + (1 - |x|^2)\nabla\varphi) = -2\varphi. \tag{4.2}$$

Moreover,

$$\Delta u = -2n\varphi - 4x \cdot \nabla\varphi + (1 - |x|^2)\Delta\varphi. \tag{4.3}$$

Hence

$$\Delta u = -2n\varphi - 4\varphi_\nu \qquad \text{on } \partial B. \tag{4.4}$$

¿From (4.3) we get for $i = 1, \ldots, n$,

$$(\Delta u)_{x_i} = -(2n + 4)\varphi_{x_i} - 4\sum_{j=1}^n x_j \varphi_{x_j x_i} - 2x_i\Delta\varphi + (1 - |x|^2)\Delta\varphi_{x_i},$$

and therefore

$$(\Delta u)_{x_i x_i} = -2(n + 4)\varphi_{x_i x_i} - 4x \cdot \nabla(\varphi_{x_i x_i}) - 2\Delta\varphi - 4x_i(\Delta\varphi)_{x_i} + (1 - |x|^2)\Delta\varphi_{x_i x_i}.$$

Summing with respect to i and recalling that u is biharmonic in B, we obtain

$$\begin{aligned} 0 = \Delta^2 u &= -2(n + 4)\Delta\varphi - 4x \cdot \nabla\Delta\varphi - 2n\Delta\varphi - 4x \cdot \nabla\Delta\varphi + (1 - |x|^2)\Delta^2\varphi \\ &= (1 - |x|^2)\Delta^2\varphi - 8x \cdot \nabla\Delta\varphi - 4(n + 2)\Delta\varphi. \end{aligned} \tag{4.5}$$

Writing (4.5) as an equation in $w = \Delta\varphi$, we get

$$(1-|x|^2)\Delta w - 8x\cdot\nabla w - 4(n+2)w = 0 \qquad \text{in } B,$$

so that

$$\begin{aligned} 0 &= -(1-|x|^2)^4\Delta w + 8(1-|x|^2)^3 x\cdot\nabla w + 4(n+2)(1-|x|^2)^3 w \\ &= -\text{div}[(1-|x|^2)^4\nabla w] + 4(n+2)(1-|x|^2)^3 w. \end{aligned} \tag{4.6}$$

Multiplying the right hand side of (4.6) by w and integrating by parts over B, we obtain

$$\int_B (1-|x|^2)^4|\nabla w|^2 + 4(n+2)\int_B (1-|x|^2)^3 w^2 = \int_{\partial B}(1-|x|^2)^4 w w_\nu = 0.$$

Hence $\Delta\varphi = w \equiv 0$ in B. Now from (1.1), (4.2) and (4.4) we get

$$\varphi_\nu = \frac{d-n}{2}\varphi \qquad \text{on } \partial B. \tag{4.7}$$

Therefore, we obtained the following result:

Lemma 4. *The number d is an eigenvalue of (1.1) with corresponding eigenfunction u if and only if φ defined by (4.1) is an eigenfunction of the boundary eigenvalue problem*

$$\begin{cases} \Delta\varphi = 0 & \text{in } B \\ \varphi_\nu = a\varphi & \text{on } \partial B, \end{cases} \tag{4.8}$$

where $a = \frac{d-n}{2}$.

So we are led to study the eigenvalues of the second order Stekloff problem (4.8). Since we were unable to find an explicit reference, we quickly explain how to obtain them. In radial and angular coordinates (r,θ), the equation in (4.8) reads

$$\frac{\partial^2\varphi}{\partial r^2} + \frac{n-1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\Delta_\theta\varphi = 0\ , \tag{4.9}$$

where $-\Delta_\theta$ denotes the Laplace-Beltrami operator on ∂B. From [4, p.160] we quote

Lemma 5. *The Laplace-Beltrami operator $-\Delta_\theta$ admits a sequence of eigenvalues $\{\lambda_k\}$ having multiplicity μ_k equal to the number of independent harmonic homogeneous polynomials of degree $k-1$. Moreover, $\lambda_k = (k-1)(n+k-3)$.*

