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SEMICLASSICS OF THE QUANTUM CURRENT IN A STRONG CONSTANT MAGNETIC FIELD

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Abstract. We study the current of the Pauli operator in a strong constant magnetic field. We prove that in the semi-classical limit the persistent current and the current from the interaction of the spin with the magnetic field cancel, in the case where the magnetic field is very strong. Furthermore we calculate the next term in the asymptotics and estimate the error. Finally, we discuss the connection between this work and the semi-classical estimate of the energy in strong magnetic fields proved by Lieb, Solovej and Yngvason [LSY94].

Contents

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1. Introduction

In recent years physicists have been very interested in understanding the current in quantum systems such as the quantum Hall systems and different types of nanostructures that experimental advances have made possible. In contrast, the current has been studied very little in the mathematics litterature. The current, however, is as natural a quantity as the density which has been studied to a great extent in the mathematics litterature, in particular, the integral of the density, i.e. the particle number (for fixed chemical potential), obeys the celebrated Weyl law in the semiclassical limit. In the semiclassical limit one cannot expect to see a static current since there is no classical, persistent or diamagnetic current. In quantum mechanics, however, there may be a static current. In [Fou98] the semiclassical limit of this current was studied and it was indeed found that the first term in the semiclassical expansion vanishes. This might be the reason why the quantum current has not attracted much attention in the mathematics community.

In this paper we study a different type of semiclassical limit in which the magnetic field strength may vary as the semiclassical parameter h tends to zero. If the field strength increases when h decreases in such a way that the magnetic length scale is comparable to the Planck scale, one should expect to see the effect of the current. In fact, in quantum Hall systems one has magnetic field strength that make the magnetic field length of the order of the Planck scale. This type of semiclassical limit was studied by Lieb, Solovej and Yngvason in [LSY94] and [LSY95], where the limits of the energy and the density were studied. The purpose of this paper is to extend this analysis to include the persistent quantum current.

It should be noted that this paper deals solely with static situations. This is different from the situation in quantum Hall systems, where a constant voltage drop creates a stationary and not just static situation.

The object of study in this paper is the Pauli operator:

$$\mathbf{P} = \mathbf{P}(h, \vec{A}, V) = (-ih\nabla - \vec{A})^2 + V(x) - h\vec{\sigma}\cdot\vec{B},$$

acting in $L^2(\mathbf{R}^3;\mathbf{C}^2)$. Here $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $\vec{B} = \nabla \times \vec{A}$. This operator has, in general, infinitely many negative eigenvalues, even for V smooth and compactly supported (and negative), but it was proved in [LSY94] (see also [ES97] for the case of non-constant magnetic fields) that the sum of the negative eigenvalues $\mathrm{tr}[\mathbf{P}1_{(-\infty,0]}(\mathbf{P})]$ is finite. The sum of the negative eigenvalues represents the energy E of a noninteracting electron gas (of chemical potential 0) in the external electric potential V and magnetic potential $\vec{A}$. Furthermore, they proved a semi-classical formula for the energy, uniformly in the magnetic field strength, i.e. an expression $E_{scl} = E_{scl}(h, \vec{A}, V)$ (see (1.1) below), such that if $E = E(h, \vec{A}, V) = \mathrm{tr}[\mathbf{P}1_{(-\infty,0]}(\mathbf{P})]$ then

$$\frac{E(h, \vec{A}, V)}{E_{scl}(h, \vec{A}, V)} \to 1,$$

as $h \to 0$, uniformly in $\vec{A}$.

Given the energy, two quantities can be calculated: the *density* and the *current*. The density ρ is defined, as a distribution, as the variational derivative of E with respect to V, i.e.

$$\int \rho\phi \, dx = \frac{d}{dt} E(h, \vec{A}, V + t\phi)|_{t=0}.$$

In the context of strong magnetic fields, this has been studied in [Sob94], and a formula for the highest order term in the semi-classical limit was given, with good control of the error term.

The current $\vec{j}$ is the variational derivative of E with respect to the vector potential $\vec{A}$.

$$\int \vec{j} \cdot \vec{a} \, dx = \frac{d}{dt} E(h, \vec{A} + t\vec{a}, V)|_{t=0},$$

where the left hand side is to be understood in the sense of distributions. It will be the objective of this paper to obtain a semi-classical formula for this quantity when the magnetic field is strong, but constant. By a strong, constant magnetic field we mean that we take the limit $h \to 0$ but with a magnetic field $\vec{B} = (0, 0, \mu)$ so strong that $\mu h \geq c > 0$ as $h \to 0$.

To get an idea of what to expect, let us first look at the semiclassical energy:

1.1. **The Semiclassical Formula in** [LSY94].

The semi-classical formula for the energy given in [LSY94] is:

$$E_{scl} = -h^{-3} \int P(h|\vec{B}(x)|, [V(x)]_-) dx, \tag{1.1}$$

where

$$P(B, W) = \frac{2}{3\pi} \sum_{n=0}^{\infty} d_n B [2nB - W]_-^{3/2},$$

and

$$[x]_- = \begin{cases} 0 & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Here $d_0 = \frac{1}{2\pi}$ and $d_n = \frac{1}{\pi}$ for $n \geq 1$. If this semiclassical formula contains most of the *physics* of the problem then it should also give the current to highest order, so we try to calculate its functional derivative with respect to the vector potential. Let thus $\vec{a}$ be a test function. Then we have:

$$\begin{aligned} \int \vec{j}_{scl} \cdot \vec{a} \, dx &= \frac{d}{dt} E_{scl}(\vec{A} + t\vec{a})|_{t=0} \\ &= \frac{-2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2 - \partial_{x_2} a_1) \\ &\quad \times \left([2nh\mu + V(x)]_-^{3/2} - 3nh\mu [2nh\mu + V(x)]_-^{1/2} \right) dx, \end{aligned} \tag{1.2}$$

or

$$\vec{j}_{scl} = \frac{1}{\pi h^2} \sum_{n=0}^{\infty} d_n \left([2nh\mu + V(x)]_-^{1/2} - nh\mu [2nh\mu + V(x)]_-^{-1/2} \right) \begin{pmatrix} \partial_{x_2} V \\ -\partial_{x_1} V \\ 0 \end{pmatrix}. \tag{1.3}$$

In the special case where $\mu h \to \infty$ we get:

$$\vec{j}_{scl} = \frac{1}{2\pi^2 h^2} [V(x)]_-^{1/2} \begin{pmatrix} \partial_{x_2} V \\ -\partial_{x_1} V \\ 0 \end{pmatrix}.$$

We will prove that the above formulas for the current are correct to highest order, and we will estimate the error.

Remark 1.1. The corresponding formulas in 2-dimensions are:

$$E_{scl}^{(2)} = -h^{-1} \int \sum_{n=0}^{\infty} d_n |\vec{B}(x)| [2nh|\vec{B}(x)| + V(x)]_- dx,$$

$$\vec{j}_{scl}^{(2)} = -h^{-1} \int \sum_{n=0}^{\infty} d_n (\partial_{x_1} a_2 - \partial_{x_2} a_1) \left([2nh\mu + V(x)]_- - 2nh\mu [2nh\mu + V(x)]_-^0 \right).$$

1.2. **Statement of the Results.**

We will fix $\vec{A} = \mu(-x_2, 0, 0)$ in the rest of this paper. We will thus write $\mathbf{P} = \mathbf{P}(h, \mu, V)$ instead of $\mathbf{P}(h, \vec{A}, V)$. A formal computation gives

$$\frac{d}{dt} E(h, \vec{A} + t\vec{a}, V)|_{t=0} = -\mathrm{tr}[\mathbf{B} 1_{(-\infty,0]}(\mathbf{P})], \tag{1.4}$$

where

$$\mathbf{B} = 2\vec{a} \cdot (-ih\nabla - \vec{A}) - ih \mathrm{div} \vec{a} + h\sigma_3 (\partial_{x_1} a_2 - \partial_{x_2} a_1).$$

We will take this as our starting point i.e. *define* the current as

$$\int \vec{j} \cdot \vec{a} \; dx = -\mathrm{tr}[\mathbf{B} 1_{(-\infty,0]}(\mathbf{P})].$$

We shall allow V to have a Coulomb singularity, i.e. suppose

$$V(x) = \frac{q}{|x|} + o(|x|^{-1}) \tag{1.5}$$

as $x \to 0$, and

$$|\partial^m V(x)| \leq C_{m,V} |x|^{-1-|m|}, \tag{1.6}$$

$\forall x \in B(8)$.
Suppose furthermore that $\exists C = C(h, \mu)$ such that

$$\mathbf{P}(h, \mu, V) \geq -C.$$

Then we have the following:

Theorem 1.2. *Let the above conditions on V be satisfied. Suppose*

- $\exists c_{\mu,1} > 0$ *such that* $\mu h \geq c_{\mu,1}$,
- $\exists c_{\mu,2} > 0$ *such that* $\mu h^3 \leq c_{\mu,2}$,

then

$$\vec{j} \overset{h \to 0}{\to} \vec{j}_{scl},$$

in the sense of distributions in the coordinates orthogonal to the magnetic field, i.e.:

$$\int \vec{j} \cdot \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} dx \overset{h \to 0}{\to} \int \vec{j}_{scl} \cdot \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} dx,$$

for all $a_1, a_2 \in C_0^\infty(B(1))$.

Remark 1.3. The condition $\exists c_{\mu,2} > 0$ such that $\mu h^3 \leq c_{\mu,2}$ is only necessary if we have a singularity. In the case where V is smooth we can allow μ to be of any order in h, see Theorem 2.3 or its improvement Theorem 7.1.

If the potential is confining in the direction parallel to the magnetic field, we can also calculate the current in that direction:

Theorem 1.4. *Let the assumptions be as in Thm 1.2. Assume furthermore that* $V(x_1,x_2,x_3) \geq c_V > 0$*, for* $1 \leq |x_3| \leq 3$*, and that the spectrum of* $\mathbf{P}$ *below* 0 *is discrete, then*

$$\vec{j} \overset{h\to 0}{\to} \vec{j}_{scl},$$

in the sense of distributions, i.e.:

$$\int \vec{j} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} dx \overset{h\to 0}{\to} \int \vec{j}_{scl} \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} dx,$$

for all $a_1, a_2, a_3 \in C_0^\infty(B(1))$.

Apart from its obvious physical relevance, the Coulomb potential is mathematically interesting in this kind of problem, since a correct analysis demands asymptotic estimates in both weak and strong magnetic fields. To see this, one has to realise, that magnetic effects are important if $\frac{[V(x)]_-}{\mu h} \ll 1$ and neglectable if $\frac{[V(x)]_-}{\mu h} \gg 1$. This can, for example, be seen from the semiclassical formula for the energy. Thus we will need to split in two regions, one, close to the singularity, where $\frac{[V(x)]_-}{\mu h}$ is big, and one outside, where the ratio is small. In the first region, we have standard semi-classics, and the analysis from [Fou98] suffices. In the outer region no analysis of the current exists, therefore the main part of this paper, sections 2 - 8 will deal with finding the correct estimates in this region. Finally, in section 9, we will prove a more precise version of theorem 1.2 above. The proof of theorem 1.4 is identical to the proof of theorem 2.6 below given in section 8 and will therefore be omitted.

1.3. **Notations.** It will be convenient to use the functions:

$$\begin{aligned} g_0(\tau) &= 1_{(-\infty,0]}(\tau), \\ g_1(\tau) &= (-\tau) g_0(\tau). \end{aligned}$$

For shortness we will sometimes write the current trace as

$$\mathrm{tr}[\mathbf{B} g_0(\mathbf{P})] = \mathcal{J}(h,\mu,\vec{a},V),$$

and the asymptotic term, as:

$$\begin{aligned} \mathcal{A}(h,\mu,\vec{a},V) &= \frac{2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2(x) - \partial_{x_2} a_1(x)) \\ &\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h [2nh\mu + V(x)]_-^{1/2} \right) dx. \end{aligned}$$

We will write $\hat{x} = (x_1, x_3)$ and $\hat{\xi} = (\xi_1, \xi_3)$.
Apart from the parameters h, μ we will need two other scales:

$$\alpha = h/\mu, \qquad \epsilon = \frac{1}{\mu^2}.$$

We will denote by $\mathcal{B}^\infty(\Omega)$ the set of smooth functions f on the open set Ω satisfying

$$|\partial^m f| \leq C_m.$$

for all m.

It is an elementary fact that:

$$L^2(\mathbf{R}^3) = L^2(\mathbf{R}^2_{(x_1,x_3)}) \otimes L^2(\mathbf{R}_{x_2}).$$

It is this splitting of $L^2(\mathbf{R}^3)$ that all tensor products will refer to.
Finally, it should be pointed out that the notation $\partial^\alpha f(z)$ is shorthand for $\partial^\alpha f|_z$ all through this paper.

2. The Local Asymptotics

Our strategy to prove the main theorems of this paper will be that of V. Ivrii: obtain good local results in regions where everything is smooth, and then use "scaling" to put the pieces together. This last "cutting-and-pasting" technique has been refined (by Ivrii and others, see [Ivr98], [IS93], [Sob95]) into what is usually called the "multiscaling" technique and will be discussed in the last sections of the paper. Here we will just remind the reader that it is *absolutely crucial* for the technique to work, that the estimates obtained are indeed *local* i.e. depend only on local bounds on, for instance, the potential. The only *global* assumption, we need, (and are allowed to impose) is the semi-boundedness (and self-adjointness) of the operator in question, end even here it is important that the local estimates only depend on the existence of a lower bound, *not* on the size of it.

The local result is:
Let $E \in \mathbf{R}_+$, $\vec{a} \in C_0^\infty(B(E/2))$. Let furthermore $H_0 = (-ih\nabla - \vec{A})^2 - \mu h$. Assume V satisfies:

Assumption 2.1 (See [Sob94][Assumption 1.1).]

- V is a real-valued function such that the self-adjoint operator $H = H_0 + V$ is well defined on the domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ and is semibounded from below;
- $V \in C^\infty(B(4E))$.

Remark 2.2. The introduction of this kind of assumption in semi-classical problems is due to Ivrii [Ivr98].

Let finally

$$\mathbf{B} = 2\vec{a} \cdot (-ih\nabla - \vec{A}) - ih\mathrm{div}\vec{a} + h\sigma_3(\partial_{x_1} a_2 - \partial_{x_2} a_1).$$

Then we have:

Theorem 2.3. *Let* $\vec{a} = (a_1, a_2, 0)$. *Suppose that*

$$|\partial_{x_1} V(x)|^2 + |\partial_{x_3} V(x)|^2 + |V(x)| \geq c_{N.C.} > 0 \tag{2.1}$$

for all $x \in B(2E)$. *Suppose further that* $0 < h \leq h_0$, $\mu \leq C_\mu h^{-\zeta}$ *for some* $\zeta > 0$ *and that there exists* $\rho \in (0, 1]$ *such that* $\mu \geq c_\rho h^{-\rho}$. *Suppose finally that*

$$|\partial^m \vec{a}(x)| + |\partial^m V(x)| \leq C_m$$

on $B(8E)$. *Then*

$$\begin{aligned}
\mathrm{tr}[\mathbf{B} g_0(\mathbf{P})] &= \frac{2}{3\pi h^2} \sum_{n=0}^{\infty} d_n \int (\partial_{x_1} a_2(x) - \partial_{x_2} a_1(x)) \\
&\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h [2nh\mu + V(x)]_-^{1/2} \right) dx \\
&\quad + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1}),
\end{aligned}$$

where O *is uniform in the constants* $\{C_m\}, c_{N.C.}, c_\rho, C_\mu, \rho, \zeta, E$.

Remark 2.4. The theorem is still true without the "non-critical" condition (2.1). This will be proved in Section 7.

First we want to prove this in the case where $\mu h \geq C$ where C is some sufficiently big constant (i.e. $\rho = 1$ and c_ρ sufficiently big). This is mainly for pedagogical reasons. When μh is big we only have to consider the lowest Landau level. This implies a greater simplicity in the exposition. Since furthermore, the persistent current and the spin-current cancel on the lowest Landau level, it becomes clear, why we have to make a somewhat finer analysis, than what is needed to find the density and the energy.
Thus, we will first prove Theorem 2.5 below, then, in section 6, we will put in the few remaining arguments to prove Theorem 2.3.

Theorem 2.5. *Let $\vec{a} = (a_1, a_2, 0)$. Suppose that*

$$|\partial_{x_1} V(x)|^2 + |\partial_{x_3} V(x)|^2 + |V(x)| \geq c_{N.C.} > 0$$

for all $x \in B(2E)$. Suppose further that $0 < h \leq h_0$, $\mu \leq C_\mu h^{-\zeta}$ for some $\zeta > 0$ and that $\mu h \geq C$. Suppose finally that

$$|\partial^m \vec{a}(x)| + |\partial^m V(x)| \leq C_m$$

on $B(8E)$. Then there exists C_0 such that if $C > C_0$ we get

$$\mathrm{tr}[\mathbf{B} g_0(\mathbf{P})] = \frac{1}{3\pi^2}\frac{1}{h^2}\int(\partial_{x_1} a_2 - \partial_{x_2} a_1)[V(x)]_-^{3/2} dx + O(h^{-1}),$$

where O is uniform in the constants $\{C_m\}, c_{N.C.}, C_\mu, \zeta, E$.

Let us also state a version of 1.4 in the setup of the two theorems above:

Theorem 2.6. *Let $a_3 \in C_0^\infty(B(E))$ and define $\vec{a} = (0, 0, a_3)$. Suppose $V \in C^\infty(\mathbf{R}^3)$ and that there exists $\gamma > 0$ such that $\liminf_{|x|\to\infty} V(x) > \gamma$. Suppose further that $\mu h \geq c > 0$ as $h \to \infty$. Then*

$$\mathrm{tr}[\mathbf{B}(h, \mu, \vec{a}) g_0(\mathbf{P})] = O(h^{-1}).$$

Finally a few words about the following sections. Section 3 below recalls the results from [Sob94] that we will need in the rest of the paper. Sections 4 and 5 contain the proof of Theorem 2.5.

3. The Birkhoff Normal Form

Let $W \in C_0^\infty(\mathbf{R}^3)$, $W(x) = V(x)$ on $B(3E)$. We will perform some reductions on

$$H_W = (-ih\partial_{x_1} + \mu x_2)^2 - h^2\partial_{x_2}^2 - h^2\partial_{x_3}^2 - \mu h + W(x).$$

We will later in this section also have to use H from Assumption 2.1, which we will then write as H_V. Outside this section H will always refer to H_V.

3.1. 1st Reduction.

Let

$$(\Phi_0 f)(x) = \frac{1}{(2\pi\alpha)^3}\int e^{i/\alpha[(x-y)\xi + \xi_1\xi_2]} f(y) dy d\xi.$$

Then it is easy to see that Φ_0 is unitary and that

$$\epsilon \Phi_0^* H_W \Phi_0 = Op_\alpha^w h_\epsilon - \alpha,$$

where

$$h_\epsilon(x, \xi) = (\xi_2^2 + x_2^2) + \xi_3^2 + \epsilon W(x_1 - \xi_2, x_2 - \xi_1, x_3).$$

In general:

$$\Phi_0^* Op_\alpha^w a \Phi_0 = Op_\alpha^w \tilde{a}, \tag{3.1}$$

where $\tilde{a}(x, \xi) = a(x_1 - \xi_2, x_2 - \xi_1, x_3, \xi)$.

3.2. **2nd Reduction.**

Using a sequence of canonical transformations Sobolev transforms h_ϵ to a form where the variables (x_2, ξ_2) almost separate. Then he constructs an almost unitary transformation which realizes the canonical transformation κ on the symbol level. The result is the Theorem 3.1 below ([Sob94][Thm 7.6]). Before we state it we need a bit more notation.
Let K_α be the operator on $L^2(\mathbf{R})$:

$$(K_\alpha u)(t) = (-\alpha^2 \partial_t^2 + t^2) u(t),$$

below K_α will be acting in the x_2 variable. We denote by u the three variables (x_1, x_3, ξ_1), and by v the remaining variables i.e. $v = (x_2, \xi_2, \xi_3)$. We denote by $\tau_N = \tau_N(x, \xi, \epsilon)$ any function in $\mathcal{B}^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_\xi \times [-\epsilon_0, \epsilon_0])$, which satisfies

$$|\partial_u^{m_1} \partial_v^{m_2} \partial_\epsilon^{m_3} \tau_N(x,\xi,\epsilon)| \leq C_N (|\epsilon| + v^2)^{(N - |m_2|/2 - |m_3|)_+}.$$

Finally, we choose a $C_0^\infty(\mathbf{R})$ function $\sigma(t)$, satisfying $\sigma(t) = 1$ for $|t| \leq 1/4$ and $\sigma(t) = 0$ for $|t| \geq 1$, for $R > 0$ we write $\sigma_R(t) = \sigma(t/R)$.

