SOME IRREDUCIBLE CHARACTERS OF GROUPS WITH BN PAIRS

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Summary

Let G be a finite group with a split BN pair at characteristic p (as defined in [16]), let $H = B \cap N$, $W = N/H$ and X the Sylow p-subgroup of B. Thus G may be any of the finite Chevalley groups, including the twisted types. We make an additional assumption concerned with the derived group (commutator subgroup) of X, and show (theorem A) that with a few exceptions the Chevalley groups and twisted types do satisfy this condition. This thesis is chiefly concerned with irreducible characters of G which are components of the character 1_X^G (induced from the principal character of X).

Specifically, let λ be a linear character of H and extend λ to B by defining $\lambda(x) = 1$ for all $x \in X$. Let μ be a linear character of Y (the product of the negative root subgroups) which is nontrivial on the root subgroup X_{-r} whenever r is fundamental. There exists an irreducible character $\chi(\lambda,\mu)$ of G which has multiplicity 1 in λ^G (theorem B; these characters were discovered by Dagger [9]). Generalizing the isometry argument used by Curtis in [6] another character $\zeta(\lambda,\mu)$ is constructed which also has multiplicity one in λ^G (theorem C). As a by-product we derive a formula for the multiplicity with which a linear character of a Sylow p-subgroup occurs in the restriction of an irreducible component of λ^G (theorem D). It is shown that any component of 1_X^G with degree prime to p is of the form $\zeta(\lambda,\mu)$ (theorem E).

Let k be the complex field and B_λ the primitive idempotent in kB affording the character λ. We use the technique (used by Curtis, Iwahori and Kilmoyer [8]) of investigating components of λ^G by investigating characters of the Hecke algebra $B_\lambda kGB_\lambda$. Irreducible components of λ^G with multiplicity m restrict to irreducible characters of $B_\lambda kgB_\lambda$ of degree m (Curtis and Fossum [7]). Thus the existence of the characters $\chi(\lambda,\mu)$ and $\zeta(\lambda,\mu)$ guarantees the existence of linear representations of $B_\lambda kGB_\lambda$. The structure of $B_\lambda kGB_\lambda$ is closely related to that of kSH_λ where $S = \{w \in W | \lambda^w = \lambda\}$ and $H_\lambda = \Sigma\lambda(h^{-1})h$, and we are able to deduce the existence of a linear representation of SH which extends λ (theorem F).

It is also proved (theorem G) that S is the split extension of W_S by D where D is an abelian p'-group and W_S a Weyl subgroup of W, and we give a set of relations which determine the multiplication of basis elements of $B_\lambda kGB_\lambda$. In theorem H we obtain a formula for the degrees of components of multiplicity one in λ^G and prove that in most cases there are precisely $|D|$ components with degree prime to p, all having the same degree. For any parabolic subgroup G_J and any component ψ of λ^{G_J} there exists a corresponding irreducible character ξ of $N \cap G_J$; the correspondence $\psi^G \leftrightarrow \xi^N$ is an isometry between the spaces generated by these characters (as J,λ vary) (theorem I).

Finally, an automorphism of order 2 of $\bar{X}kG\bar{X}$ is obtained which provides an alternative method of constructing the $\zeta(\lambda,\mu)$ from the $\chi(\lambda,\mu)$ (theorem J), and shows that for each component of λ^G there is a "dual" component.

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, contains no material previously published or written by another person, except when due reference is made in the text of the thesis.

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R. HOWLETT.

CHAPTER I

STATEMENTS OF THE MAIN THEOREMS

THEOREM A Let G be any finite Chevalley group (of normal or twisted type) other than $B_\ell(2)$, $C_\ell(2)$, $F_4(2)$, $G_2(2)$, $G_2(3)$ or $F_4^1(2)$ (in the notation of [2]). Let $X_1, X_2, \ldots X_n$ be the fundamental root subgroups of G. Then the natural map $\Pi X_i/X_i' \to X/X'$ is an isomorphism, and all root subgroups for non-fundamental positive roots are contained in X'. (The prime denotes "commutator subgroup").

In all the remaining theorems G will be an arbitrary finite group with split BN pair satisfying this condition on X', and λ will be a linear character of B with kernel containing X. For each $w \in W$ define $\lambda^w(hx) = \lambda(whw^{-1})$ $(h \in H,\ x \in X)$ so that λ^w is another such linear character of B. Let $S = \{w \in W \mid \lambda^w = \lambda\}$.

THEOREM B For each fundamental root r_i choose a nontrivial linear character μ_i of Y_i, the root subgroup corresponding to $-r_i$. Let $J \subseteq \{1,2,\ldots n\}$, G_J the corresponding parabolic subgroup, Y the product of the negative root subgroups in G_J, and μ_J the linear character of Y extending each μ_i for $i \in J$ and trivial on Y_i for $i \notin J$. Then for each $w \in W$, $(\lambda^w)^{G_J}$ and $(\mu_J)^{G_J}$ have a unique common component χ_{Jw}, and it has multiplicity one in each.

THEOREM C For any subset J of $\{1,2,\ldots n\}$ let W_J be the corresponding parabolic subgroup and define $\varphi_J = \Sigma \chi_{Jv}^G$, where v runs through a set of representatives of the $S\backslash W/W_J$ cosets. Then $\zeta = \Sigma(-1)^{|J|}\varphi_J$ (summation over all subsets J) is an irreducible character of G occurring with multiplicity 1 in λ^G.

<u>THEOREM D</u> Let σ be any component of λ^G and α the restriction of σ to Y (with Y as defined in theorem B). Then for any $J \subseteq \{1,2,\ldots n\}$,

$$(\alpha,\mu_J) = (\sigma,\varphi_J).$$

(n.b. this is the usual inner product for characters).

<u>THEOREM E</u> Any irreducible component of 1_X^G which cannot be obtained by the method given in theorem C has degree divisible by p.

<u>THEOREM F</u> The character λ of H may be extended to a linear character of SH. (One particular extension will be denoted by "λ").

<u>THEOREM G</u> (i) S is the split extension of W_S by D, where W_S is a Weyl subgroup of W and D an abelian p'-group.

(ii) Let Γ be the root system of W_S. Then $B_\lambda kGB_\lambda$ has a basis $\{\gamma_w | w \in S\}$ such that if $w \in S$, $v \in D$, and r is a fundamental root of Γ with corresponding reflection s,

(a) $\gamma_v \gamma_w = \gamma_{vw}$ and $\gamma_w \gamma_v = \gamma_{wv}$,

(b) $\gamma_w \gamma_s = \gamma_{ws}$ if w(r) is positive

$\gamma_w \gamma_s = p_r \gamma_{ws} + (p_r - 1)\gamma_w$ if w(r) is negative,

(c) $\gamma_s \gamma_w = \gamma_{sw}$ if $w^{-1}(r)$ is positive

$\gamma_s \gamma_w = p_r \gamma_{sw} + (p_r - 1)\gamma_w$ if $w^{-1}(r)$ is negative.

The constants p_r are nonnegative integral powers of p such that $p_a = p_b$ if a and b are in the same S-orbit of Γ.

THEOREM H (i) Let ν be any linear character of W_S and η any linear character of D. Then there exists an irreducible character $\psi(\nu,\eta,\lambda)$ of G which has multiplicity 1 in λ^G and degree given by $|D|^{-1}W(\underset{\sim}{q})/W_S(\underset{\sim}{\theta})$. Here $W(\underset{\sim}{t})$ and $W_S(\underset{\sim}{u})$ are the Poincare polynomials of W and W_S. Thus $\underset{\sim}{u}$ has one component for each W_S-orbit of Γ; $\underset{\sim}{\theta}$ is calculated by setting the component corresponding to $r \in \Gamma$ equal to p_r if $\nu(s) = 1$ and equal to p_r^{-1} if $\nu(s) = -1$ (where s is the reflection corresponding to r). The components of $\underset{\sim}{q}$ are just the orders of corresponding root subgroups. Any component of multiplicity 1 in λ^G is of the form $\psi(\nu,\eta,\lambda)$.

(ii) There are precisely $|D|$ components of λ^G with degree prime to p, namely the characters $\psi(1,\eta,\lambda)$ as η varies through all linear characters of D, except that if W_S has a Weyl subgroup which is dihedral of order 4m and $p_a | m$ for all roots a of this subgroup, then λ^G may have further components with degree prime to p.

THEOREM I Corresponding to the character $\psi(\nu,\eta,\lambda)$ of G there is an irreducible character $\xi(\nu,\eta,\lambda)$ of N, given by inducing the character $\nu\eta\lambda$ of SH. Similarly for any parabolic subgroup G_J and any component ψ of multiplicity 1 in λ^{G_J} there exists an irreducible character ξ of $N \cap G_J$, and the correspondence $\psi^G \leftrightarrow \xi^N$ is an isometry between the inner product spaces generated by these characters (as J,λ vary).