In the sequel, we denote by e_k^ℓ ($\ell = 1, ..., \mu_k$) the independent normalized eigenfunctions corresponding to λ_k. Then, one seeks functions $\varphi = \varphi(r,\theta)$ of the kind

$$\varphi(r,\theta) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\mu_k}\varphi_k^\ell(r)e_k^\ell(\theta)\ .$$

Hence, by differentiating the series, we obtain

$$\Delta\varphi(r,\theta) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\mu_k}\left(\frac{d^2}{dr^2}\varphi_k^\ell(r) + \frac{n-1}{r}\frac{d}{dr}\varphi_k^\ell(r) - \frac{\lambda_k}{r^2}\varphi_k^\ell(r)\right)e_k^\ell(\theta) = 0\ .$$

Therefore, we must solve the equations

$$\frac{d^2}{dr^2}\varphi_k^\ell(r)+\frac{n-1}{r}\frac{d}{dr}\varphi_k^\ell(r)-\frac{\lambda_k}{r^2}\varphi_k^\ell(r)=0 \qquad k=1,2... \quad \ell=1,...,\mu_k. \tag{4.10}$$

With the change of variables $r=e^t$ $(t\le 0)$, equation (4.10) becomes a linear constant coefficients equation. It has two linearly independent solutions, but one is singular. Hence, up to multiples, the only regular solution of (4.10) is given by $\varphi_k^\ell(r)=r^{k-1}$ because

$$\frac{2-n+\sqrt{(n-2)^2+4\lambda_k}}{2}=k-1\ .$$

Since the boundary condition in (4.8) reads $\frac{d}{dr}\varphi_k^\ell(1)=a\varphi_k^\ell(1)$ we immediately infer that $a=\bar{k}-1$ for some $\bar{k}$. In turn, Lemma 4 tells us that

$$d_{\bar{k}}=n+2(\bar{k}-1).$$

The proof of Theorem 3 is so complete.

5 Proof of Fichera's principle of duality (1.11)

We say that δ is an eigenvalue relative to problem (1.10) if there exists $g\in\mathbf{H}$ such that

$$\delta\int_\Omega gv=\int_{\partial\Omega}gv \qquad \text{for all } v\in\mathbf{H}\ .$$

Clearly, δ_1 is the least eigenvalue. We prove (1.11) by showing that both $\delta_1\ge d_1$ and $\delta_1\le d_1$.

Proof of $\delta_1\ge d_1$. Let h be a minimizer for δ_1, then

$$\delta_1\int_\Omega hv=\int_{\partial\Omega}hv \qquad \text{for all } v\in\mathbf{H}\ . \tag{5.1}$$

Let $u\in\mathcal{H}(\Omega)$ be the unique solution of

$$\begin{cases}\Delta u=h & \text{in } \Omega\\ u=0 & \text{on } \partial\Omega\ .\end{cases} \tag{5.2}$$

Integrating by parts we have

$$\int_\Omega hv=\int_\Omega v\Delta u=\int_{\partial\Omega}vu_\nu \quad \text{for all } v\in\mathbf{H}\cap C^2(\overline{\Omega})\ .$$

By a density argument, the latter follows for all $v\in\mathbf{H}$. Inserting this into (5.1) gives

$$\delta_1\int_{\partial\Omega}vu_\nu=\int_{\partial\Omega}v\Delta u \qquad \text{for all } v\in\mathbf{H}\ .$$

This yields $\Delta u = \delta_1 u_\nu$ on $\partial\Omega$. Therefore,

$$\delta_1 = \frac{\int_{\partial\Omega} h^2}{\int_\Omega h^2} = \frac{\int_{\partial\Omega} |\Delta u|^2}{\int_\Omega |\Delta u|^2} = \delta_1^2 \frac{\int_{\partial\Omega} u_\nu^2}{\int_\Omega |\Delta u|^2} .$$