Theorem 3.1. *For any positive integers N, M, L there exists an operator $T = T_{N,M,L}(\alpha, \epsilon)$ satisfying the following properties:*
(1) It is almost unitary:

$$\begin{aligned} T^*(\alpha,\epsilon) T(\alpha,\epsilon) &= I + O(\alpha^L), \\ T(\alpha,\epsilon) T^*(\alpha,\epsilon) &= I + O(\alpha^L). \end{aligned}$$

(2) The representation

$$T^* Op_\alpha^w h_\epsilon T = B = B_0 + B_1 \tag{3.2}$$

holds. Here

$$\begin{aligned} B_0 &= B_0(\alpha,\epsilon) \\ &= -\alpha^2 \partial_{x_3}^2 + (I \otimes K_\alpha) + \epsilon \sum_{n=0}^{N} \alpha^n \\ &\quad \times \sum_{0 \leq m+l+j \leq M} \epsilon^m \{ Op_\alpha^w W_{mlj}^{(n)} \otimes K_\alpha^l \sigma(K_\alpha) \}, \\ W_{mlj}^{(n)} &= \sigma(\xi_3^2) [Y_{mlj}^{(n)}(u) \xi_3^{2j} + Z_{mlj}^{(n)}(u) \xi_3^{2j+1}], \end{aligned}$$

with some $Y_{mlj}^{(n)}, Z_{mlj}^{(n)} \in \mathcal{B}^\infty(\mathbf{R}^3)$. In particular, the functions $Y_{mlj}^{(0)}, Z_{mlj}^{(0)}$ are defined in [Sob94]*[Theorem 7.4], and*

$$Y_{mlj}^{(1)} = Z_{mlj}^{(1)} = 0$$

for all m, l, j.
The operator $B_1 = B_1(\alpha,\epsilon)$ in (3.2) has the form $B_1 = \epsilon B_2 + \alpha^{N+1} B_3$. Here $B_2 = B_2(\alpha,\epsilon) = Op_\alpha^w \tau_{M+1}$ and the operator $B_3 = B_3(\alpha,\epsilon)$ can be represented for any integer $N_1 > 0$ as

$$\begin{aligned} B_3(\alpha,\epsilon) &= \epsilon \sum_{n=0}^{N_1} \alpha^n Op_\alpha^w b_{3n} + O(\alpha^{N_1+1}), \\ b_{3n} &\in \mathcal{B}^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_\xi \times [-\epsilon_0, \epsilon_0]). \end{aligned}$$

(3) Let κ be the canonical transformation constructed in [Sob94]*[Thm 7.4].Then for any symbol $\psi \in \mathcal{B}^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$*

$$T^* Op_\alpha^w \psi T = Op_\alpha^w (\psi \circ \kappa) + O(\epsilon^2) + O(\alpha^2).$$

(4) Let $\psi_1, \psi_2 \in \mathcal{B}^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$ *be two symbols* $\psi_j = \psi_j(x,\xi,\epsilon), j = 1,2, \epsilon \in [-\epsilon_0, \epsilon_0]$, *such that*

$$\text{dist}\{\text{supp}\,\psi_1, \text{supp}\,\psi_2\} \geq c > 0,$$

when ϵ_0 *is small enough. Then for any* $N_1 > 0$

$$Op^w_\alpha \psi_1 T Op^w_\alpha \psi_2 = O(\alpha^{N_1}).$$

Remark 3.2. The idea of reducing our operator to this form is due to Ivrii (see [Ivr98]).

3.3. 3rd Reduction.

We define

$$(\mathcal{U}_\mu f)(x) = \mu^{1/4} f(x_1, \frac{x_2}{\sqrt{\mu}}, x_3).$$

For any symbol a we then have

$$\mathcal{U}^*_\mu Op^w_\alpha a \mathcal{U}_\mu = Op^w_h \tilde{a}, \tag{3.3}$$

where

$$\tilde{a}(x,\xi) = a(x_1, \frac{x_2}{\sqrt{\mu}}, x_3, \frac{\xi_1}{\mu}, \frac{\xi_2}{\sqrt{\mu}}, \frac{\xi_3}{\mu}).$$

With the T from Theorem 3.1 above we define

$$\Phi = \Phi_0 T \mathcal{U}_\mu,$$

and we get ([Sob94][Theorem 7.7]):

Theorem 3.3. *Let* $R > 0$ *be an arbitrary number. Suppose* $\mu \geq R$. *Then*

$$\Phi^* H_W \Phi = P = P_0 + P_1,$$

and for any $g \in C^\infty_0(\mathbf{R})$,

$$\Phi^* g(H_W)\Phi = g(P) + O(\alpha^L).$$

Here

$$P_0 = P_0(h,\mu) = -h^2\partial^2_{x_3} + \mu K_h - \mu h + \mathcal{W}_{M,N}(h,\mu),$$

where

$$\begin{aligned}\mathcal{W}_{M,N}(h,\mu) &= \sum_{n=0}^{N} (h/\mu)^n \\ &\times \sum_{0\leq m+l+j\leq M} \mu^{-2m-l-2j}\{Op^w_h(p^{(n)}_{mlj} + \mu^{-1}q^{(n)}_{mlj}) \otimes K^l_h \sigma(\mu^{-1}K_h)\},\end{aligned}$$

with

$$\begin{aligned} p^{(n)}_{mlj}(\hat{x},\hat{\xi}) &= Y^{(n)}_{mlj}(\hat{x}, \mu^{-1}\xi_1)\xi_3^{2j}\sigma_R(\xi_3^2), \\ q^{(n)}_{mlj}(\hat{x},\hat{\xi}) &= Z^{(n)}_{mlj}(\hat{x}, \mu^{-1}\xi_1)\xi_3^{2j+1}\sigma_R(\xi_3^2),\end{aligned}$$

where $Y^{(n)}_{mlj}, Z^{(n)}_{mlj}$ *are from Theorem 3.1. In particular,*

$$\begin{aligned} p^{(0)}_{000}(\hat{x},\hat{\xi}) &= W(x_1, -\mu^{-1}\xi_1, x_3)\sigma_R(\xi_3^2), \\ p^{(0)}_{100}(\hat{x},\hat{\xi}) &= -\frac{1}{4}(\nabla_2 W(x_1, -\mu^{-1}\xi_1, x_3))^2\sigma_R(\xi_3^2), \\ p^{(0)}_{010}(\hat{x},\hat{\xi}) &= \frac{1}{4}\Delta_2 W(x_1, -\mu^{-1}\xi_1, x_3)\sigma_R(\xi_3^2), \\ p^{(0)}_{001}(\hat{x},\hat{\xi}) &= q^{(0)}_{000}(\hat{x},\hat{\xi}) = 0,\end{aligned}$$

and

$$p^{(1)}_{mlj} = q^{(1)}_{mlj} = 0,$$

for all m, l, j*. The operator* $P_1 = P_1(h,\mu)$ *above has the form* $P_1(h,\mu) = P_2 + P_3 + O(\alpha^{N+1}) + O(\alpha^{L-1})$*. Here* $P_2 = P_2(h,\mu) = Op_h^w p_2$ *is an operator whose symbol* $p_2 \in \mathcal{B}^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$ *satisfies the bound*

$$|\partial_x \partial_\xi p_2(x,\xi,\alpha,\epsilon)| \leq C_M \mu^{-2(M+1)},$$

for $x_3^2 + x_2^2 + \xi_2^2 \leq C$*, and* $P_3 = P_3(h,\mu)$ *is an operator which can be presented in the form* $P_3 = (Op_h^w \zeta)\tilde{P}_3$*, where* $\|\tilde{P}_3\| \leq C$ *and* $\zeta \in \mathcal{B}^\infty(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$ *is a function such that* $\zeta(x,\xi) = 0$ *for* $|\xi_3| \leq R/2$*.*

3.4. **Consequences.**

Of course, this reduction is not worth anything if the operator P_1 is not "small" in some sense. This is indeed the case. Sobolev proves the following consequences of the reduction:

Theorem 3.4. *(*[Sob94]*[Cor.8.5])*
Let Φ *and* H_W *be as above and let* $g \in C_0^\infty(\mathbf{R})$*. Then*

$$\Phi^* g(H_W)\Phi = g(P_0) + \omega(h,\mu),$$

where we have introduced the notation

$$\omega(h,\mu) = O(\mu^{-2(M+1)} + (h/\mu)^{N+1} + (h/\mu)^{L-1} + h^{N_1})$$

for all $N_1 > 0$*.*

Theorem 3.5. *(*[Sob94]*[Theorem 10.2])*
Let $\psi \in C_0^\infty(B(E/2))$ *and* $g_1 \in C_0^\infty(\mathbf{R})$*. Suppose* $\mu \geq \mu_1, h \in (0, h_0]$ *and* $\mu \leq ch^{-\zeta}$ *for some* $\zeta \geq 1$*. Then there exists* $T > 0$ *such that for all* $|t| \leq T$*,*

$$\|\psi g_1(H_V)e^{-itH_V/h} - \psi\Phi g_1(P_0)e^{-itP_0/h}\Theta^{(\mu)}\Phi^*\|_1 \leq Ch^{-\frac{3}{2}(1+\zeta)}\omega(h,\mu).$$

Here $\Theta^{(\mu)}$ *is a pseudodifferential cut-off in the variables* (x_1, x_3, ξ_1, ξ_3) *defined just before the theorem.*

Finally we notice [Sob94][(8.5)]

$$g(P_0) = \sum_{0 \leq k \leq C/(\mu h)} \oplus (g(P_0^{(k)}) \otimes \Pi_k),$$

where Π_k is the projection in $L^2(\mathbf{R}_{x_2})$ on the k-th eigenvalue of K_h. In particular we get when $\mu h \to \infty$:

$$g(P_0) = g(P_0^{(0)}) \otimes \Pi_0.$$

4. An Equivalent Operator on the Lowest Landau Level

In this section we assume that $\mu h > C$, where C is some sufficiently big constant. This assures that only the lowest Landau level plays a role. We find an equivalent operator on this level which has much nicer a priori properties than $\mathbf{B}$. This is the statement of Lemma 4.1 below. The whole section is devoted to the proof of this Lemma, which is the key to the calculation of the current.

Since

$$\mathbf{P} = \begin{pmatrix} H_0 + V & 0 \\ 0 & H_0 + V + 2\mu h \end{pmatrix},$$

we get:

$$g_0(\mathbf{P}) = \begin{pmatrix} g_0(H_0 + V) & 0 \\ 0 & g_0(H_0 + V + 2\mu h) \end{pmatrix}.$$

Now, $H_0 + V$ is assumed to be bounded below, thus $H_0 + V + 2\mu h > 0$ when μh is sufficiently big. Therefore,

$$\mathrm{tr}[\mathbf{B} g_0(\mathbf{P})] = \mathrm{tr}[B(\mu,h) g_0(H)], \tag{4.1}$$

where $B(\mu,h) = Op_h^w(2\vec{a}(\xi - \vec{A}) + h(\partial_{x_1} a_2 - \partial_{x_2} a_1)$, and $H = H_0 + V$.

Let $\psi \in C_0^\infty(\mathbf{R}^3)$, $\psi \equiv 1$ on a neighborhood of $\mathrm{supp}\,\vec{a}$. We may choose it such that $\mathrm{supp}\,\psi \subset B(E/2)$. Choose also $f \in C_0^\infty(\mathbf{R})$, $f \equiv 1$ on a neighborhood of 0.

Lemma 4.1. *Suppose* $\vec{a} = (a_1, a_2, 0)$. *Let*

$$b(x,\xi) = [a_2(x)\partial_{x_1} V(x) - a_1(x)\partial_{x_2} V(x)]\, f\left((\xi_1 + x_2)^2 + \xi_2^2 + \xi_3^2 + \frac{V(x)}{\mu^2}\right).$$

Then

$$\mathrm{tr}[B(\mu,h) g_0(H)] = \frac{1}{\mu}\mathrm{tr}[\psi (Op_\alpha^w b)\psi g_0(H)] + O(1/h).$$

Remark 4.2. The assumption $a_3 \equiv 0$ is very important for the Lemma.

Proof. We write

$$h_\alpha(x,\xi) = (\xi_1 + x_2)^2 + \xi_2^2 + \xi_3^2 + \epsilon V(x) - \alpha.$$

Then $Op_\alpha^w h_\alpha = \frac{1}{\mu^2} H$, and since $g_0(ct) = g_0(t)$ for all $c > 0$, we get

$$\mathrm{tr}[B(\mu,h) g_0(H)] = \mathrm{tr}[\psi B(\mu,h) f(Op_\alpha^w h_\alpha)\psi g_0(Op_\alpha^w h_\alpha)] + O(h^\infty), \tag{4.2}$$

because $f(Op_\alpha^w h_\alpha) g_0(Op_\alpha^w h_\alpha) = g_0(Op_\alpha^w h_\alpha)$ for μ sufficiently big[1]. Since

$$B(\mu,h) = \mu\Big(Op_\alpha^w(2\vec{a}\cdot(\xi_1 + x_2, \xi_2, \xi_3)) + \alpha(\partial_{x_1} a_2 - \partial_{x_2} a_1)\Big),$$

we have

$$B(\mu,h) f(Op_\alpha^w h_\alpha) = \mu Op_\alpha^w(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3) + O(\mu\alpha^3),$$

Here

$$\gamma_0 = 2\vec{a}\cdot(\xi_1 + x_2, \xi_2, \xi_3) f(h_\alpha + \alpha), \tag{4.3}$$

$$\gamma_1 = 2\Big(\vec{a}\cdot(\xi_1 + x_2, \xi_2, \xi_3) + \frac{1}{2i}\big[\nabla_\xi(\vec{a}\cdot(\xi_1 + x_2, \xi_2, \xi_3))\cdot\nabla_x(h_\alpha) - \nabla_x(\vec{a}\cdot(\xi_1 + x_2, \xi_2, \xi_3))\cdot\nabla_\xi(h_\alpha)\big]\Big) f'(h_\alpha + \alpha), \tag{4.4}$$

$$\gamma_2 = (\partial_{x_1} a_2 - \partial_{x_2} a_1) f(h_\alpha + \alpha) \tag{4.5}$$

and

$$\gamma_3 = \gamma_{3,1}(x,\xi) f'(h_\alpha + \alpha) + \gamma_{3,2}(x,\xi) f''(h_\alpha + \alpha), \tag{4.6}$$

where $\gamma_{3,1}, \gamma_{3,2} \in C_0^\infty(\mathbf{R}_x^3 \times \mathbf{R}_\xi^3)$. Since $\|\psi g_0(H)\|_1 = O((\mu/h)^{3/2})$ (see [Sob94][Cor.2.14]), we get:

$$\mathrm{tr}[B(\mu,h) g_0(H)] = \mathrm{tr}[\mu\psi Op_\alpha^w(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\psi g_0(H)] + O(\mu\alpha^{3/2}). \tag{4.7}$$

Let now $g \in C_0^\infty(\mathbf{R})$ such that

$$g(H) g_0(H) = g_0(H),$$

[1]The equation (4.2) seems very innocent, but a priori we will need global restrictions on V to prove it, see [Fou98]. It is easy to extend the localisation argument in [Sob95] to the present situation though. This is done in Appendix A.

i.e. $g \equiv 1$ on $[\inf \operatorname{Spec} V, 0]$. Then

$$\begin{aligned} & \operatorname{tr}[\mu\psi Op_\alpha^w(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\psi g_0(H)] \\ = \; & \operatorname{tr}[\mu\psi Op_\alpha^w(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\psi g(H) g_0(H) g(H)] \\ = \; & \mu \operatorname{tr}[g(H)\psi Op_\alpha^w(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\psi g(H) g_0(H)]. \end{aligned}$$

According to Theorem 3.5:

$$\|\psi g(H) - \psi \Phi g(P_0)\Theta^{(\mu)}\Phi^*\|_1 \leq c h^{-\frac{3}{2}(1+\zeta)}\omega(h,\mu), \tag{4.8}$$

and, when μh is sufficiently big:

$$g(P_0) = g(P_0^{(0)}) \otimes \Pi_0.$$

Thus

$$\begin{aligned} \operatorname{tr}[B(h,\mu)g_0(H)] \; = \; & \mu \operatorname{tr}\Big[\Phi\left(Op_h^w(\theta^{(\mu)})g(P_0^{(0)}) \otimes \Pi_0\right)\Phi^*\psi Op_\alpha^w(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\psi\Phi \\ & \times \left(g(P_0^{(0)})Op_h^w(\theta^{(\mu)}) \otimes \Pi_0\right)\Phi^* g_0(H)\Big] + O(\mu\alpha^{3/2}). \end{aligned} \tag{4.9}$$

Now we can apply Cor. 4.10 below to conclude:

$$\begin{aligned} \operatorname{tr}[B(h,\mu)g_0(H)] \; = \; & \frac{1}{\mu}\operatorname{tr}[\Phi\left(Op_h^w(\theta^{(\mu)})g(P_0^{(0)}) \otimes \Pi_0\right)(Op_h^w(r) \otimes \Pi_0) \\ & \times \left(g(P_0^{(0)})Op_h^w(\theta^{(\mu)}) \otimes \Pi_0\right)\Phi g_0(H)] + O(1 + \frac{1}{\mu h^2}), \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} r(\hat{x},\hat{\xi}) = \; & \left(a_2(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_1}V(x_1, \frac{-\xi_1}{\mu}, x_3) - a_1(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_2}V(x_1, \frac{-\xi_1}{\mu}, x_3)\right) \\ \times \; & f\left(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}\right) \end{aligned}$$

Here the error term was estimated using the fact that

$$\|Op_h^w(\theta^{(\mu)}) \otimes \Pi_0\|_1 = O(\mu/h^2).$$

In the same way we can calculate:

$$\begin{aligned} \operatorname{tr}[\psi(Op_\alpha^w b)\psi g_0(H)] \; = \; & \operatorname{tr}[g(H)\psi(Op_\alpha^w b)\psi g(H) g_0(H)] \\ \approx \; & \operatorname{tr}[\Phi\left(Op_h^w(\theta^{(\mu)})g(P_0^{(0)}) \otimes \Pi_0\right)\Phi^*\psi Op_\alpha^w(b)\psi\Phi \\ & \times \left(g(P_0^{(0)})Op_h^w(\theta^{(\mu)}) \otimes \Pi_0\right)\Phi g_0(H)], \end{aligned}$$

using (4.8). We apply Lemma 4.4 and get:

$$\begin{aligned} \operatorname{tr}[\psi(Op_\alpha^w b)\psi g_0(H)] \; = \; & \operatorname{tr}[\Phi\left(Op_h^w(\theta^{(\mu)})g(P_0^{(0)}) \otimes \Pi_0\right)(Op_h^w(r) \otimes \Pi_0) \\ & \times \left(g(P_0^{(0)})Op_h^w(\theta^{(\mu)}) \otimes \Pi_0\right)\Phi g_0(H)] + O(h^{-1}). \end{aligned}$$

Comparing with (4.10) we get the lemma. □

Lemma 4.3. *Let $\nu \in C_0^\infty(\mathbf{R}_x^3 \times \mathbf{R}_\xi^3)$, and let κ be the canonical transformation constructed in Theorem 3.1 then*

$$\nu(\kappa(x,\xi)) = \nu(x,\xi) + \epsilon\nu_1 + O(\epsilon^2),$$

where

$$\nu_1 = \begin{pmatrix} \partial_x \nu \\ \partial_\xi \nu \end{pmatrix} \cdot \begin{pmatrix} -\partial_\xi A \\ \partial_x A \end{pmatrix}.$$

where A is given in (4.11) below[2].