THEOREM J There exists an automorphism f of $\bar{X}kG\bar{X}$ of order two such that for all λ,μ_i if e is a primitive idempotent in $\bar{X}kG\bar{X}$ affording the irreducible character χ of G defined in theorem B, then f(e) affords the character ζ of theorem C.

CHAPTER 2

FINITE GROUPS GENERATED BY REFLECTIONS

In this chapter some standard results on root systems and reflection groups are listed. More detailed descriptions can be found in [3] and [18].

Let V be a real n-dimensional Euclidean space with inner product $(\ , \ )$. For $r \in V$ the orthogonal linear transformation

$$s: v \mapsto v - \frac{2(v,r)}{(r,r)} r$$

is the reflection in the hyperplane orthogonal to r. A root system Δ in V is a finite set of vectors which generate V such that:

(1) For each $r \in \Delta$, $-r \in \Delta$, but no other multiple of r is contained in Δ.

(2) If $r \in \Delta$ and s is the reflection in the hyperplane orthogonal to r, $s(\Delta) = \Delta$.

The elements of Δ are called <u>roots</u> and the reflections corresponding to the roots generated a finite group W, called a <u>finite group generated by reflections</u> or g.g.r. If x is a fixed but arbitrary vector in V satisfying $(x,r) \neq 0$ for all $r \in \Delta$, we define

$$\Delta^+ = \{r \in \Delta \mid (x,r) > 0\}$$

the set of positive roots, and

$$\Delta^- = \{r \in \Delta \mid (x,r) < 0\}$$

the set of negative roots. Any root system has a <u>base</u> which is a subset π of Δ^+ satisfying

(3) $\pi = \{r_1, r_2, \ldots r_n\}$ is a basis of V,

(4) If $r = \sum_{i=1}^{n} t_i r_i$ is an element of Δ, then all the t_i are nonnegative or all nonpositive.

The elements of π are called fundamental roots, and the corresponding reflections $s_1, s_2, \ldots s_n$ fundamental reflections. It can be proved that any root is the image of a fundamental root under the action of some $w \in W$.

2.1 THEOREM (Coxeter [5, §9.3]) W is generated by $s_1, s_2, \ldots s_n$ subject to the defining relations

$$(s_i s_j)^{n_{ij}} = 1 \quad \text{for all } i,j \quad 1 \le i \le j \le n$$

where n_{ij} is the order of $s_i s_j$ in W.

For $w \in W$ define $\ell(w)$ to be the least m such that there exists an expression

$$w = w_1 w_2 \ldots w_m \quad (w_i \in \{s_1, s_2 \ldots s_n\}$$

for each i)

for w as the product of fundamental reflections. Such an expression with $m = \ell(w)$ is called reduced.

2.2 LEMMA (Solomon [17 lemma 1]) If $w \in W$ and $1 \le i \le n$ then $\ell(ws_i) = \ell(w) + 1$ if $w(r_i)$ is a positive root, and $\ell(ws_i) = \ell(w) - 1$ if $w(r_i)$ is a negative root. Similarly $\ell(s_i w) = \ell(w) + 1$ if $w^{-1}(r_i) \in \Delta^+$ and $\ell(s_i w) = \ell(w) - 1$ if $w^{-1}(r_i) \in \Delta^-$.

2.3 COROLLARY If $w_1 w_2 \ldots w_m$ is a reduced expression for w, and if $a_1, a_2, \ldots a_m$ are the fundamental roots corresponding to $w_1, w_2, \ldots w_m$ (i.e. if $w_i = s_j$ then $a_i = r_j$) then the positive roots r such that $w(r)$ is negative are:

$$a_m, w_m(a_{m-1}), \quad w_m w_{m-1}(a_{m-2}), \ldots, w_m w_{m-1} \ldots w_2(a_1).$$

In particular for each $w \neq 1$ there is a fundamental root r_j (= a_m) such that $w(r_j)$ is negative.

Let Γ be a set of roots and suppose that the corresponding reflections generate a subgroup T of W. Let $\Omega = T(\Gamma)$. Then Ω is a root system for T acting on the subspace of V spanned by Ω. The positive roots of Ω can be chosen to be those which are positive in Δ. Hence if P is the set of fundamental roots for Ω $P \subseteq \Delta^+$.

<u>2.4 LEMMA</u> Each coset wT $(w \in W)$ contains unique elements v_1 and v_2 such that $v_1(r)$ is positive and $v_2(r)$ negative for all positive roots $r \in \Omega$.

<u>Proof</u> It is well known that if r is a fundamental root and s the corresponding reflection then s permutes the positive roots other than r. Let $w \in W$ and choose $v_1 \in wT$ which negatives a minimal number of positive roots in Ω, and suppose that $v_1(r)$ is negative for some r in P. Then if s is the reflection corresponding to r,

$$v_1 s(r) = v_1(-r) = -v_1(r) \quad \text{is positive,}$$

and since

$\{a \mid a \in \Omega, \text{ a is positive, } a \neq r\} = \{s(a) \mid a \in \Omega, \text{ a is positive, } a \neq r\}$

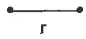
it follows that the number of positive $a \in \Omega - \{r\}$ such that $v_1(a)$ is negative equals the number of positive $a \in \Omega - \{r\}$ such that $v_1 s(a)$ is negative. Thus $v_1 s$ negatives fewer positive roots in Ω than does v_1, a contradiction since $v_1 s \in wT$. Therefore v_1 negatives no element of P, and therefore no positive linear combination of elements of P either, as required. To prove uniqueness, assume $v \in T$ and that $v_1 v$ also does not negative any positive root in Ω. If $v \neq 1$ there exists a positive root $r \in \Omega$ with $v(r)$ negative. Since $v_1 v(r)$ is positive, $-v(r)$ is a positive root in Ω negatived by v_1. This is a contradiction, and so $v=1$ and hence $v_1 v = v_1$.

A similar proof applies for v_2, which is characterized as the element of wT which negatives a maximal number of positive roots in Ω.

2.5 LEMMA Let $J \subseteq \{1,2,...n\}$ and W_J the group generated by the s_i for $i \in J$. Then $\{r_i | i \in J\}$ is a base for the root system of W_J. Each coset wW_J $(w \in W)$ contains exactly one v such that $v(r_i)$ is positive for all $i \in J$. In particular W_J contains a unique involution w_J which maps the roots $\{r_i | i \in J\}$ to the roots $\{-r_i | i \in J\}$, and there exists $w_0 \in W$ mapping $\{r_i | 1 \leq i \leq n\}$ to $\{-r_i | 1 \leq i \leq n\}$.

The proof of this (which is similar to the proof of 2.4) follows from 1.12 and 1.16 of [18].

A g.g.r. is said to be reducible if its root system can be divided into two nonempty subsets such that all the roots in one are orthogonal to all the roots in the other. In this case the group is a nontrivial direct product of two smaller g.g.r.'s. Conversely if W_1 and W_2 are two g.g.r's acting on Euclidean spaces V_1 and V_2 respectively, then $W_1 \times W_2$ is a g.g.r. acting on the direct sum of V_1 and V_2 (making V_1 orthogonal to V_2). The root system of $W_1 \times W_2$ is the union of the root systems of W_1 and W_2. The irreducible g.g.r.'s have been classified by Coxeter. Using the notation of [5] they are:

Symbol	Lie Algebra	Number of generators	Diagram
$[3^{n-1}]$	A_n	$n \geq 1$	
$[3^{n-2},4]$	B_n and C_n	$n \geq 2$	
$[3^{n-3,1,1}]$	D_n	$n \geq 4$	
$[r]$	G_2 if $r=6$	2	
$[3,5]$	-	3	
$[3,3,5]$	-	4	
$[3,4,3]$	F_4	4	
$[3^{2,2,1}]$	E_6	6	
$[3^{3,2,1}]$	E_7	7	
$[3^{4,2,1}]$	E_8	8	

Certain g.g.r.'s are Weyl groups of Lie algebras and for convenience the correspondence is given in the table. The correspondence is relevant since Chevalley groups (which are the topic of chapter 4) are constructed from Lie algebras. Furthermore there is a theorem of Feit and Higman [10] that only those g.g.r.'s which are Weyl groups of Lie algebras and the dihedral group of order 16 (i.e. [8] in the notation of the above table) can be Weyl groups of BN pairs (BN pairs will be defined in chapter 3). The diagram of a g.g.r is obtained from the generators and relations given in theorem 2.1 by placing one node for each generator and joining the ith and jth nodes by a bond of strength n_{ij}. Bonds of strength two are omitted, and unmarked bonds are understood to have strength 3.