In turn, this implies that

$$\delta_1 = \frac{\int_\Omega |\Delta u|^2}{\int_{\partial\Omega} u_\nu^2} \geq \min_{v\in\mathcal{H}(\Omega)} \frac{\int_\Omega |\Delta v|^2}{\int_{\partial\Omega} v_\nu^2} = d_1 .$$

Proof of $\delta_1 \leq d_1$. Let u be a minimizer for d_1 in (1.3), then $\Delta u = d_1 u_\nu$ on $\partial\Omega$ so that $\Delta u \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ and

$$\int_{\partial\Omega} v\Delta u = d_1 \int_{\partial\Omega} v u_\nu \qquad \text{for all } v \in \mathbf{H} . \tag{5.3}$$

Let $h := \Delta u$ so that $h \in L^2(\Omega) \cap L^2(\partial\Omega)$. Moreover, $\Delta h = \Delta^2 u = 0$ (in distributional sense) and hence $h \in \mathbf{H}$. Two integrations by parts (and a density argument) yield

$$\int_\Omega hv = \int_{\partial\Omega} v u_\nu \qquad \text{for all } v \in \mathbf{H} .$$

Replacing this into (5.3) gives

$$\int_{\partial\Omega} hv = d_1 \int_\Omega hv \qquad \text{for all } v \in \mathbf{H} .$$

This proves that h is an eigenfunction relative to problem (1.10) with corresponding eigenvalue d_1. Since δ_1 is the least eigenvalue, we obtain $d_1 \geq \delta_1$.

6 Proof of Theorem 4

Let φ be a first eigenfunction of (1.1) such that $\varphi > 0$ in Ω and $\varphi_\nu < 0$ on $\partial\Omega$ (see Lemma 2). The boundary condition $\Delta\varphi = d_1\varphi_\nu$ on $\partial\Omega$ also reads

$$\varphi_{\nu\nu} + (n-1)\,K\varphi_\nu = d_1\varphi_\nu \qquad \text{on } \partial\Omega \tag{6.1}$$

(see e.g. (4.68) p.62 in [16]). Therefore

$$\left(\varphi_\nu^2\right)_\nu = 2\varphi_{\nu\nu}\varphi_\nu = 2\left[d_1 - (n-1)\,K\right]\varphi_\nu^2$$

so that if we put $D^2\varphi D^2\varphi = \sum_{i,j=1}^n (\partial_{ij}\varphi)^2$, by (1.1) and integration by parts, we obtain

$$2\int_{\partial\Omega} \left[d_1 - (n-1)\,K\right]\varphi_\nu^2\, dS = \int_{\partial\Omega} \left(\varphi_\nu^2\right)_\nu\, dS = \int_{\partial\Omega} \left(|\nabla\varphi|^2\right)_\nu dS = \int_\Omega \Delta\left(|\nabla\varphi|^2\right)\, dx$$

$$= 2\int_\Omega \nabla\left(\Delta\varphi\right)\nabla\varphi\ dx + 2\int_\Omega D^2\varphi D^2\varphi\ dx = -2\int_\Omega \varphi\Delta^2\varphi\ dx + 2\int_{\partial\Omega}\varphi\left(\Delta\varphi\right)_\nu dS + 2\int_\Omega D^2\varphi D^2\varphi\ dx$$

$$= 2\int_\Omega D^2\varphi D^2\varphi\ dx \geq \frac{2}{n}\int_\Omega |\Delta\varphi|^2\,dx.$$

Finally, by (1.2) we have

$$2\int_{\partial\Omega}\left[d_1 - (n-1)\,K\right]\varphi_\nu^2\ dS \geq \frac{2d_1}{n}\int_{\partial\Omega}\varphi_\nu^2 dS$$

from which we obtain

$$\int_{\partial\Omega}\left(\frac{d_1}{n} - K\right)\varphi_\nu^2\,dS \geq 0, \tag{6.2}$$

which implies at once that $d_1 \geq n\underline{K}$.

