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REPUNIT LEHMER NUMBERS

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Abstract A Lehmer number is a composite positive integer n such that $\phi(n)|n-1$. In this paper, we show that given a positive integer $g>1$ there are at most finitely many Lehmer numbers which are repunits in base g and they are all effectively computable. Our method is effective and we illustrate it by showing that there is no such Lehmer number when $g \in [2, 1000]$.

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1. Introduction

Let $\phi(n)$ be the Euler function of the positive integer n. Clearly, $\phi(n) = n-1$ if n is a prime. Lehmer [**4**] (see also [**3**, Problem B37]) conjectured that if $\phi(n)|n-1$, then n is prime. To this day, no counter-example to this conjecture has been found. A composite number m such that $\phi(m)|m-1$ is called a *Lehmer number*. Thus, Lehmer's conjecture is that Lehmer numbers do not exist, but it is not even known if there should be at most finitely many of them.

Given an integer $g>1$, a base g repunit is a number of the form $m = (g^n-1)/(g-1)$ for some integer $n \geqslant 1$. We will refer to such numbers simply as repunits without mentioning the dependence on g. It is not known whether, given g, there are infinitely many repunit primes. When $g=2$ such primes are better known as Mersenne primes. In [**5**], it was shown that there is no Lehmer number in the Fibonacci sequence. Here, we use some ideas from [**5**] together with finer arguments to prove the following results. In what follows, we write $u_n = (g^n-1)/(g-1)$.

Theorem 1.1. *For each fixed $g>1$, there are only finitely many positive integers n such that u_n is a Lehmer number, and all are effectively computable.*

Theorem 1.2. *There is no Lehmer number of the form u_n when $2 \leqslant g \leqslant 1000$.*

2. Preliminaries

For a prime q and a non-zero integer m we write $\nu_q(m)$ for the exponent of q in the factorization of m. We start by collecting some elementary and well-known properties of the sequence of general terms $u_n = (g^n - 1)/(g - 1)$ for $n \geqslant 1$.

Lemma 2.1.

(i) $u_n = g^{n-1} + \cdots + g + 1$. *In particular,* u_n *is coprime to* g.

(ii) *The sequence* u_n *satisfies the linear recurrence*

$$u_1 = 1, \quad u_n = g u_{n-1} + 1, \quad n \geqslant 2. \tag{2.1}$$

(iii) *If* $d|n$, *then* $u_d|u_n$.

(iv) *Let* q *be a prime. If* $q|n$, *then* $q|\phi(u_n)$.

(v) *Let* q *be a prime not dividing* g. *If* $q|n$, *then* $\nu_q(u_{n-1}) \leqslant \nu_q(u_f) \leqslant \nu_q(u_{q-1})$, *where* f *is the order of* g *modulo* q.

(vi) *If* u_n *is a Lehmer number, then* $(u_n, g-1) = 1$.

Proof. Parts (i) and (ii) are obvious. For (iii), we observe that

$$u_n = \frac{g^n - 1}{g - 1} = \frac{(g^d)^{n/d} - 1}{g^d - 1}\,\frac{g^d - 1}{g - 1} = ((g^d)^{(n/d)-1} + \cdots + 1)u_d.$$

(iv) If $q = 2$, then $u_n \geqslant u_2 = g + 1 > 2$; therefore $\phi(u_n)$ is even. Assume now that q is odd. Let p be a prime which divides u_q. Then, $g^q \equiv 1 \pmod p$, so the order of g modulo p is 1 or q. If it is q, then $q|p-1|\phi(u_q)$. Since by (iii) we know that $u_q|u_n$, we get that $q|\phi(u_q)|\phi(u_n)$, which is what we wanted. Assume now that the order of g modulo p is 1 for all primes p dividing u_q. Let us show that this cannot happen. If it could, then $p|g-1$ for all such primes p. Since also $p|u_q$, we have

$$0 \equiv u_q \equiv \frac{g^q - 1}{g - 1} = g^{q-1} + \cdots + g + 1 \equiv 1 + \cdots + 1 + 1 \equiv q,$$