2.6 DEFINITION Let W be a g.g.r. with root system Δ. Let the orbits of W on Δ be $\Omega_1, \Omega_2, \ldots \Omega_m$. For each $w \in W$ let $N_i(w)$ be the number of positive roots in Ω_i

negatived by w. The Poincare polynomial of W is defined to be

$$W(\underset{\sim}{t}) = \Sigma\, t_1^{N_1(w)} t_2^{N_2(w)} \ldots\ldots t_m^{N_m(w)} \qquad (w\in W)$$

(where $\underset{\sim}{t} = (t_1, t_2, \ldots t_m)$).

For the purpose of computing values taken by these polynomials it is useful to be able to factorize them, and for this the reader is referred to [14]. In fact the Poincare polynomial of a reducible g.g.r. is the product of the Poincare polynomials of the component irreducible g.g.r.'s, and the factors of each of the Poincare polynomials of irreducible g.g.r.'s are listed in [14].

Now suppose that S is any subgroup of the g.g.r.W, and that for each root r there exists $\theta_r \in k$ (where k is any field) such that

(i) $\theta_r = \theta_{w(r)}$ for all $r\in\Delta$ and $w\in S$

(ii) $\theta_r = 1$ if the reflection s corresponding to r is not in S.

Let $\theta_w = \Pi\theta_r$ where the product is over all positive roots r negatived by W. Let W_S be the subgroup of S generated by the reflections corresponding to roots r such that $\theta_r \neq 1$. Then we have the following result, which will be used in chapter 8:

<u>2.7 LEMMA</u> (i) Let $D = \{v\in S \mid v(r)$ is positive for all positive roots r of $W_S\}$. Then D is a subgroup of S which normalizes W_S, $S=DW_S$, and $D\cap W_S=1$

(ii) $\sum_{w\in S} \theta_w = [S:W_S]W_S(\underset{\sim}{\theta})$ (i.e. the value taken by $W_S(\underset{\sim}{t})$ when t_i is replaced by θ_r, where r is any root in the corresponding W_S-orbit).

Proof (i) From 2.4 it is clear that $S=DW_S$ and $D\cap W_S=1$. Let $v,w\in D$ and let r be a positive root of W_S. Then $w(r) > 0$ (since $w\in D$), and $w(r)$ is a root of W_S since $\theta_{w(r)} = \theta_r \neq 1$. Hence $v(w(r)) > 0$, since $v\in D$. It follows that $vw(r) > 0$ for all positive roots r of W_S, and so $vw\in D$. Hence D is a subgroup. Furthermore, if $v\in D$ and $s\in W_S$ is the reflection corresponding to the root r, then vsv^{-1} is the reflection corresponding to $v(r)$, and it follows that $vsv^{-1}\in W_S$. Since W_S is generated by such reflections s, D normalizes W_S.

(ii) Obviously $W_S(\underset{\sim}{\theta}) = \Sigma\theta_w$ $(w\in W_S)$. Let $v\in D$, $w\in W_S$, and r a positive root of W_S. Then $vw(r)$ is negative if and only if $w(r)$ is negative, and so vw and w negative the same positive roots of W_S. But $\theta_r = 1$ for all other positive roots and so $\theta_{vw} = \theta_w$. The rest is obvious.

From the generators and relations given in theorem 2.1 it follows that a g.g.r. W has a linear character ε such that $\varepsilon(s) = -1$ for each fundamental reflection s. For each subset J of $\{1,2,...n\}$ let W_J be as in 2.5 and let δ_J be the character of W induced from the principal character (1-character) of W_J.

<u>2.8 THEOREM</u> (Solomon [17, Theorem 2])

$$\varepsilon = \Sigma(-1)^{|J|}\delta_J$$

where the summation is over all subsets J of $\{1,2,...n\}$.

CHAPTER 3

SPLIT BN PAIRS

3.1 DEFINITION (Tits [20]) A finite group G has a BN pair if there exist subgroups B and N of G which generate G, $H = B \cap N$ is a normal subgroup of N, and $W = N/H$ is generated by involutions $s_1, s_2, \ldots s_n$, and

(1) $s_i Bw \subseteq BwB \cup Bs_i wB$

(2) $s_i Bs_i \neq B$

for all $w \in W$ and $1 \leq i \leq n$.

W is called the Weyl group of the BN pair, and n its rank.

The elements $w \in W$ are cosets of H in N. We will choose a fixed but arbitrary set of coset representatives, and following the notation of Richen [16] (w) will be the coset representative corresponding to $w \in W$. The parentheses are omitted when the choice of coset representative does not alter the object in question (e.g. 'wB' for '$(w)B$', as in the above definition.

3.2 THE BRUHAT THEOREM (Tits [20]). If G has a BN pair then

(1) $G = \cup BwB$ (union over all $w \in W$)

(2) If $BwB = Bw'B$ for $w, w' \in W$, then $w = w'$.

(3) If $\ell(s_i w) > \ell(w)$ for $1 \leq i \leq n$ and $w \in W$ then $s_i Bw \subseteq Bs_i wB$.

3.3 THEOREM (Iwahori and Matsumoto [15]).

If $W = \langle s_1, s_2, \ldots, s_n \rangle$ is the Weyl group of a finite BN pair then the relations

$$(s_i s_j)^{n_{ij}} = 1 \quad \text{for all } i,j \quad 1 \leq i \leq j \leq n$$

(where n_{ij} is the order of $s_i s_j$ in W) are defining relations for W.

<u>3.4 COROLLARY</u> The Weyl group of a finite BN pair of rank n is isomorphic to a finite group generated by reflections in n-dimensional Euclidean space.

As a consequence of this corollary we may use the notation of chapter 2: Δ is a root system for W and $\pi = \{r_1, r_2, \ldots r_n\}$ a base for Δ.

<u>3.5 DEFINITION</u> (Richen [16]) G is said to have a split BN pair of rank n at characteristic p (where p is any prime number) if G has a BN pair of rank n, $H = B \cap N = \cap w^{-1}Bw$ $(w \in W)$ is an abelian p'-group, and $B = XH$ where X is a normal p-subgroup of B.

<u>3.6 THEOREM</u> (Richen [16, theorem 2.12]). For each $w \in W$ (the Weyl group of a split BN pair), let $X^w = w^{-1}Xw$ and define $X_w = X \cap X^w$ and $X_i = X_w$ when $w = w_o s_i$ (w_o as defined in 2.5). Then W acts as a permutation group on $\Sigma = \{wX_i w^{-1} \mid w \in W,\ 1 \le i \le n\}$ under

$$w\colon Z \mapsto wZw^{-1} \quad (\text{for each } Z \in \Sigma)$$

and $wX_i w^{-1} \mapsto w(r_i)$ is a well defined isomorphism $(W, \Sigma) \cong (W, \Delta)$. (In effect, Σ is a root system for W).

<u>3.7 DEFINITION</u> Let $r \in \Delta$. The root subgroup X_r of G is defined by

$$X_r = wX_i w^{-1}$$

where $w \in W$ and $1 \le i \le n$ such that $r = w(r_i)$.

This definition is justified by 3.6 and the fact that any root is the image of some fundamental root under the action of some $w \in W$.

The proofs of all the following facts can also be found in Richen's paper.

3.8 LEMMA Let G be a finite group with a split BN pair. With the notation as above:

(1) H normalizes each root subgroup.

(2) If $w_1 w_2 \dots w_m$ is a reduced expression for $w \in W$ and if we let $v_j = w_m w_{m-1} \dots w_j$ $(j=1,2,\dots n)$ then $B \cap w^{-1} w_0^{-1} B w_0 w = H Z_m (v_m Z_{m-1} v_m^{-1})(v_{m-1} Z_{m-2} v_{m-1}^{-1}) \dots\dots (v_2 Z_1 v_2^{-1})$ where $Z_1, Z_2, \dots Z_m$ are the fundamental root subgroups corresponding to the fundamental reflections $w_1, w_2, \dots w_m$. Thus we see that

$$X_{w_0 w} = X_{a_m} X_{v_m(a_{m-1})} X_{v_{m-1}(a_{m-2})} \dots X_{v_2(a_1)}$$

where $a_1, a_2, \dots a_m$ are the roots corresponding to $w_1, w_2, \dots w_m$ (so that $X_{a_1} = Z_1$, etc.) Notice that $X_{w_0 w}$ is a product of the root subgroups corresponding to the positive roots negatived by w (see 2.3). In particular $X (= X_{w_0 w_0})$ is a product of the root subgroups corresponding to the positive roots.

(3) For all $w \in W$

$$X = X_{w_0 w} X_w \quad \text{and} \quad X_{w_0 w} \cap X_w = \{1\}$$

In chapter 5 it will be necessary to deal with linear characters of X, and this will involve investigation of the derived group X' of X. In the case of the Chevalley groups (see section 4) this is accomplished by means of the Chevalley commutator formula, and since the axioms for a split BN pair have no analogue of this formula it seems necessary to assume another axiom.