Proof. We have from [Sob94][Equation (7.9)] that if $(y,\eta)=\kappa(x,\xi)$ then

$$\begin{aligned} y &= x-\epsilon\sum_{j=0}^{M}\partial_\xi\left(\sigma(\xi_3^2)\sigma(x_2^2+\xi_2^2)A_j(x,\xi)\right)+O(\epsilon^2),\\ \eta &= \xi+\epsilon\sum_{j=0}^{M}\partial_x\left(\sigma(\xi_3^2)\sigma(x_2^2+\xi_2^2)A_j(x,\xi)\right)+O(\epsilon^2). \end{aligned}$$

where the A_j are given in the definition of κ. So

$$\begin{aligned} y &= x-\epsilon\partial_\xi A(x,\xi)+O(\epsilon^2),\\ \eta &= \xi+\epsilon\partial_x A(x,\xi)+O(\epsilon^2), \end{aligned}$$

where

$$\begin{aligned} A(x,\xi) &= \sigma(\xi_3^2)\sigma(x_2^2+\xi_2^2)\times\\ &\left(\frac{\xi_2}{2}\partial_z W(x_1,z,x_3)|_{z=-\xi_1}+\frac{x_2}{2}W(x_1,-\xi_1,x_3)+\sum_{2\le l+n+k\le 2M+1}a_{n,k}^{0,l}(\hat{x},\xi_1)\xi_2^n x_2^k\xi_3^l\right). \end{aligned} \tag{4.11}$$

Here the $a_{n,k}^{0,l}$ lie in $\mathcal{B}^\infty$ and are part of the definition of κ [Sob96b][Thm. 7.4]. Thus the lemma follows by taking a Taylor expansion to second order. □

Lemma 4.4. *Let $\nu\in C_0^\infty(\mathbf{R}_x^3\times\mathbf{R}_\xi^3)$, then*

$$(I\otimes\Pi_0)\Phi^*Op_\alpha^w(\nu)\Phi(I\otimes\Pi_0)=Op_h^w(e)\otimes\Pi_0+\epsilon Op_h^w(e_1)\otimes\Pi_0+O(\alpha^2),$$

where

$$e(\hat{x},\hat{\xi})=\nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})+\frac{h}{4\mu}(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{\xi_2}^2-2\partial_{x_1}\partial_{\xi_2})\nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}),$$

and where

$$\begin{aligned} e_1(\hat{x},\hat{\xi}) &= \left[\frac{1}{2}\left(-\partial_{x_1}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})+\partial_{\xi_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})\right)\right.\\ &\times\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3)\\ &\left.-\frac{1}{2}\partial_{x_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})\partial_z W(x_1,z,x_3)|_{z=\frac{-\xi_1}{\mu}}\right]\sigma(\frac{\xi_3^2}{\mu^2})+\frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}), \end{aligned}$$

where $\zeta\in C_0^\infty(\mathbf{R}_{\hat{x}}^2\times\mathbf{R}_{\hat{\xi}}^2)$.

Proof. From (3.1) it follows that

$$\Phi^*Op_\alpha^w\nu\Phi=U_\mu^*T^*Op_\alpha^w\tilde{\nu}TU_\mu$$

where $\tilde{\nu}(x,\xi)=\nu(x_1-\xi_2,x_2-\xi_1,x_3;\xi)$. Part (3) of Theorem 3.1 now tells us that:

$$T^*(Op_\alpha^w\tilde{\nu})T=Op_\alpha^w(\tilde{\nu}\circ\kappa)+O(\alpha^2),$$

and from Lemma 4.3 we get

$$\tilde{\nu}(\kappa(x,\xi))=\tilde{\nu}(x,\xi)+\epsilon\tilde{\nu}_1(x,\xi)+O(\epsilon^2),$$

[2]Remember that $W(x)=V(x)$ on $B(3E)$, and that $W\in C_0^\infty$.

Because of (3.3) we therefore conclude that

$$\Phi^* Op_\alpha^w \nu \Phi = Op_h^w \bar{\nu} + \epsilon Op_h^w \bar{\nu}_1 + O(\alpha^2),$$

where

$$\begin{aligned}\bar{\nu}(x,\xi) &= \tilde{\nu}(x_1, \frac{x_2}{\sqrt{\mu}}, x_3; \mu^{-1}\xi_1, \frac{\xi_2}{\sqrt{\mu}}, \mu^{-1}\xi_3)\\ \bar{\nu}_1(x,\xi) &= \tilde{\nu}_1(x_1, \frac{x_2}{\sqrt{\mu}}, x_3; \mu^{-1}\xi_1, \frac{\xi_2}{\sqrt{\mu}}, \mu^{-1}\xi_3).\end{aligned}$$

So

$$(I \otimes \Pi_0)\Phi^* Op_\alpha^w(\nu)\Phi(I \otimes \Pi_0) = (Op_h^w(e) + \epsilon Op_h^w(e_1)) \otimes \Pi_0 + O(\alpha^2),$$

where e, e_1 have symbols

$$e(\hat{x},\hat{\xi}) = \frac{1}{2\pi h}\iiint \mathcal{H}_0(x_2) e^{ih^{-1}(x_2-y_2,\xi_2)}\bar{\nu}(\hat{x},\hat{\xi},\frac{x_2+y_2}{2},\xi_2)\mathcal{H}_0(y_2)\quad dx_2\,dy_2\,d\xi_2, \tag{4.12}$$

and

$$e_1(\hat{x},\hat{\xi}) = \frac{1}{2\pi h}\iiint \mathcal{H}_0(x_2) e^{ih^{-1}(x_2-y_2,\xi_2)}\bar{\nu}_1(\hat{x},\hat{\xi},\frac{x_2+y_2}{2},\xi_2)\mathcal{H}_0(y_2)\quad dx_2\,dy_2\,d\xi_2.$$

Let us first analyze e:

We can look upon the expression (4.12) as the expectation value of the operator $Op_h^w s(x_2,\xi_2)$ in the state $\mathcal{H}_0$. Here the symbol s depends on the parameters $(\hat{x},\hat{\xi})$ in the sense that

$$s(x_2,\xi_2) = \bar{\nu}(x,\xi).$$

Since $\bar{\nu}$ depends on (x_2,ξ_2) in the form $(\frac{x_2}{\sqrt{\mu}}, \frac{\xi_2}{\sqrt{\mu}})$, we get from the laws for changing symbol types:

$$\begin{aligned}e(\hat{x},\hat{\xi}) &= \langle \mathcal{H}_0, Op_{h,0} s \mathcal{H}_0\rangle + h\langle \mathcal{H}_0, Op_{h,0} s_1 \mathcal{H}_0\rangle + O(\alpha^2)\\ &= I_1 + I_2 + O(\alpha^2),\end{aligned}$$

where

$$s_1(x,\xi) = \frac{1}{2i}\partial_{x_2}\partial_{\xi_2}\bar{\nu}.$$

Let us remember that

$$\begin{aligned}\mathcal{H}_0(x) &= \frac{1}{\sqrt[4]{\pi h}} e^{-x^2/(2h)},\\ \mathcal{H}_0(\xi) &= \frac{1}{\sqrt{2\pi h}}\int e^{-ix\xi/h}\mathcal{H}_0(x)dx.\end{aligned}$$

So if we look at

$$I_1 = \langle \mathcal{H}_0, Op_{h,0} s \mathcal{H}_0\rangle,$$

we get

$$\begin{aligned}I_1 &= \frac{1}{2\pi h}\iiint \mathcal{H}_0(x) e^{ih^{-1}(x-y,\xi)} s(x,\xi)\mathcal{H}_0(y)\quad dx\,dy\,d\xi\\ &= \frac{1}{\sqrt{2}\pi h}\iint e^{-x^2/(2h)} e^{ih^{-1}x\xi} s(x,\xi) e^{-\xi^2/(2h)} dx d\xi\\ &= \frac{1}{\sqrt{2}\pi h}\iint e^{ih^{-1}(2x\xi + i(x^2+\xi^2)/2)} s(x,\xi) dx d\xi\\ &= \frac{1}{\sqrt{2}\pi h}\iint e^{ih^{-1}\langle (x,\xi), A(x,\xi)\rangle/2} s(x,\xi) dx d\xi,\end{aligned}$$

where A is the matrix:

$$A = \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

From the theorem of stationary phase ([Ḧ90][Lemma 7.7.3]) we get:

$$I_1 \approx \frac{1}{\sqrt{2}\pi h}\left(\sqrt{\det(h^{-1}A/(2\pi i)}\right)^{-1}\sum_{j=0}^{\infty}\frac{h^j}{(2i)^j}\frac{(A^{-1}D,D)^j}{j!}s|_{(0,0)},$$

in the sense of an asymptotic series. We easily see that

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix},$$

and therefore

$$(A^{-1}D,D) = \frac{1}{2}[i\Delta - 2\partial_x\partial_\xi],$$

and

$$\left(\sqrt{\det(h^{-1}A/(2\pi i)}\right)^{-1} = \sqrt{2}\pi h.$$

So we get:

$$I_1 = s(0,0) + \frac{h}{4i}[i\Delta - 2\partial_x\partial_\xi]s|_{(0,0)} + O(\alpha^2).$$

By the same method

$$\begin{aligned} I_2 &= \langle \mathcal{H}_0, hOp_{h,0}s_1\mathcal{H}_0\rangle \\ &= hs_1(0,0) + O(\alpha^2), \end{aligned}$$

so

$$I_1 + I_2 = s(0,0) + \frac{h}{4}\Delta s(0,0) + O(\alpha^2).$$

Thus

$$\begin{aligned} e(\hat{x},\hat{\xi}) &= \tilde{\nu}(x_1,0,x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + \frac{h}{4\mu}\Delta_{(x_2,\xi_2)}\tilde{\nu}(x_1,0,x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + O(\alpha^2) \\ &= \nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + \frac{h}{4\mu}(\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{\xi_2}^2 - 2\partial_{x_1}\partial_{\xi_2})\nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) \\ &+ O(\alpha^2). \end{aligned} \tag{4.13}$$

In the same way we get:

$$e_1(\hat{x},\hat{\xi}) = \tilde{\nu}_1(x_1,0,x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}) + O(\alpha)$$

Here

$$\tilde{\nu}_1 = \begin{pmatrix} \partial_x\tilde{\nu} \\ \partial_\xi\tilde{\nu} \end{pmatrix}\cdot\begin{pmatrix} -\partial_\xi A \\ \partial_x A \end{pmatrix}.$$

We can thus calculate:

$$
\begin{aligned}
\tilde{\nu}_1|_{x_2=\xi_2=0} = \; & -\partial_{x_1}\tilde{\nu} \sum_{2\leq l\leq 2M+1} \xi_3^l \partial_{\xi_1} a_{0,0}^{0,l}(\hat{x},\xi_1)\sigma(\xi_3^2) \\
& -\partial_{x_2}\tilde{\nu}\left(\frac{1}{2}\partial_z W(x_1,z,x_3)|_{z=-\xi_1} + \sum_{2\leq l+1\leq 2M+1} \xi_3^l a_{1,0}^{0,l}(\hat{x},\xi_1)\right)\sigma(\xi_3^2) \\
& -\partial_{x_3}\tilde{\nu}\left(\sum_{2\leq l\leq 2M+1} l\xi_3^{l-1} a_{0,0}^{0,l}(\hat{x},\xi_1)\sigma(\xi_3^2) + \sigma'(\xi_3^2)2\xi_3 \sum_{2\leq l\leq 2M+1} \xi_3^l a_{0,0}^{0,l}(\hat{x},\xi_1)\right) \\
& +\partial_{\xi_1}\tilde{\nu} \sum_{2\leq l\leq 2M+1} \xi_3^l \partial_{x_1} a_{0,0}^{0,l}(\hat{x},\xi_1)\sigma(\xi_3^2) \\
& +\partial_{\xi_2}\tilde{\nu}\left(\frac{1}{2}\partial_{x_1} W(x_1,-\xi_1,x_3) + \sum_{2\leq l+1\leq 2M+1} \xi_3^l a_{1,0}^{0,l}(\hat{x},\xi_1)\right)\sigma(\xi_3^2) \\
& +\partial_{\xi_3}\tilde{\nu} \sum_{2\leq l\leq 2M+1} \partial_{x_3} a_{0,0}^{0,l}(\hat{x},\xi_1)\xi_3^l\sigma(\xi_3^2) \\
= \; & \left(\frac{1}{2}(\partial_{\xi_2}\tilde{\nu})\partial_{x_1} W(x_1,z,x_3) - \frac{1}{2}(\partial_{x_2}\tilde{\nu})\partial_z W(x_1,z,x_3)|_{z=-\xi_1}\right)\sigma(\xi_3^2) \\
& +\xi_3\zeta(\hat{x},\hat{\xi}),
\end{aligned}
$$

where $\zeta \in C_0^\infty(\mathbf{R}_x^2 \times \mathbf{R}_\xi^2)$. Now $\tilde{\nu}(x,\xi) = \nu(x_1-\xi_2, x_2-\xi_1, x_3, \xi)$ so we get:

$$
\begin{aligned}
\tilde{\nu}_1|_{x_2=\xi_2=0} = \; & \Big[\frac{1}{2}(-\partial_{x_1}\nu(x_1,-\xi_1,x_3,\hat{\xi}) + \partial_{\xi_2}\nu(x_1,-\xi_1,x_3,\hat{\xi}))\partial_{x_1}W(x_1,-\xi_1,x_3) \\
& -\frac{1}{2}(\partial_{x_2}\nu(x_1,-\xi_1,x_3,\hat{\xi}))\partial_z W(x_1,z,x_3)|_{z=-\xi_1}\Big]\sigma(\xi_3^2) + \xi_3\zeta(\hat{x},\hat{\xi}).
\end{aligned}
$$

Finally, we get

$$
\begin{aligned}
\tilde{e_1}(\hat{x},\hat{\xi}) = \; & \Big[\frac{1}{2}(-\partial_{x_1}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu}) + \partial_{\xi_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu}))\partial_{x_1}W(x_1,-\xi_1,x_3) \\
& -\frac{1}{2}(\partial_{x_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu}))\partial_z W(x_1,z,x_3)|_{z=\frac{-\xi_1}{\mu}}\Big]\sigma(\frac{\xi_3^2}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}),
\end{aligned}
$$

where $\zeta \in C_0^\infty(\mathbf{R}_{\hat{x}}^2 \times \mathbf{R}_{\hat{\xi}}^2)$. □

Corollary 4.5. *Suppose $\vec{a} = (a_1,a_2,0)$. Then*

$$
(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\gamma_0)\Phi(I\otimes\Pi_0) = \frac{h}{\mu}Op_h^w(\tilde{d})\otimes\Pi_0 + \epsilon Op_h^w\tilde{d}_1\otimes\Pi_0 + O(\alpha^2),
$$

where

$$
\tilde{d} = \left(\partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3) - \partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3)\right) f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}),
$$

and

$$
\begin{aligned}
\tilde{d}_1(\hat{x},\hat{\xi}) \; & = \left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3) - a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\right) \\
& \times f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}).
\end{aligned}
$$

Proof. From (4.3) we know that

$$\gamma_0 = 2\left(a_1(\xi_1 + x_2) + a_2\xi_2\right) f(h_\alpha + \alpha),$$

so

$$\begin{aligned}
\gamma_0(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 0,\\
\partial^2_{x_1}\gamma_0(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 0,\\
\partial^2_{x_2}\gamma_0(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 4\partial_{x_2} a_1(x_1, -\frac{\xi_1}{\mu}, x_3) f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2})\\
&\quad + 4\epsilon a_1(x_1, -\frac{\xi_1}{\mu}, x_3) f'(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2})\partial_{x_2} V(x_1, -\frac{\xi_1}{\mu}, x_3),\\
\partial^2_{\xi_2}\gamma_0(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 0,\\
-2\partial_{x_1}\partial_{\xi_2}\gamma_0(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= -4\partial_{x_1} a_2(x_1, -\frac{\xi_1}{\mu}, x_3) f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2})\\
&\quad - 4\epsilon a_2(x_1, -\frac{\xi_1}{\mu}, x_3) f'(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2})\partial_{x_1} V(x_1, -\frac{\xi_1}{\mu}, x_3).
\end{aligned}$$

Thus

$$\tilde{d} = \left(\partial_{x_2} a_1(x_1, -\frac{\xi_1}{\mu}, x_3) - \partial_{x_1} a_2(x_1, -\frac{\xi_1}{\mu}, x_3)\right) f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}).$$

We can also calculate:

$$\begin{aligned}
\partial_{x_2}\gamma_0(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 2a_1(x_1, \frac{-\xi_1}{\mu}, x_3) f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2})\\
\partial_{\xi_2}\gamma_0(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 2a_2(x_1, \frac{-\xi_1}{\mu}, x_3) f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2})\\
\partial_{x_1}\gamma_0(x_1, \frac{-\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu}) &= 0,
\end{aligned}$$

so we get:

$$\begin{aligned}
\tilde{d}_1(\hat{x}, \hat{\xi}) &= \left(a_2(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_1} W(x_1, \frac{-\xi_1}{\mu}, x_3) - a_1(x_1, \frac{-\xi_1}{\mu}, x_3)\partial_{x_2} W(x_1, \frac{-\xi_1}{\mu}, x_3)\right)\\
&\quad \times f(\frac{\xi_3^2 + V(x_1, -\frac{\xi_1}{\mu}, x_3)}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x}, \frac{\hat{\xi}}{\mu}).
\end{aligned}$$

□

Remark 4.6. Notice that if a_3 had not been zero then $\gamma_0(x_1, -\frac{\xi_1}{\mu}, x_3, \frac{\xi_1}{\mu}, 0, \frac{\xi_3}{\mu})$ would not have vanished.

Lemma 4.7. *Let $\tilde{\theta} \in C_0^\infty(\mathbf{R}^4_{\hat{x},\hat{\xi}})$, and let $\tilde{\theta}^{(\mu)}(\hat{x}, \hat{\xi}) = \tilde{\theta}(\hat{x}, \frac{\xi_1}{\mu}, \xi_3)$, then*

$$Op_h^w(\frac{\xi_3}{\mu}\zeta(\hat{x}, \frac{\hat{\xi}}{\mu}))Op_h^w(\tilde{\theta}^{(\mu)}) = O(1/\mu).$$

Proof. This is an easy consequence of the symbolic calculus and the compactness of the support of $\tilde{\theta}$. □

Corollary 4.8. *Let* $\tilde\theta \in C_0^\infty(\mathbf{R}^4_{\hat x,\hat\xi})$, *and let* $\tilde\theta^{(\mu)}(\hat x,\hat\xi) = \tilde\theta(\hat x, \frac{\xi_1}{\mu}, \xi_3)$, *then*

$$(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\gamma_1)\Phi(Op_h^w(\tilde\theta^{(\mu)})\otimes\Pi_0) = O(\epsilon)$$

and

$$(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\gamma_3)\Phi(Op_h^w(\tilde\theta^{(\mu)})\otimes\Pi_0) = O(\epsilon).$$

Proof. This follows easily from Lemma 4.4 because $f'\equiv 0$ on a neighborhood of 0, and therefore

$$f'(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}) \equiv 0$$

on the support of $\tilde\theta^{(\mu)}$ for μ sufficiently big. Same argument works for f''. ☐

Corollary 4.9.

$$(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\gamma_2)\Phi(I\otimes\Pi_0) = Op_h^w(\tilde d_2)\otimes\Pi_0 + O(\epsilon),$$

where

$$\tilde d_2(\hat x,\hat\xi) = \left(\partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3) - \partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3)\right) f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}).$$

To summarize the content of the above we have:

Corollary 4.10.

$$\begin{aligned}&(I\otimes\Pi_0)\Phi^* Op_\alpha^w(\gamma_0+\alpha\gamma_1+\alpha\gamma_2+\alpha^2\gamma_3)\Phi(Op_h^w\theta^{(\mu)}\otimes\Pi_0)\\ =\;& \epsilon(Op_h^w(r)Op_h^w(\theta^{(\mu)}))\otimes\Pi_0 + O(\alpha^2+\epsilon/\mu),\end{aligned}$$

where

$$\begin{aligned}r(\hat x,\hat\xi) = \;& \left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3) - a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\right)\\ &\times f(\frac{\xi_3^2+V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}).\end{aligned}$$

5. Calculation of the Current

With the reduced operator it is rather easy to calculate the current:
Choose $f_1, f_2 \in C_0^\infty(\mathbf{R})$ such that:

- $(f_1^2(H)+f_2^2(H))g_0(H) = g_0(H)$.
- $|\partial_{x_1}V(x)|^2 + |\partial_{x_3}V(x)|^2 + |V(x)-\lambda| \geq c > 0$ for all $(x,\lambda)\in B(2E)\times \operatorname{supp} f_2$.

Then

$$\begin{aligned}\operatorname{tr}[\psi(Op_\alpha^w b)\psi g_0(H)] &= \operatorname{tr}[\psi(Op_\alpha^w b)\psi f_1^2(H)g_0(H)]\\ &\quad+\operatorname{tr}[\psi(Op_\alpha^w b)\psi f_2^2(H)g_0(H)].\end{aligned}$$

The first part, $\operatorname{tr}[\psi(Op_\alpha^w b)\psi f_1^2(H)g_0(H)]$, will be calculated directly in Thm 5.1 below. To handle the second term, $\operatorname{tr}[\psi(Op_\alpha^w b)\psi f_2^2(H)g_0(H)]$, we need a Tauberian argument. The theorems 5.2 and 5.4 will carry this through. From the Theorems 5.1, 5.2 and 5.4 together we get Theorem 2.5 by a simple integration by parts.

