

On Recent Existence Theorems in the Theory of Optimization¹

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Abstract. A condition recently proposed is shown to imply the weak compactness in $H^{1,1}$ and actually is equivalent to another condition previously proposed by the authors. Once compactness is proved, then existence theorems follow from lower closure theorems also previously proved by the authors, and extended to Pareto problems. The present analysis adds to the recent work of Goodman concerning the equivalence of seminormality conditions with concepts of convex analysis and lattice theory.

Key Words. Lower closure, weak compactness, existence theorems, convex duality, property (Q), Lipschitz type conditions.

1. Introduction

In a recent paper by Cecconi, a weak compactness theorem in $H^{1,1}$ is reported as having been proved by Rockafellar on the basis of convex analysis and functional analysis, and based on a condition that we call here (G3). Actually, this condition is a variant of the Nagumo type and analogous growth conditions, and indeed we first prove (Section 2) that condition (G3) is equivalent to a condition of that type, say (G2), that we have used for the same scope in previous papers.

Once the weak compactness theorem is proved, the ensuing existence theorems in the theory of optimization are well-known consequences of lower-closure theorems which we proved in the past years (e.g., Refs. 1, 2), where (among other points) no condition (Q) is explicitly needed (see Section 3). Actually we further extended these lower-closure theorems and ensuing existence theorems to functionals taking values in any R^n , or any reflexive Banach space, where a partial ordering has been defined by a given closed convex cone (Refs. 3–6).

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Actually, as a consequence of the *weak convergence of the derivatives* property, and a well-known remark by De la Valle Poussin, a Nagumo-type growth condition holds, under which suitable auxiliary sets possess *property (Q)*, and this condition—whether we explicitly name it or not—enters in the proof of the lower-closure theorems. We point out here that property (Q) has been equivalently expressed by Cesari (Refs. 7, 8) in terms of *Tonelli and McShane seminormality conditions* and by Goodman (Ref. 9) in terms of duality operations in convex analysis and lattice theory in point-set topology (see Section 5 for Goodman's equivalence statements).

In recent work of Berkovitz and Bates (see Remark 6.1), certain Lipschitz-type conditions are used in place of property (Q), claiming that these conditions are independent of property (Q). In Remark 6.1, we point out that the contrary is true, since their Lipschitz condition is a particular case of an analytic condition, namely, property (D) (see Section 6) studied long ago by Cesari and Suryanarayana (Ref. 10), which implies a suitable form of property (Q). It is relevant that this property (D) is valid even in Hilbert spaces and Banach spaces (Ref. 3). Finally, in Remark 3.2, we correct an erroneous statement by Olech in one of his recent reviews.

2. Weak Compactness

Let us consider any family

$$\mathfrak{E} = \{\eta(t), x(t), a \leq t \leq b\}$$

of pairs of functions, $\eta(t)$ scalar and L -integrable, and $x(t) = (x_1, \dots, x_n)$ absolutely continuous in intervals $[a, b]$, not necessarily the same, but within a fixed finite interval or $-\infty < a_0 \leq a < b \leq b_0 < +\infty$. There are well-known *growth conditions* which guarantee compactness properties of the class $\{x\}$ of all elements $x(t)$, $a \leq t \leq b$, appearing in \mathfrak{E} . We name here a few.

(G1) There are a constant $M \geq 0$ and a scalar function $\Phi(\xi)$, $0 \leq \xi < +\infty$, bounded below and with $\Phi(\xi)/\xi \rightarrow \infty$ as $\xi \rightarrow \infty$, such that

$$(a) \int_a^b \eta(t) dt \leq M$$

and

$$(b) \eta(t) \geq \Phi(|x'(t)|), \quad a \leq t \leq b,$$

for all pairs η, x of class \mathfrak{E} .

(G2) There are a constant $M \geq 0$ and, for every $\epsilon > 0$, a scalar function $\Psi_\epsilon(t) \geq 0$, L -integrable in $[a_0, b_0]$, such that

$$(a) \int_a^b \eta(t) dt \leq M, \text{ and}$$

$$(b) |x'(t)| \leq \Psi_\epsilon(t) + \epsilon \eta(t), \quad a \leq t \leq b,$$

for all pairs η, x in the class \mathfrak{C} .

(G3) There are a constant $M \geq 0$ and, for every $p = (p_1, \dots, p_n) \in R^n$, a scalar function $\phi_p(t) \geq 0$, L -integrable in $[a_0, b_0]$, such that

$$(a) \int_a^b \eta(t) dt \leq M, \text{ and}$$

$$(b) \eta(t) \geq (p, x'(t)) - \phi_p(t), \quad a \leq t \leq b,$$

for all pairs η, x of the class \mathfrak{C} .

Theorem 2.1. Under either condition (G1), (G2), or (G3), the class $\{x\}$ is equiabsolutely continuous and the class $\{x'\}$ is equiabsolutely integrable. Thus if, in addition, the class $\{x\}$ is equibounded, then the same class $\{x\}$ is relatively weakly compact in $H^{1,1}$.

This is essentially the Tonelli–Nagumo statement under condition (G1) (see, for example, Ref. 11). The same statement was essentially proved by Cesari, La Palm, and Nishiura (Ref. 12) under condition (G2), and for the sake of simplicity we give a direct proof of this in the Appendix below. (See also Ref. 15.) Condition (G3) is the one mentioned by Cecconi (Ref. 13, Theorem 6, p. 296) as reported by Rockafellar (Ref. 14). Here, under (i) we give a simple direct proof of Theorem 2.1 under condition (G3). Under (ii), we prove that (G3) is equivalent to (G2). Condition (G1) implies (G2), but (G2) is actually more general than (G1). For instance, for $\eta(t) = t^\alpha x'^2$, $0 \leq t \leq 1$, $0 < \alpha < 1$, condition (G2) is satisfied, but (G1) is not. Condition (G2) has been extensively used in Refs. 3, 12, 16, and elsewhere. Thus, condition (G2) applied to the integral

$$I = \int_0^1 t^\alpha x'^2 dt, \quad x(0) = 1, \quad x(1) = 0, \quad t x'^2,$$

allows one to separate the case $\alpha \geq 1$ for which I has no absolute minimum from the case $0 < \alpha < 1$ for which I has an absolute minimum. Condition (G3) also was used in Ref. 16 and other papers.