3.9 AXIOM The natural map $\prod_{i=1}^{n} X_i / X_i' \to X/X'$ is an isomorphism. All root subgroups X_r corresponding to nonfundamental positive roots r are contained in X'.

In chapter 4 theorem A will be proved; that is that with a few exceptions 3.9 is satisfied by the Chevalley groups, including the twisted types.

3.10 LEMMA (Richen [16, definition 3.7]) Let $x \in X_i$, $x \neq 1$. Then there exist unique elements $f_i(x) \in X_i$, $h_i(x) \in H$, and $g_i(x) \in X$ with

$$(s_i)x(s_i)^{-1} = g_i(x)\, h_i(x)\, (s_i) f_i(x)$$

(This differs slightly from Richen's notation: '$f_i(x)$' replaces '$f_i(x^{-1})^{-1}$' etc.) The equations given in 3.10 are called by Richen the structural equations of G.

CHAPTER 4

CHEVALLEY GROUPS

The Chevalley groups are our chief object of interest; for their construction the reader is referred to [2] and the references given there. In this chapter we will prove that 3.9 holds and obtain the structural equations for these groups.

The normal types.

Let L be a simple Lie algebra over the complex field, with Weyl group W and root system Δ. (Δ is also a root system for W. The lengths of the roots are specified, and in such a way that a non-integral linear combination of linearly independent roots cannot be a root). An ordering of Δ is fixed in the usual way. If q is a power of a prime p there exists a Chevalley group $G = L(q)$, which has a split BN pair at characteristic p. For each $r \in \Delta$ there is an isomorphism $t \mapsto x_r(t)$ from the additive group of $GF(q)$ to the root subgroup X_r of G. For linearly independent roots r and s we have the Chevalley commutator formula:

$$[x_s(u), x_r(t)] = x_s(u)^{-1} x_r(t)^{-1} x_s(u) x_r(t) = \Pi x_{ir+js}(c_{ij;rs} t^i u^j)$$

where the $c_{ij;rs}$ are certain integers. The product, over positive integers i,j such that $ir+js \in \Delta$ is taken in the order of increasing roots.

Let P be the free abelian group generated by the roots, and let $\chi: P \to GF(q)^*$ (the nonzero elements of $GF(q)$) be a linear character of P. Then there exists an automorphism of G such that

$$x_r(t) \mapsto x_r(\chi(r)t) \quad \text{for all } r \in \Delta,\ t \in GF(q)$$

The subgroup H of G consists of elements $h(\chi)$ such that

$$h(\chi)x_r(t)h(\chi)^{-1} = x_r(\chi(r)t)$$

and $$h(\chi_1)h(\chi_2) = h(\chi_1\chi_2)$$

where the product of the characters χ_1, χ_2 of P is defined by $\chi_1\chi_2(a) = \chi_1(a)\chi_2(a)$ $(a \in P)$.

Indeed there exists a group $\hat{G}$ containing G and a subgroup $\hat{H}$ comprising elements $h(\chi)$ for all characters $\chi : P \to GF(q)^*$.

If r is any root and s the corresponding reflection,

$$(s)x_r(t)(s)^{-1} = x_{-r}(-t) = x_r(-t^{-1})h(\chi_{r,t})^{-1}(s)x_r(-t^{-1})$$

(where (s) is n_r in the notation of [2]), the character $\chi_{r,t} : P \to GF(q)^*$ being defined by

$$\chi_{r,t}(a) = t^{\frac{2(r,a)}{(rr)}} \qquad (t \neq 0)$$

This gives the structural equations for the non-twisted Chevalley groups, and the next theorem shows that 3.9 is also satisfied, except for $B_\ell(2)$, $C_\ell(2)$, $F_4(2)$, $G_2(2)$ and $G_2(3)$.

<u>4.1 THEOREM</u> (Howlett [12 lemma 7]) Let Z be the subgroup of X generated by the X_r for $r \in \Delta^+ - \pi$. Then with the above exceptions, $Z = U'$, the derived group of U.

<u>Proof</u> We use the Chevalley commutator formula and the fact (see [3]) that if $r-s$ is not a root then $c_{1,j;r,s} = \pm 1$.

It is clear that $U' \leqslant Z$. Let $r \in \Delta^+ - \pi$. Then there exist positive roots a and b (different from r) such that r is in the root system spanned by a and b, and $a-b$ is not a root. (i.e. a and b are fundamental roots for the root system they span).

<u>Case 1.</u> If a and b span a root system of type A_2 then

$[x_a(t), x_b(u)] = x_{b+a}(ctu)$ $(c=\pm 1, t, u \in GF(q))$

and thus $x_{b+a}(t) \in U'$ for all $t \in GF(q)$. Therefore

$X_r \leqslant U'$ in this case.

Case 2. If a and b span a root system of type B_2 then L is of type B_ℓ, C_ℓ or F_4, and

$[x_a(t), x_b(u)] = x_{b+a}(ctu)x_{b+2a}(dt^2u)$ $(c,d=\pm 1, t, u \in GF(q))$.

Replacing u by $t^{-1}u$ and t by $-t$ if necessary gives

$$x_{a+b}(u)x_{2a+b}(tu) \in U' \qquad (t \neq 0,\ u \in GF(q)) \tag{1}$$

Therefore $x_{a+b}(u)x_{2a+b}(tu)(x_{a+b}(u)x_{2a+b}(u))^{-1} \in U'$

and so $x_{2a+b}(u(t-1)) \in U'$

So $X_{2a+b} \leqslant U'$ if $GF(q)$ contains an element $t \neq 0,1$.

Then $X_{a+b} \leqslant U'$ also (from (1) above)

$X_r \leqslant U'$ if $q \neq 2$.

Case 3. If a and b span a root system of type G_2 then $L = G_2$ and $\Delta^+ = \{a, b, a+b, 2a+b, 3a+b, 3a+2b\}$. Then b and $3a+b$ span a system of type A_2, and so $X_{3a+2b} \leqslant U'$ by case 1. Now

$[x_a(t), x_b(u)] = x_{b+a}(c_1tu)x_{b+2a}(c_2t^2u)x_{b+3a}(c_3t^3u)x_{2b+3a}(c_4t^3u^2)$

$(c_1, c_2, c_3 = \pm 1,\ c_4, t, u \in GF(q))$

$$x_{a+b}(c_1tu)x_{2a+b}(c_2t^2u)x_{3a+b}(c_3t^3u) \in U' \tag{2}$$

$$x_{a+b}(c_1u)x_{2a+b}(c_2tu)x_{3a+b}(c_3t^2u) \in U' \tag{3}$$

$(t \neq 0,\ u \in GF(q))$

$$x_{2a+b}(c_2u(t_1-t_2))x_{3a+b}(c_3u(t_1^2-t_2^2)) \in U' \tag{4}$$

$(t_1, t_2 \neq 0,\ u \in GF(q))$

Suppose first of all that q is even and greater than 2. Then $(t_1-t_2)^2 = t_1^2-t_2^2$, and each element of $GF(q)$ may be written as t_1-t_2 for $t_1, t_2 \neq 0$. Hence

$$x_{2a+b}(tu)x_{3a+b}(t^2u) \in U'$$

and by (3)

$$X_{a+b} \leqslant U' . \qquad (5)$$

Now (2) gives $x_{2a+b}(t^2u)x_{3a+b}(t^3u) \in U'$, and replacing u by $t^{-2}u$ gives

$$x_{2a+b}(u)x_{3a+b}(tu) \in U' \qquad (t\neq 0)$$

$$x_{3a+b}(u(t-1)) \in U' \qquad (t\neq 0)$$

$$X_{3a+b} \leqslant U'$$

since t can be chosen so that $t-1 \neq 0$.

Using (5) and (2), $X_{2a+b} \leqslant U'$ also. Hence $X_r \leqslant U'$.

Suppose alternatively that q is odd and greater than 3. Then (4) gives (on replacing t_2 by t and t_1 by $t+1$)

$$x_{2a+b}(c_2u)x_{3a+b}(c_3u(2t+1)) \in U' \qquad (t\neq 0,-1)$$

$$x_{2a+b}(c_2u)x_{3a+b}(c_3u(2t+1)(x_{2a+b}(c_2u)x_{3a+b}(3c_3u))^{-1} \in U'$$

$$x_{3a+b}(2c_3u(t-1)) \in U' \qquad (t\neq 0,-1)$$

$$X_{3a+b} \leqslant U'$$

Now (2) and the argument used in case 2 gives X_{a+b} and $X_{2a+b} \leqslant U'$ as well.

So except in the cases $B_\ell(2)$, $C_\ell(2)$, $F_4(2)$, $G_2(2)$ and $G_2(3)$, $X_r \leqslant U'$ for all $r \in \Delta^+ - \pi$. Thus $Z \leqslant U'$ and so $Z = U'$.