Theorem 5.1.

$$\frac{1}{\mu}\mathrm{tr}[\psi(Op_\alpha^w b)\psi f_1^2(H)g_0(H)] =$$
$$\frac{1}{4\pi^2h^2}\iint (a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3) - a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3))$$
$$\times f_1^2(\xi_3^2 + V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi} + O(h^{-1}).$$

Proof. $f_1^2(H)g_0(H) = f_1^2(H)$ and we get:

$$\begin{aligned}
\frac{1}{\mu}\mathrm{tr}[\psi(Op_\alpha^w b)\psi f_1^2(H)] &= \frac{1}{\mu}\mathrm{tr}[f_1(H)\psi(Op_\alpha^w b)\psi f_1(H)] \\
&= \frac{1}{\mu}\mathrm{tr}[\Phi\left((Op_h^w\theta^{(\mu)})f_1(P_0^{(0)})Op_h^w(r)f_1(P_0^{(0)})(Op_h^w\theta^{(\mu)})\otimes\Pi_0\right)\Phi^*] \\
&\quad +O(h^{-1}) \\
&= \frac{1}{\mu}\mathrm{tr}[(Op_h^w\theta^{(\mu)})^2 f_1(P_0^{(0)})Op_h^w(r)f_1(P_0^{(0)})] + O(h^{-1}) \\
&= \frac{1}{(2\pi h)^2}\iint r(\hat{x},\mu\hat{\xi})f_1^2(\xi_3 + V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi} + O(h^{-1}),
\end{aligned}$$

where we used [Sob94][Lemma 9.2] to get the last equality. □

We need to make a standard "smoothing out": Let $\hat{\chi} \in C_0^\infty(-T,T)$ for a "sufficiently small [3]" T satisfy

- $\hat{\chi}(t) = \hat{\chi}(-t)$,
- $\hat{\chi}(t) = 1/\sqrt{2\pi}$,
- $\hat{\chi} \geq 0$.

Then we define

$$\chi(\tau) = \frac{1}{\sqrt{2\pi}} \in \hat{\chi}(t)e^{i\tau t}dt.$$

We assume that $\chi \geq 0$, this is possible since we could have replaced $\hat{\chi}$ by $\hat{\chi} * \hat{\chi}$. Finally, we define

$$\chi_h(\tau) = \frac{1}{h}\chi(\frac{\tau}{h}),$$

and

$$g_0^{(h)}(\tau) = \int g_0(\sigma)\chi_h(\tau-\sigma)d\sigma.$$

Now we can state:

Theorem 5.2.

$$\mathrm{tr}[\psi(Op_\alpha^w b)\psi f_2^2(H)g_0(H)] = \mathrm{tr}[\psi(Op_\alpha^w b)\psi f_2^2(H)g_0^{(h)}(H)] + O(\mu/h).$$

We will need the following lemma:

Lemma 5.3. $\exists\epsilon > 0$ *such that*

$$|g_0(\tau) - g_0(\tau-\rho)| \leq ch\chi_h(\tau),$$

for all $|\rho| \leq h\epsilon$. *Here* c *is a constant independent of* h *and* τ.

[3] See [Sob94][Sec.4] for a more precise statement.

Proof. Choose 2ϵ such that $\chi_1(\tau) \geq \tilde{c} > 0$ for $|\tau| \leq 2\epsilon$. Then, for $|\tau| \leq 2\epsilon h$ and $|\rho| \leq h\epsilon$ we have

$$|g_0(\tau) - g_0(\tau - \rho)| \leq \frac{1}{\tilde{c}}\chi_1(\tau/h) = \frac{h}{\tilde{c}}\chi_h(\tau),$$

and for $|\tau| \geq 2\epsilon h$ and $|\rho| \leq h\epsilon$ we have

$$|g_0(\tau) - g_0(\tau - \rho)| = 0.$$

□

Now we can prove Theorem 5.2.

Proof. In this proof (and only here) we will use the notation: $[x]$ = the integral part of x $= \inf\{n \in \mathbf{Z} | n \leq x\}$. By cyclicity of trace it is enough to prove

$$\|Op_\alpha^w(b)\psi f_2(H)\left(g_0(H) - g_0^{(h)}(H)\right) f_2(H)\psi\|_1 = O(\mu/h).$$

Because $\|Op_\alpha^w(b)\| = O(1)$ it is thus enough to prove

$$\|\psi f_2(H)\left(g_0(H) - g_0^{(h)}(H)\right) f_2(H)\psi\|_1 = O(\mu/h).$$

We now estimate using the lemma above:

$$\begin{aligned}
& \|\psi f_2(H)\left(g_0(H) - g_0^{(h)}(H)\right) f_2(H)\psi\|_1 \\
= \; & \|\psi f_2(H) \int \chi_h(\rho)\left(g_0(H) - g_0(H-\rho)\right) d\rho f_2(H)\psi\|_1 \\
= \; & \|\psi f_2(H) \int_{-\delta}^{\delta} \chi_h(\rho)\left(g_0(H) - g_0(H-\rho)\right) d\rho f_2(H)\psi\|_1 + O(h^\infty) \\
\leq \; & \int_0^\delta \operatorname{tr}\left(\psi f_2(H)\chi_h(\rho)\left(g_0(H-\rho) - g_0(H)\right) f_2(H)\psi\right) d\rho \\
& + \int_{-\delta}^0 \operatorname{tr}\left(\psi f_2(H)\chi_h(\rho)\left(g_0(H) - g_0(H-\rho)\right) f_2(H)\psi\right) d\rho + O(h^\infty) \\
\leq \; & \int_{-\delta}^{\delta} \chi_h(\rho) \operatorname{tr}\left(\psi f_2(H)\left(\sum_{j=0}^{[\frac{|\rho|}{h\epsilon}]} ch\chi_h(H - \operatorname{sign}(\rho) jh\epsilon) + ch\chi_h(H + |\rho| - \operatorname{sign}(\rho)[\frac{|\rho|}{h\epsilon}]h\epsilon)\right)\right. \\
& \left. \times f_2(H)\psi \right) d\rho + O(h^\infty) \\
\leq \; & c\frac{\mu}{h} \int \chi_h(\rho)(\frac{|\rho|}{h} + 1) d\rho + O(h^\infty) \\
= \; & O(\mu/h),
\end{aligned}$$

where we used

$$\|\psi f_2(H)\chi_h(H - \tau)\|_1 = O(\mu/h^2)$$

in the end. That inequality comes from [Sob94][Theorem 10.4]. □

Theorem 5.4. *Suppose that (2.1) is satisfied. Then*

$$\begin{aligned}&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)]\\ =\ &\frac{1}{4\pi^2h^2}\int\Big(a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3)-a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3)\Big)\\ &\times(f_2^2g_0)(\xi_3^2+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi}+O(h^{-1}).\end{aligned}$$

Proof.

$$\begin{aligned}\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)] &= \frac{1}{\mu}\int g_0(\tau)\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)\chi_h(H-\tau)]d\tau\\ &= \frac{1}{\sqrt{2\pi}\mu h}\iint g_0(\tau)\hat{\chi}(t)\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)e^{-it(H-\tau)/h}]dt\,d\tau\\ &= \frac{1}{\sqrt{2\pi}\mu h}\iint g_0(\tau)\hat{\chi}(t)e^{it\tau/h}\mathrm{tr}[f_2(H)\psi(Op^w_\alpha b)\psi f_2(H)e^{-itH/h}]dt\,d\tau.\end{aligned}$$

Notice, that since χ_h is a Schwarz function, we can replace g_0 by $1_{[-E_0,0]}$. This will only introduce an error of order $O(h^\infty)$, and makes the integral absolutely convergent. Now we apply Theorem 3.5:

$$\|\psi f_2(H)e^{-itH/h}-\psi\Phi f_2(P_0)e^{-itP_0/h}\Theta^{(\mu)}\Phi^*\|_1\leq ch^{\frac{3}{2}(1+\zeta)}\omega(h,\mu).$$

Since $\Phi^*\Phi\approx I$ and μh is large, we thus get:

$$\begin{aligned}&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi f_2^2(H)g_0^{(h)}(H)]\\ =\ &\frac{1}{\mu}\mathrm{tr}[(Op^w_h\theta^{(\mu)})^2f_2(P_0^{(0)})(Op^w_h r)f_2(P_0^{(0)})g_0^{(h)}(P_0^{(0)})]+O(\frac{1}{\mu h}),\end{aligned}$$

and we conclude using [Sob94][Lemma 9.3]. □

Now we can prove Theorem 2.5:

Proof. From the Theorems 5.1, 5.2 and 5.4 together we get

$$\begin{aligned}&\frac{1}{\mu}\mathrm{tr}[\psi(Op^w_\alpha b)\psi g_0(H)]=\\ &\frac{1}{4\pi^2h^2}\iint\Big[a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3)-a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3))\Big]\\ &\times g_0(\xi_3^2+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi}+O(h^{-1}).\end{aligned}$$

Now we calculate:

$$\frac{1}{4\pi^2h^2}\iint\Big[a_2(x_1,-\xi_1,x_3)\partial_{x_1}V(x_1,-\xi_1,x_3)-a_1(x_1,-\xi_1,x_3)\partial_{x_2}V(x_1,-\xi_1,x_3))\Big]$$

$$\times\, g_0(\xi_3^2+V(x_1,-\xi_1,x_3))d\hat{x}d\hat{\xi}$$

$$=\frac{1}{4\pi^2h^2}\int_{\{V(x)\leq 0\}}\Big[a_2(x)\partial_{x_1}V(x)-a_1(x)\partial_{x_2}V(x)\Big]2\sqrt{-V(x)}dx$$

$$=\frac{-1}{2\pi^2h^2}\int_{\{V(x)\leq 0\}}a_2\frac{2}{3}\partial_{x_1}(\sqrt{-V(x)})^3-a_1\frac{2}{3}\partial_{x_2}(\sqrt{-V(x)})^3dx$$

$$=\frac{1}{3\pi^2h^2}\int_{\{V(x)\leq 0\}}(\partial_{x_1}a_2-\partial_{x_2}a_1)(\sqrt{-V(x)})^3dx.$$

This finishes the proof of Theorem 2.5. □

6. The Current for Bounded μh.

In the case where $\mu h\leq C$, $\mu\geq ch^{-\rho}$ for a $\rho\in(0,1]$ we can use the same type of analysis as in the case of the very strong magnetic field.

6.1. Projection on the Landau Levels.

Lemma 6.1. *Let* $\nu\in C_0^\infty(\mathbf{R}^3_x\times\mathbf{R}^3_\xi)$ *and let* $K\geq 0$. *Then*

$$\sup_{k:k\mu h\leq K}\|(I\otimes\Pi_k)\Phi^*Op^w_\alpha(\nu)\Phi(I\otimes\Pi_k)-Op^w_h(e^{(k)})\otimes\Pi_k+\epsilon Op^w_h(e_1^{(k)})\otimes\Pi_k\|=O(\epsilon^2+\alpha^2)$$

where

$$\begin{aligned}e^{(k)}(\hat{x},\hat{\xi}) &= \nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu})\\ &\quad+\frac{(2k+1)h}{4\mu}(\partial_{x_1}^2+\partial_{x_2}^2+\partial_{\xi_2}^2-2\partial_{x_1}\partial_{\xi_2})\nu(x_1,-\frac{\xi_1}{\mu},x_3,\frac{\xi_1}{\mu},0,\frac{\xi_3}{\mu}),\end{aligned}$$

and where

$$\begin{aligned}e_1^{(k)}(\hat{x},\hat{\xi}) = &\Bigg[\frac{1}{2}\left(-\partial_{x_1}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu})+\partial_{\xi_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu})\right)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3)\\ &-\frac{1}{2}\partial_{x_2}\nu(x_1,\frac{-\xi_1}{\mu},x_3,\frac{\hat{\xi}}{\mu})\partial_z W(x_1,z,x_3)|_{z=\frac{-\xi_1}{\mu}}\Bigg]\sigma(\frac{\xi_3^2}{\mu^2})+\frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}),\end{aligned}$$

where $\zeta\in C_0^\infty(\mathbf{R}^2_{\hat{x}}\times\mathbf{R}^2_{\hat{\xi}})$.

Proof. As in the proof of Lemma 4.4 we get:

$$\Phi^*Op^w_\alpha\nu\Phi=Op^w_h\bar{\nu}+\epsilon Op^w_h\bar{\nu_1}+O(\alpha^2+\epsilon^2),$$

with notation as in that lemma. We now appeal to [Sob94][Lemma A.1] (stated below as Lemma 6.2) to conclude that

$$(I\otimes\Pi_k)\Phi^*Op^w_\alpha(\nu)\Phi(I\otimes\Pi_k)=(Op^w_h(\bar{\nu}_{0,k})+\epsilon Op^w_h(\bar{\nu}_{1,k}))\otimes\Pi_k+O(\epsilon^2+\alpha^2),$$

where

$$\bar{\nu}_{0,k}(\hat{x},\hat{\xi})=\bar{\nu}_{sym}(x_1,\sqrt{\frac{(2k+1)h}{2\mu}},x_3,\xi_1,\sqrt{\frac{(2k+1)h}{2\mu}},\xi_3),$$

and

$$\bar{\nu}_{1,k}(\hat{x},\hat{\xi}) = \bar{\nu}_{1,sym}(x_1, \sqrt{\frac{(2k+1)h}{2\mu}}, x_3, \xi_1, \sqrt{\frac{(2k+1)h}{2\mu}}, \xi_3),$$

By a Taylor expansion of $\bar{\nu}_{0,k}$ we get

$$\bar{\nu}_{0,k}(\hat{x},\hat{\xi}) = \bar{\nu}(x_1,0,x_3,\xi_1,0,\xi_3) + \frac{(2k+1)h}{4\mu}(\partial^2_{x_2,x_2} + \partial^2_{\xi_2,\xi_2})\bar{\nu}(x_1,0,x_3,\xi_1,0,\xi_3) + O(h^4),$$

where the error was estimated using $O(k^2h^2/\mu^2) = O(h^4)$. If we compare this with eq.(4.13) we see that the expression for $e^{(k)}$ above is correct.
A Taylor expansion of $\bar{\nu}_{1,k}$ to first order and comparison with the proof of Lemma 4.4 finishes the proof. □

We used the following lemma:

Lemma 6.2. *Let $\nu \in \mathcal{B}(\mathbf{R}^3_x \times \mathbf{R}^3_\xi)$, $\delta \in (0,1)$ and define*

$$\nu^{(\delta)}(x,\xi) = \nu(x_1, \delta x_2, x_3, \xi_1, \delta\xi_2, \xi_3)$$

Then the following bound holds:

$$\sup_{k:hk\leq C\delta^2} \|(I\otimes\Pi_k)Op^w_h(\nu^{(\delta)})(I\otimes\Pi_k) - Op^w_h(\nu_k)(I\otimes\Pi_k)\| = O(\delta^8 + h^4),$$

where

$$\nu_k(\hat{x},\hat{\xi}) = \nu_{sym}((x_1, \sqrt{\frac{2k+1}{2}}\delta, x_3, \xi_1, \sqrt{\frac{2k+1}{2}}\delta, \xi_3).$$

Here we used the notation:

$$\begin{aligned} a_{sym}(x,\xi) \quad &= \quad \frac{1}{4}\Big(a(x_1,x_2,x_3,\xi_1,\xi_2,\xi_3) + a(x_1,-x_2,x_3,\xi_1,\xi_2,\xi_3) \\ &\quad + a(x_1,x_2,x_3,\xi_1,-\xi_2,\xi_3) + a(x_1,-x_2,x_3,\xi_1,-\xi_2,\xi_3)\Big). \end{aligned}$$

We get the following corollary (compare with Cor. 4.5).

Corollary 6.3. *Suppose $\vec{a} = (a_1,a_2,0)$. Then*

$$(I\otimes\Pi_k)\Phi^* Op^w_\alpha(\gamma_0)\Phi(I\otimes\Pi_k) = \frac{h}{\mu}(2k+1)Op^w_h(\tilde{d})\otimes\Pi_k + \epsilon Op^w_h\tilde{d}_1 + O(\epsilon^2+\alpha^2),$$

where

$$\tilde{d} = \left(\partial_{x_2}a_1(x_1,-\frac{\xi_1}{\mu},x_3) - \partial_{x_1}a_2(x_1,-\frac{\xi_1}{\mu},x_3)\right) f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}),$$

and

$$\begin{aligned} \tilde{d}_1(\hat{x},\hat{\xi}) \quad &= \left(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3) - a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\right) \\ &\quad \times f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}) + \frac{\xi_3}{\mu}\zeta(\hat{x},\frac{\hat{\xi}}{\mu}). \end{aligned}$$

And we can conclude the projection by stating:

Corollary 6.4. *Suppose $\vec{a} = (a_1,a_2,0)$, then we get the following estimate uniformly in k where $k\mu h \leq K$.*

$$\begin{aligned} &(I\otimes\Pi_k)\Phi^* Op^w_\alpha(\gamma_0 + \alpha\gamma_1 + \alpha\gamma_2 + \alpha^2\gamma_3)\Phi(Op^w_h\theta^{(\mu)}\otimes\Pi_k) \\ &= \left(\left(\frac{h}{\mu}2kOp^w_h(\tilde{d}) + \epsilon Op^w_h(r)\right)Op^w_h(\theta^{(\mu)})\right)\otimes\Pi_k + O(\alpha^2+\epsilon/\mu), \end{aligned}$$

where $\tilde{d}$ *was defined above, and where*

$$r(\hat{x},\hat{\xi}) = \Big(a_2(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_1}W(x_1,\frac{-\xi_1}{\mu},x_3) - a_1(x_1,\frac{-\xi_1}{\mu},x_3)\partial_{x_2}W(x_1,\frac{-\xi_1}{\mu},x_3)\Big) \times f(\frac{\xi_3^2 + V(x_1,-\frac{\xi_1}{\mu},x_3)}{\mu^2}).$$

6.2. **Calculation of the Current for the Spin-down Part.**
With the notation from Lemma 4.1 and section 5 we have:

$$\begin{aligned}\mathrm{tr}[B(\mu,h)g_0(H)] &= \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1^2(H)] + \\ &\quad \mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_2^2(H)g_0(H)] + O(\mu\alpha^{3/2}).\end{aligned}$$

We now analyze each term separately.

Theorem 6.5. *Suppose* $\vec{a} = (a_1,a_2,0)$ *then*

$$\begin{aligned}&\mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1^2(H)] \\ &= \sum_{0\leq k\leq c/(\mu h)} \frac{1}{4\pi^2h^2}\iint\Big(\mu h2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)] + [a_2(x)\partial_{x_1}W(x) - a_1(x)\partial_{x_2}W(x)]\Big) \\ &\quad \times f_1^2(\xi_3^2+2k\mu h+W(x))dxd\xi_3 + O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).\end{aligned}$$

Proof. We can calculate:

$$\begin{aligned}&\mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1^2(H)] \\ \simeq\ &\mu\mathrm{tr}[f_1(H)\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1(H)] \\ \simeq\ &\mu\mathrm{tr}[\Phi\Theta^{(\mu)}f_1(H)\Phi^*\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\Phi f_1(H)\Theta^{(\mu)}\Phi^*] \\ \simeq\ &\mu\sum_{0\leq k\leq c/(\mu h)}\mathrm{tr}[Op^w_h\theta^{(\mu)}f_1(P_0^{(k)})\Phi^*Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\Phi f_1(P_0^{(k)})Op^w_h\theta^{(\mu)}] \\ =\ &\mu\sum_{0\leq k\leq c/(\mu h)}\mathrm{tr}[Op^w_h\theta^{(\mu)}f_1(P_0^{(k)})\Phi^*\left(\frac{h}{\mu}2kOp^w_h(\tilde{d})+\epsilon Op^w_h(r)\right)\Phi f_1(P_0^{(k)})Op^w_h\theta^{(\mu)}] \\ &+O(\mu\frac{c}{\mu h}(\alpha^2+\epsilon/\mu)\mu/h^2)\end{aligned}$$

We used that $P_0^{(k)} \geq 2k\mu h - c$. The error can be written as $O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2})$, so with the definitions of r and $\tilde{d}$ we get from [Sob94][Lemma 9.2]

$$\begin{aligned}&\mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_1^2(H)] \\ &= \mu\sum_{0\leq k\leq c/(\mu h)}\frac{\mu}{4\pi^2h^2}\iint\left(\frac{h}{\mu}2k[\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x)] + \epsilon[a_2(x)\partial_{x_1}W(x)-a_1(x)\partial_{x_2}W(x)]\right) \\ &\quad \times f_1^2(\xi_3^2+2k\mu h+W(x))dxd\xi_3 + O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).\end{aligned}$$

□

Theorem 6.6.

$$\mu\mathrm{tr}[\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi f_2^2(H)(g_0(H)-g_0^{(h)}(H))] = O(h^{-1}+\mu).$$

Proof. We cannot use the argument from Theorem 5.2 right away, since $\|\psi Op^w_\alpha(\gamma_0+\alpha(\gamma_1+\gamma_2)+\alpha^2\gamma_3)\psi\| = O(1)$, and using this estimate would lead to a too big error. Therefore we have to try to improve the estimate:
Let

- $\tilde{f} \in C_0^\infty(\mathbf{R})$, $\tilde{f} \equiv 1$ on supp f_2.
- $\tilde{\psi} \in C_0^\infty(\mathbf{R}^3)$, $\tilde{\psi} \equiv 1$ on supp ψ.