It remains to prove that equality holds if and only if Ω is a ball. If $d_1 = n\underline{K}$, then $d_1 \leq nK(x)$ for $x \in \partial\Omega$ and since $\varphi_\nu < 0$ on $\partial\Omega$, by (6.2) we infer that $K(x) \equiv \frac{d_1}{n}$. This proves that Ω is a ball in view of Alexandrov's characterization of spheres [1].

7 Proof of Theorem 5

Assume that equality holds in (1.12). Then, $h \equiv 1$ is a minimizer for (1.10) and, according to Fichera's principle (see (5.2)), the minimizer u of (1.3) is the *stress function* for Ω (the solution of the torsion problem), namely

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\ . \end{cases}$$

Since u also solves the Euler equation (1.1) with $d = d_1$, we have a solution to the problem

$$-\Delta u = 1 \text{ in } \Omega\ , \qquad u_\nu = -d_1^{-1} \text{ on } \partial\Omega\ , \qquad u = 0 \text{ on } \partial\Omega\ .$$

By a result of Serrin [17], this shows that Ω is a ball and completes the proof.

8 Proof of (1.15)

In view of the results in [8], we may argue as for (5.2) in order to show that $d_1 \leq \delta_1$.

For our convenience, we translate the square $Q_{\sqrt{\pi}}$ and consider instead

$$Q := \left(-\frac{\sqrt{\pi}}{2}, \frac{\sqrt{\pi}}{2}\right)^2\ .$$

For all $k \in \mathbb{R}$, consider the harmonic function $h_k(x,y) := x^4 + y^4 - 6x^2y^2 + k$. Then by (1.10)-(1.11) we have

$$d_1(Q) \leq \delta_1(Q) \leq \frac{\displaystyle\int_{\partial Q} h_k^2\,dS}{\displaystyle\int_Q h_k^2\,dxdy} \qquad \text{for all } k \in \mathbb{R}. \tag{8.1}$$

Via direct computation we obtain

$$\int_{\partial Q} h_k^2\, dS = \sqrt{\pi}\left(4k^2 - \frac{2}{5}\pi^2 k + \frac{59}{1260}\pi^4\right) \qquad \text{and} \qquad \int_Q h_k^2\, dxdy = \pi k^2 - \frac{1}{30}\pi^3 k + \frac{59}{25200}\pi^5.$$

We recall that $3.141592 < \pi < 3.141593$. Hence, if we choose $k = 2.69$ we obtain

$$\int_{\partial Q} h_k^2\, dS < \sqrt{3.141593}\left(4 \cdot (2.69)^2 - \frac{2}{5} \cdot (2.69) \cdot (3.141592)^2 + \frac{59}{1260} \cdot (3.141593)^4\right) < 40.56426$$

and

$$\int_Q h_k^2\, dxdy > (3.141592) \cdot (2.69)^2 - \frac{(3.141593)^3}{30} \cdot (2.69) + \frac{59}{25200} \cdot (3.141592)^5 > 20.66911.$$

By inserting these estimates into (8.1) we obtain (1.15).

9 Proof of Theorem 6

Let $u \in H^2 \cap H_0^1(R_{\pi/2})$ be a (positive) minimizer for (1.3) when $\Omega = R_{\pi/2}$:

$$d_1(R_{\pi/2}) = \frac{\displaystyle\int_{R_{\pi/2}} |\Delta u|^2}{\displaystyle\int_{\partial R_{\pi/2}} u_\nu^2}.$$

By uniqueness of the minimizer, u is symmetric in $R_{\pi/2}$ so that

$$\int_0^{\pi/2} u_x^2(0, y)\, dy = \int_0^{\pi/2} u_x^2\Big(\frac{\pi}{2}, y\Big)\, dy = \int_0^{\pi/2} u_y^2(x, 0)\, dx = \int_0^{\pi/2} u_y^2\Big(x, \frac{\pi}{2}\Big)\, dx. \tag{9.1}$$