The twisted types.

Let G be one of the groups $A_\ell(q^2)$ $(\ell \geq 2)$ $D_\ell(q^2)$ $(\ell \geq 4)$ or $E_6(q^2)$. Then the diagram corresponding to the root system has a symmetry of order two which gives rise to an automorphism $r \mapsto \bar{r}$ of the root system. The field $GF(q^2)$ also has an automorphism of order two, namely $t \mapsto \bar{t} = t^q$. It can be shown that

$$\sigma: \ x_r(t) \mapsto x_{\bar{r}}(\bar{t}) \qquad r \in \pi$$

extends to an automorphism of G. Define X^1, Y^1 to be the sets of elements of $X, Y = w_0^{-1} X w_0$ respectively which are invariant under σ, and G^1 to be the subgroup of G generated by X^1 and Y^1.

Similarly when $G = D_4(q^3)$ there is a symmetry of the diagram which has order 3, and the automorphism $t \mapsto \bar{t} = t^q$ of $GF(q^3)$ also has order 3. These yield an automorphism

$$\sigma: \ x_r(t) \mapsto x_{\bar{r}}(\bar{t})$$

of G. Define X^2, Y^2 to be the sets of elements of X, Y invariant under σ, and G^2 to be the subgroup of G generated by X^2 and Y^2. These groups G^1 and G^2 are the twisted types which were discovered by Steinberg.

In a similar fashion it is possible to construct twisted types (discovered by Ree and Suzuki) from the groups $B_2(q)$ when $q = 2^{2n+1}$, $F_4(q)$ when $q = 2^{2n+1}$, and $G_2(q)$ when $q = 3^{2n+1}$. Again it is possible (see [2]) to construct a permutation $r \mapsto \bar{r}$ of the root system, such that $\bar{\bar{r}} = r$, and there exists an automorphism of G

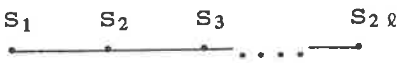
$$\sigma: \ x_r(t) \mapsto x_{\bar{r}}(t^{\lambda(\bar{r})\theta}) \qquad r \in \pi$$

where $\lambda(r) = (r,r)$ and $t^\theta = t^{2^n}$ if $q = 2^{2n+1}$ and $t^\theta = t^{3^n}$ if $q = 3^{2n+1}$. Using this automorphism σ the group G^1 is constructed as in the other cases.

The twisted types all have split BN pairs, and in particular the Weyl group W^1 (or W^2) is

$$\{w \in W \mid w(\bar{r}) = \overline{w(r)} \ \text{ for all } \ r \in \Delta\}$$

(1) $A^{1}_{2\ell}(q^2)$



In this case W^1 is isomorphic to the Weyl group of type B_ℓ, and the fundamental reflections are

$S_1=s_1 s_{2\ell}$, $S_2=s_2 s_{2\ell-1}, \ldots, S_{\ell-1}=s_{\ell-1}s_{\ell+2}$, $S_\ell=s_\ell s_{\ell+1} s_\ell$

The root subgroup corresponding to S_1 is $X_1^1 \cap S_1 Y^1 S_1$, which is clearly equal to $(X \cap s_1 s_{2\ell} Y s_1 s_{2\ell}) \cap G^1$

$$= \{x_a(t)x_b(\bar{t}) \mid t \in GF(q^2)\}$$

where $a = r_1$ and $b = r_{2\ell}$

The structural equation is (for $t\neq 0$)

$$(S_1)x_a(t)x_b(\bar{t})(S_1)^{-1} = x_{-a}(-t)x_{-b}(-\bar{t})$$

$$= x_{-a}(-t^{-1})x_{-b}(-\bar{t}^{-1})h(\chi_{a,t})^{-1}h(\chi_{b,\bar{t}})^{-1}(S_1)x_{-a}(-t^{-1})x_{-b}(-\bar{t}^{-1})$$

where $(S_1) = (s_1)(s_{2\ell})$. The same formulae hold when S_1 is replaced by any of $S_2, S_3, \ldots S_{\ell-1}$, and a and b are appropriately chosen. The root subgroup corresponding to S_ℓ is $(X \cap S_\ell Y S_\ell) \cap G^1$

$$= X_a X_b X_{a+b} \cap G^1 \quad \text{where} \quad a = r_\ell,\ b = r_{\ell+1}$$

$$= \{x_a(t)x_b(\bar{t})x_{a+b}(u) \mid t,u \in GF(q^2),\ u+\bar{u} = ct\bar{t}\}$$

where c may be either $+1$ or -1. The structural equation is (for $u\neq 0$)

$$(S_\ell)x_a(t)x_b(\bar{t})x_{a+b}(u)(S_\ell)^{-1} = x_{-b}(-t)x_{-a}(-\bar{t})x_{-a-b}(u)$$

$$= x_a(-ctu^{-1})x_b(-c\bar{t}\bar{u}^{-1})x_{a+b}(u^{-1})h(\chi)(S_\ell)x_a(-ct\bar{u}^{-1})x_b(-c\bar{t}u^{-1}).$$

$$.x_{a+b}(u^{-1})$$

where $h(\chi) = h(\chi_{a,cu})^{-1}h(\chi_{b,c\bar{u}})^{-1}$ and $(S_\ell)=(s_\ell)(s_{\ell+1})(s_\ell)$.

Next we show that the derived group Z of X^1 contains all nonfundamental positive root subgroups T of $A^1_{2\ell}(q^2)$. The group T is either of the form $X_r X_{\bar{r}} \cap X^1$ or $X_r X_{\bar{r}} X_{r+\bar{r}} \cap X^1$ where r is a root of $A_{2\ell}$. In either case r can be written as the sum of two positive roots in $A_{2\ell}$, and there is a corresponding expression for $\bar{r}$. The roots of $A_{2\ell}$ thus obtained generate a subsystem of type $A_2 \times A_2$ or A_4, with fundamental roots a,b,c and d.

$A_2 \times A_2$

A_4

For $A_2 \times A_2$ we have $[x_a(t)x_d(\bar{t}),\ x_b(1)x_c(1)]$

$$= [x_a(t), x_b(1)][x_d(\bar{t}),\ x_c(1)]$$

$$= x_{a+b}(\alpha t)x_{c+d}(\alpha\bar{t}) \qquad \alpha = \pm 1$$

and since $a+b=r$ it follows that in this case T is contained in Z.

For A_4, $\quad [x_a(t)x_d(\bar{t}),\ x_b(u)x_c(\bar{u})x_{b+c}(v)] \quad (v+\bar{v}=\pm u\bar{u})$

$$= x_d(t)^{-1}[x_a(t),\ x_c(\bar{u})x_b(u)x_{b+c}(-\bar{v})]x_d(\bar{t})[x_d(\bar{t}),x_b(u)x_c(\bar{u}).$$
$$.x_{b+c}(v)]$$

$$= x_d(\bar{t})^{-1}[x_a(t),x_{b+c}(-\bar{v})][x_a(t),x_b(u)]x_d(\bar{t})[x_d(\bar{t}),x_{b+c}(v)]\ .$$
$$.\ [x_d(\bar{t}),x_c(\bar{u})].$$

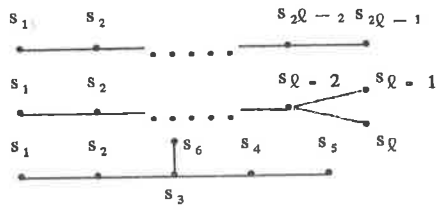
Call this formula (A). Putting $u=0$ and using the Chevalley commutator formula we see that Z contains $x_{a+b+c}(t\bar{v})x_{b+c+d}(\bar{t}v)x_{a+b+c+d}(\pm t\bar{t}v)$ where $v+\bar{v} = 0$. This same formula with v replaced by $t\bar{t}v$ and t by 1 gives that Z contains $x_{a+b+c}(t\bar{v}-t\bar{t}\bar{v})x_{b+c+d}(\bar{t}v-t\bar{t}v)$. Therefore $T \leqslant Z$ if $r = a+b+c$. The other possibility is $r=a+b$, and for this case set $u=1$ in (A) above. Combining this with $[x_a(t),x_{b+c}(-\bar{v})][x_d(\bar{t}),x_{b+c}(v)] \in Z$ (which follows from what we have just proved) we get

$$x_{a+b+c+d}(\pm\bar{t}\bar{v})x_{a+b}(t)x_{c+d}(\bar{t}) \in Z \qquad (v+\bar{v} = \pm 1)$$

and so $T \leqslant Z$ in this case too.

It can be proved readily that if $T=X_a X_b X_{a+b} \cap G^1$ is the root subgroup corresponding to S_ℓ then the derived group of T is $X_{a+b} \cap G^1$. It is now clear that $Z=X' \cap X^1$ and that $A^1_{2\ell}(q^2)$ satisfies 3.9.