Proof. (i). *Direct Proof of Theorem 2.1 under Condition (G3).* Let $\varphi(t), \psi(t), a_0 \leq t \leq b_0$, be the nonnegative L -integrable functions in (G3)(b)

corresponding to the two unit vectors

$$p = u_1 = (1, 0, \dots, 0) \quad \text{and} \quad p = -u_1 = (-1, 0, \dots, 0).$$

Then, for any pair η, x in \mathfrak{C} , we have

$$x'_1(t) \leq \eta(t) + \varphi(t), \quad -x'_1(t) \leq \eta(t) + \psi(t), \quad a \leq t \leq b,$$

and hence

$$|x'_1(t)| \leq \eta(t) + \varphi(t) + \psi(t), \quad a \leq t \leq b.$$

Thus,

$$\eta(t) + \varphi(t) + \psi(t) \geq 0, \quad a \leq t \leq b. \tag{1}$$

Let

$$M_1 = \int_a^b (\varphi(t) + \psi(t)) dt.$$

Now, given $\epsilon > 0$, let N be an integer, such that

$$N^{-1}nM \leq \epsilon/3, \quad N^{-1}nM_1 \leq \epsilon/3.$$

If u_i, v_i denote the unit vectors

$$u_i = (\delta_{ij}, j = 1, 2, \dots, n), \quad v_i = (-\delta_{ij}, j = 1, \dots, n),$$

then, for $p = Nu_i$ and $p = Nv_i$, there are functions

$$\Phi_i(t) \geq 0, \quad \Psi_i(t) \geq 0,$$

L -integrable in $[a_0, b_0]$, such that

$$Nx'_i(t) \leq \eta(t) + \Phi_i(t), \quad -Nx'_i(t) \leq \eta(t) + \Psi_i(t), \quad a \leq t \leq b, \quad i = 1, \dots, n,$$

and hence

$$N|x'_i(t)| \leq \eta(t) + \Phi_i(t) + \Psi_i(t), \quad a \leq t \leq b, \quad i = 1, \dots, n.$$

Then,

$$\Phi(t) = \sum_{i=1}^n \Phi_i(t), \quad \Psi(t) = \sum_{i=1}^n \Psi_i(t)$$

are L -integrable in $[a_0, b_0]$, and we also have

$$N|x'(t)| \leq n\eta(t) + \Phi(t) + \Psi(t), \quad a \leq t \leq b. \tag{2}$$

If E denotes any measurable subset of $[a, b]$, we have, from inequalities (1)–(2),

$$\begin{aligned} \int_E |x'(t)| dt &\leq N^{-1}n \int_E \eta(t) dt + N^{-1}n \int_E (\Phi(t) + \Psi(t)) dt \\ &\leq N^{-1}n \int_E [\eta(t) + \varphi(t) + \psi(t)] dt + N^{-1}n \int_E (\Phi(t) + \Psi(t)) dt \\ &\leq N^{-1}n \int_a^b \eta(t) dt + N^{-1}n \int_a^b (\varphi(t) + \psi(t)) dt \\ &\quad + N^{-1}n \int_E (\Phi(t) + \Psi(t)) dt. \end{aligned}$$

Since $\Phi + \Psi$ is L -integrable, there is $\delta > 0$, such that $\text{meas } E \leq \delta$ implies that

$$\int_E (\Phi + \Psi) dt \leq \epsilon/3,$$

and hence

$$\int_E |x'(t)| dt \leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

We have proved the equiabsolute integrability of the class $\{x'\}$. From here, the equiabsolute continuity of the class $\{x\}$ follows immediately. If we know that the class $\{x\}$ is equibounded, then the compactness of $\{x\}$ in C follows from the Ascoli theorem, and the weak compactness of $\{x\}$ in $H^{1,1}$ follows from the Dunford–Pettis theorem.

Proof. (ii). *Proof that (G3) implies (G2).* If (G3) is satisfied, then, given $\epsilon > 0$, let us consider the $2n$ points $p \in R^n$ defined by $p = \pm \epsilon^{-1}n$ (δ_{is} , $s = 1, \dots, n$), $i = 1, \dots, n$. Let $\phi_p(t) \geq 0$ be the corresponding $2n$ locally integrable functions, let $\phi(t)$ denote their sum, and take

$$\Psi_\epsilon(t) = \phi(t).$$

Then

$$\eta(t) \geq \pm \epsilon^{-1}n x'_i(t) - \phi_p(t),$$

for each of the $2n$ points above, and hence

$$|x'(t)| \leq \sum_i |x'_i(t)| \leq \epsilon \eta(t) + \Psi_\epsilon(t)$$

for all t . Thus, (G3) implies (G2).

Proof that (G2) implies (G3). Given $p \in R^n$ take $\epsilon > 0$ such that $1/\epsilon \geq |p|$. By (G2) there is $\Psi_\epsilon(t) \geq 0$, Ψ_ϵ locally integrable, such that $|x'(t)| \leq \Psi_\epsilon(t) + \epsilon \eta(t)$, that is,

$$\eta(t) \geq (1/\epsilon)|x'(t)| - (1/\epsilon)\Psi_\epsilon(t) \geq (p, x'(t)) - \phi_p(t),$$

where $\phi_p(t) = (1/\epsilon)\Psi_\epsilon(t)$. Thus (G2) implies (G3).

A direct proof of Theorem 2.1 under condition (G2) is given in the Appendix.