Then we have from [Sob94][Cor.2.14] that $\|\psi \tilde{f}(H)(1-\tilde{\psi})\|_1 = O(h^\infty)$. Thus we get:

$$\begin{aligned}
& \mu \mathrm{tr}[\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2\gamma_3)\psi f_2^2(H)(g_0(H) - g_0^{(h)}(H))] \\
\simeq\ & \mu \mathrm{tr}[\tilde{f}(H)\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2\gamma_3)\psi \tilde{f}(H)\tilde{\psi} f_2(H)(g_0(H) - g_0^{(h)}(H))f_2(H)\tilde{\psi}] \\
\leq\ & \mu \|\tilde{f}(H)\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2\gamma_3)\psi \tilde{f}(H)\| \|\tilde{\psi} f_2(H)(g_0(H) - g_0^{(h)}(H))f_2(H)\tilde{\psi}\|_1.
\end{aligned}$$

The trace norm was estimated as $O(\mu/h)$ in the proof of Theorem 5.2, so let us look at the operator norm. Splitting into Landau levels as in the proof of the last theorem we get:

$$\begin{aligned}
& \|\tilde{f}(H)\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2\gamma_3)\psi \tilde{f}(H)\| \\
\simeq\ & \|\sum_{0\leq k \leq c/(\mu h)} \left(Op_h^w \theta^{(\mu)} \tilde{f}(P_0^{(k)}) \Phi^* \left(\frac{h}{\mu} 2k Op_h^w(\tilde{d}) + \epsilon Op_h^w(r) \right) \Phi \tilde{f}(P_0^{(k)}) Op_h^w \theta^{(\mu)} \right) \otimes \Pi_k \| \\
=\ & O(h/\mu + \mu^{-2}).
\end{aligned}$$

This finishes the proof. ∎

Theorem 6.7. *Assume $\vec{a} = (a_1, a_2, 0)$ and that (2.1) is satisfied, then*

$$\begin{aligned}
& \mu \mathrm{tr}[\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2\gamma_3)\psi f_2^2(H) g_0^{(h)}(H)] \\
=\ & \sum_{0\leq k\leq c/(\mu h)} \frac{1}{4\pi^2 h^2} \iint \Big(\mu h 2k[\partial_{x_2} a_1(x) - \partial_{x_1} a_2(x)] + [a_2(x)\partial_{x_1} W(x) - a_1(x)\partial_{x_2} W(x)] \Big) \\
& \times (f_2^2 g_0)(\xi_3^2 + 2k\mu h + W(x)) dx d\xi_3 + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2}).
\end{aligned}$$

Proof. We calculate as usual:

$$\begin{aligned}
& \mu \mathrm{tr}[\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2\gamma_3)\psi f_2^2(H) g_0^{(h)}(H)] \\
=\ & \frac{\mu}{\sqrt{2\pi} h} \iint g_0(\tau) \hat{\chi}(\tau e^{it\tau/h} \mathrm{tr}[f_2(H)\psi Op_\alpha^w(\gamma_0 + \alpha(\gamma_1+\gamma_2) + \alpha^2 c\gamma_3)\psi f_2(H) e^{-itH/h}] \quad dt\, d\tau \\
\simeq\ & \mu \sum_{0\leq k\leq c/(\mu h)} \mathrm{tr}[(Op_h^w(\theta^{(\mu)})^2 f_2(P_0^{(k)}) \left(\frac{h}{\mu} 2k Op_h^w \tilde{d} + \epsilon Op_h^w r \right) f_2(P_0^{(k)}) g_0^{(h)}(P_0^{(k)}) \\
\simeq\ & \mu \sum_{0\leq k\leq c/(\mu h)} \frac{\mu}{4\pi^2 h^2} \iint \left(\frac{h}{\mu} 2k[\partial_{x_2} a_1(x) - \partial_{x_1} a_2(x)] + \epsilon[a_2(x)\partial_{x_1} W(x) - a_1(x)\partial_{x_2} W(x)] \right) \\
& \times (f_2^2 g_0)(\xi_3^2 + 2k\mu h + W(x)) dx d\xi_3 + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2}).
\end{aligned}$$

∎

We can thus conclude that for $\mu h \leq C$, $\mu \geq ch^{-\rho}$ and the noncritical condition (2.1) satisfied, we get up to an error of order $O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1} + \mu)$:

$$
\begin{aligned}
& \operatorname{tr}[B(\mu,h)g_0(H)] \\
= & \sum_{0\leq k \leq c/(\mu h)} \frac{1}{4\pi^2 h^2} \iint \Big(\mu h 2k [\partial_{x_2} a_1(x) - \partial_{x_1} a_2(x)] + [a_2(x)\partial_{x_1} W(x) - a_1(x)\partial_{x_2} W(x)] \Big) \\
& \times g_0(\xi_3^2 + 2k\mu h + W(x)) dx d\xi_3 \\
= & \sum_{0\leq k \leq c/(\mu h)} \frac{1}{4\pi^2 h^2} \Big(\int \mu h 2k [\partial_{x_2} a_1(x) - \partial_{x_1} a_2(x)] 2\sqrt{[V(x) + 2\mu h k]_-} dx \\
& + \int [a_2(x)\partial_{x_1} W(x) - a_1(x)\partial_{x_2} W(x)] 2\sqrt{[V(x) + 2\mu h k]_-} dx \Big) \\
= & \sum_{0\leq k \leq c/(\mu h)} \frac{1}{4\pi^2 h^2} \Big(\int \mu h 4k [\partial_{x_2} a_1(x) - \partial_{x_1} a_2(x)] \sqrt{[V(x) + 2\mu h k]_-} dx \\
& + \frac{4}{3} \int [\partial_{x_1} a_2(x) - \partial_{x_2} a_1(x)][V(x) + 2\mu h k]_-^{3/2} dx \Big) \qquad (6.1)
\end{aligned}
$$

6.3. **The Spin-up Part.** Remember from (1.4) and (4) that the current is given by:

$$
\begin{aligned}
\int \vec{j} \cdot \vec{a} dx \; = \; & \operatorname{tr}[\Big(-2\vec{a} \cdot (-ih\nabla - \vec{A}) + ih \operatorname{div}\vec{a} + h(\partial_{x_1} a_2 - \partial_{x_2} a_1) \Big) g_0(H_0 + V + 2\mu h)] \\
+ \; & \operatorname{tr}[\Big(-2\vec{a} \cdot (-ih\nabla - \vec{A}) + ih \operatorname{div}\vec{a} - h(\partial_{x_1} a_2 - \partial_{x_2} a_1) \Big) g_0(H_0 + V)].
\end{aligned}
$$

When μh is finite we cannot disregard the first term. Having calculated the spin-down part of the current it is easy to treat the spin-up part though: Define $\tilde{V} = V + 2\mu h$. Then we get

$$
\begin{aligned}
& -\operatorname{tr}[\Big(-2\vec{a} \cdot (-ih\nabla - \vec{A}) + ih \operatorname{div}\vec{a} + h(\partial_{x_1} a_2 - \partial_{x_2} a_1) \Big) g_0(H_0 + V + \mu h)] \\
= \; & \operatorname{tr}[\tilde{B}(\mu,h) g_0(H_0 + \tilde{V} - \mu h)],
\end{aligned}
$$

where

$$
\tilde{B}(\mu,h) = \mu Op^w_{h/\mu}(2\vec{a} \cdot (\xi_1 + x_2, \xi_2, \xi_3) - h/\mu(\partial_{x_1} a_2 - \partial_{x_2} a_1)).
$$

It is easy to see that this change of sign on $(\partial_{x_1} a_2 - \partial_{x_2} a_1)$ (compare with (4.1)) only has as consequence that the factor k on the first term in (6.1) should be changed to $(k+1)$. Therefore we get:

$$\begin{aligned}
&\operatorname{tr}[\tilde{B}(\mu,h)g_0(H_0+\tilde{V}-\mu h)] \\
=\ &\sum_{0\le k\le c/(\mu h)} \frac{1}{4\pi^2h^2}\Bigg(\int \mu h 4(k+1)(\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x))\sqrt{[\tilde{V}(x)+2\mu hk]_-}\,dx \\
&+\frac{4}{3}\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x))[\tilde{V}(x)+2\mu hk]_-^{3/2}dx\Bigg)+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}) \\
=\ &\sum_{1\le k\le c/(\mu h)} \frac{1}{4\pi^2h^2}\Bigg(\int \mu h 4k(\partial_{x_2}a_1(x)-\partial_{x_1}a_2(x))\sqrt{[V(x)+2\mu hk]_-}\,dx \\
&+\frac{4}{3}\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x))[V(x)+2\mu hk]_-^{3/2}dx\Bigg)+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}).
\end{aligned}$$

Adding this to (6.1) we get the theorem 2.3.

7. Multiscaling: The Non-critical Condition

In this section we will prove that Theorem 2.3 holds without the non-critical condition (2.1):

Theorem 7.1. *Let $\vec{a}=(a_1,a_2,0)$. Suppose that $0<h\le h_0$, $\mu\le C_\mu h^{-\zeta}$ for some $\zeta>0$ and that there exists $\beta\in(0,1]$ such that $\mu\ge c_\beta h^{-\beta}$. Suppose finally that*

$$|\partial^m\vec{a}(x)|+|\partial^m V(x)|\le C_m$$

on $B(8E)$. Then

$$\begin{aligned}
\operatorname{tr}[\mathbf{B}g_0(\mathbf{P})] &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty}d_n\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x)) \\
&\quad\times\left([2n\mu h+V(x)]_-^{3/2}-3n\mu h[2nh\mu+V(x)]_-^{1/2}\right)dx \\
&\quad+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}+h^{-1}),
\end{aligned}$$

where O is uniform in the constants $\{C_m\},c_\beta,C_\mu,\beta,\zeta,E$.

To prove this we will need the following version of Theorem 2.3, where the non-criticality assumption has been slightly modified:

Lemma 7.2 (Reference Problem 1). *Let $\vec{a}=(a_1,a_2,0)$. Suppose that*

$$|\nabla V(x)|^2+|V(x)|\ge c_{N.C.}>0 \tag{7.1}$$

for all $x\in B(2E)$. Suppose further that $0<h\le h_0$, $\mu\le C_\mu h^{-\zeta}$ for some $\zeta>0$ and that there exists $\beta\in(0,1]$ such that $\mu\ge c_\beta h^{-\beta}$. Suppose finally that

$$|\partial^m\vec{a}(x)|+|\partial^m V(x)|\le C_m$$

on $B(8E)$. Then

$$\begin{aligned}
\operatorname{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})] &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty}d_n\int(\partial_{x_1}a_2(x)-\partial_{x_2}a_1(x)) \\
&\quad\times\left([2n\mu h+V(x)]_-^{3/2}-3n\mu h[2nh\mu+V(x)]_-^{1/2}\right)dx \\
&\quad+O(h^{-1}\mu^{-1}+h^{-3}\mu^{-2}+h^{-1}),
\end{aligned}$$

where O is uniform in the constants $\{C_m\}, c_{N.C.}, c_\beta, C_\mu, \beta, \zeta, E$.

The proof follows essentially by a change of gauge:

Proof. Write $\mathbf{P} = \mathbf{P}(\vec{A}, V)$. Let U_1 be the unitary gauge transformation given by:

$$(U_1 f)(x) = e^{-i\mu x_1 x_2/h} f(x),$$

and let U_2 be the unitary change of variables:

$$(U_2 f)(x) = f(x_2, -x_1, x_3).$$

Notice the following relations:

$$\begin{aligned}
(U_2^* f)(x) &= f(-x_2, x_1, x_3) \\
U_2^* \nabla U_2 &= \begin{pmatrix} -\partial_{x_2} \\ \partial_{x_1} \\ \partial_{x_3} \end{pmatrix} \\
U_2^* V U_2 &= \tilde{V},
\end{aligned}$$

where $\tilde{V}(x) = V(-x_2, x_1, x_3)$. Then:

$$\begin{aligned}
U_2^* U_1^* \mathbf{P}(\vec{A}, V) U_1 U_2 &= U_2^* \mathbf{P}(\vec{A} + \mu \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix}, V) U_2 \\
&= U_2^* \left(-h^2 \partial_{x_1}^2 + (-ih\partial_{x_2} - \mu x_1)^2 - h^2 \partial_{x_3}^2 - \mu h \sigma_3 + V(x) \right) U_2 \\
&= -h^2 (-\partial_{x_2})^2 + (-ih\partial_{x_1} + \mu x_2)^2 - h^2 \partial_{x_3}^2 - \mu h \sigma_3 + \tilde{V}(x) \\
&= \mathbf{P}(\vec{A}, \tilde{V}).
\end{aligned}$$

Similarly

$$\begin{aligned}
U_2^* U_1^* \vec{a} \cdot (-ih\nabla - \vec{A}) U_1 U_2 &= U_2^* \vec{a} \cdot (-ih\nabla - \mu \begin{pmatrix} 0 \\ x_1 \\ 0 \end{pmatrix}) U_2 \\
&= U_2^* \vec{a} U_2 \cdot \left(-ih \begin{pmatrix} -\partial_{x_2} \\ \partial_{x_1} + \mu x_2 \\ \partial_{x_3} \end{pmatrix} \right) \\
&= \tilde{a} \cdot (-ih\nabla - \vec{A}),
\end{aligned}$$

where $\tilde{a}(x) = \begin{pmatrix} a_2(-x_2, x_1, x_3) \\ -a_1(-x_2, x_1, x_3) \\ a_3(-x_2, x_1, x_3) \end{pmatrix}$.

Let us finally notice that:

$$U_2^* (\partial_{x_1} a_2 - \partial_{x_2} a_1) U_2 = \partial_{x_1} \tilde{a}_2 - \partial_{x_2} \tilde{a}_1.$$

Now we are ready to prove Lemma 7.2:

Choose a partition of unity $\{\phi_j\}$ on $B(E)$ such that $\operatorname{supp} \phi_j \subset B(x_j, E_j/2)$ and that on $B(x_j, 8E_j)$ we have either

$$|\partial_{x_1} V(x)|^2 + |\partial_{x_3} V(x)|^2 + |V(x)| \geq c_{N.C.}/4, \tag{7.2}$$

or

$$|\partial_{x_2} V(x)|^2 + |\partial_{x_3} V(x)|^2 + |V(x)| \geq c_{N.C.}/4. \tag{7.3}$$

This can obviously be done uniformly in $c_{N.C.}$ and the C_m's. Now we write: $\mathcal{J}(h,\mu,\vec{a},V) = \mathrm{tr}[\mathbf{B}(h,\mu,\vec{a},V)g_0(\mathbf{P})]$, and notice that

$$\mathcal{J}(h,\mu,\vec{a},V) = \sum_j \mathcal{J}(h,\mu,\phi_j\vec{a},V).$$

Likewise, we write:

$$\begin{aligned}\mathcal{A}(h,\mu,\vec{a},V) &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}a_2(x) - \partial_{x_2}a_1(x)) \\ &\quad\times\left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2}\right)dx,\end{aligned}$$

and notice the same linearity:

$$\mathcal{A}(h,\mu,\vec{a},V) = \sum_j \mathcal{A}(h,\mu,\phi_j\vec{a},V).$$

Now, if (7.2) is satisfied on $\operatorname{supp}\phi_j$ we can use Theorem 2.3 to estimate:

$$|\mathcal{J}(h,\mu,\phi_j\vec{a},V) - \mathcal{A}(h,\mu,\phi_j\vec{a},V)| \le O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1}).$$

On the other hand, if (7.3) is satisfied on $\operatorname{supp}\phi_j$ we conjugate by (U_1U_2), and find ourselves, once again, in a situation where Theorem 2.3 is applicable: The above calculation shows that

$$\mathcal{J}(h,\mu,\phi_j\vec{a},V) = \mathcal{J}(h,\mu,\widetilde{\phi_j\vec{a}},\tilde{V}),$$

and we see that

$$|\partial_{x_1}\tilde{V}|^2 + |\partial_{x_3}\tilde{V}|^2 + |V(x)| \ge c_{N.C.}/4$$

on $B(x_j, 8E_j)$. Thus we can apply Theorem 2.3. If we finally notice that

$$\mathcal{A}(h,\mu,\vec{a},V) = \mathcal{A}(h,\mu,\tilde{a},\tilde{V}),$$

we can put the pieces together and obtain Lemma 7.2. □

Remark 7.3. Note that the lemma remains true if (7.1) is replaced by:

$$|\partial_x V|^2 + |V(x)| + h \ge c > 0. \tag{7.4}$$

This is the condition that we will use in the following.

Having cast the reference problem in this form we are facing very much the same problem as treated in [Sob95][section 5,6]. Our treatment will also be very similar.

Proof. We choose

$$f(x) = l(x) = A^{-1}\left[V(x)^2 + (\partial_x V)^4 + h^2\right]^{1/4},$$

where A is a sufficiently big constant to be determined below. Then

$$\begin{aligned} f(x), l(x) &> 0, \\ |\partial_x l(x)| \le \rho &< 1 \\ c \le \frac{f(x)}{f(y)} &\le C \qquad \forall x \in B(8)\cap B(y,l(y)), \end{aligned} \tag{7.5}$$

if A is sufficiently big. Furthermore, there exist constants c_α, independent of h, such that

$$\begin{aligned} |\partial^\alpha V(x)| &\le c_\alpha f(x)^2 l(x)^{-|\alpha|} \\ |\partial^\alpha \vec{a}(x)| &\le c_\alpha l(x)^{-|\alpha|} \end{aligned}$$

on $B(8)$. Now, if we choose a sequence of points $\{x_k\}$ such that

- $B(1) \subset \cup_k B(x_k, l(x_k)) \equiv \cup_k B_k$,
- $\cup_k B(x_k, 8l(x_k)) \subset B(8)$,
- the intersection of more than $N = N(\rho)$ balls is empty (this is possible due to $\rho < 1$ in (7.5), see [Sob95], [H$\ddot{9}$0]),

and a corresponding partition of unity:

- $\psi_k \in C_0^\infty(B_k)$,
- $|\partial_\alpha \psi_k(x)| \leq c_\alpha l_k^{-|\alpha|}$, where $l_k = l(x_k)$,
- $\sum \psi_k \equiv 1$ on $B(1)$,

then

$$\mathcal{J}(h, \mu, \vec{a}, V) = \mathcal{J}(h, \mu, \sum_k \psi_k \vec{a}, V) = \sum_k \mathcal{J}(h, \mu, \psi_k \vec{a}, V).$$

Since also the asymptotic term satisfies

$$\mathcal{A}(h, \mu, \vec{a}, V) = \mathcal{A}(h, \mu, \sum_k \psi_k \vec{a}, V) = \sum_k \mathcal{A}(h, \mu, \psi_k \vec{a}, V),$$

we can write

$$\mathcal{J}(h, \mu, \vec{a}, V) - \mathcal{A}(h, \mu, \vec{a}, V) = \sum_k \left(\mathcal{J}(h, \mu, \psi_k \vec{a}, V) - \mathcal{A}(h, \mu, \psi_k \vec{a}, V)\right).$$

Now by scaling, dilatation and a gauge-transformation in the $\mathcal{J}$-term:

$$\mathcal{J}(h, \mu, \psi_k \vec{a}, V) - \mathcal{A}(h, \mu, \psi_k \vec{a}, V) = f_k \left[\mathcal{J}(\frac{h}{f_k l_k}, \frac{\mu l_k}{f_k}, \hat{a}_k, \hat{V}) - \mathcal{A}(\frac{h}{f_k l_k}, \frac{\mu l_k}{f_k}, \hat{a}_k, \hat{V})\right],$$

where $\hat{a}_k(x) = (\psi_k \vec{a})(l_k x + x_k)$ and $\hat{V}(x) = f_k^{-2} V(l_k x + x_k)$ (see [Sob95]). We want to apply the reference problem to $\mathcal{J}(\frac{h}{f_k l_k}, \frac{\mu l_k}{f_k}, \hat{a}_k, \hat{V})$, so we have to check that this is allowed. Let us notice that by continuity of V; f, l are bounded on $B(8)$. Therefore it is easy to see that

$$\begin{aligned} |\partial^\alpha \hat{a}(x)| &\leq C_\alpha \\ |\partial^\alpha \hat{V}(x)| &\leq C_\alpha, \end{aligned}$$

where the C_α's are independent of k. Let us check the non-critical condition (7.4):

$$\begin{aligned} |\partial_x \hat{V}|^2 + |\hat{V}(x)| + \frac{h}{l_k f_k} &= \frac{|(\partial_x V)(l_k x + x_k)|^2 + V(l_k x + x_k) + h}{f_k^2} \\ &\geq \frac{cA^2 f(l_k x + x_k)}{f_k^2} \\ &\geq c, \end{aligned}$$

for $x \in B(1)$. We also have to check that $\frac{h}{f_k l_k}$ is bounded above, and that $\mu = \frac{\mu l_k}{f_k} \geq c_\mu (\frac{h}{f_k l_k})^{-\beta}$. This is easily seen to be the case.