(2) $A^1_{2\ell-1}(q^2), D^1_\ell(q^2), E^1_6(q^2)$



In these cases W is isomorphic to the Weyl groups of types B_ℓ, $B_{\ell-1}$, and F_4 respectively. The fundamental reflections are:

(for $A_{2\ell-1}$) $S_1=s_1 s_{2\ell-1}$, $S_2=s_2 s_{2\ell-2}, \ldots, S_{\ell-1}=s_{\ell-1} s_{\ell+1}, S_\ell=s_\ell$

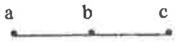
(for D_ℓ) $S_1=s_1$, $S_2=s_2, \ldots S_{\ell-2}=s_{\ell-2}, S_{\ell-1}=s_{\ell-1} s_\ell$

(for E_6) $S_1=s_1 s_5$, $S_2=s_2 s_4$, $S_3=s_3$, $S_4=s_6$

The root subgroups are either of the form $X_r X_{\bar{r}} \cap G^1$ or $X_r \cap G^1$ if $r=\bar{r}$. For those of the latter kind the structural equations are as for the Chevalley groups of normal type, for the former kind the structural equations are the same as for the root subgroups of $A^1_{2\ell}(q^2)$ which are of the same form.

To show that Z (derived group of X^1) contains all positive nonfundamental root subgroups we proceed in the same manner as for $A^1_{2\ell}(q^2)$. In this case we obtain a subsystem which may be of type A_2, $A_2 \times A_2$ or A_3. $A_2 \times A_2$ is treated exactly as before. The other possibilities are

where $a = \bar{a}$ and $b = \bar{b}$



where $c = \bar{a}$ and $b = \bar{b}$

For A_2 we have $x_{a+b}(\pm t) = [x_a(t), x_b(1)] \in Z$ (where $t=\bar{t}$).

For A_3, $[x_a(t)x_c(\bar{t}),\ x_b(u)]$ $(u=\bar{u})$

$$= x_c(\bar{t})^{-1}[x_a(t), x_b(u)]x_c(\bar{t})[x_c(\bar{t}), x_b(u)]$$

and so

$$x_{a+b}(tu)x_{b+c}(\bar{t}u)x_{a+b+c}(\pm t\bar{t}u) \in Z \qquad \text{(B)}$$

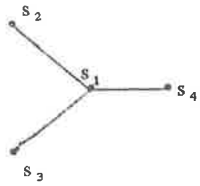
Replacing u by $t\bar{t}u$ and t by 1 in (B) gives

$$x_{a+b}(t\bar{t}u)x_{b+c}(t\bar{t}u)x_{a+b+c}(\pm t\bar{t}u) \in Z$$

Hence $x_{a+b}(t\bar{t}u - tu)x_{b+c}(t\bar{t}u - \bar{t}u) \in Z$.

It follows that $x_{a+b}(v)x_{b+c}(\bar{v}) \in Z$ for all $v \in GF(q^2)$ (since any element $(\neq 1)$ of $GF(q^2)$ can be written in the form $t\bar{t}u - tu$) and hence $x_{a+b+c}(w) \in Z$ if $w=\bar{w}$ (using (B) again). Thus 3.9 is also satisfied for these groups.

(3) $D_4^2(q^3)$



In this case W^2 is isomorphic to the Weyl group of type G_2. The fundamental reflections are $S_1 = s_1$ and $S_2 = s_2 s_3 s_4$. The root subgroup corresponding to S_1 is $X_a \cap G^2 = \{x_a(t) \mid t = \bar{t}\}$ where $a = r_1$. The root subgroup corresponding to S_2 is $X_b X_c X_d \cap G^2$

$$= \{x_b(t)x_c(\bar{t})x_d(\bar{\bar{t}}) \mid t \in GF(q^3)\}$$

where $b=r_2$, $c=r_3$, $d=r_4$. The structural equations are obvious.

Let Z be the derived group of X^2. Let $t \in GF(q^3)$ such that $\bar{t}=t$. Then Z contains

$$[x_a(t), x_{a+b+c+d}(1)] = x_{2a+b+c+d}(\pm t),$$

showing that one of the nonfundamental root subgroups is contained in Z. For the others note that by the Chevalley commutator formula, $[x_a(u),\ x_b(t)\ x_c(\bar{t})\ x_d(\bar{\bar{t}})]$

$= x_{a+d}(\alpha\bar{\bar{t}}u)x_{a+c}(\alpha\bar{t}u)x_{a+b}(\alpha tu)x_{a+c+d}(\beta\bar{\bar{t}}\bar{t}u)x_{a+b+d}(\beta\bar{\bar{t}}tu)$.

$.x_{a+b+c}(\beta\bar{t}tu)x_{a+b+c+d}(\gamma t\bar{t}\bar{\bar{t}}u)$ where $\alpha, \beta, \gamma = \pm 1$, and $u = \bar{u}$. Call this formula (C). In (C) replace t by tu and u by 1. The result, together with (C) itself, gives formula (D):

$x_{a+c+d}(\bar{\bar{t}}\bar{t}(u^2-u))x_{a+b+d}(\bar{\bar{t}}t(u^2-u))x_{a+b+c}(\bar{t}t(u^2-u)$.

$x_{a+b+c+d}(\pm t\bar{t}\bar{\bar{t}}(u^3-u)) \in Z$.

Similarly we can now replace u by $-u+1$ and prove that Z contains $x_{a+b+c+d}(t\bar{t}\bar{\bar{t}}(2u^3-3u^2+u))$. Thus clearly Z contains all elements of the form $x_{a+b+c+d}(v)$ where $v = \bar{v}$, and substituting back in (D) and (C) it follows that Z also contains

$x_{a+c+d}(t)x_{a+b+d}(\bar{t})x_{a+b+c}(\bar{\bar{t}})$ and $x_{a+b}(t)x_{a+c}(\bar{t})x_{a+d}(\bar{\bar{t}})$ for all $t \in GF(q^3)$. And hence $D_4^2(q^3)$ satisfies 3.9.

(4) $B_2^1(q)$, $F_4^1(q)$. $q = 2^{2n+1}$

B_2 $\lambda(r_1)=1$, $\lambda(r_2)=2$

F_4 $\lambda(r_1)=\lambda(r_2)=1$, $\lambda(r_3)=\lambda(r_4)=2$

For B_2 W^1 has order 2, generated by $(s_1s_2)^2$. For F_4 it is dihedral of order 16, the fundamental reflections being $S_1 = s_1s_4$ and $S_2 = (s_2s_3)^2$. Define elements $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$ in $F_4^1(q)$ $(i=1,2,3,4,\ t \in GF(q))$ by

$$\alpha_i(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})$$

$$\beta_i(t) = x_{a+b}(t^\theta)x_{2a+b}(t)$$

$$\gamma_i(t) = x_c(t^\theta)x_d(t)$$

for the following values of a, b, c and d:

i	a	b	c	d
1	r_2	r_3	r_1	r_4
2	r_1+r_2	r_3+r_4	$r_1+2r_2+r_3$	$2r_2+2r_3+r_4$
3	$r_1+r_2+r_3$	$2r_2+r_3+r_4$	$r_1+2r_2+r_3+r_4$	$2r_1+2r_2+2r_3+r_4$
4	$r_2+r_3+r_4$	$2r_1+2r_2+r_3$	$r_1+2r_2+2r_3+r_4$	$2r_1+4r_2+2r_3+r_4$

The root subgroup corresponding to S_1 is $\{\gamma_1(t) \mid t \in GF(q)\}$, and the structural equation is obvious in this case. The root subgroup corresponding to S_2 is $\{\alpha_1(t)\beta_1(u) \mid t,u \in GF(q)\}$, and the root subgroup in $B_2^{\frac{1}{2}}(q)$ is of this form also.

PROPOSITION Let $t,u \in GF(q)$ and let $v=t^{2\theta+2}+u^{2\theta}+tu$. Then $v \neq 0$ if either $t \neq 0$ or $u \neq 0$, and each nonzero v occurs for exactly $q+1$ pairs (t,u).

Proof Define $f(u) = u^{2\theta}+u+1$. Comparing $f(u)$ and $f(u)^{\theta}$ and using the fact that $2\theta^2=1$ it is clear that f has no zeros in $GF(q)$. Now if $t \neq 0$

$$t^{2\theta+2}+u^{2\theta}+tu = t^{2\theta+2}f(t^{-1-2\theta}u)$$

which is nonzero and takes all nonzero values with equal frequency since $t^{2\theta+2}$ takes all nonzero values as t varies. Similarly if $t=0$,

$$t^{2\theta+2}+u^{2\theta}+tu = u^{2\theta}$$

and all nonzero values occur once.