Remark 2.1. The arguments underlying Theorem 2.1 apply as well to any family

$$\mathfrak{C} = \{\eta(t), \xi(t), t \in G\}$$

of pairs of functions

$$\eta \in L_1(G), \xi \in (L_1(G))^n, \quad \text{with } \int_G \eta(t) dt \leq M,$$

where G is any measurable bounded subset of R^v . Condition (G1) then becomes

$$\eta(t) \geq \phi(|\xi(t)|), \quad t \in G,$$

for all pairs (η, ξ) in \mathfrak{C} . Condition (G2) then becomes: given $\epsilon > 0$, there is some $\psi_\epsilon(t) \geq 0$, $t \in G$, $\psi_\epsilon \in L_1(G)$, such that

$$|\xi(t)| \leq \psi_\epsilon(t) + \epsilon \eta(t), \quad t \in G,$$

for all pairs (η, ξ) in \mathfrak{C} . Condition (G3) then becomes: given $p \in R^n$, there is $\phi_p(t) \geq 0$, $t \in G$, $\phi_p \in L_1(G)$, such that

$$\eta(t) \geq (p, \xi(t)) - \phi_p(t), \quad t \in G,$$

for all pairs (η, ξ) in \mathfrak{C} . Under either hypothesis (G1), (G2), or (G3), the class $\{\xi(t)\}$ is equiabsolutely integrable in G and hence relatively weakly compact in $L_1(G)$.

3. Some Lower-Closure and Existence Theorems

We briefly state here a simple version of a lower-closure theorem. For $t \in [a, b]$, let $A(t)$ denote a closed subset of the y -space R^n ; let

$$A = \{(t, y) | a \leq t \leq b, y \in A(t)\} \subset R^{1+n},$$

for every $(t, y) \in A$ let $Q(t, y)$ denote a nonempty closed subset of the z -space R^n ; and let

$$M = \{(t, y, z) | (t, y) \in A, z \in Q(t, y)\} \subset R^{1+2n}.$$

Let $F_0(t, y, z)$ denote a real-valued function defined on M , $F_0 < +\infty$ on M , and let $F_0(t, y, z)$ denote also the extended function defined by taking $F_0 = +\infty$ in $R^{1+2n} - M$. Let us assume that the extended function F_0 is measurable in t for all (y, z) and that, for almost all \bar{t} , $F_0(\bar{t}, y, z)$ is lower semicontinuous in (y, z) and convex in z .

Theorem 3.1. Lower Closure Theorem. If $\xi(t), y(t), \eta_k(t), \xi_k(t), y_k(t), \lambda(t), \lambda_k(t), t \in G = [a, b], k = 1, 2, \dots$, are measurable functions, $\xi, \xi_k \in (L_1(G))^r, \eta_k \in L_1(G)$, such that $y_k \rightarrow y$ in measure in $G, \xi_k \rightarrow \xi$ weakly in $(L_1(G))^r$ as $k \rightarrow \infty$, with

$$y_k(t) \in A(t), \quad \xi_k(t) \in Q(t, y_k(t)), \quad \eta_k(t) \geq F_0(t, y_k(t), \xi_k(t)),$$

$$t \in G \text{ a.e., } \quad k = 1, 2, \dots,$$

$$-\infty \leq i = \liminf_{k \rightarrow \infty} \int_G \eta_k(t) dt < +\infty,$$

$$\eta_k(t) \geq \lambda_k(t), \quad \lambda, \lambda_k \in L_1(G), \quad \lambda_k \rightarrow \lambda \text{ weakly in } L_1(G),$$

then there is a function $\eta(t), t \in G, \eta \in L_1(G)$, such that

$$y(t) \in A(t), \quad \xi(t) \in Q(t, y(t)), \quad \eta(t) \geq F_0(t, y(t), \xi(t)),$$

$$t \in G \text{ a.e., } \quad \int_G \eta(t) dt \leq i.$$

For this theorem, see Cesari (Ref. 1) and Cesari and Suryanarayana (Ref. 2). For extensions to multidimensional t or t, y, f_0 valued in Banach spaces, see Refs. 2, 3, 5.

For every $(t, y) \in A$, let $\tilde{Q}(t, y)$ denote the set of all $(z^0, z) \in R^{n+1}$, with

$$z^0 \geq F_0(t, y, z), \quad z \in Q(t, y).$$

The condition concerning F_0 above can be equivalently expressed by requiring that F_0 be measurable in t for all (y, z) , that A be closed, and that, for almost all \bar{t} , the sets $\tilde{Q}(\bar{t}, y)$ have property (K) with respect to y in $A(\bar{t})$ and are convex (see Section 4 below).

For lower-closure theorems for Banach valued functions $y(t)$ see Cesari (Refs. 3, 5). Of course, Theorem 3.1 holds even for functions defined on any bounded measurable subset G of R^r or any finite measure space.

Remark 3.1. Note that, in the lower-closure Theorem 3.1, if we know that the functions $\lambda_k(t), t \in G$, can be taken in the form

$$\lambda_k(t) = (p, \xi_k(t)) - \phi_p(t), \quad t \in G, \quad k = 1, 2, \dots,$$

for every fixed $p \in R^n$ and corresponding

$$\phi_p(t) \geq 0, \quad t \in G, \phi_p \in L_1(G),$$

then, by Remark 2.1, the sequence λ_k certainly has a subsequence which is weakly convergent in $(L_1(G))^n$ as required.

Remark 3.2. Note that the lower-closure Theorem 3.1 has been proved in Ref. 1 for functions $\xi(t)$, $\xi_k(t)$ having their values in a finite-dimensional space R^r , as stated. For functions ξ , ξ_k having their values in a real Banach space B , we have proved in Ref. 2, Refs. 3, Part I, and Refs. 4, 16, 17 an analogous statement, namely, the same as Theorem 3.1 with the following modified requirements:

$$\xi(t), y(t), \eta_k(t), \xi_k(t), y_k(t), \mu(t), \mu_k(t), p_k(t), \lambda(t), \lambda_k(t), t \in G, \\ k = 1, 2, \dots,$$

are measurable functions, $\xi, \xi_k, p_k \in L_1(G, B)$, $\eta_k, \mu_k, \mu, \lambda_k, \lambda \in L_1(G, R)$, $y(t), y_k(t) \in A(t)$, such that

$$\eta_k(t) \geq F_0(t, y_k(t), \xi_k(t)), \quad \mu_k(t) \geq F_0(t, y_k(t), p_k(t)), \\ \xi_k(t) \in Q(t, y_k(t)), \quad p_k(t) \in Q(t, y_k(t)), \quad t \in G \text{ a.e.}, \quad k = 1, 2, \dots, \\ -\infty < i = \liminf_{k \rightarrow \infty} \int_G \eta_k(t) dt < +\infty, \quad \eta_k(t) \geq \lambda_k(t), \\ t \in G, \quad k = 1, 2, \dots,$$

$$\xi_k(t) \rightarrow \xi(t) \text{ weakly in } L_1(G, B), \quad y_k(t) \rightarrow y(t) \text{ in measure in } G, \\ p_k(t) \rightarrow p(t) \text{ strongly in } L_1(G, B), \quad \mu_k(t) \rightarrow \mu(t) \text{ weakly in } L_1(G, R), \\ \lambda_k(t) \rightarrow \lambda(t) \text{ weakly in } L_1(G, R).$$