Now, since we can use the reference problem, we get:

$$\begin{aligned} |\mathcal{J}(h, \mu, \psi_k \vec{a}, V) - \mathcal{A}(h, \mu, \psi_k \vec{a}, V)| &\leq C f_k \left(\frac{f_k l_k}{h} \frac{f_k}{\mu l_k} + \frac{f_k^3 l_k^3}{h^3} \frac{f_k^2}{\mu^2 l_k^2} + \frac{f_k l_k}{h}\right) \\ &= C \int_{B_k} f_k \left(\frac{f_k l_k}{h} \frac{f_k}{\mu l_k} + \frac{f_k^3 l_k^3}{h^3} \frac{f_k^2}{\mu^2 l_k^2} + \frac{f_k l_k}{h}\right) l_k^{-3} dx \\ &\leq C \int_{B_k} \left(\frac{1}{h\mu} + \frac{l(x)^4}{h^3 \mu^2} + \frac{1}{h}\right) dx, \end{aligned}$$

where we used that $f(x) = l(x)$ in the last inequality.

Thus,

$$\begin{aligned} |\mathcal{J}(h,\mu,\vec{a},V)-\mathcal{A}(h,\mu,\vec{a},V)| &\leq CN_\rho \int_{B(8)} \left(\frac{1}{h\mu}+\frac{l(x)^4}{h^3\mu^2}+\frac{1}{h}\right) dx \\ &\leq C\left(\frac{1}{h\mu}+\frac{1}{h^3}\mu^2+\frac{1}{h}\right). \end{aligned}$$

□

8. The Current Parallel to the Magnetic Field

In this section we prove Theorem 2.6. We will first prove that the current parallel to the magnetic field is constant in the x_3-variable. This allows us to move the test-function a_3 out where the potential is positive, and here the current vanishes to all orders in h.

Lemma 8.1. *Suppose*

$$\int_{-\infty}^{\infty} a_3(x_1,x_2,x_3)\ dx_3 = 0,$$

for all (x_1,x_2)*. Then*

$$\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_3))g_0(\mathbf{P})] = O(h^{-1}).$$

Proof. Define $\vec{a}=(a_1,a_2,0)\in C_0^\infty(\mathbf{R}^3)$ as

$$\begin{aligned} a_1(x) &= -\int_{-\infty}^{x_3} \partial_{x_1} a_3(x_1,x_2,y)\ dy \\ a_2(x) &= -\int_{-\infty}^{x_3} \partial_{x_2} a_3(x_1,x_2,y)\ dy \end{aligned}$$

Then $\nabla\times\vec{a}=\nabla\times(0,0,a_3)$ and therefore we get by the result from Appendix D that:

$$\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_3))g_0(\mathbf{P})] = \mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})].$$

Theorem 2.5 now gives the conclusion of the lemma. □

Let now $a_{3,T}(x)\equiv a_3(x_1,x_2,x_3-T)$. The lemma above then says that

$$\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_3))g_0(\mathbf{P})] = \mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_{3,T}))g_0(\mathbf{P})]$$

locally uniformly in T.
Let $T\in\mathbf{R}$ be so big that $V>\gamma/2$ on $B(4E)+T\vec{e}_3$. The next lemma proves that then $\mathrm{tr}[\mathbf{B}(h,\mu,(0,0,a_{3,T}))g_0(\mathbf{P})]=O(h^\infty)$, which finishes the proof of the theorem.

Lemma 8.2. *Suppose* $\vec{a}\in C_0^\infty(B(E))$*, that* $V\geq\gamma>0$ *on* $B(4E)$ *and that the hypothesis of Theorem 2.5 are fullfilled, then*

$$\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})] = O(h^\infty).$$

Proof. Choose $\tilde{V}$ satifying

- $\tilde{V}\equiv V$ on $B(4E)$.
- $\tilde{V}(x)\geq\gamma$ for all x.
- $\tilde{V}-\gamma\in C_0^\infty(\mathbf{R}^3)$.

Choose furthermore $f \in C_0^\infty(\mathbf{R})$, with $\sup(\operatorname{supp} f) \leq \gamma/2$ such that $f(\mathbf{P})g_0(\mathbf{P}) = g_0(\mathbf{P})$. Let $\tilde{\mathbf{P}}$ denote the Pauli-operator with V exchanged with $\tilde{V}$. Then we get:

$$\begin{aligned}
|\operatorname{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})]| &= |\operatorname{tr}[\mathbf{B}(h,\mu,\vec{a})f(\mathbf{P})g_0(\mathbf{P})]| \\
&\leq \|\mathbf{B}(h,\mu,\vec{a})f(\mathbf{P})\|_1 \\
&= \|\mathbf{B}(h,\mu,\vec{a})f(\tilde{\mathbf{P}})\|_1 + O(h^\infty) \\
&= O(h^\infty).
\end{aligned}$$

The last equality is due to the fact that $\tilde{\mathbf{P}} \geq \gamma$ and therefore $f(\tilde{\mathbf{P}}) = 0$. The next to last equality is a consequence of localisation, see for example [Sob94][Thm 2.13]. □

9. Multiscaling

In this section we finally want to prove the following more precise version of Theorem 1.2:

Theorem 9.1. *Suppose*

$$V(x) = \frac{q}{|x|} + o(|x|^{-1}) \tag{9.1}$$

as $x \to 0$, and

$$|\partial^m V(x)| \leq C_{m,V}|x|^{-1-|m|}, \tag{9.2}$$

$\forall x \in B(8)$.
Suppose furthermore that $\exists C = C(h,\mu)$ such that

$$\mathbf{P}(h,\mu,V) \geq -C.$$

Suppose

- *$\exists c_{\mu,1} > 0$ such that $\mu h \geq c_{\mu,1}$,*
- *$\exists c_{\mu,2} > 0$ such that $\mu h^3 \leq c_{\mu,2}$*

Let finally $\vec{a} = (a_1, a_2, 0) \in C_0^\infty(B(1))$ satisfy

$$|\partial^m \vec{a}| \leq C_{m,\vec{a}},$$

then for all $\nu > 0$

$$\begin{aligned}
\operatorname{tr}[\mathbf{B}g_0(\mathbf{P})] &= \frac{2}{3\pi h^2}\sum_{n=0}^\infty d_n \int (\partial_{x_1}a_2(x) - \partial_{x_2}a_1(x)) \\
&\quad \times \left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2}\right)dx \\
&\quad + O(h^{-1} + \frac{1}{\mu^{1/3}h^{2+\nu}}),
\end{aligned}$$

where O is uniform in the constants $\{C_{m,V}\}, \{C_{m,\vec{a}}\}, c_{\mu,1}, c_{\mu,2}$.

Remark 9.2. The constants $\{C_{m,\vec{a}}\}, c_{\mu,1}, c_{\mu,2}$ do not depend on $\vec{a}, \mu$. The index is only there to distinguish them from each other and the other constants in the theorem.

Remark 9.3. The asymptotics does not depend on the lower bound $-C$ of $\mathbf{P}$.

For the parallel current the corresponding result is

Theorem 9.4. *Let the assumptions be as in Thm 9.1, but with $\vec{a} = (a_1, a_2, a_3)$. Assume that $V(x) \geq c_V > 0$, for $1 \leq |x| \leq 3$, and that the spectrum of* $\mathbf{P}$ *below 0 is discrete, then for all $\nu > 0$*

$$|\mathcal{J}(h, \mu, \vec{a}, V) - \mathcal{A}(h, \mu, \vec{a}, V)| = O(h^{-1} + \frac{1}{\mu^{1/3} h^{2+\nu}}),$$

where O is uniform in the constants $\{C_{m,V}\}, \{C_{m,\vec{a}}\}, c_{\mu,1}, c_{\mu,2}, c_V$.

We are going to perform a so-called *multiscale analysis* invented by Ivrii et al.([Ivr98], [IS93], see also [Sob94]) Since our problem is very similar to the problem analyzed in [Sob96b] our choices of scaling functions will be the same.

We will divide space into several regions and obtain asymptotic extimates in each of them. This is due to the fact that as far as magnetic effects are concerned there is an enormous difference between the vicinity of the singularity and the rest of the space. Close to the singularity V is much bigger than μh and therefore magnetic effects are neglectable. In this region the analysis performed in [Fou98] is applicable. Further out, μh and V become comparable and we see a current.

Let us write $\vec{a} = \chi_1 \vec{a} + \chi_2 \vec{a} = \vec{a}_1 + \vec{a}_2$, where $\chi_1(x) = \chi(x/r^2)$ and $\chi_2 = 1 - \chi_1$ (here and in what follows χ will denote a standard smooth cut-off function around 0). The exact choice of r will be made in the end of this section, here we will just remark that we impose:

$$r^2 \leq \frac{1}{\mu h}, \tag{9.3}$$

which, in a sense, is the condition that, on the support of χ_1, the electric potential dominates.

9.1. **The Inner Region** $\{|x| \leq r^2\}$.

In the innermost region, we do not see a current. This will be the result of Cor. 9.7 below.
We have to evaluate the trace $\mathrm{tr}[\mathbf{B}(h, \mu, \vec{a}_1) g_0(\mathbf{P})]$, with $\vec{a}_1$ supported on a region of radius r^2. This we can write as

$$\mathbf{B}(h, \mu, \vec{a}_1) = Op_h^w \left(2\tilde{a}(\frac{x}{r^2}) \cdot (\xi - \mu \vec{A}) \right) + h/r^2 b(\frac{x}{r^2}) \sigma_3,$$

where $\tilde{a}$ and $b = \nabla \times \tilde{a}$ are now supported on a region of radius 1.

Lemma 9.5. *We have*

$$\mathrm{tr}[\mathbf{B}(h, \mu, \vec{a}_1) g_0(\mathbf{P})] = O(h^{-1} + \frac{\mu r^3}{h} + \frac{r}{h^2} + \frac{\mu r^6}{h^2} + \frac{r^3}{h^2}).$$

Lemma 9.5 follows upon collecting the results of the Lemmas 9.9, 9.11 and 9.12 below.
Let us look at the asymptotic term:

Lemma 9.6.

$$\mathcal{A}(h, \mu, \vec{a}_1, V) = O(\frac{r\mu}{h}).$$

Proof. We write $\vec{a}_1(x) = \tilde{a}(\frac{x}{r^2})$, and $V(x) = \frac{\Phi(x)}{|x|}$. Then we can calculate:

$$\begin{aligned}
\mathcal{A}(h,\mu,\tilde{a}(\frac{x}{r^2}),V) &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int \frac{1}{r^2}(\partial_{x_1}\tilde{a}_2(\frac{x}{r^2}) - \partial_{x_2}\tilde{a}_1(\frac{x}{r^2})) \\
&\quad \times\Bigg([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2}\Bigg)\,dx \\
&= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n r^4 \int (\partial_{x_1}\tilde{a}_2(y) - \partial_{x_2}\tilde{a}_1(y)) \\
&\quad \times\Bigg([2n\mu h + \frac{\Phi(r^2 y)}{r^2|y|}]_-^{3/2} - 3n\mu h[2nh\mu + [\frac{\Phi(r^2 y)}{r^2|y|}]_-^{1/2}\Bigg)\,dy \\
&= \frac{2r}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}\tilde{a}_2(y) - \partial_{x_2}\tilde{a}_1(y)) \\
&\quad \times\Bigg([2n\mu h r^2 + \frac{\Phi(r^2 y)}{|y|}]_-^{3/2} - 3n\mu h r^2[2nh\mu r^2 + \frac{\Phi(r^2 y)}{|y|}]_-^{1/2}\Bigg)\,dy.
\end{aligned}$$

Now we use Prop. C.1 to conclude:

$$\begin{aligned}
\mathcal{A}(h,\mu,\tilde{a}(\frac{x}{r^2}),V) &= O\left(\frac{r}{h^2}\int |\partial_{x_1}\tilde{a}_2 - \partial_{x_2}\tilde{a}_1|(y)\left((h\mu r^2)^{3/2} + \sqrt{h\mu r^2}\frac{1}{|y|} + h\mu r^2\frac{1}{|y|}\right)dy\right) \\
&= O(\frac{r}{h^2}\sqrt{h\mu r^2}),
\end{aligned}$$

since $h\mu r^2 \leq 1$. □

From Lemma 9.5 and Lemma 9.6 we get, upon noticing that $\mu r^2 \leq h^{-1}$ and $r^3 \leq r$:

Corollary 9.7.

$$|\mathcal{J}(h,\mu,\vec{a}_1,V) - \mathcal{A}(h,\mu,\vec{a}_1,V)| = O(h^{-1} + \frac{r}{h^2} + \frac{\mu r^6}{h^2}).$$

Remark 9.8. Notice that we prove that $|\mathcal{J} - \mathcal{A}|$ is small by proving that both $|\mathcal{J}|$ and $|\mathcal{A}|$ are small.

To prove Lemma 9.5 let us first look at the part of the trace involving b, i.e. the spin current:

Lemma 9.9.

$$\operatorname{tr}[h/r^2 b(x/r^2)\sigma_3 g_0(\mathbf{P})] = O(h^{-1} + \frac{\mu r^3}{h} + \frac{r}{h^2}).$$

Proof. If we write $V(x) = \frac{\Phi(x)}{|x|}$ and make the change of variables $y = x/r^2$ we get, on the spin down subspace,

$$h/r^2\operatorname{tr}\left[b(y)g_0\left((-ih/r\nabla - \mu r^3\vec{A}(y))^2 - (\mu r^3)h/r - \frac{\Phi(r^2 y)}{|y|}\right)\right],$$

and correspondingly on the spin up subspace. Let us first concentrate on the spin-down case. Since $(\mu r^3)h/r = \mu h r^2 \leq 1$ by (9.3), this trace is of the type analyzed in [Sob96a] (It is his case number 5). Therefore we get:

$$
\begin{aligned}
& h/r^2 \mathrm{tr}\left[b(y)g_0\left((-ih/r\nabla - \mu r^3\vec{A}(y))^2 - (\mu r^3)h/r - \frac{\Phi(r^2y)}{|y|}\right)\right] \\
= \quad & c(h/r)^{-3}\int b(y)\left[-\mu h r^2 - \frac{\Phi(r^2y)}{|y|}\right]_-^{3/2} dy + O((h/r)^{-2}(1+\mu r^3)),
\end{aligned}
$$

where the constant c is explicit. If we analyze the spin-up part in the same way, we get:

$$
\begin{aligned}
& \mathrm{tr}[h/r^2 b(x/r^2)\sigma_3 g_0(\mathbf{P})] \\
& = cr/h^2 \int b(y)\left(\left[-\mu h r^2 - \frac{\Phi(r^2y)}{|y|}\right]_-^{3/2} - \left[\mu h r^2 - \frac{\Phi(r^2y)}{|y|}\right]_-^{3/2}\right) dy \\
& + O\left(\frac{1+\mu r^3}{h}\right).
\end{aligned}
$$

□

Remark 9.10. Notice that the result depends only on how the potential V behaves on a region of size r^2.

Now we look at the remaining term in the trace. Here we have to split into two regions. This is not due to any fundamental difference between this part and the part considered above. In fact this splitting is essentially the same as Sobolev uses in his paper, but in the case considered above we could just go in and use the final result.
The two regions are:

$$\Omega_1 = \{|x| \le h^2/\theta\}$$

and

$$\Omega_2 = \{h^2/\theta \le |x| \le r^2\},$$

where θ is a sufficiently small constant (independent of h, μ) which will be chosen below. Write $\vec{a}_1 = \phi_1\vec{a}_1 + \phi_2\vec{a}_1$, where ϕ_1, ϕ_2 are smooth cut-offs to the regions Ω_1, Ω_2, respectively.
On Ω_1 we have to analyze

$$\mathrm{tr}[Op_h^w(\bar{a}(\frac{x}{h^2/\theta})\cdot(\xi - \mu\vec{A}))g_0(\mathbf{P})],$$

where $\bar{a}$ is supported on a ball of radius 1.

Lemma 9.11.

$$\mathrm{tr}[Op_h^w(\bar{a}(\frac{x}{h^2/\theta})\cdot(\xi - \mu\vec{A}))g_0(\mathbf{P})] = O(h^{-1}).$$

Proof. We will only look at the spin down part, the other case follows easily. After the change of variable $y = \theta x/h^2$ the expression becomes:

$$h^{-1}\mathrm{tr}\left[Op_\theta^w\left(\bar{a}\cdot(\xi - \frac{\mu h^3}{\theta}\vec{A})\right)g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2y/\theta)}{|y|}\right)\right].$$

Choose a function $\psi \in C_0^\infty$, $0 \leq \psi$, $\psi \bar{a} = \bar{a}$ then we get (using the spectral theorem and the cyclicity of trace)

$$\begin{aligned}
&\frac{1}{h}\text{tr}\left[Op_\theta^w\left(\bar{a}\cdot(\xi - \frac{\mu h^3}{\theta}\vec{A})\right) g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right)\right] \\
\leq\; & \frac{1}{h}\left\| Op_\theta^w(\xi - \frac{\mu h^3}{\theta}\vec{A}) g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right)\right\| \\
&\times \text{tr}\left[\psi g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right)\right] \\
=\; & \frac{1}{h}O(\theta^{-3}),
\end{aligned}$$

where we used the estimates:

$$\left\| Op_\theta^w(\xi - \frac{\mu h^3}{\theta}\vec{A}) g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right)\right\| \leq C, \tag{9.4}$$

and

$$\text{tr}\left[\psi g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right)\right] \leq C\theta^{-3}. \tag{9.5}$$

We prove (9.5) by applying once more the result from [Sob96a]. This is possible since μh^3 is bounded. The result is, for θ sufficiently small:

$$\begin{aligned}
&\text{tr}\left[\psi g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right)\right] \\
=\; & c\theta^{-3}\int \psi(y)\left[-\mu h^3 - \frac{\Phi(h^2 y/\theta)}{|y|}\right]_-^{3/2} dy + O(\theta^{-2}(1 + \frac{\mu h^3}{\theta})).
\end{aligned}$$

This proves (9.5). Notice, that here we only use properties of V on a region of size h^2/θ.
To prove (9.4) we take $W(y) = \zeta(y)\frac{\Phi(h^2 y/\theta)}{|y|}$, where ζ is some C_0^∞ function, which is 1 on $B(1)$. Using results from Appendix B, we only have to prove the estimate, with $\frac{\Phi(h^2 y/\theta)}{|y|}$ replaced by W. Now take $\phi \in \text{Ran}(g_0\left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - W(y)\right)$ with $\|\phi\| = 1$, and write

$$\begin{aligned}
&\left\|\left(-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y)\right)\phi\right\|^2 \\
\leq\; & \langle \phi, \left((-i\theta\nabla - \frac{\mu h^3}{\theta}\vec{A}(y))^2 - \mu h^3 - W(y)\right)\phi\rangle + \mu h^3 + \langle\phi, W(y)\phi\rangle \\
\leq\; & \mu h^3 + \langle \phi, W(y)\phi\rangle
\end{aligned}$$

and we finish using the infinitesimal boundedness of the potential. □

Lemma 9.12.

$$\text{tr}[Op_h^w(\phi_2\vec{a}_1\cdot(\xi - \mu\vec{A}))g_0(\mathbf{P})] = O(\frac{\mu r^6}{h^2} + \frac{r^3}{h^2}).$$

In the proof below, we will write $\vec{a}$ instead of $\phi_2\vec{a}_1$. On Ω_2 we need to multiscale: We have the following reference problem:

Theorem 9.13 (Reference Problem). *If $\vec{a} \in C_0^\infty(B(0,1))$ and $\vec{a}$, V, satisfy the following bounds: $|\partial^\alpha \vec{a}| \leq C_\alpha$, $|\partial^\alpha V| \leq C_\alpha$ on $B(8)$, and $\mu h \leq 1$, $h \leq h_0$. Then we have*

$$\text{tr}[Op_h^w(\vec{a}\cdot(\xi - \mu\vec{A}))g_0(H)] = O((\mu + 1)h^{-2}),$$

where the O is uniform in the constants bounding the derivatives of $\vec{a}$, V.