We can now give the structural equation for the second kind of root subgroup. If $t \neq 0$ or $u \neq 0$ and if v is as defined in the proposition,

$$(S)\alpha(t)\beta(u)(S)^{-1}=\alpha(t^{1+2\theta}v^{-1}+uv^{-1})\beta(uv^{-2\theta})h(\chi)(S)\alpha(uv^{-1})\beta(tv^{-1})$$

where $\chi(a) = v^{1-2\theta}$, $\chi(b) = v^{2\theta-2}$ and S is the reflection. (In $B_2^{\frac{1}{2}}(q)$, $(S) = [(s_1)(s_2)]^2$; similarly in the other cases.)

Note that $B_2^1(q)$ trivially satisfies 3.9 since there is only one positive root subgroup. If $q\neq 2$, $F_4^1(q)$ also satisfies 3.9; we must prove that the derived group Z of X^1 contains the elements $\alpha_i(t)$, $\beta_i(t)$, $\gamma_i(t)$ ($i=2,3$ and 4), since these generate the non-fundamental positive root subgroups. The commutator relations that we make use of follow readily from the Chevalley commutator formula. Firstly, for each i,

$$[\alpha_i(t^{-1}),\ \alpha_i(t^{2\theta})] = \beta_i(t+1) \qquad (t\neq 0)$$

and since $q>2$ it follows that $\beta_i(t)\in Z$ for all t.

$$[\alpha_3(t),\ \alpha_1(1)] = \gamma_2(t)$$

$$[\alpha_4(t),\ \alpha_2(1)] = \gamma_3(t)$$

$$[\alpha_4(t),\ \alpha_3(1)] = \gamma_4(t)$$

and so $\gamma_i(t)\in Z$ for $i=2,3$ and 4.

$$[\alpha_2(t),\alpha_1(u)] = \gamma_2(tu^{2\theta+1})\gamma_3(t^{2\theta+1}u)\alpha_4(t^{2\theta}u)\alpha_3(tu^{2\theta})$$

$$[\alpha_1(t),\gamma_1(u)] = \alpha_2(tu)\alpha_4(t^{2\theta+1}u^{2\theta})\beta_4(t^{4\theta+3}u^{2\theta+2})\gamma_2(t^{2\theta+2}u)\ .\ \beta_3(t^{4\theta+3}u^{2\theta+1})$$

and hence

$$\alpha_4(t^{2\theta}u)\alpha_3(tu^{2\theta}) \in Z$$

$$\alpha_2(tu)\alpha_4(t^{2\theta+1}u^{2\theta}) \in Z$$

The latter of these two yields

$$\alpha_2(1)\alpha_4(1) \in Z$$

and

$$\alpha_2(1)\alpha_4(t) \in Z \qquad (t\neq 0)$$

in the cases $t=u=1$ and $u=t^{-1}$.

Hence $\alpha_4(t+1)\in Z$ if $t\neq 0$. It follows that $\alpha_4(t)\in Z$ for all t, and accordingly that $\alpha_2(t)\in Z$ and $\alpha_3(t)\in Z$ also.

(5) $G_2^1(q)$ $\qquad q = 3^{2n+1}$

$s_1 \overset{6}{\text{———}} s_2$ $\qquad \lambda(r_1)=1,\ \lambda(r_2)=3.$

In this case W^1 has order 2, generated by $(s_1s_2)^3$. Again 3.9 is trivially satisfied. The root subgroup is

$$\{\alpha(t)\beta(u)\gamma(v) \mid t,u,v \in GF(q)\}, \text{ where}$$

$$\alpha(t) = x_a(t^\theta)x_b(t)x_{a+b}(t^{\theta+1})x_{2a+b}(t^{2\theta+1})$$
$$\beta(t) = x_{a+b}(t^\theta)x_{3a+b}(t)$$
$$\gamma(t) = x_{2a+b}(t^\theta)x_{3a+2b}(t)$$

$(a=r_1,\ b=r_2)$.

The structural equation is

$$(S)\alpha(t)\beta(u)\gamma(v)(S)^{-1}=\alpha(x_1d^{-1})\beta(x_2d^{-3\theta})\gamma(x_3d^{-3\theta-1})(S)h(\chi).$$
$$\alpha(x_4d^{-1})\beta(x_5d^{-1})\gamma(x_6d^{-2})$$

where $\chi(a) = d^{2\theta-1}$, $\chi(b) = d^{3\theta-2}$, $(S) = [(s_1)(s_2)]^3$, and

$$d = u^{3\theta+1}+tvu+t^{3\theta+3}u-t^{6\theta+4}-tv^{3\theta}+v^2$$
$$x_1 = t^{3\theta}u^{3\theta}-t^{6\theta+3}-t^{3\theta+1}v+t^{3\theta+2}u-vu+v^{3\theta}$$
$$x_2 = -u^{3\theta+2}+t^2u^3+v^2u^{3\theta}-t^{6\theta+4}u^{3\theta}-t^{6\theta+3}v-tv^{3\theta}u^{3\theta}-v^{3\theta+1}-tvu^2-t^{3\theta+3}u^2$$
$$x_3 = u\,x_1x_2 + (tu-v)d^{3\theta}$$
$$x_4 = t^{6\theta+3}+v^{3\theta}+tu^2+vu$$
$$x_5 = -u^{3\theta}-tv-t^{3\theta+3}$$
$$x_6 = -t^{9\theta+6}-t^{6\theta+5}u-t^{6\theta+3}u^\theta+vu^{3\theta+1}-v^3-v^{3\theta}u^\theta-tv^2u+t^{3\theta+3}vu-t^2v^{3\theta}u-t^{3\theta+3}v^{3\theta}$$

Note that the transformation

$$t \to t$$
$$u \to u + t^{3\theta+1}$$
$$v \to v + ut + t^{3\theta+2}$$

transforms d to $u^{3\theta+1}-v^{3\theta}t+v^2-t^2u^2-t^{6\theta+4}$, which is zero only when $t=u=v=0$ (see [19] p.186). It is not true that each nonzero value of d occurs for the same number of triples (t,u,v). For example if $q=3$ then d takes the value -1 for 16 triples (t,u,v) and $+1$ for 10 triples.

CHAPTER 5

THE CHARACTERS DISCOVERED BY DAGGER

Let k be a field of characteristic zero which contains a pth root of unity, and let G be a group with a split BN pair of rank n at characteristic p. Let μ_i be a nontrivial linear character of $X_i^{s_i}$ $(=X_{-r_i})$ for $i=1,2,\ldots n$. Since we assume 3.9 it follows that

$$X^{w_J}/(X^{w_J})' \cong \prod_{i=1}^{n} X_i^{w_J}/(X_i^{w_J})'$$

where w_J is as defined in 2.5. (J is any subset of $\{1,2,\ldots n\}$). Now as a consequence of 3.6 the subgroups $\{X_i^{w_J} \mid i\in J\}$ equal the subgroups $\{X_{-r_i} \mid i\in J\}$ in some order (see 2.5); so we may define a linear character μ_J of X^{w_J} which coincides with μ_i on X_{-r_i} if $i\in J$ and is trivial on $X_i^{w_J}$ for $i\notin J$. Indeed μ_J is trivial on all root subgroups $X_{w_J(r)}$ for r positive and $r\notin\{r_i \mid i\in J\}$ since $X_{w_J(r)} = X_r^{w_J}$ is contained in the derived group of X^{w_J} if r is not fundamental. In the case $J = \{1,2,\ldots n\}$ we write simply "μ" for "μ_J".

Let us adopt the following notation: If A is a subgroup of G and α a linear character of A, let

$$\overline{A} = |A|^{-1}\Sigma x \qquad (x\in A)$$

and

$$A_\alpha = |A|^{-1}\Sigma\alpha(x^{-1})x \qquad (x\in A)$$

Throughout the rest of this thesis λ will be a linear character of B with kernel containing X. G_J will be the <u>parabolic subgroup</u> $\cup BwB$ $(w\in W_J)$ corresponding to the subset J of $\{1,2,\ldots n\}$.

<u>5.1 LEMMA</u> (i) The induced characters λ^{G_J} and $\mu_J^{G_J}$ have a unique common component, and it occurs with multiplicity one in each.