Then, there is a function $\eta(t)$, $t \in G$, $\eta \in L_1(G, R)$, such that

$$\eta(t) \geq F_0(t, x(t), \xi(t)), \quad t \in G \text{ a.e.}, \quad \int_G \eta(t) dt \leq i.$$

As we proved in the quoted papers, we may allow G to be any metric space and also a finite complete measure space (G, α, μ) , and thus $d\mu$ may replace dt above, and y, y_k may take their values in any metric space (Y, d) .

Remark 3.3. Olech (Ref. 19), in his review of Ref. 2, has stated that the existence of sequences μ_k, p_k (with $\mu_k \rightarrow \mu$ weakly in L_1 , and $p_k \rightarrow p$ strongly in L_1) is a consequence of the other hypotheses in Theorem 4.1, p. 172, Ref. 2, that is, the proposition in Remark 2.1 above. Olech's statement is not true, as Example 5.4, p. 178 of the same paper (Ref. 2), exactly shows.

However, in finite-dimensional spaces, as shown in Ref. 4, it is possible to prove the lower-closure theorem without this specific assumption regarding the existence of a sequence $\mu_k(t), p_k(t)$, with

$$p_k(t) \in Q(t, y_k(t)), \quad \mu_k(t) \geq F_0(t, y_k(t), p_k(t)),$$

such that p_k is strongly convergent and μ_k is weakly convergent. Such sequences may not exist even if the upper semicontinuity property (K) in y of the sets $Q(t, y)$ is assumed. If such sequences do exist, then the original minimizing sequences η_k, ξ_k can be modified suitably so as to lie in equibounded sets $Q^*(t, y_k(t))$, which would then have property (Q).

Remark 3.4. As in Remark 2.1, we mention here another version of the lower-closure Theorem 3.1, which we shall use in Section 5 below, and for which we refer to Ref. 10 and Ref. 3, Part I, for proofs. Namely, the same Theorem 3.1 holds with the following modified requirements:

$\xi(t), y(t), \eta_k(t), \xi_k(t), y_k(t), \bar{\eta}_k(t), \bar{\xi}_k(t), \lambda(t), \lambda_k(t), t \in G, k = 1, 2, \dots$, are measurable functions, $\xi, \xi_k, \bar{\xi}_k \in L_1(G, B), \eta_k, \bar{\eta}_k, \lambda_k, \lambda \in L_1(G, R), y(t), y_k(t) \in A(t)$, such that

$$\begin{aligned} \eta_k(t) &\geq F_0(t, y_k(t), \xi_k(t)), & \bar{\eta}_k(t) &\geq F_0(t, y(t), \bar{\xi}_k(t)), \\ \xi_k(t) \in Q(t, y_k(t)), & \bar{\xi}_k(t) \in Q(t, y(t)), & \eta_k(t), \bar{\eta}_k(t) &\geq \lambda_k(t), \quad t \in G, \end{aligned}$$

$$-\infty < i = \liminf_{k \rightarrow \infty} \int_G \eta_k(t) dt < +\infty,$$

$$\xi_k(t) \rightarrow \xi(t) \text{ weakly in } L_1(G, B), \quad y_k(t) \rightarrow y(t) \text{ in measure in } G,$$

$$\lambda_k(t) \rightarrow \lambda(t) \text{ weakly in } L_1(G, R),$$

$$\delta_k(t) = \xi_k(t) - \bar{\xi}_k(t) \rightarrow 0 \text{ strongly in } L_1(G, B),$$

$$\delta_k^0(t) = \eta_k(t) - \bar{\eta}_k(t) \rightarrow 0 \text{ strongly in } L_1(G, R).$$

Then, there is a function $\eta(t), t \in G, \eta \in L_1(G, R)$, such that

$$\eta(t) \geq F_0(t, y(t), \xi(t)), \eta(t) \in Q(t, y(t)), \quad t \in G \text{ a.e.}, \quad \int_G \eta(t) dt \leq i.$$

We can now state an existence theorem, in which again for the sake of simplicity we assume t scalar, $y \in R^n, A$ compact, and F_0 also scalar. Thus, we are concerned here with a functional

$$I[y] = g(t_1, y(t_1), t_2, y(t_2)) + \int_{t_1}^{t_2} F_0(t, y(t), y'(t)) dt, \quad (3)$$

where $g(t_1, y_1, t_2, y_2)$ is a scalar function defined on a subset B of R^{2+2n} ,

$F_0(t, y, z)$ is a scalar function defined on a set

$$M = \{(t, y, z) \mid (t, y) \in A, z \in Q(t, y)\}$$

of the tyz -space and extended to R^{1+2n} as stated above. We may require on F_0 one of the following growth conditions:

(g1) there is a scalar function $\phi(\xi)$, $0 \leq \xi < +\infty$, bounded below with $\phi(\xi)/\xi \rightarrow \infty$, such that $F_0(t, y, z) \geq \phi(|z|)$ for all (t, y, z) ;

(g2) for every $\epsilon > 0$, there is a scalar function $\psi_\epsilon(t) \geq 0$, locally integrable, such that $|z| \leq \psi_\epsilon(t) + \epsilon F_0(t, y, z)$;

(g3) for every $p \in R^n$, there is a scalar function $\phi_p(t)$, locally integrable, such that $F_0(t, y, z) \geq (p, z) - \phi_p(t)$ for all (t, y, z) .