where all congruences above are modulo p. Thus, $p|q$, and therefore $p = q$. Hence, $u_q = q^\alpha$ for some positive integer α. However, writing $g - 1 = q\lambda$ with some positive integer λ, we get

$$\begin{aligned} u_q &= (1+q\lambda)^{q-1} + (1+q\lambda)^{q-2} + \cdots + (1+q\lambda) + 1 \\ &\equiv (1 + (q-1)q\lambda) + (1 + (q-2)q\lambda) + \cdots + (1 + q\lambda) + 1 (\bmod q^2) \\ &\equiv q + q\lambda((q-1) + \cdots + 1)(\bmod q^2) \\ &\equiv q + \tfrac{1}{2}q^2(q-1)\lambda(\bmod q^2) \\ &\equiv q(\bmod q^2). \end{aligned}$$

In the above chain of congruences, we have used the fact that q is odd, and therefore $(q-1)/2$ is an integer. The above argument shows that $q\|u_q$; hence, $\alpha = 1$. So, $u_q = q$. However, we clearly have $u_q \geqslant 2^q - 1 > q$, which is a contradiction.

(v) We may also assume that $q|u_{n-1}$, otherwise $\nu_q(u_{n-1}) = 0$ and the first inequality is clear. Now $g^{n-1} \equiv 1 \pmod q$, and so $f|n-1$. We now write

$$u_{n-1} = ((g^f)^{(n-1)/f-1} + \cdots + 1)u_f.$$

The quantity in brackets above is not divisible by q since it is congruent to $(n-1)/f$ modulo q and $q|n$. Thus, $\nu_q(u_{n-1}) \leqslant \nu_q(u_f) \leqslant \nu_q(u_{q-1})$, where the last inequality follows because $f|q-1$; so, $u_f|u_{q-1}$ by (iii).

(vi) Suppose that q is a prime dividing both u_n and $g-1$. We then have that $g \equiv 1 \pmod q$ and $u_n = g^{n-1} + \cdots + 1 \equiv n \pmod q$. Thus, $q|n$. By (iv), we know that $q|\phi(u_n)$. Since u_n is a Lehmer number, we know that $\phi(u_n)|u_n - 1 = gu_{n-1}$. Since q divides $g-1$, it cannot divide g; therefore, $q|u_{n-1}$. Hence, $q|u_n - u_{n-1} = g^{n-1}$, which is not possible. □

In the next lemma, we gather some known facts about Lehmer numbers.

Lemma 2.2.

(i) *Any Lehmer number must be odd and square-free.*

(ii) *If* $m = p_1 \cdots p_K$ *is a Lehmer number, then* $K^{2^K} > m$.

(iii) *If* $m = p_1 \cdots p_K$ *is a Lehmer number, then* $K \geqslant 14$.

Proof. (i) If $m > 2$, then $\phi(m)$ is even, and since $\phi(m)|m-1$ we get that m must be odd. If $p^2|m$, then $p|\phi(m)$, and since $\phi(m)|m-1$ we have $p|m-1$, which is not possible. Part (ii) was proved in [**6**], while part (iii) was proved in [**2**]. □

Lemma 2.3. *Theorems 1.1 and 1.2 hold when* g *is even.*

Proof. Note that

$$2^K|(p_1-1)\cdots(p_K-1) = \phi(u_n)|u_n - 1 = gu_{n-1}.$$

We observe that if g is even, then u_{n-1} is odd. In that case, we have

$$K \leqslant \nu_2(\phi(u_n)) \leqslant \nu_2(gu_{n-1}) = \nu_2(g), \tag{2.2}$$

implying, by Lemma 2.2 (ii), that

$$g^{n-1} < u_n < K^{2^K} \leqslant (\nu_2(g))^{2^{\nu_2(g)}} \leqslant (\nu_2(g))^g.$$

Thus,

$$n \leqslant 1 + \left\lfloor \frac{g\log(\nu_2(g))}{\log g} \right\rfloor.$$

For Theorem 1.2, we observe that $\nu_2(g) \leqslant 9$ for any $g \leqslant 1000$, and we obtain a contradiction from (2.2) and Lemma 2.2 (iii). □