Using the localisation arguments in Appendix A this theorem is a consequence of the results of [Fou98]. The proof is identical to sections 4,5 in [Sob95] and will therefore be omitted.

We now define functions $f = \frac{1}{\sqrt{|x|}}$, $l(x) = \rho|x|$ where $\rho < 1/16$. Notice that $|\partial^\alpha \vec{a}| \leq c_\alpha l(x)^{-|\alpha|}$ and $|\partial^\alpha V| \leq c_\alpha f(x)^2 l(x)^{-|\alpha|}$, on Ω_2. Since $|\partial_x l(x)| \leq \rho < 1$ we can find a sequence of points (See [Hö90] or [Sob95]) $x_k \subset \Omega_2$ such that

$$\cup_{x\in \text{supp}\,\vec{a}} B(x, l(x)) \subset \Omega_2 \subset \cup_k B(x_k, 8l(x_k))$$

and a number $N = N(\rho)$ (independent of h) such that the intersection of more than $N(\rho)$ balls is empty, and furthermore a corresponding partition of unity $\{\psi_k\}$ satisfying:

- $\psi_k \in C_0^\infty(B(x_k, 8l(x_k)))$,
- $|\partial^m \psi_k| \leq C(\rho) l(x_k)^{-|m|}$,
- $\sum \psi_k = 1$ on Ω_2.

Using this partition of unity we write

$$\begin{aligned} \text{tr}[Op_h^w(\vec{a}\cdot(\xi - \mu\vec{A}))g_0(H)] &= \sum \text{tr}[Op_h^w(\psi_k \vec{a}\cdot(\xi - \mu\vec{A}))g_0(H)] \\ &\equiv \sum_k T_k. \end{aligned}$$

Now we have

Lemma 9.14.

$$|T_k| \leq C \int_{B_k} \left(\frac{\mu}{h^2} f(x)^2 + \frac{f(x)^3}{l(x)h^2} \right) dx$$

This will be proved below. We first prove Lemma 9.12 using Lemma 9.14:

Proof. Because only a finite (fixed) number of balls can intersect we thus get that:

$$\begin{aligned} |\text{tr}[Op_h^w(\vec{a}\cdot(\xi - \mu\vec{a}))g_0(H)]| &\leq C \int_{\Omega_2} \left(\frac{\mu}{h^2} f(x)^2 + \frac{f(x)^3}{l(x)h^2} \right) dx \\ &= Ch^{-2} \int_{h^2/\theta}^{r^2} (\mu|x| + \frac{1}{\sqrt{|x|}}) d|x| \\ &= O(\frac{\mu r^4}{h^2} + \frac{r}{h^2}). \end{aligned}$$

In the final estimate we used that θ is a *constant*. This proves Lemma 9.12. ☐

Now we prove Lemma 9.14:

Proof. First we notice the following scaling relations: Let l, f be positive scalars, $z \in \mathbf{R}^3$ and define $\mathcal{U}_l u(x) = l^{3/2} u(lx)$, $T_z u(x) = u(x+z)$, then:

$$f^{-2}\mathcal{U}_l T_z H(A, V, h, \mu) T_z^* \mathcal{U}_l^* = H(\hat{A}, \hat{V}, \alpha, \nu),$$

where

- $\hat{A}(x) = l^{-1} A(lx + z) = (-x_2 - z_2/l, 0, 0)$
- $\hat{V}(x) = f^{-2} V(lx + z)$
- $\alpha = h/(fl)$, $\nu = \mu l / f$.

Let now Φ be the gaugetransformation $\Phi u(x) = e^{i\frac{h}{fl}x_1 z_2/l}u(x)$ and let $\mathcal{U}(l,f,z)$ be the unitary transformation $\mathcal{U}(l,f,z) = \Phi\mathcal{U}_l T_z$, then

$$f^{-2}\mathcal{U}(l,f,z)H(A,V,h,\mu)\mathcal{U}(l,f,z)^* = H(A,\hat{V},h/(fl),\mu l/f).$$

Let

$$J(A,V,h,\mu,\vec{a}) = \mathrm{tr}[Op_h^\alpha(\vec{a}\cdot(\xi-\mu A)g_0(H(A,V,h,\mu))],$$

then the above proves that $J(A,V,h,\mu,\vec{a}) = fJ(A,\hat{V},\frac{h}{fl},\frac{\mu l}{f},\hat{a})$, where $\hat{a} = \vec{a}(lx+z)$. Now $T_k = J(A,V,h,\mu,\psi_k\vec{a})$, which thus means that:

$$T_k = f_k J\left(A, \frac{V(l\cdot+x_k)}{f_k^2}, \frac{h}{f_k l_k}, \frac{\mu l_k}{f_k}, (\psi_k\vec{a})(l_k\cdot+x_k)\right).$$

The following conditions are satisfied:

- $\frac{h}{f_k l_k}\frac{\mu l_k}{f_k} = \frac{h\mu}{f_k^2} = h\mu|x_k| \leq h\mu r^2 \leq 1$.
- $\frac{h}{f_k l_k} = \frac{h}{\rho\sqrt{|x_k|}} \leq \frac{\sqrt{\theta}}{\rho} \leq h_0$ if θ,ρ are chosen properly.
- $|\partial^\alpha f_k^{-2}V(l_k\cdot+x_k)| \leq c_\alpha$, where c_α is some constant independent of f,l,k.
- $|\partial^\alpha(\psi_k\vec{a})(l_k\cdot+x_k)| \leq C_\alpha$ where the same remark applies to C_α.

Therefore we can apply the reference problem (9.13) to conclude that

$$\begin{aligned}
|T_k| &\leq f_k C\left((\frac{\mu l_k}{f_k}+1)\frac{f_k^2 l_k^2}{h^2}\right) \\
&= C'\int_{B_k} f_k\left((\frac{\mu l_k}{f_k}+1)\frac{f_k^2 l_k^2}{h^2}\right)l_k^{-3}dx \\
&\leq C''\int_{B_k} f(x)\left((\frac{\mu l(x)}{f(x)}+1)\frac{f(x)^2 l(x)^2}{h^2}\right)l(x)^{-3}dx \\
&= C''\int_{B_k}\left(\frac{\mu}{h^2}f(x)^2+\frac{f(x)^3}{l(x)h^2}\right)dx.
\end{aligned}$$

□

9.2. The Outer Region.

In the outer region the result is the following[4]

Lemma 9.15. *Let the assumption be as in Section 1. Then*

$$|\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a}_2)g_0(\mathbf{P}(h,\mu,V))] - \mathcal{A}(h,\mu,\vec{a}_2,V(x))| = O(\frac{1+r^{-7}\mu^{-4}}{\mu h})$$

In the outer region, $\mathcal{D} = \{|x| \geq r^2\}$, magnetic effects become important and we see a current.

In $\mathcal{D}$ we perform a multiscaling with the same scaling functions $f(x) = |x|^{-1/2}$ and $l(x) = \rho|x|$, $\rho < \frac{1}{16}$ as in Ω_2, but now we use the asymptotics for the current in a strong magnetic field as reference problem.

We will write $\vec{a}$ instead of $\vec{a}_2$.

Theorem 9.16 (Ref. Problem in $\mathcal{D}$.)**.** *Let $\vec{a}\in C_0^\infty(B(0,1))$, $A(x) = (-x_2,0,0)$, and V be a function such that*

$$\mathbf{P} = \mathbf{P}(h,\mu,A,V) = [\vec{\sigma}\cdot(-ih\nabla-\mu A)]^2 + V$$

is self adjoint and bounded below. Suppose that $\exists c_\alpha, m, M, \zeta, \beta, h_0 > 0$ such that

- $|\partial^\alpha\vec{a}| \leq c_\alpha$, $|\partial^\alpha V| \leq c_\alpha$ *on* $B(0,8)$,
- $0 < h \leq h_0$,

[4] Remember that $\vec{a}_2$ is the testfunction $\vec{a}$ cut smoothly down to the region $\{|x| > r^2\}$

- $h^\zeta \mu \leq m$,
- $h^\beta \mu \geq M$,

then

$$\mathrm{tr}[\mathbf{B}(h,\mu,\vec{a})g_0(\mathbf{P})] = \mathcal{A} + O(h^{-1}\mu^{-1} + h^{-3}\mu^{-2} + h^{-1}).$$

where

$$\begin{aligned}\mathcal{A} &= \mathcal{A}(h,\mu,\vec{a},V)\\ &= \frac{2}{3\pi h^2}\sum_{n=0}^{\infty} d_n \int (\partial_{x_1}a_2(x) - \partial_{x_2}a_1(x))\\ &\quad \times\left([2n\mu h + V(x)]_-^{3/2} - 3n\mu h[2nh\mu + V(x)]_-^{1/2}\right)dx.\end{aligned}$$

This is the statement of Thm 7.1. We will use this with $\zeta = 3$, and β such that

$$\mu h^\beta r^{3-\beta} \geq 1. \tag{9.6}$$

That it is possible to find such a β for our choice of r will be proved at the end of this section.

On $\mathcal{D}$ we have

- $|\partial^\alpha \vec{a}| \leq c_\alpha l(x)^{-|\alpha|}$,
- $|\partial^\alpha V| \leq c_\alpha f(x)^2 l(x)^{-|\alpha|}$.

Again we can find a partition of unity $\{\psi_k\}$ as in the previous multiscaling. We write:

$$\begin{aligned}\mathcal{J}(h,\mu,\vec{a},V) &= \mathcal{J}(h,\mu,\sum_k \psi_k\vec{a},V)\\ &= \sum_k \mathcal{J}(h,\mu,\psi_k\vec{a},V),\end{aligned}$$

and also

$$\begin{aligned}\mathcal{A}(h,\mu,\vec{a},V) &= \mathcal{A}(h,\mu,\sum_k \psi_k\vec{a},V)\\ &= \sum_k \mathcal{A}(h,\mu,\psi_k\vec{a},V).\end{aligned}$$

We want to prove that

$$\begin{aligned}&|\mathcal{J}(h,\mu,\psi_k\vec{a},V) - \mathcal{A}(h,\mu,\psi_k\vec{a},V)|\\ &\leq C\int_{B_k} f(x)\left(\frac{f(x)l(x)}{h}\frac{f(x)}{\mu l(x)} + \frac{f(x)^3 l(x)^3}{h^3}\frac{f(x)^2}{\mu^2 l(x)^2} + \frac{f(x)l(x)}{h}\right) l(x)^{-3}dx.\end{aligned}$$

This is proved as the lemma of the previous multiscaling argument and the proof will therefore be omitted. First we have to check:

$$\frac{\mu l}{f}\cdot\frac{h^3}{l^3 f^3} \approx \mu h^3 \frac{|x|^{1-3}}{|x|^{-(1+3)/2}} = \mu h^3 \leq m,$$

and

$$\frac{\mu l}{f}\cdot\frac{h^\beta}{l^\beta f^\beta} = \mu h^\beta \frac{l^{1-\beta}}{f^{1+\beta}} \approx \mu h^\beta \frac{|x|^{1-\beta}}{|x|^{-(1+\beta)/2}} = \mu h^\beta |x|^{3-\beta/2} \geq \mu h^\beta r^{3-\beta}$$

Thus we get:

$$\begin{aligned}|\mathcal{J}(h,\mu,\vec{a},V)-\mathcal{A}(h,\mu,\vec{a},V)| &= O\left(\int_{\mathcal{D}_1} f\left(\frac{fl}{h}\frac{f}{\mu l}+\frac{f^3 l^3}{h^3}\frac{f^2}{\mu^2 l^2}\right) l^{-3}dx\right)\\ &= O\left(\int_{r^2}^{1}\frac{1}{\mu h}|x|^{-5/2}+\frac{1}{h^3\mu^2}\frac{1}{|x|^3}d|x|\right)\\ &= O\left(\frac{1}{\mu h}+\frac{1}{h^3\mu^2}\frac{1}{\mu h}r^{-3}\frac{h^3}{\mu^2}r^{-4}\right)\\ &= \frac{1}{\mu h}O(1+r^{-7}\mu^{-4}).\end{aligned}$$

Finally, we can finish the proof of theorem 9.1.

Proof. We have the following conditions on r i.e. equations (9.3) and (9.6):

$$\mu h r^2 \leq 1 \tag{9.7}$$

$$\exists \beta\in(0,3] \text{ such that } \mu h^{\beta} r^{3-\beta} \geq 1, \tag{9.8}$$

and since we want the error terms to be small we need

$$\begin{aligned} r &\ll 1\\ \mu r^6 &\ll 1\\ h r^{-7}\mu^{-5} &\ll 1 \end{aligned}\tag{9.9}$$

To make the optimal choice of r let $\delta > 0$ and write

$$\begin{aligned}\mu &= h^{-\gamma}\\ r &= h^{\gamma/3-\delta(3-\gamma)}.\end{aligned}$$

This defines γ and r. Choose

$$\beta = \frac{9\delta}{1+3\delta}.$$

Then (9.8) is satisfied, since:

$$\begin{aligned}\mu h^{\beta} r^{3-\beta} &= h^{-\gamma+\beta+(3-\beta)(\gamma/3-\delta(3-\gamma))}\\ &= h^{-\gamma+\beta+\gamma-3\delta(3-\gamma)-\beta\gamma/3+\beta\delta(3-\gamma)}\\ &= h^{\beta(1-\gamma/3)+\beta 3\delta(1-\gamma/3)-9\delta(1-\gamma/3)}\\ &= h^{(1-\gamma/3)(\beta(1+3\delta)-9\delta)}\\ &= 1.\end{aligned}$$

The other equation, (9.7), holds if just $\delta < 1/6$ since:

$$\begin{aligned}\mu h r^2 &= h^{-\gamma+1+2\gamma/3-2\delta(3-\gamma)}\\ &= h^{1-\gamma/3-6\delta(1-\gamma/3)}\\ &= h^{(1-\gamma/3)(1-6\delta)}.\end{aligned}$$

The conditions (9.9) become

$$\begin{aligned}h^{\gamma/3-\delta(3-\gamma)} &\ll 1\\ h^{\gamma-6\delta(3-\gamma)} &\ll 1\\ h^{1+8\gamma/3+7\delta(3-\gamma)} &\ll 1.\end{aligned}$$

The first two of these get better for small δ, and the first is the largest term of the three. This finishes the proof of theorem 9.1. □

Appendix A. Some Localisation Arguments

In this section we will prove the following localisation result:
Let $E > 0$.

Assumption A.1. • $\mathcal{H}$ is a self adjoint operator which is bounded below on $L^2(\mathbf{R}^d)$.
- $\exists a_l\ l = 1,..d\ ,V$ all in $C_0^\infty(\mathbf{R}^d)$, such that for all $u \in C_0^\infty(B(4E))$:

$$\mathcal{H}u = Hu,$$

where we have used the notation

$$H = \sum(-ih\partial_l - a_l)^2 + V.$$

Let C_α be the constants such that

$$\begin{aligned} |\partial^\alpha V| &\leq C_\alpha \\ |\partial^\alpha a_l| &\leq C_\alpha, \end{aligned} \tag{A.1}$$

on $B(8E)$.
Under this assumption we have:

Theorem A.2. *Let $\chi \in C_0^\infty(B(E))$ and $g \in C_0^\infty(\mathbf{R})$, then*

$$\|\chi(-ih\partial_l - a_l)[g(\mathcal{H}) - g(H)]\|_1 = O(h^\infty),$$

where the O is uniform in E, g, χ and the constants C_α in (A.1).

Remark A.3. Let C_r be constants so that $|g^{(r)}| \leq C_r$. By *uniform* we mean that if $\tilde{\mathcal{H}}$, $\tilde{H} = \sum(-ih\partial_l - \tilde{a}_l)^2 + \tilde{V}$ satisfy the above assumptions with the same constants C_α and the same E, χ, and if $\tilde{g} \in C_0^\infty(\mathbf{R})$ with $|\tilde{g}^{(r)}| \leq C_r$ (the same constants as in the bounds on $|g^{(r)}|$) and $\sup \operatorname{supp} \tilde{g} \leq \sup \operatorname{supp} g$, then

$$\|\chi(-ih\partial_l - \tilde{a}_l)[g(\tilde{\mathcal{H}}) - g(\tilde{H})]\|_1 \leq C_N h^N,$$

$\forall N \in \mathbf{N}$, where the constants C_N are the same as in Theorem A.2. Observe, that we do not assume $\operatorname{supp} \tilde{g} \subset \operatorname{supp} g$.

Notation:
We will need some results from [Sob95], so we introduce the notation used in that paper:
Let $\lambda_0 \geq 1 + 2\sup|V(x)|$ then we define $d(z) = \operatorname{dist}(z, [-\lambda_0, \infty))$.
Let furthermore $<z>= (1 + |z|^2)^{1/2}$. Finally we will write $Q_l = (-ih\partial_l - a_l)$.
We start the proof with the following lemma:

Lemma A.4. *Suppose $\chi \in C_0^\infty(B(3E))$. Then for any $N > d/2$:*

$$\left\|\chi Q_l \left\{(\mathcal{H} - z)^{-1} - (H - z)^{-1}\right\}\right\|_1 \leq C_N \left[\frac{<z>^{1/2}}{h}\right]^d \left[\frac{h^2 <z>}{d(z)^2}\right]^{N+1/2} \left\{\frac{<z>^{1/2}}{|\Im z|} + h^{-1}\right\},$$

where $\Im(z)$ is the imaginary part of z.