Let Ω be a nonempty closed class of absolutely continuous functions $y(t) = (y_1, \dots, y_n)$, $t_1 \leq t \leq t_2$, with $(t, y(t)) \in A$, $(t_1, y(t_1), t_2, y(t_2)) \in B$; $F_0(\cdot, y(\cdot), y'(\cdot))$ is L -integrable in $[t_1, t_2]$. The property of closedness needed here is a very mild one, that is, every limit element y in the weak topology of $H^{1,1}$ which has the properties just stated belongs to Ω . Thus, the class Ω of all y with the properties stated is closed. The existence theorem below certainly holds in the class of all y with the properties stated.

Theorem 3.2. *Existence Theorem for Problems of Calculus of Variations and Optimal Control Theory.* Let A be compact, B closed, and let us assume that g is lower semicontinuous on B , that the extended function $F_0(t, y, z)$ is measurable in t for all (y, z) , and that, for almost all t , $F_0(t, y, z)$ is lower semicontinuous in (y, z) and convex in z . Let us assume that one of the growth hypotheses (G1), (G2), or (G3) holds. Then, the functional (3) has an absolute minimum in Ω .

If no condition (G) holds, the conclusion is still valid provided we know that the class Ω is weakly compact in $H^{1,1}$ and F_0 satisfies one of the following conditions:

(L1) $F_0(t, y, z) \geq -\psi(t)$, for some locally integrable function $\psi \geq 0$;

(L2) $F_0(t, y, z) \geq -\psi(t) - c|z|$, for some constant c and ψ as in (L1);

(L3) $F_0(t, y, z) \geq -\psi(t) - (\varphi(t), z)$, for ψ as in L1 and some bounded measurable φ ;

(L4) $F_0(t, y, z)$ is defined at least in $A \times B_0$, where B_0 is the closed unit ball in R^n and $F_0(t, y, z) \geq \beta$ in $A \times B_0$ and $F_0(t, y, 0) \leq \alpha$ in A for some constants $\alpha \leq \beta$.

The condition (L4) is certainly satisfied if F_0 is continuous on $A \times B_0$.

The extension of Theorem 3.2 to the case of A being not compact, but closed, is only technical. For extensions to the case where t is multi-dimensional or t, y, F_0 take their values in Banach spaces, we refer to Cesari (Ref. 3, Part I) and Cesari and Suryanarayana (Refs. 2, 4).

4. Some Upper Semicontinuity Properties of Set-Valued Maps

Given any set Z in a linear topological space Y , we denote by $\text{cl } Z$, $\text{bd } Z$, $\text{co } Z$ the closure of Z , the boundary of Z , and the convex hull of Z , respectively. Thus, $\text{cl co } Z$ denotes the closure of the convex hull of Z .

Let $x \rightarrow Q(x)$, $x \in X$, $Q(x) \subset Y$ be a set-valued map from a metric space X to a linear topological space Y . Let x_0 be a point of X .

Kuratowski's concept of upper semicontinuity is relevant. We say that the map $x \rightarrow Q(x)$ has property (K) at x_0 (Kuratowski), provided

$$Q(x_0) = \bigcap_{\delta > 0} \text{cl } \cup [Q(x), x \in N_\delta(x_0)].$$

Here, $N_\delta(x_0)$ denotes the δ -neighborhood of x_0 in X . Note that $Q(x_0)$, as the intersection of closed sets, is certainly closed.

We shall need also the following variant. We say that the map $x \rightarrow Q(x)$ has property (Q) at x_0 , provided

$$Q(x_0) = \bigcap_{\delta > 0} \text{cl co } \cup [Q(x), x \in N_\delta(x_0)]$$

(Cesari, Ref. 20). Here, $Q(x_0)$, as the intersection of closed convex sets, is certainly closed and convex.

We say that the map has property (K) or (Q) in X if it has such property at every point x_0 of X . For brevity, we may also say that the sets $Q(x)$ have property (K) or (Q) . The indication "with respect to," will be needed if the sets depend also on other parameters which then are kept constant. It is well known that property (Q) implies property (K) . Moreover, the map $x \rightarrow Q(x)$ has the property (K) in X if and only if its graph

$$[(x, y) | x \in X, y \in Q(x)]$$

is closed in the product space $X \times Y$.

We are concerned here with the situation in which a closed set A is given, say in R^v , for every $x \in A$ a subset $Q(x)$ of R^n is given, $M \subset R^{v+n}$ is the set of all (x, u) with $x \in A$, $z \in Q(x)$, and $T(x, u)$ is a real-valued function on M . For every $x \in A$, let $\tilde{Q}(x)$ denote the subsets of R^{n+1} defined by

$$\tilde{Q}(x) = [(z^0, u): z^0 \geq T(x, u), u \in Q(x)].$$

We may extend the function T outside M by taking $T = +\infty$ in $R^{v+n} - M$. Cesari has proved that the sets $\tilde{Q}(x)$ have property (Q) (with respect to x) in A if and only if $T(x, u)$ has everywhere in M the Tonelli-McShane seminormality property expressed as usual in terms of supporting hyperplanes to $\tilde{Q}(x)$ in R^{n+1} [Cesari, Refs. 7, 15, 16, 20; see also Goodman (Ref. 9), who has used these equivalence properties in his paper]. Moreover,

Cesari has proved that, under alternate growth assumptions, the sets $\tilde{Q}(t, x)$ have property (Q). For instance (see Refs. 15, 16), if there is a scalar function $\phi(\xi)$, $0 \leq \xi < +\infty$, bounded below, such that $\phi(\xi)/\xi \rightarrow \infty$ as $\xi \rightarrow \infty$, $z^0 \geq \phi(|x|)$ for all $(z^0, z) \in \tilde{Q}(x)$, and the sets $\tilde{Q}(x)$ are convex and satisfy property (K), then the same sets $\tilde{Q}(x)$ also satisfy property (Q).