From Lemma 2.1 (i), we see that if g is odd and n is even, then u_n is even, so Lemma 2.2 (i) shows that u_n cannot be a Lehmer number. From now on, we shall assume that both g and n are odd and larger than 1 and that $u_n = (g^n-1)/(g-1)$ is a Lehmer number. We also keep the following notation:

$$n = q_1^{\alpha_1}\cdots q_s^{\alpha_s}, \quad \text{where } 2 < q_1 < \cdots < q_s, \tag{2.3}$$

are primes and $\alpha_1,\ldots,\alpha_s$ are positive integers, and

$$u_n = p_1\cdots p_K, \qquad \text{where } 2 < p_1 < \cdots < p_K, \tag{2.4}$$

are also primes.

3. Proof of Theorem 1.1

3.1. Primitive divisors

Let $(A_n)_{n\geqslant 1}$ denote a sequence with integer terms. We say that a prime p is a *primitive divisor* of A_n if $p|A_n$ and $\gcd(p,A_m)=1$ for all non-zero terms A_m with $1\leqslant m<n$.

In 1886, Bang [**1**] showed that if $g>1$ is any fixed integer, then the sequence $(A_n)_{n\geqslant 1}$ of nth term $A_n = g^n-1$ has a primitive divisor for any index $n>6$.

We will apply this important theorem to our sequence u_n.

Lemma 3.1. *If $d>1$ is odd, then u_d has a primitive divisor p_d. Furthermore, $p_d\equiv 1 (\bmod 2d)$.*

Proof. We revisit the argument used in Lemma 2.1 (iv). We write $v_n = g^n-1$. It is well known that $\gcd(v_n,v_m) = v_{\gcd(n,m)}$. Observe also that

$$\frac{v_d}{v_1} = u_d = g^{d-1}+\cdots+1 \equiv d (\bmod g-1).$$

Therefore, if d is a prime not dividing $g-1$, then v_d has primitive divisors. If $d>2$ is a prime dividing $g-1$, then the above argument, or the argument from the proof of Lemma 2.1 (iv), shows that $\gcd(v_d,v_1)$ is a power of d. Write $g-1=d\lambda$ and observe that

$$\begin{aligned}\frac{v_d}{v_1} &= (1+d\lambda)^{d-1}+(1+d\lambda)^{d-2}+\cdots+1\\ &\equiv (1+(d-1)d\lambda)+(1+(d-2)d\lambda)+\cdots+1\\ &= d+d\lambda((d-1)+(d-2)+\cdots+1) \pmod{d^2}\\ &\equiv d+\tfrac{1}{2}d^2(d-1)\lambda (\bmod d^2) \equiv d \pmod{d^2}.\end{aligned}$$

Thus, $d\|v_d/v_1$, and therefore

$$\frac{v_d}{dv_1} = \frac{1}{d}(g^{d-1}+\cdots+1) > 1$$

is an integer coprime to v_1, so v_d again has primitive divisors. Thus, v_3 and v_5 (and, of course, v_1 if $g > 2$) have primitive divisors. The fact that v_d has primitive divisors for all odd $d \geqslant 7$ follows from Bang's result.

We now note that if p is a primitive prime divisor of v_d for $d > 1$, then $g^d \equiv 1 \pmod{p}$, and d is the order of $g \pmod{p}$. Indeed, for if not, then $f < d$ and $p|v_f$, contradicting the fact that p is primitive for v_d. So, $d|p-1$, and since d is odd, we get that $d|(p-1)/2$. Thus, $p \equiv 1 (\operatorname{mod} 2d)$.

Since a prime factor of $g-1$ cannot be a primitive divisor for v_d except for $d = 1$, we deduce that if $d > 1$, then the primitive prime divisors for v_d are exactly those of $u_d = v_d/(g-1)$, and we get the first assertion of the lemma. □

In what follows, for a positive integer m we use $\omega(m)$ and $\tau(m)$ for the number of prime divisors and the total number of divisors of m, respectively.

