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Bilinear fractal interpolation and box dimension

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Abstract

In the context of general iterated function systems (IFSs), we introduce bilinear fractal interpolants as the fixed points of certain Read–Bajraktarević operators. By exhibiting a generalized "taxi-cab" metric, we show that the graph of a bilinear fractal interpolant is the attractor of an underlying contractive bilinear IFS. We present an explicit formula for the box-counting dimension of the graph of a bilinear fractal interpolant in the case of equally spaced data points.

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1. Introduction

Bilinear filtering or bilinear interpolation is used in computer graphics to compute intermediate values for a two-dimensional regular grid. One of the main objectives is the smoothening

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of textures when they are enlarged or reduced in size. In mathematical terms, the interpolation technique is based on finding a function $f(x, y)$ of the form $f(x, y) = a + bx + cy + dxy$, where $a, b, c, d \in \mathbb{R}$, that passes through prescribed data points.

As textures reveal, in general, a non-smooth or even fractal characteristic, a description in terms of fractal geometric methods seems reasonable. To this end, the classical bilinear approximation method is replaced by a bilinear fractal interpolation procedure. The latter allows for additional parameters, such as the box dimension, that are related to the regularity and appearance of an underlying texture pattern.

We introduce a class of fractal interpolants that are based on bilinear functions of the above form. We do this by considering a more general class of iterated function systems (IFSs) and by using a more general definition of attractor of an IFS. These more comprehensive concepts are primarily based on topological considerations. In this context, we extend and correct some known results from [15] concerning fractal interpolation functions that are fixed points of so-called Read–Bajraktarević operators. Theorem 4 relates the fixed point in Theorem 3 to the attractor of an IFS and generalizes known results to the case where the IFS is not contractive.

As a special example of the preceding theory we introduce bilinear fractal interpolants and show that their graphs are the attractors of an underlying contractive bilinear IFS. Such bilinear IFSs have been investigated in [5] in connection with fractal homeomorphisms and address structures underlying an IFS. Finally, we present an explicit formula for the box dimension of the graph of a bilinear fractal interpolant in the case where the data points are equally spaced.

2. General iterated function systems

The terminology here for iterated function system, attractor, and contractive iterated function system is from [4]. Throughout this paper, $(\mathbb{X}, d)$ denotes a complete metric space with metric $d = d_{\mathbb{X}}$.

Definition 1. Let $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$. If $f_n : \mathbb{X} \to \mathbb{X}$, $n = 1, 2, \ldots, N$, are continuous mappings, then $\mathcal{F} = (\mathbb{X}; f_1, f_2, \ldots, f_N)$ is called an *iterated function system* (IFS).

By slight abuse of terminology we use the same symbol $\mathcal{F}$ for the IFS, the set of functions in the IFS, and for the following mappings. We define $\mathcal{F} : 2^{\mathbb{X}} \to 2^{\mathbb{X}}$ by

$$\mathcal{F}(B) := \bigcup_{f \in \mathcal{F}} f(B)$$

for all $B \in 2^{\mathbb{X}}$, the set of subsets of $\mathbb{X}$. Let $\mathbb{H} = \mathbb{H}(\mathbb{X})$ be the set of nonempty compact subsets of $\mathbb{X}$. Since $\mathcal{F}(\mathbb{H}) \subset \mathbb{H}$, we can also treat $\mathcal{F}$ as a mapping $\mathcal{F} : \mathbb{H} \to \mathbb{H}$. When $U \subset \mathbb{X}$ is nonempty, we may write $\mathbb{H}(U) = \mathbb{H}(\mathbb{X}) \cap 2^U$. We denote by $|\mathcal{F}|$ the number of distinct mappings in $\mathcal{F}$.

Let $d_{\mathbb{H}}$ denote the Hausdorff metric on $\mathbb{H}$, defined in terms of $d_{\mathbb{X}}$. A convenient definition (see for example [7, p. 66]) is

$$d_{\mathbb{H}}(B, C) := \inf\{r > 0 : B \subset C + r, C \subset B + r\},$$

for all $B, C \in \mathbb{H}$. For $S \subset \mathbb{X}$ and $r > 0$, $S + r$ denotes the set $\{y \in \mathbb{X} : \exists x \in S \text{ so that } d_{\mathbb{X}}(x, y) < r\}$.

We say that a metric space $\mathbb{X}$ is *locally compact* to mean that if $C \subset \mathbb{X}$ is compact and r is a positive real number then $\overline{C + r}$ is compact. Here, $\overline{S}$ denotes the closure of a set S. (For an equivalent definition of local compactness, see for instance [8, 3.3].)