We know from the Dunford–Pettis theorem and a remark of De La Valle Poussin that, if $\xi_k(t) \rightarrow \xi(t)$ weakly in $L_1(G, B)$ as $k \rightarrow \infty$, then necessarily there are a convex function ϕ as above and scalar L -integrable functions $\sigma(t)$, $\sigma_k(t)$, $t \in G$, such that $\sigma_k(t) \geq \phi(|\xi_k(t)|)$ and $\sigma_k(t) \rightarrow \sigma(t)$ weakly in $L_1(G, R)$. If the sets $\tilde{Q}(x)$ above are convex and have property (K), then the sets $\tilde{Q}^*(x) = \{(z^0, v, u) : z^0 \geq T(x, u), v \geq \phi(u), u \in Q(x)\}$ are also convex and, by (Refs. 15, 16), have property (Q).

Recently, Suryanarayana (Ref. 21) has shown that any maximal monotone map on a Hilbert space has property (Q).

5. Duality Operation

Given an extended real function Tu , $u \in R^n$ (that is, T may take values $+\infty$ and $-\infty$) we denote by $\text{epi } T$ (or epigraph of T) the subset of R^{n+1} defined by

$$\text{epi } T = \{(z^0, u) \in R^{n+1} \mid +\infty > z^0 \geq Tu\}.$$

Thus, the projection of $\text{epi } T$ on R^n is the set U of all u where $Tu < +\infty$. The following well-known statement is needed.

Theorem 5.1. *Epi T is closed (in R^{n+1}) if and only if the extended function Tu is lower semicontinuous in R^n ; epi T is convex if and only if Tu is convex in R^n .*

If Tu , $u \in R^n$, is any extended, real-valued function in R^n , then, for every $v \in R^n$, we consider all r real, if any, such that $-r + (v, u) \leq Tu$, for all $u \in R^n$, and we take $T^*v = \inf \{r\}$. In other words, we take

$$T^*v = \sup \{(v, u) - Tu \mid u \in R^n\}, \quad v \in R^n.$$

Indeed, if $r = T^*v$, then $r \geq (v, u) - Tu$ for all u , that is, $Tu \geq -r + (v, u)$ for all u , and $r = T^*v$ is the inf of all numbers r for which this holds. Note that, if the statement

$$-r + (v, u) \leq Tu, \quad \text{for all } u,$$

holds for no $r \in R$, that is, the class of such r is empty, then $T^*v = -\infty$, according to the usual conventions. The following further statement is needed. Here, T^* is said to be the dual of T .

Theorem 5.2. If $Tu, u \in \mathbb{R}^n$, is an extended real-valued function, $Tu \neq +\infty, Tu \neq -\infty$ for all u , and Tu is convex and lower semicontinuous in \mathbb{R}^n , then

$$\text{epi } T = \bigcap_{v \in \mathbb{R}^n} [(z^0, u) | z^0 \geq (v, u) - T^*y].$$

Moreover, $T^*v, v \in \mathbb{R}^n$, is also an extended real-valued function, $T^*v \neq +\infty, T^*v \neq -\infty$ for all v , and T^*v is convex and lower semicontinuous in \mathbb{R}^n . Finally, $(T^*)^* = T$, or briefly $T^{**} = T$.

For functions $T(x, u)$ depending on u and also on the parameter x , with the usual conventions, we shall understand that all duality operations are made with respect to the second variable. On the other hand, as before, we shall denote by $\tilde{Q}(x)$ the set of all $(z^0, u) \in \mathbb{R}^{n+1}$ with $+\infty > z^0 \geq T(x, u)$.

For integrand functions $F(t, x, u)$ of problems of optimization, the dual with respect to u , namely $F^*(t, x, p)$, is of course the Hamiltonian.

Given any function T as above, we denote by $\text{cl } T$ the new analogous function such that

$$\text{epi } (\text{cl } T) = \text{cl } (\text{epi } T).$$

Property (Q) of the sets $Q(x)$ can now be equivalently expressed in terms of duality operations as Goodman has recently proved (Ref. 9).

Theorem 5.3. (Goodman). If $T(x, u) \neq -\infty$ for all x, u , then the sets $\tilde{Q}(x)$ have property (Q) at x_0 if and only if

$$T^*(x_0, v) = \text{cl } [\limsup_{x \rightarrow x_0} T^*(x, v)].$$

Theorem 5.4. (Goodman). If $T(x, u) \neq -\infty$ for all x and u , then the sets $\tilde{Q}(x)$ have property (Q) at x_0 if and only if

$$T(x_0, y) = [\limsup_{x \rightarrow x_0} T^*(x, y)]^*.$$

For further necessary and sufficient conditions for property (Q) in terms of duality and lattice operations, we refer to Ref. 9 and to the exposition in Ref. 15.

For integrand functions $F_0(t, x, z)$ which are convex in (x, z) , then, for every t , the dual operation above in the variable (x, z) yields a dual function $F_0^*(t, p, w)$ with $F_0^{**} = F_0$. The corresponding dual problems (say, I and I^*) possess formal symmetric dual properties in the class of functions, say, $y(t) = y_1(t) + y_2(t)$, y_1 absolutely continuous as usual, while y_2 is of bounded variation and singular. Lower-closure theorems hold with suitable topologies as mentioned in Section 3, and the dual properties of I and I^* are then proved as usual.

6. Analytical Property (D)

We shall use here the actual notations of optimal control theory, where explicit use is made of control parameters and control functions or strategies. Let A be a given closed subset of the ty -space R^{n+1} , whose projection on the t -axis is an interval I finite or infinite. For every $(t, y) \in A$, let $U(t, y)$ be a given subset of the u -space R^m , and let M denote the set of all (t, y, u) with $(t, y) \in A$, $u \in U(t, y)$. We denote by $A(t)$ the sections of A , that is, the sets

$$A(t) = [x | (t, y) \in A] \subset R^n, \quad t \in I.$$

Let

$$f_0(t, y, u), f(t, y, u) = (f_1, \dots, f_n)$$

be given functions defined on M . Here, u denotes the control variable and $U(t, y)$ the control set function. Let \tilde{f} denote the function