Lemma 3.2. *If u_n is square-free, n is odd and $(u_n, g-1) = 1$, then*

$$\log\left(\frac{u_n}{\phi(u_n)}\right) < \frac{\omega(n)}{2q}\left(1+\log\left(\frac{q\log g}{\log(2q+1)}\right)\right) + \frac{\tau(n)-2}{2q^2}\left(1+\log\left(\frac{q^2\log g}{\log(2q^2+1)}\right)\right),$$

where q is the smallest prime dividing n.

Proof. We write $\mathcal{P}_d = \{p \text{ is primitive prime divisor for } u_d\}$. We shall first prove that

$$\prod := \prod_{1<d|n}\prod_{p\in\mathcal{P}_d} p = u_n.$$

To prove the above formula, we observe that if $p|u_d$ and $p \nmid g-1$, then $p \in \mathcal{P}_d$ for some divisor $d > 1$ of n. Since u_n is square-free, we have that $u_n|\prod$. On the other hand, the sets $\mathcal{P}_d$ are disjoint, and if $p \in \mathcal{P}_d$, then $p|u_d|u_n$. Thus, $\prod|u_n$.

Now, since u_n is square-free,

$$\phi(u_n) = \prod_{1<d|n}\prod_{p\in\mathcal{P}_d}(p-1),$$

and then

$$\log\left(\frac{u_n}{\phi(u_n)}\right) < \sum_{\substack{d|n\\ d>1}}\sum_{p\in\mathcal{P}_d}\frac{1}{p-1}.$$

Since all the primes $p \in \mathcal{P}_d$ are congruent to $1(\operatorname{mod} 2d)$, we have

$$S_d := \sum_{p\in\mathcal{P}_d}\frac{1}{p-1} \leqslant \frac{1}{2d}\sum_{j=1}^{\#\mathcal{P}_d}\frac{1}{j} \leqslant \frac{1}{2d}(1+\log\#\mathcal{P}_d).$$

To bound the cardinality of $\mathcal{P}_d$, we observe that $(2d+1)^{\#\mathcal{P}_d} \leqslant u_d < g^d$, so

$$\#\mathcal{P}_d < \frac{d\log g}{\log(2d+1)}.$$

We observe that $d \geqslant q$ and if d is not a prime, then $d \geqslant q^2$. Then

$$\begin{aligned}\sum_{1<d|n} S_d &= \sum_{\substack{d|n\\ d \text{ prime}}} S_d + \sum_{\substack{d|n\\ d \text{ composite}}} S_d\\ &\leqslant \omega(n)\frac{1}{2q}\left(1+\log\left(\frac{q\log g}{\log(2q+1)}\right)\right) + (\tau(n)-2)\frac{1}{2q^2}\left(1+\log\left(\frac{q^2\log g}{\log(2q^2+1)}\right)\right).\end{aligned}$$

□

3.2. Bounds for q_1 and $\tau(n)$

Recall that we keep the notation from (2.3) and (2.4).

Lemma 3.3. *If u_n is a Lehmer number and n is odd, then*

$$\begin{aligned}\tau(n/q_i) &\leqslant \tfrac{1}{2}\alpha_i(\alpha_i+1)\tau(n/q_i^{\alpha_i})\\ &\leqslant \nu_{q_i}(\phi(u_n))\\ &\leqslant \nu_{q_i}(gu_{n-1})\\ &\leqslant \begin{cases}\nu_{q_i}(g) & \text{if } q_i|g,\\ \nu_{q_i}(u_{q_i-1}) & \text{if } q_i\nmid g\end{cases}\end{aligned} \tag{3.1}$$

for all $i = 1, \ldots, s$.