Proof. Define $\chi_1 \in C_0^\infty(B(20E/6))$ satisfying: $\chi_1(x) = 1$ on $|x| \leq 19E/6$. Thus $\chi_1\chi = \chi$. Furthermore we will write $\phi = 1 - \chi_1$. Writing $(H - z)^{-1} = (H - z)^{-1}\chi_1 + (H - z)^{-1}\phi$, we get

$$\begin{aligned} &\chi Q_l[(\mathcal{H} - z)^{-1} - (H - z)^{-1}] \\ =\ &\chi Q_l[\chi_1(\mathcal{H} - z)^{-1} - (H - z)^{-1}\chi_1] - \chi Q_l(H - z)^{-1}\phi \\ =\ &T_1 + T_2. \end{aligned}$$

The last term is easily estimated using [Sob95][Lemma 3.6] as

$$\|T_2\|_1 = \|\chi Q_l (H-z)^{-1}\phi\|_1 \leq C_N \frac{<z>^{1/2}}{d(z)} \left[\frac{<z>^{1/2}}{h}\right]^d \left[\frac{h^2 <z>}{d(z)^2}\right]^N,$$

which is seen to fit the estimate we want to prove.
Using the identity:

$$\chi(\mathcal{H}-z)^{-1} = (H-z)^{-1}\chi - (H-z)^{-1} Z (\mathcal{H}-z)^{-1},$$

where

$$Z = -[H,\chi] = \sum_{j=1}^n ih(Q_j^*(\partial_j \chi) + (\partial_j \chi) Q_j),$$

we get that the first term is

$$T_1 = \sum_{j=1}^n \left(-\chi Q_l (H-z)^{-1} ih (Q_j^*(\partial_j \chi_1) + (\partial_j \chi_1) Q_j)(\mathcal{H}-z)^{-1}\right).$$

This we can estimate as:

$$\begin{aligned}
\|T_1\|_1 &\leq \sum_{j=1}^n h \Big\{ \big\|\chi Q_l (H-z)^{-1} Q_j^* (\partial_j \chi_1)(\mathcal{H}-z)^{-1}\big\|_1 \\
&\quad + \big\|\chi Q_l (H-z)^{-1} (\partial_j \chi_1) Q_j (\mathcal{H}-z)^{-1}\big\|_1 \Big\} \\
&\leq \sum_{j=1}^n 2h \|\chi Q_l (H-z)^{-1} Q_j^*(\partial_j \chi_1)\|_1 \frac{1}{|\Im z|} + \sum_{j=1}^n h^2 \|\chi Q_l (H-z)^{-1} (\partial_j^2 \chi_1)\|_1 \frac{1}{|\Im z|} \\
&\leq C_N h \frac{<z>}{d(z)} \left[\frac{<z>^{1/2}}{h}\right]^d \left[\frac{<z> h^2}{d(z)^2}\right]^N \frac{1}{|\Im z|},
\end{aligned}$$

where we used [Sob95][Lemma 3.6] to get the last estimate. □

Now we can prove the theorem:

Proof. We use the representation:

$$\begin{aligned}
g(A) &= \sum_{j=0}^m \int (\partial^j g)(\lambda) \Im[i^j (A-\lambda-i)^{-1}] d\lambda \\
&\quad \frac{1}{\pi(m-1)!} \int_0^1 \tau^{m-1} \int_{\mathbf{R}} (\partial^m g)(\lambda) \Im[i^m (A-\lambda-i\tau)^{-1}] d\lambda \; d\tau,
\end{aligned}$$

which holds for all self adjoint operators A, $g \in C_0^\infty$, $m \geq 2$ (See [Sob95], [AdMBG91]). Writing

$$\delta(\lambda,\tau) = (\mathcal{H}-\lambda-i\tau)^{-1} - (H-\lambda-i\tau)^{-1},$$

we thus get:

$$\begin{aligned}
\chi Q_l \{g(\mathcal{H}) - g(H)\} &= \sum_{j=0}^m \frac{1}{\pi(m-1)!} \int_{\mathbf{R}} (\partial^j g)(\lambda) \chi Q_l \Im[i^j \delta(\lambda,1)] d\lambda \\
&\quad + \frac{1}{\pi(m-1)!} \int_0^1 \tau^{m-1} \int_{\mathbf{R}} (\partial^m g)(\lambda) \chi Q_l \Im[i^m \delta(\lambda,\tau)] d\lambda d\tau.
\end{aligned}$$

Choose $m = 2N+3$. Using the Lemma the first term is easily estimated by $O(h^{2N+1-d})$:

$$\|\chi Q_l \Im(i^j \delta(\lambda, 1)\|_1 \leq ch^{-d-1}(\sqrt{2+\lambda^2})^{d/2+N+1/2} h^{2N+1} \frac{1}{(1+|\lambda|)^{2N+1}}.$$

For N sufficiently big, this is integrable in λ, and we get

$$\| \sum_{j=0}^{m} \frac{1}{\pi(m-1)!} \int_{\mathbf{R}} (\partial^j g)(\lambda) \chi Q_l \Im[i^j \delta(\lambda, 1)] d\lambda \|_1 \leq c \sup_{j=0..m} \{|g^{(j)}|\} h^{2N-d}. \tag{A.2}$$

The second integral we split in two:

$$I_1 = \frac{1}{\pi(m-1)!} \int_0^1 \tau^{m-1} \int_{-2\lambda_0}^{\infty} (\partial^m g)(\lambda) \chi Q_l \Im[i^m \delta(\lambda, \tau)] d\lambda d\tau,$$

and

$$I_2 = \frac{1}{\pi(m-1)!} \int_0^1 \tau^{m-1} \int_{-\infty}^{-2\lambda_0} (\partial^m g)(\lambda) \chi Q_l \Im[i^m \delta(\lambda, \tau)] d\lambda d\tau.$$

Inside the integral in I_1 we estimate:

$$\|\chi Q_l \Im(i^j \delta(\lambda, \tau)\|_1 \leq ch^{-d-1}(\sqrt{2+\lambda^2})^{d/2+N+1/2} h^{2N+1} \tau^{-2N-2}.$$

Using our choice of m, I_1 is easily estimated. I_2 is estimated just like (A.2). □

As a corollary we get the following generalisation of the result in [Fou98]:

Lemma A.5. *Let the notation be as above. Then the currents of $\mathcal{H}$ and of H on the set $B(E)$ are the same up to an error of order $O(h^{1-n})$, i.e. for all $\chi \in C_0^\infty(B(E))$ and for all l we have:*

$$\mathrm{tr}[\chi Q_l (g_0(\mathcal{H}) - g_0(H))] = O(h^{1-n}).$$

Again this is uniform in E, χ and the C_α's.

Proof. Choose $g \in C_0^\infty(\mathbf{R})$ such that $gg_0 = g_0$ on $\mathrm{Spec}\,\mathcal{H}$. Notice, that the bounds on $|g^{(r)}|$ do not depend on $\inf \mathrm{Spec}\,\mathcal{H}$. Write, using the spectral theorem:

$$\begin{aligned} \mathrm{tr}[\chi Q_l g_0(\mathcal{H})] &= \mathrm{tr}\,[\chi Q_l g(\mathcal{H}) g_0(\mathcal{H})] \\ &= \mathrm{tr}\,[\chi Q_l g(H) g_0(\mathcal{H})] + O(h^\infty). \end{aligned}$$

Now we get from [Fou98] that $\chi Q_l g(H)$ is h-admissible. By an expansion of this operator in powers of h we get:

$$\mathrm{tr}[\chi Q_l g_0(\mathcal{H})] = \mathrm{tr}[Op_h^w \theta g_0(\mathcal{H})] + O(h^{1-n}),$$

where $\theta(x,\xi) = \chi(\xi_l - a_l) g((\xi - a_l)^2 + V(x)) \in C_0^\infty(\mathbf{R}^n)$. That this is $O(h^{1-n})$ follows from [Sob95] and the Tauberian argument given in [Fou98]. □

We will now prove the equation (4.2):

Proof. Let h_W be h_α with V changed into W. Remember that W is a locally C_0^∞ version of V. Then Theorem A.2 proves that

$$\mathrm{tr}[B(\mu, h) g_0(H)] = \mathrm{tr}[B(\mu, h) f(Op_\alpha^w h_W) g_0(H)].$$

Since $B(\mu, h) f(Op_\alpha^w h_W)$ is an α-admissible operator (in the sense of [Rob87]) and $(1-\psi)(x)$ vanishes on a neighborhood of the support of the symbol of $B(\mu, h) f(Op_\alpha^w h_W)$, the equation (4.2) is now obvious. □

Appendix B. Localisation in a Neighborhood of a Singularity

In this appendix we will prove that to study the current close to, for example, a Coulomb singularity, only the local behaviour of the singularity matters. The result below can be rephrased as follows:
Let $\chi \in C_0^\infty(B(1))$ and let V be a potential, such that, if $\zeta \in C_0^\infty(B(2))$, then ζV is bounded relatively to the kinetic energy $(-i\nabla - \vec{A})^2$. Then $\exists C > 0$ such that:

$$\|\chi(-i\nabla - \vec{A})[g_0((-i\nabla - \vec{A})^2 + V) - g_0((-i\nabla - \vec{A})^2 + \zeta V)]\| \leq C,$$

where C only depends on local information, i.e. on ζV.

Let us now be more precise: Let V (playing the role of ζV in the discussion above) be a multiplication operator such that $\exists\ 0 < \epsilon < 1$ and $M > 0$:

$$\langle u, |V| u\rangle \leq \epsilon \langle u, -\Delta u\rangle + M\|u\|^2, \tag{B.1}$$

for all $u \in C_0^\infty$. Observe that this implies, by the diamagnetic inequality, that

$$\langle u, |V| u\rangle \leq \epsilon \langle u, (-i\nabla - \vec{A})^2 u\rangle + M\|u\|^2,$$

with the same constants ϵ, M. Denote by H the selfadjoint operator $(-i\nabla - \vec{A})^2 + V$.

Assumption B.1. Let $\mathcal{H}$ be a selfadjoint operator in $L^2(\mathbf{R}^3)$, $\mathcal{H} \geq -\lambda_0$ for some $\lambda_0 > 1$ and satisfying for all $\phi \in C_0^\infty(B(2))$:

- $\forall u \in \mathcal{D}[\mathcal{H}]$ (the form domain of $\mathcal{H}$) we have $\phi u \in \mathcal{D}[\mathcal{H}]$ and $\exists \phi_1 \in C_0^\infty(B(2))$ such that $\langle u, \mathcal{H}(\phi v)\rangle = \langle (\phi_1 u), H(\phi v)\rangle$ for all $u, v \in \mathcal{D}[\mathcal{H}]$.

Remark B.2. The application in this article is to decompose Coulomb singularities, but the assumption is by far more general.

The result is the following:

Lemma B.3. *Let $\chi \in C_0^\infty(B(1))$, then*

$$\|\chi(-i\partial_{x_j} - A_j)[g_0(\mathcal{H}) - g_0(H)]\| \leq C,$$

where C depends only on χ and on ϵ, M in (B.1).

Remark B.4. C does *not* depend on the lower bound λ_0.

The main ingredient to prove the lemma is the following:

Lemma B.5. *Let $\chi \in C_0^\infty(B(1))$, and let $z \in \mathbf{C}$ with $0 < |\Im(z)| \leq 1$, then for all $N > 0$ there exists $C_N > 0$ such that*

$$\|\chi(-i\partial_{x_j} - A_j)[(\mathcal{H} - z)^{-1} - (H - z)^{-1}]\| \leq C_N \frac{M + 1 + |z|}{d_M(z)} \left[\frac{M + |z|}{d_M(z)}\right]^N \frac{1}{|\Im(z)|},$$

where $d_M(z) = \operatorname{dist}(z, [-M, \infty))$.

Proof. Choose $\chi_1 \in C_0^\infty(B(2))$, $\chi_1 \equiv 1$ on $B(3/2)$, and write

$$\begin{aligned} \chi(-i\partial_{x_j} - A_j)[(\mathcal{H} - z)^{-1} - (H - z)^{-1}] &= \chi(-i\partial_{x_j} - A_j)[\chi_1(\mathcal{H} - z)^{-1} - (H - z)^{-1}\chi_1] \\ &\quad + \chi(-i\partial_{x_j} - A_j)(H - z)^{-1}\phi, \end{aligned}$$

where $\phi = 1 - \chi_1$. Now the lemma follows from the identity

$$(\mathcal{H} - z)^{-1} - (H - z)^{-1}\chi_1 = -2i\sum_{l=1}^{3}(-i\partial_{x_l} - A_l)(\partial_{x_l}\chi_1) + \Delta\chi_1,$$

and the following result from [Sob96a][Lemma 3.3]:

$$\|\chi(-i\partial_{x_j} - A_j)^{m_1}(H-z)^{-1}(-i\partial_{x_l} - A_l)^{m_2}\phi\| \leq C\frac{(M+|z|)^{\frac{m_1+m_2}{2}}}{d_M(z)}\left[\frac{M+|z|}{d_M(z)^2}\right]^N,$$

where $m_1, m_2 \in \{0,1\}$. □

The lemma B.3 now follows, using almost analytic extensions, just like in the previous appendix.

Appendix C. A Calculation with Poisson Summation

Let us write $t = \frac{[V(x)]_-}{2\mu h}$, and

$$\begin{aligned} S = S([V(x)]_-, h\mu) &= \sum_{n=0}^{\infty} d_n \left([2h\mu n - [V(x)]_-]_-^{3/2} - 3/2(2h\mu)n[2h\mu n - [V(x)]_-]_-^{1/2}\right) \\ &= (2h\mu)^{3/2}\sum_{n=0}^{\infty} d_n\left([n-t]_-^{3/2} - \frac{3}{2}n[n-t]_-^{1/2}\right). \end{aligned}$$

In this appendix we want to prove the following computational result:

Proposition C.1.

$$S([V(x)]_-, h\mu) = O((h\mu)^{3/2} + \sqrt{h\mu}[V(x)]_- + h\mu[V(x)]_-),$$

uniformly in x.

Proof. Let us write $F_t(\alpha) = \left([\alpha - t]_-^{3/2} - \frac{3}{2}\alpha[\alpha-t]_-^{1/2}\right)$, then

$$S = \frac{(2h\mu)^{3/2}}{\pi}\left(\frac{F_t(0)}{2} + \sum_{k=1}^{\infty} F_t(k)\right).$$

We use Poisson Summation and get:

$$S = \frac{(2h\mu)^{3/2}}{\pi}\left\{\int_0^\infty F_t(\alpha)d\alpha + 2\Re\left(\sum_{k=1}^{\infty}\int_0^\infty F_t(\alpha)e^{i2\pi k\alpha}d\alpha\right)\right\}.$$

Let us look at the first term:

$$\begin{aligned} \int_0^\infty F_t(\alpha)d\alpha &= \int_0^t (t-\alpha)^{3/2}d\alpha - \frac{3}{2}\left\{[\alpha\frac{2}{3}(t-\alpha)^{3/2}]_{\alpha=0}^t - \frac{2}{3}\int_0^t (t-\alpha)^{3/2}d\alpha\right\} \\ &= 0. \end{aligned}$$

One part of $\Re\left(\int_0^\infty F_t(\alpha)e^{i2\pi k\alpha}d\alpha\right)$ was calculated in [Sob96b][p.399]:

$$\begin{aligned} &\Re\left(\int_0^\infty (t-\alpha)^{3/2}e^{i2\pi k\alpha}d\alpha\right) \\ &= \frac{3}{8\pi^2k^2}t^{1/2} - \frac{3}{16\pi^2k^{5/2}}\left(\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right), \end{aligned}$$

where

$$\mathcal{C}(x) = \int_0^x \cos(\pi u^2/2)du,$$

and

$$\mathcal{S}(x) = \int_0^x \sin(\pi u^2/2)du.$$

What we have left to calculate is thus $-\frac{3}{2}\int_0^t \alpha\sqrt{t-\alpha}e^{i2\pi k\alpha}d\alpha$. This we do explicitly:

$$\begin{aligned}
-\frac{3}{2}\int_0^t \alpha\sqrt{t-\alpha}e^{i2\pi k\alpha}d\alpha &= -\frac{3}{4\pi i}\frac{d}{dk}\int_0^t \sqrt{t-\alpha}e^{i2\pi k\alpha}d\alpha \\
&= -\frac{1}{2\pi i}\frac{d}{dk}\int_0^t \frac{d}{dt}(t-\alpha)^{3/2}e^{i2\pi k\alpha}d\alpha \\
&= -\frac{1}{2\pi i}\frac{d}{dk}\left\{\frac{d}{dt}\left(\int_0^t (t-\alpha)^{3/2}e^{i2\pi k\alpha}d\alpha\right) - (t-\alpha)^{3/2}e^{i2\pi k\alpha}\Big|_{\alpha=t}\right\} \\
&= -\frac{1}{2\pi i}\frac{d}{dk}\frac{d}{dt}\int_0^t (t-\alpha)^{3/2}e^{i2\pi k\alpha}d\alpha \\
&= -\frac{1}{2\pi i}\frac{d}{dk}\frac{d}{dt}\left\{\frac{t^{3/2}}{2\pi ik} + \frac{3t^{1/2}}{8\pi^2k^2} - \frac{3e^{i2\pi kt}}{16\pi^2k^{5/2}}\int_0^{2\sqrt{kt}} e^{-i\pi u^2/2}du\right\} \\
&= \frac{1}{2\pi i}3/2\frac{t^{1/2}}{2\pi ik^2} + \frac{1}{2\pi i}\frac{3}{8\pi^2k^3\sqrt{t}} + \\
&\quad \frac{1}{2\pi i}\frac{d}{dk}\left(\frac{3e^{i2\pi kt}}{16\pi^2k^{5/2}}\left\{2\pi ik\int_0^{2\sqrt{kt}} e^{-i\pi u^2/2}du + \sqrt{k}t^{-1/2}e^{-i\pi 2kt}\right\}\right).
\end{aligned}$$

Here we used the calculation from [Sob96b][p.399] to get the next to last equality. We calculate the real part and get:

$$\begin{aligned}
&-\frac{3t^{1/2}}{8\pi^2k^2} + \frac{3}{16\pi^2}\Re\left\{\frac{d}{dk}\left(k^{-3/2}e^{i2\pi kt}\int_0^{2\sqrt{kt}} e^{-i\pi u^2/2}du\right)\right\} \\
= \; &-\frac{3t^{1/2}}{8\pi^2k^2} + \frac{3}{16\pi^2}\frac{d}{dk}\left\{k^{-3/2}\left[\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right]\right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
S &= \sum_{k=1}^{\infty}\frac{(2h\mu)^{3/2}}{\pi}2\Big\{\frac{-15}{32\pi^2k^{5/2}}\left[\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right] \\
&\quad + \frac{3}{16\pi^2k^{3/2}}\frac{d}{dk}\left[\cos(2\pi kt)\mathcal{C}(2\sqrt{kt}) + \sin(2\pi kt)\mathcal{S}(2\sqrt{kt})\right]\Big\}.
\end{aligned}$$

Using, that $\mathcal{C}$ and $\mathcal{S}$ are bounded with bounded first derivatives, we thus see that

$$\begin{aligned}
S &= O\left((h\mu)^{3/2}\sum_{k=1}^{\infty}(k^{-5/2} + \frac{t}{k^{3/2}} + \frac{\sqrt{t}}{k^2})\right) \\
&= O((h\mu)^{3/2}(1 + t + \sqrt{t})) \\
&= O((h\mu)^{3/2} + \sqrt{h\mu}[V]_- + h\mu[V]_-).
\end{aligned}$$

□

Appendix D. Gauge Invariance of the Current

In this appendix we will prove that the current $\mathcal{J}(h,\mu,\vec{a},V)$ as a function of $\vec{a}$ only depends on the magnetic field $\vec{b} = \nabla\times\vec{a}$ generated by $\vec{a}$, i.e. that if $\vec{a} = \tilde{a} + \nabla\phi$ then $\mathcal{J}(h,\mu,\vec{a},V) = \mathcal{J}(h,\mu,\tilde{a},V)$:

Lemma D.1. *Suppose V is relatively bounded with respect to $-h^2\Delta$ and that* $\mathrm{Spec}(\mathbf{P}(h,\mu,V))$ *below zero is discrete. Then $\forall \phi \in C_0^\infty(\mathbf{R}^3)$ we have $\mathcal{J}(h,\mu,\nabla\phi,V)=0$.*

Proof. Let ψ be an eigenfunction of $\mathbf{P}(h,\mu,V)$ with eigenvalue $\lambda<0$. We may, with a slight abuse of notation assume that

$$(H+W)\psi=\lambda\psi,$$

where $W=V\pm\mu h$ and $H=(-ih\nabla-\vec{A})^2$. We have to prove that

$$\langle\psi,(\nabla\phi)\cdot(-ih\nabla-\vec{A})\psi\rangle+\langle\psi,(-ih\nabla-\vec{A})\cdot(\nabla\phi)\psi\rangle=0,$$

or equivalently

$$\langle\psi,(-ih\nabla\phi)\cdot(-ih\nabla-\vec{A})\psi\rangle+\langle\psi,(-ih\nabla-\vec{A})\cdot(-ih\nabla\phi)\psi\rangle=0.$$

Notice that $(-ih\partial_{x_j}-A_j)\phi=\phi(-ih\partial_{x_j}-A_j)+(-ih\partial_{x_j}\phi)$, thus we get, using the self-adjointness of $(-ih\partial_{x_j}-A_j)$ and the relative boundedness of W:

$$\begin{aligned}
&\langle\psi,(-ih\nabla\phi)\cdot(-ih\nabla-\vec{A})\psi\rangle+\langle\psi,(-ih\nabla-\vec{A})\cdot(-ih\nabla\phi)\psi\rangle\\
=\;&\langle\psi,\Big[(-ih\nabla-\vec{A})\phi-\phi(-ih\nabla-\vec{A})\Big]\cdot(-ih\nabla-\vec{A})\psi\rangle\\
&+\langle\psi,(-ih\nabla-\vec{A})\cdot\Big[(-ih\nabla-\vec{A})\phi-\phi(-ih\nabla-\vec{A})\Big]\,\psi\rangle\\
=\;&\langle(-ih\nabla-\vec{A})\psi,\phi(-ih\nabla-\vec{A})\psi\rangle-\langle\psi,\phi\lambda\psi\rangle+\langle\psi,\phi W\psi\rangle\\
&+\langle\lambda\psi,\phi\psi\rangle-\langle W\psi,\phi\psi\rangle-\langle(-ih\nabla-\vec{A})\psi,\phi(-ih\nabla-\vec{A})\psi\rangle.
\end{aligned}$$

□

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