$$\tilde{f}(t, y, u) = (f_0, f_1, \dots, f_n).$$

For any $(t, y) \in A$, let $Q(t, y)$ and $\tilde{Q}(t, y)$ denote the sets

$$Q(t, y) = f(t, y, U(t, y)) = [z | z = f(t, y, u), u \in U(t, y)] \subset R^n,$$

$$\tilde{Q}(t, y) = [(z^0, z) | z^0 \geq f_0(t, y, u), z = f(t, y, u), u \in U(t, y)] \subset R^{n+1},$$

so that $Q(t, y)$ is the projection of $\tilde{Q}(t, y)$ on the z -space R^n . Note that, if

$$T(t, y, z) = \inf [z^0 | (z^0, z) \in \tilde{Q}(t, y)], \quad (t, y, z) \in M, \quad (4)$$

then $T(t, y, z) < +\infty$ in M . We may extend T to all of R^{1+2n} by taking $T(t, y, z) = +\infty$ in $R^{1+2n} - M$. Thus, T is one of the functions that we considered above. Note that, if all the sets $\tilde{Q}(t, y)$ are closed, then whenever $T(t, y, z)$ is finite, then \min can replace \inf in Eq. (4) or, equivalently,

$$\tilde{Q}(t, y) = [(z^0, z) | z^0 \geq T(t, y, z), z \in Q(t, y)].$$

A number of criteria for property (Q) of the sets $\tilde{Q}(t, y)$ have been proved by Cesari and others (Refs. 7, 8, 15, 16). Below, we present a set of analytical considerations in finite-dimensional spaces which imply property (Q), and which have natural extensions in Banach spaces. We shall assume here that the set $U(t)$ depend on t only.

Let $u_k(t)$, $u(t)$, $y_k(t)$, $y(t)$, $t \in G = [a, b]$, $k = 1, 2, \dots$, be measurable functions, $u_k(t) \in U(t)$, $y_k(t) \in A(t)$, and let $\delta_k = (\delta_k^0, \delta_k)$ be defined by

$$\tilde{\delta}_k(t) = \tilde{f}(t, y_k(t), u_k(t)) - \tilde{f}(t, y(t), u_k(t)), \quad t \in [a, b].$$

Proposition 6.1. $\tilde{\delta}_k(t) \rightarrow 0$ strongly in $(L_1(G))^{n+1}$ if and only if the same functions $\tilde{\delta}_k(t)$ are equiabsolutely integrable in G .

This statement was proved in Ref. 10 in a slightly more elementary situation, but the proof is the same.

Proposition 6.2. $\tilde{\delta}_k(t) \rightarrow 0$ strongly in $(L_1(G))^{n+1}$ implies the following weak form of property (Q); namely, there is a subsequence $[k_s]$ such that

$$\tilde{Q}(t, y(t)) \supset \bigcap_{h=1}^{\infty} \text{cl co} \left[\bigcup_{s=h}^{\infty} \tilde{f}(t, y_{k_s}(t), u_{k_s}(t)) \right], \quad t \in G \text{ a.e.}$$

This statement was proved in Refs. 10 and 3, Part I, in various settings. The proof in the present situation is the same. In Ref. 1, p. 352 (cf. also Ref. 10), we have pointed out that, for lower closure theorems, it is enough to assume only that $\delta_k \rightarrow 0$ strongly in $(L_1(G))^n$ and that

$$\infty > \liminf_{k \rightarrow \infty} \int_G \delta_k^0(t) dt \geq 0.$$

For mere closure theorems, $f_0 = 0$, and the last requirement is trivial.

We denote by property (D) of a sequence $y_k(t), u_k(t), t \in G, k = 1, 2, \dots$, as above the requirement $\tilde{\delta}_k(t) \rightarrow 0$ strongly in $(L_1(G))^{n+1}$, or $\|\tilde{\delta}_k\| \rightarrow 0$.

Below we give several specific criteria for property (D). This property implies the weak form of property (Q) stated above. We have mentioned these criteria in Refs. 2, 10, 22, and Ref. 3, Part I. They are repeated here in the present context, since they hold not only in Euclidean spaces but also in Banach space situations. [see Ref. 10, pp. 446–447, properties (F_p) and (F_∞)].

(Lg) Lipschitz type conditions, geometric viewpoint.

$(F_p), 1 \leq p < \infty$. Here, B and Y are Banach spaces, U a metric space. If $1 \leq p < \infty$, let us assume that there are constants $c, \gamma, \xi_0, p', c \geq 0, \xi_0 \geq 0, 0 < \gamma \leq p, p' = p(p - \gamma)^{-1}$ if $0 < \gamma < p, p' = \infty$ if $\gamma = p$, and functions $F(t) \geq 0, t \in G, F \in L_{p'}(G, R), h(\xi), 0 \leq \xi < \infty, h$ monotone nondecreasing, $h(0+) = 0, h(\xi) \leq c|\xi|^\gamma$ for $\xi \geq \xi_0$, such that, for all $(t, y_1, u), (t, y_2, u) \in M, t \in G - T_0, \text{meas } T_0 = 0$, we have

$$\|\tilde{f}(t, y_1, u) - \tilde{f}(t, y_2, u)\|_B \leq F(t)h(\|y_1 - y_2\|_Y). \tag{5}$$

Let $\xi(t), y(t), \eta_k(t), \bar{\eta}_k(t), \xi_k(t), \bar{\xi}_k(t), y_k(t), u_k(t), t \in G, k = 1, 2, \dots$, be such that

$$y(t), y_k(t) \in A(t), \quad u(t), u_k(t) \in U(t),$$

$$\eta_k(t) \geq f_0(t, y_k(t), u_k(t)), \quad \xi_k(t) = f(t, y_k(t), u_k(t)),$$

$$\bar{\eta}_k(t) \geq f_0(t, y(t), u_k(t)), \quad \bar{\xi}_k(t) = f(t, y(t), u_k(t)), \quad t \in G, \quad k = 1, 2, \dots,$$

$\xi_k \rightarrow \xi$ weakly in L_1 .

Let us assume that, for almost all $t \in G$, the sets $Q(t, y(t))$ are closed and convex and that $\|y_k - y\|_p \rightarrow 0$ as $k \rightarrow \infty$. Then, property (Q) holds as in Proposition 6.2 for almost all $t \in G$, and the lower-closure property holds as in Theorem 3.1.