Proof. Lemma 3.1 implies that for each divisor of n of the form $q_i^{\alpha}d$ with $1 \leqslant \alpha \leqslant \alpha_i$ and $d|(n/q_i^{\alpha_i})$, the divisor $u_{q_i^{\alpha}d}$ of u_n has a primitive prime factor $p_{q_i^{\alpha}d} \equiv 1 (\mathrm{mod}\, dq_i^{\alpha})$. In particular, $q_i^{\alpha}|p_{dq_i^{\alpha}}-1$, and the primes $p_{dq_i^{\alpha}}$ are distinct as d ranges over the divisors of $n/q_i^{\alpha_i}$. Thus,

$$q_i^{(1+\cdots+\alpha_i)\tau(n/q_i^{\alpha_i})}\Bigg|\prod_{1\leqslant\alpha\leqslant\alpha_i}\prod_{d|n/q_i^{\alpha_i}}(p_{dq_i^{\alpha}}-1)\Bigg|\prod_{p|u_n}(p-1) = \phi(u_n)|u_n-1|gu_{n-1},$$

which gives the two central inequalities. The first inequality is trivial and the equality holds when $\alpha_i = 1$. When $q_i|g$, the last inequality follows from Lemma 2.1 (i), while when $q_i \nmid g$, then $\nu_{q_i}(gu_{n-1}) = \nu_{q_i}(u_{n-1})$, and we apply Lemma 2.1 (v) to get the desired conclusion. □

Lemma 3.4. *Let u_n be a Lehmer number with both n and g odd. If $q_i > \sqrt{g}$, then*

$$\tau(n/q_i) \leqslant q_i - 2.$$

Proof. If $q_i|g$ and $q_i > \sqrt{g}$, then $\nu_{q_i}(g) = 1$, and Lemma 3.3 gives

$$\tau(n/q_i) \leqslant \nu_{q_i}(g) = 1 \leqslant q_i - 2. \tag{3.2}$$

If $q_i \nmid g$, then, again by Lemma 3.3, we have

$$\tau(n/q_i) \leqslant \nu_{q_i}(u_{q_i-1}).$$

Observe that

$$u_{q_i-1}|g^{q_i-1}-1=(g^{(q_i-1)/2}-1)(g^{(q_i-1)/2}+1).$$

Since q_i cannot divide both factors above, we have that

$$\tau(n/q_i)\leqslant \nu_{q_i}(g^{(q_i-1)/2}+\epsilon)\quad \text{for some } \epsilon\in\{-1,+1\}.$$

If $\tau(n/q_i)\geqslant q_i-1$, then

$$q_i^{q_i-1}\leqslant q_i^{\tau(n/q_i)}\leqslant g^{(q_i-1)/2}+1\leqslant (q_i^2-1)^{(q_i-1)/2}+1, \tag{3.3}$$

and we get a contradiction for $q_i>3$, because

$$q_i^{q_i-1}=((q_i^2-1)+1)^{(q_i-1)/2}$$

and the expression on the right is larger than $(q_i^2-1)^{(q_i-1)/2}+1$ except when $q_i=3$.

Finally, if $q_i=3$, the only odd $g<q_i^2$ with $q_i\nmid g$ are $g=5$ and $g=7$. But in both cases we have $\tau(n/3)\leqslant \nu_3(u_2)\leqslant 1\leqslant q_i-2$, which completes the proof of this lemma. □

Lemma 3.5. *Let u_n be a Lehmer number with both n and g odd. Then*

$$q_1\leqslant \max\{\sqrt{g},19\}. \tag{3.4}$$

Proof. Assume that the above inequality does not hold. Then $q_1\geqslant 23$, $g\leqslant q_1^2-1$, and since $q_1>\sqrt{g}$ we can apply Lemma 3.4 to deduce that $\tau(n)\leqslant 2\tau(n/q_1)\leqslant 2q_1-4$. We also observe that $\tau(n)\geqslant 2^{\omega(n)}$, so $\omega(n)\leqslant \log(2q_1-4)/\log 2$.