Fix $a \in (0, \frac{\pi}{2})$, let $R_a := (0, \pi - a) \times (0, a)$ and consider the function $v \in H^2 \cap H_0^1(R_a)$ defined by

$$v(x, y) = u\left(\frac{\pi\, x}{2(\pi - a)}, \frac{\pi\, y}{2a}\right) .$$

Then,

$$d_1(R_a) \le \frac{\displaystyle\int_{R_a} |\Delta v|^2}{\displaystyle\int_{\partial R_a} v_\nu^2} \tag{9.2}$$

and we estimate the two integrals in the right hand side of (9.2). We have

$$v_x(x, y) = \frac{\pi}{2(\pi - a)} \cdot u_x\left(\frac{\pi\, x}{2(\pi - a)}, \frac{\pi\, y}{2a}\right) , \qquad v_{xx}(x, y) = \left(\frac{\pi}{2(\pi - a)}\right)^2 \cdot u_{xx}\left(\frac{\pi\, x}{2(\pi - a)}, \frac{\pi\, y}{2a}\right) ,$$

$$v_y(x, y) = \frac{\pi}{2a} \cdot u_y\left(\frac{\pi\, x}{2(\pi - a)}, \frac{\pi\, y}{2a}\right) , \qquad v_{yy}(x, y) = \left(\frac{\pi}{2a}\right)^2 \cdot u_{yy}\left(\frac{\pi\, x}{2(\pi - a)}, \frac{\pi\, y}{2a}\right) .$$

Hence, applying (9.1) and with obvious changes of variables, we obtain

$$\int_0^a v_x^2(\pi-a,y)\,dy = \int_0^a v_x^2(0,y)\,dy = \frac{\pi^2}{4(\pi-a)^2}\int_0^a u_x^2\left(0,\frac{\pi\, y}{2a}\right)\,dy =$$

$$= \frac{a\pi}{2(\pi-a)^2}\int_0^{\pi/2} u_x^2(0,y)\,dy = \frac{a\pi}{8(\pi-a)^2}\int_{\partial R_{\pi/2}} u_\nu^2\ ,$$

$$\int_0^{\pi-a} v_y^2(x,a)\,dx = \int_0^{\pi-a} v_y^2(x,0)\,dx = \frac{\pi^2}{4a^2}\int_0^{\pi-a} u_y^2\left(\frac{\pi\, x}{2(\pi-a)},0\right)\,dx =$$

$$= \frac{\pi(\pi-a)}{2a^2}\int_0^{\pi/2} u_y^2(x,0)\,dx = \frac{\pi(\pi-a)}{8a^2}\int_{\partial R_{\pi/2}} u_\nu^2\ .$$

Therefore, we infer

$$\int_{\partial R_a} v_\nu^2 = \frac{\pi}{4}\left(\frac{a}{(\pi-a)^2}+\frac{\pi-a}{a^2}\right)\int_{\partial R_{\pi/2}} u_\nu^2\ . \tag{9.3}$$

Moreover, with a change of variables, we also obtain

$$\int_{R_a}|\Delta v|^2 = \int_{R_a}\left[\left(\frac{\pi}{2(\pi-a)}\right)^2\cdot u_{xx}\left(\frac{\pi\, x}{2(\pi-a)},\frac{\pi\, y}{2a}\right)+\left(\frac{\pi}{2a}\right)^2\cdot u_{yy}\left(\frac{\pi\, x}{2(\pi-a)},\frac{\pi\, y}{2a}\right)\right]^2 dx\,dy =$$

$$= 4a(\pi-a)\pi^2\int_{R_{\pi/2}}\left(\frac{u_{xx}^2(x,y)}{16(\pi-a)^4}+\frac{u_{xx}(x,y)\cdot u_{yy}(x,y)}{8a^2(\pi-a)^2}+\frac{u_{yy}^2(x,y)}{16a^4}\right)dx\,dy\ . \tag{9.4}$$

Next, as noticed by Kuttler [10, p.334], we recall that two integration by parts yield

$$\int_{R_{\pi/2}} u_{xx}(x,y)\cdot u_{yy}(x,y)\,dx\,dy = \int_{R_{\pi/2}} u_{xy}^2(x,y)\,dx\,dy > 0\ .$$