Analogously, we have property (F_∞) :

(F_∞) Let B, U, Y be as above. Let us assume that there are functions $F(t) \geq 0, t \in G, F \in L_1(G, R)$, such that (5) holds. Let $\xi(t), y(t), \eta_k(t), \bar{\eta}_k(t), \xi_k(t), \bar{\xi}_k(t), y_k(t), u_k(t), t \in G, k = 1, 2, \dots$, be as in Remark 3.4; assume that, for a.a. $t \in G$, the set $\tilde{Q}(t, y(t))$ is closed and convex and that $\|y_k - y\|_\infty \rightarrow 0$ as $k \rightarrow \infty$. Then, property (D) holds, and then also property (Q) holds as in Proposition 6.2 for almost all $t \in G$, and the lower-closure theorem holds as in Theorem 3.1.

(Lf) Lipschitz type conditions, functional viewpoint.

$(F'_p), 1 \leq p < \infty$. Let $\{u(t)\}$ be a family of control functions, and let us assume that, for every control function $u = u(t)$ of the class, there is a function $F_u(t)$ with the following properties. First, let $p, c, \gamma, \xi_0, p', h$ as in (F_p) , and let $F_u(t) \geq 0, t \in G, F_u \in L_{p'}(G, R), \|F_u\|_{p'} \leq M$, and

$$\|\tilde{f}(t, y_1, u(t)) - \tilde{f}(t, y_2, u(t))\|_B \leq F_u(t)h(\|y_1 - y_2\|_Y). \tag{6}$$

Let $\xi(t), y(t), \eta_k(t), \bar{\eta}_k(t), \xi_k(t), \bar{\xi}_k(t), y_k(t), u_k(t), t \in G, k = 1, 2, \dots$, be as in (Lg) with $u_k \in \{u(t)\}$ for all k ; assume that, for all $t \in G$, the sets $Q(t, y(t))$ are closed and convex and that $\|y_k - y\|_p \rightarrow 0$ as $k \rightarrow \infty$. Then, property (Q) holds as in Proposition 6.2, and the lower-closure theorem holds as in Theorem 3.1.

In particular F may be simply a function of (t, u) , say $F = F(t, u)$, or $F_u(t) = F(t, u(t))$, and in this case we need require that

$$\int_G \|F(t, u(t))\|^{p'} dt \leq M.$$

Analogously, we have property (F'_∞) :

(F'_∞) Let B, Y, U be as above. Let us assume that there is a function $h(\xi), 0 \leq \xi < +\infty, h$ monotone nondecreasing, $h(0+) = 0$, and, for every $u \in \{u(t)\}$, a function $F_u(t) \geq 0, F_u \in L_1(G, R), \|F_u\|_1 \leq M$, such that (6) holds for all $u \in \{u(t)\}$. Let $\xi(t), y(t), \eta_k(t), \bar{\eta}_k(t), \xi_k(t), \bar{\xi}_k(t), y_k(t), u_k(t), t \in G, k = 1, 2, \dots$, be as in property (F'_p) , with $u_k \in \{u(t)\}$ for all k , and assume that, for a.a. $t \in G$, the set $Q(t, y(t))$ is closed and convex and that $\|y_k - y\|_\infty \rightarrow 0$ as $k \rightarrow \infty$. Then, property (Q) holds as in Proposition 6.2, and the lower-closure theorem holds as in Theorem 3.1.

In particular, F may be simply a function of (t, u) , say $F = F(t, u)$, or $F_u(t) = F(t, u(t))$, and in this case we need require that

$$\int_G F(t, u(t)) dt \leq M.$$

For a great many other analytical criteria for property (Q), we refer to Ref. 2, Ref. 3, Part I, and Refs. 8, 15, 23.

Remark 6.1. Berkovitz (Ref. 24) and Bates (Ref. 25) state that, in their lower-closure theorems, property (Q) does not hold. They assume a Lipschitz condition

$$|\tilde{f}(t, y_1, u) - \tilde{f}(t, y_2, u)| \leq F(t, u)h(|y_1 - y_2|)$$

as above and the further assumption

$$\int_{t_1}^{t_2} F(t, u(t)) dt \leq A.$$

From the above, it is apparent that property (Q) in the weak form does hold in their situation. Both Berkovitz and Bates do not seem to have realized this in their statements. Their proofs are not much different from ours in the analogous context.

Appendix

In connection with Theorem 2.1, let us prove here directly that (G2) implies weak compactness in $H^{1,1}$. First, for $\epsilon = 1$ we have $|x'(t)| \leq \Psi_1(t) + \eta(t)$; hence, $\eta(t) \geq -\Psi_1(t)$. Let

$$M_0 = \int_{a_0}^{b_0} \Psi_1(t) dt.$$

Now, given $\epsilon > 0$, let $\sigma = \min[1, \epsilon 2^{-1}(M_0 + M_1 + 1)^{-1}]$, and note that the function $\Psi_0(t)$ is L -integrable in $[a_0, b_0]$. Hence, there is some $\delta > 0$, such that

$$\int_E \Psi_\sigma(t) dt < \epsilon/2$$

for every measurable subset E of $[a_0, b_0]$ with $\text{meas } E < \delta$. Finally, let $\eta(t)$, $x(t)$, $a \leq t \leq b$, be any pair of the family, and let E be any measurable subset of $[a, b]$ with $\text{meas } E < \delta$. Then

$$\begin{aligned} \int_E |x'(t)| dt &\leq \int_E [\Psi_\sigma(t) + \sigma\eta(t)] dt \leq \int_E [\Psi_\sigma + \sigma(\eta + \Psi_1)] dt \\ &\leq \sigma \int_b^a \eta dt + \sigma \int_a^b \Psi_1 dt + \int_E \Psi_\sigma dt \\ &\leq \sigma(M_0 + M_1) + \int_E \Psi_\sigma dt \leq \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This proves the absolute integrability of the class $\{x'\}$. The proof continues as in Section 2.

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