Since u_n is a Lehmer number, we have that $2\leqslant u_n/\phi(u_n)$. Now Lemma 3.2 and the bounds above give

$$\begin{aligned}\log 2<\frac{\log((2q_1-4)/\log 2)}{2q_1}&\left(1+\log\left(\frac{q_1\log(q_1^2-1)}{\log(2q_1+1)}\right)\right)\\ &+\frac{2q_1-6}{2q_1^2}\left(1+\log\left(\frac{q_1^2\log(q_1^2-1)}{\log(2q_1^2+1)}\right)\right),\end{aligned}$$

which is false for $q_1\geqslant 23$. □

For a given value of g, Lemma 3.5 gives us our bound for q_1 and then this is used in Lemma 3.3, since $\tau(n)\leqslant 2\tau(n/q_1)$, to give a bound for $\tau(n)$. Observe also that $\omega(n)\leqslant \log\tau(n)/\log 2$.

3.3. The conclusion of the proof of Theorem 1.1

Since we have already proved that both $s=\omega(n)$ and $\tau(n)$ are bounded by effectively computable constants depending only on g, in order to conclude the proof of Theorem 1.1 it is enough to prove that all the primes q_i with $i=1,\dots,s$ are also bounded by effectively computable constants depending on g. We shall prove this by induction on $i=1,\dots,s$, observing that this has already been achieved for $i=1$. Let $i\leqslant s-1$ and assume that

q_i has been bounded. Put $Q_i = \prod_{j=1}^{j=i} q_j^{\alpha_j}$. There are only finitely many possibilities for this number. We put $g_i = g^{Q_i}$, $n_i = n/Q_i$ and rewrite the condition that u_n is Lehmer as

$$a\phi\left(\frac{g^{Q_i}-1}{g-1}\,\frac{g_i^{n_i}-1}{g_i-1}\right) = u_n - 1 = \frac{g^{Q_i}-1}{g-1}\,\frac{g_i^{n_i}-1}{g_i-1} - 1$$

with some integer $a \geqslant 2$. We put $w_m = (g_i^m - 1)/(g_i - 1)$ for the sequence of repunits in base g_i. Then, since u_n is square-free, we get that

$$a\phi(u_{Q_i})\phi(w_{n_i}) = u_{Q_i} w_{n_i} - 1,$$

and therefore

$$a\frac{\phi(u_{Q_i})}{u_{Q_i}} = \frac{w_{n_i}}{\phi(w_{n_i})} - \frac{1}{u_{Q_i}\phi(w_{n_i})}. \tag{3.5}$$

The left-hand side takes only finitely many values, which are all effectively computable. Assume that it takes some value $\delta \leqslant 1$. Then

$$w_{n_i} - 1 < w_{n_i} - \frac{1}{u_{Q_i}} = \delta\phi(w_{n_i}) \leqslant \phi(w_{n_i}),$$

which is a contradiction. Thus, it remains to study the case when the right-hand side of (3.5) is greater than 1. Let $\delta_i > 1$ be the smallest possible value larger than 1 of the left-hand side of (3.5). Clearly, this is effectively computable. We then get

$$\delta_i < \frac{w_{n_i}}{\phi(w_{n_i})}.$$

We observe that w_{n_i} is a sequence similar to u_n but the new value of g is $g_i = g^{Q_i}$ and the new value of n is $n_i = n/Q_i$. Thus, the smallest prime factor of n_i is q_{i+1}. We also note that $\tau(n_i) = \tau(n/Q_i) < \tau(n)$, which is bounded, and that $\omega(n_i) < \omega(n)$. Finally, we observe that $(w_{n_i}, g^{Q_i} - 1) = 1$; otherwise, since $(w_{n_i}, g-1) = 1$, the number $u_n = (g^{Q_i} - 1)w_{n_i}/(g-1)$ would not be square-free.

We now apply Lemma 3.2 to obtain that

$$\log\delta_i < \frac{\omega(n_i)}{2q_{i+1}}\left(1 + \log\left(\frac{Q_i q_{i+1}\log g}{\log(2q_{i+1}+1)}\right)\right) + \frac{\tau(n_i)-2}{2q_{i+1}^2}\left(1 + \log\left(\frac{Q_i q_{i+1}^2\log g}{\log(2q_{i+1}^2+1)}\right)\right). \tag{3.6}$$

Hence, $\log\delta_i \ll (\log q_{i+1})/q_{i+1}$, where the constant implied by the Vinogradov symbol $\ll$ above depends only on g, implying that q_{i+1} must be bounded by some effectively computable constant depending only on g. This concludes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We assume that g is odd and that $3 \leqslant g \leqslant 999$, so that $3 \leqslant q_1 \leqslant 31$ by Lemma 3.5.