Hence, we may estimate (9.4) as follows

$$\int_{R_a}|\Delta v|^2 \le \frac{\pi^2(\pi-a)}{4a^3}\int_{R_{\pi/2}}|\Delta u|^2\ . \tag{9.5}$$

Inserting (9.3) and (9.5) into (9.2) yields

$$d_1(R_a) \le \frac{(\pi-a)^3}{a(\pi^2-3\pi a+3a^2)}\,d_1(R_{\pi/2})\ .$$

Letting $a\to 0$, shows that

$$\limsup_{a\to 0}[a\cdot d_1(R_a)] \le \pi\, d_1(R_{\pi/2})\ ,$$

which is precisely the upper bound in the statement of Theorem 6.

In order to prove the lower bound, we rewrite [10, (15)] as

$$d_1(R_a) \ge \frac{\pi}{2}\sqrt{\frac{1}{(\pi-a)^2}+\frac{1}{a^2}}$$

and we let $a\to 0$.

References

[1] A.D. Alexandrov, *A characteristic property of spheres*, Ann. Mat. Pura Appl. **58** (1962) 303–354

[2] L. Ambrosio, *Geometric evolution problems, distance function and viscosity solutions*, In: Calculus of variations and partial differential equations (Pisa, 1996), 5–93, Springer, Berlin (2000)

[3] E. Berchio, F. Gazzola, E. Mitidieri, *Positivity preserving property for a class of biharmonic elliptic problems*, preprint 2005

[4] M. Berger, P. Gauduchon, E. Mazet, *Le spectre d'une variété Riemannienne*, Lecture Notes Math. 194, Springer 1971

[5] F. Brock, *An isoperimetric inequality for eigenvalues of the Stekloff problem*, ZAMM **81** (2001) 69–71

[6] G. Fichera, *Su un principio di dualità per talune formole di maggiorazione relative alle equazioni differenziali*, Atti Accad. Naz. Lincei **19** (1955) 411–418

[7] F. Gazzola, G. Sweers, in preparation

[8] P. Grisvard, *Elliptic problems in nonsmooth domains*, Research Notes in Mathematics, Pitman Advanced Publishing Program (1985)

[9] J.R. Kuttler, *Remarks on a Stekloff eigenvalue problem*, SIAM J. Numer. Anal. **9** (1972) 1–5

[10] J.R. Kuttler, *Dirichlet eigenvalues*, SIAM J. Numer. Anal. **16** (1979) 332–338

[11] J.R. Kuttler, V.G. Sigillito, *Inequalities for membrane and Stekloff eigenvalues*, J. Math. Anal. Appl. **23** (1968) 148–160

[12] J.R. Kuttler, V.G. Sigillito, *Estimating eigenvalues with a posteriori/a priori inequalities*, Research Notes in Mathematics, Pitman Advanced Publishing Program (1985)

[13] R. S. Lakes, *Foam structures with a negative Poisson's ratio*, Science **235** (1987) 1038–1040

[14] J.L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications* - Vol. 1, Dunod (1968)

[15] L.E. Payne, *Some isoperimetric inequalities for harmonic functions*, SIAM J. Math. Anal. **1** (1970) 354–359

[16] R. Sperb, *Maximum principles and applications*, Academic Press (1981)

[17] J. Serrin, *A symmetry problem in potential theory*, Arch. Ration. Mech. Anal. **43** (1971) 304–318

[18] J. Smith, *The coupled equation approach to the numerical solution of the biharmonic equation by finite differences, I*, SIAM J. Numer. Anal. **5** (1968) 323–339

[19] J. Smith, *The coupled equation approach to the numerical solution of the biharmonic equation by finite differences, II*, SIAM J. Numer. Anal. **7** (1970) 104–111

[20] M.W. Stekloff, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. Ecole Norm. Sup. **19** (1902) 455–490

[21] P. Villaggio, *Mathematical models for elastic structures*, Cambridge University Press (1997)