Claim 4.1. *The fact that $\nu_{q_1}(u_{q_1-1}) \leqslant 5$ can be checked with* Mathematica. *In particular, by Lemma 3.3, we have that if $q_1 \nmid g$, then $\nu_{q_1}(\phi(u_n)) \leqslant 5$.*

Claim 4.2. *$\tau(n/q_1) \leqslant \nu_{q_1}(\phi(u_n)) \leqslant 6$ and $s \leqslant 3$.*

Suppose first that $q_1|g$. Then, by Lemma 3.3,

$$\tau(n/q_1) \leqslant \nu_{q_1}(\phi(u_n)) \leqslant \nu_{q_1}(gu_{n-1}) = \nu_{q_1}(g) \leqslant \left\lfloor \frac{\log g}{\log q_1} \right\rfloor \leqslant \left\lfloor \frac{\log 1000}{\log 3} \right\rfloor = 6.$$

In the above expression, in fact, $\nu_{q_1}(g) < 6$ unless $(q_1, g) = (3, 729)$. Then, for any q_1, by Claim 4.1, either $q_1 = 3$ and $\tau(n/q_1) \leqslant 6$, or $\tau(n/q_1) \leqslant 5$. In particular, $\tau(n) \leqslant 2\tau(n/q_1) \leqslant 12$, which shows that $s \leqslant 3$.

Claim 4.3. *$s \geqslant 2$.*

Let us see that indeed for our particular case we cannot have $s = 1$. If this were so, then $n = q_1^{\alpha_1}$. Then each prime factor p_j of u_n is primitive for some divisor $d > 1$ of n, which is a power of q_1 (again, this is because $\gcd(u_n, g-1) = 1$). Thus, $p_j \equiv 1 (\bmod\, q_1)$ for all $j = 1, \ldots, K$, showing that $\nu_{q_1}(\phi(u_n)) \geqslant K \geqslant 14$ (see Lemma 2.2 (iii)), which contradicts the fact that $\nu_{q_1}(\phi(u_n)) \leqslant 6$. Hence, $s \geqslant 2$.

Claim 4.4. *$\alpha_1 = 1$ except when $(\alpha_1, q_1, g) = (2, 3, 729)$.*

As in the proof of Theorem 1.1, again set $Q_1 = q_1^{\alpha_1}$. By Lemma 3.3 and the fact that $s \geqslant 2$, we have

$$\alpha_1(\alpha_1 + 1) \leqslant \frac{\alpha_1(\alpha_1 + 1)}{2} \tau(n/q_1^{\alpha_1}) \leqslant \nu_{q_1}(\phi(u_n)).$$

By Claims 4.1 and 4.2, we know that $\nu_{q_1}(\phi(u_n)) \leqslant 5$, except when $(\alpha_1, q_1, g) = (2, 3, 729)$. So, $\alpha_1 = 1$ except for this case.

Note that, at any rate, since $s \geqslant 2$, it follows that $2 \leqslant \tau(n/q_1) \leqslant \nu_{q_1}(gu_{q_1-1})$. A computation with Mathematica revealed 431 possibilities for the pairs (q_1, g) in our range satisfying $\nu_{q_1}(gu_{q_1-1}) \geqslant 2$.

Claim 4.5. *$q_2 \leqslant 19$.*

The smallest left-hand side of (3.5) computed over all the 432 possible pairs (Q_1, g) has $\delta_1 > 1.49$ (it was obtained for $g = 809$, $Q_1 = q_1 = 3$ and $a = 2$, for which the obtained value is greater than 1.495). Of course, we did not factor all the numbers of the form $(g^{Q_1} - 1)/(g - 1)$. If $q_1 = 31$, then the smallest prime $p_1 \equiv 1 (\bmod\, q_1)$ is 311. The number K of prime factors of u_{31} therefore satisfies

$$K < \frac{\log u_{q_1}}{\log p_1} < \frac{3 \cdot 31 \cdot \log 10}{\log 311} < 38;$$

hence,

$$a \frac{\phi(u_{q_1})}{u_{q_1}} \geqslant 2(1 - \tfrac{1}{311})^{37} > 1.7.$$

Similarly, using the fact that when $q_1 = 29$ and 23 the first two primes congruent to $1 (\bmod\, q_1)$ are 59 and 233, and 47 and 139, respectively, and

$$\frac{3 \cdot 29 \cdot \log 10}{\log 233} < 37 \quad \text{and} \quad \frac{3 \cdot 23 \cdot \log 10}{\log 139} < 33,$$

we have that

$$a\frac{\phi(u_{q_1})}{u_{q_1}} \geqslant 2\min\{(1-\tfrac{1}{59})(1-\tfrac{1}{233})^{36},(1-\tfrac{1}{47})(1-\tfrac{1}{139})^{32}\}$$
$$> 1.55,$$

whenever $q_1 \in \{23, 29\}$. Thus, we have factored only the numbers u_{Q_1} with $Q_1 \leqslant 19$. We now use inequality (3.6) for $i = 1$ to obtain

$$\log(1.49) < \frac{\omega(n_1)}{2q_2}\left(1+\log\left(\frac{Q_1 q_2 \log g}{\log(2q_2+1)}\right)\right) + \frac{\tau(n_1)-2}{2q_2^2}\left(1+\log\left(\frac{Q_1 q_2^2 \log g}{\log(2q_2^2+1)}\right)\right).$$

If $q_1 > 3$, then $Q_1 = q_1 \leqslant 31$. If $q_1 = 3$, then $Q_1 = q_1^2 = 9$. Thus, $Q_1 \leqslant 31$ in both cases. We also saw in Claims 4.1 and 4.2 that $\tau(n_1) \leqslant \tau(n/q_1) \leqslant 6$, so also $\omega(n_1) \leqslant 2$. Hence,

$$\log(1.49) < \frac{1}{q_2}\left(1+\log\left(\frac{31 q_2 \log 999}{\log(2q_2+1)}\right)\right) + \frac{2}{q_2^2}\left(1+\log\left(\frac{31 q_2^2 \log 999}{\log(2q_2^2+1)}\right)\right),$$

and this inequality does not hold when $q_2 \geqslant 23$.

4.1. The conclusion of the proof of Theorem 1.2

So far, we have shown that $3 \leqslant q_1 < q_2 \leqslant 19$. The argument showing that $\alpha_1 = 2$ except if $(q_1, g) = (3, 729)$ now shows that $\alpha_2 = 1$. We are now able to show that $s = 2$. Indeed, if it were not so, then we would have both $\tau(n/q_1) \geqslant 4$ and $\tau(n/q_2) \geqslant 4$. A quick computation with MATHEMATICA shows that while there are pairs (q, g) such that $\nu_q(gu_{q-1}) \geqslant 4$ in our ranges, there is no odd g in $[3, 999]$ that has the above property with respect to two different primes $3 \leqslant q_1 < q_2 \leqslant 19$. Thus, either $n = q_1 q_2$ or $n = 9q_2$ and $g = 729$. To test these last possiblilites, we proceeded as follows. First we detected all pairs (n, g) with $n = q_1 q_2$ with $3 \leqslant q_1 < q_2 \leqslant 19$ and odd $g \in [3, 999]$ such that $\nu_{q_i}(gu_{n-1}) \geqslant 2$ holds for both $i = 1, 2$. There are 2043 such pairs. For each one of these we checked that $\nu_2(u_{n-1}) < 14$. Similarly, when $Q_1 = 9$ and $g = 729$, the only possibility for q_2 in our range such that $\nu_{q_2}(u_{q_2-1}) \geqslant 2$ is $q_2 = 11$, but in this case $n = 99$ and $\nu_2(u_{n-1}) = 1 < 14$. This finishes the proof of Theorem 1.2.

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