(ii) $(\lambda^G, \mu_J^G) = |W:W_J|$

Proof Note first that

$$G_J = w_J^{-1} G_J = w_J^{-1}(\cup XwB) \qquad \text{(union over } w\in W_J)$$
$$= \cup X^{w_J} wB \qquad (w\in W_J)$$

and similarly $G = \cup X^{w_J} wB$ $(w\in W)$

For $w\in W$ let $\lambda^{w^{-1}}$ be the character of wBw^{-1} defined by $\lambda^{w^{-1}}(g) = \lambda(w^{-1}gw)$. The restriction of this to $X^{w_J} \cap wBw^{-1}$ is the 1-character, since λ is trivial on elements of p-power order. Now the restriction of μ_J to $X^{w_J} \cap wBw^{-1}$ is the 1-character if and only if $w^{-1}(r_i) > 0$ for all $i\in J$. For by 3.6 and 3.8 it is clear that $X^{w_J} \cap wBw^{-1}$ is a product of root subgroups, and $X_{-r_i} \subseteq wBw^{-1}$ if and only if $w^{-1}X_{-r_i}w \subseteq B$; i.e. $-w^{-1}(r_i) > 0$. It follows from 2.5 that there is exactly one w in each $W_J \backslash W$ coset such that μ_J restricted to $X^{w_J} \cap wBw^{-1}$ is the 1-character, and the inner product of $\lambda^{w^{-1}}$ and μ_J over this subgroup is 1 for these w and 0 for others. So, by a well known theorem of Mackey,

$$(\mu_J^{G_J}, \lambda^{G_J}) = \Sigma(\mu_J, \lambda^{w^{-1}}) \qquad (w\in W_J)$$
$$= 1 \text{ (since only the term for } w=1 \text{ contributes)}$$

and $$(\mu_J^G, \lambda^G) = \Sigma(\mu_J, \lambda^{w^{-1}}) \qquad (w\in W)$$
$$= |W:W_J|$$

5.2 LEMMA $B_\lambda X_{\mu_J}^{w_J}$ is a nonzero multiple of a primitive idempotent in kG_J.

Proof By the proof of 5.1 the only double coset $X^{w_J} wB$ for which μ_J and $\lambda^{w^{-1}}$ agree corresponds to the case $w=1$. Hence lemma 1 of [1] applies. (See also lemma 2 of [1] and its proof).

The irreducible character of G_J corresponding to this idempotent (i.e. the common component of λ^{G_J} and $\mu_J^{G_J}$) will be called χ_J or $\chi_J(\lambda,\mu_1,\mu_2,\ldots\mu_n)$. In the case $J = \{1,2,\ldots n\}$ we obtain an irreducible character of G (and we will write "χ" for "χ_J"). These characters, corresponding to the various λ, were discovered by Dagger [9].

For each $w \in W$ we define a linear character λ^w of B whose kernel contains X by setting

$$\lambda^w(h) = \lambda(whw^{-1}) \qquad \text{for all } h \in H.$$

Using λ^w in place of λ in the above construction yields a character of G_J which will be called $\chi_{J\,w}$. Theorem B follows directly from 5.1.

CHAPTER 6

THE HECKE ALGEBRA $\bar{X}kG\bar{X}$

Continuing with the same notation, define (for each $w\in W$)

$$q_w = |X:X_w| \qquad \text{(the index of } X_w \text{ in } X)$$

In particular for $i=1,2,\ldots n$ define

$$q_i = q_{s_i} = |X:X_{s_i}| = |X_i| \qquad \text{(see 3.8)}$$

For each $w\in W$ define also

$$\alpha_w = q_w\bar{X}(w)\bar{X}$$

$$\beta_w = q_w B_\lambda(w)B_\lambda$$

$$\hat{\alpha}_w = q_w\bar{X}(w)^{-1}\bar{X}$$

$$\hat{\beta}_w = q_w B_\lambda(w)^{-1}B_\lambda$$

Let $S = \{w\in W \mid \lambda^w=\lambda\}$.

6.1 LEMMA The set $\{\alpha_w h \mid w\in W, h\in H\}$ is a basis for $\bar{X}kG\bar{X}$, and the set $\{\beta_w \mid w\in S\}$ is a basis for $B_\lambda kGB_\lambda$. $\{\beta_w \mid w\in W_J\cap S\}$ is a basis for $B_\lambda kG_J B_\lambda$.

Proof This kind of result is well known (see theorem 2.2 of [7] for example). Since H normalizes X

$$\bar{X}h = h\bar{X} \qquad \text{for all } h\in H,$$

and so the elements $\alpha_w h$ are indeed in $\bar{X}kG\bar{X}$. The cosets $\bar{X}(w)h\bar{X}$ $(w\in W, h\in H)$ are all distinct, as an easy consequence of the split BN pair axioms and the Bruhat theorem.

6.2 PROPOSITION For $v,w\in W$,

$$B_{\lambda^v}\alpha_w = \alpha_w B_{\lambda^{vw}}.$$

If $u\in W$ such that $\lambda^u \neq \lambda^{vw}$ then

$$B_{\lambda^v}\alpha_w B_{\lambda^u} = 0$$

(In particular $\beta_w=0$ for $w\notin S$)

The proof of 6.2 is straightforward and is omitted.

Our interest in Hecke algebras is motivated by theorems such as the next two:

<u>6.3 THEOREM</u> If e and f are idempotents in kG affording characters φ and ψ respectively then

$$\mathrm{Hom}_{kG}(kGe, kGf) \cong ekGf$$

and the dimension of these vector spaces is

$$(\varphi,\psi) = |G|^{-1}\Sigma\varphi(x^{-1})\psi(x) \qquad (x\in G)$$

This theorem is well known and holds for an arbitrary group G, although in this thesis it will only be applied to groups with BN pairs. Theorem 6.4 is also a general result, adapted to apply to the parabolic subgroups G_J, where J is an arbitrary subset of $\{1,2,...n\}$.

<u>6.4 THEOREM</u> (Curtis and Fossum [7, Cor.1.2 and 2.5]).
If ψ is an irreducible character of kG_J such that $(\psi,\lambda^{G_J}) = 1$ (i.e. ψ occurs with multiplicity 1 in the induced character λ^{G_J}), then the restriction of ψ to $B_\lambda kG_J B_\lambda$ is a homomorphism

$$\theta\colon B_\lambda kG_J B_\lambda \to k.$$

Conversely every such homomorphism θ is the restriction of a unique irreducible character ψ of kG_J such that $(\psi,\lambda^{G_J}) = 1$. Under these circumstances,

$$e = \psi(1)[G_J:B]^{-1}\Sigma q_w^{-1}\theta(\hat{\beta}_w)\beta_w \qquad (w\in W_J\cap S)$$

is a primitive idempotent in kG_J such that kGe affords ψ. It is the only such idempotent in $B_\lambda kG_J B_\lambda$. Moreover,

$$1 = \theta(e) = \psi(1)[G_J:B]^{-1}\Sigma q_w^{-1}\theta(\hat{\beta}_w)\theta(\beta_w) \qquad (w\in W_J\cap S)$$

and for any $\beta\in B_\lambda kG_J B_\lambda$,

$$\beta e = \theta(\beta)e = e\beta.$$

<u>Proof</u> $\beta e = \theta(\beta)e$ for all $\beta\in B_\lambda kG_J B_\lambda$ is a consequence of the fact that $B_\lambda kG_J B_\lambda$ is one-dimensional (by 6.3). The only other point not proved explicitly by Curtis

and Fossum is uniqueness of e. But if $f \in B_\lambda kG_J B_\lambda$ is another primitive idempotent affording ψ then

$$f = \theta(e)f = ef = \theta(f)e$$

and so $e=f$.

<u>6.5 LEMMA</u> If $w \in W$, $1 \le i \le n$ and $\ell(s_i w) = \ell(w)+1$ then $q_{s_i w} = q_i q_w$ and there exists $h \in H$ with $\alpha_{s_i w} = \alpha_{s_i} \alpha_w h$. If $w_1 w_2 \dots w_m$ is a reduced expression for w (as in 2.3) then there exists $h \in H$ such that

$$\alpha_w = \alpha_{w_1} \alpha_{w_2} \dots \alpha_{w_m} h$$

<u>Proof</u> By 3.8 (3) and (2)

$$\begin{aligned} q_{s_i w} &= |X_{w_0 s_i w}| \\ &= |X_{w_0 w} X_{w^{-1}(r_i)}| \end{aligned}$$

Now since $\ell(s_i w) = \ell(w)+1$, $w^{-1}(r_i)$ is a positive root, and it is negatived by $w_0 w$. So by 3.8(2),

$$X_{w^{-1}(r_i)} \le X_{w_0(w_0 w)} = X_w .$$

Now 3.8 (3) gives

$$\begin{aligned} q_{s_i w} &= |X_{w_0 w}|\,|X_i^w| \\ &= q_w q_{s_i} . \end{aligned}$$

Let h be an arbitrary element of H. Then

$$\begin{aligned} \alpha_{s_i} \alpha_w h &= q_{s_i} (\overline{X}(s_i)\overline{X}) q_w (\overline{X}(w)\overline{X}) h \\ &= q_{s_i w} (\overline{X}(s_i)\overline{X}_{s_i} \overline{X}_i(w)\overline{X}) h \quad \text{(by 3.8(3))}. \end{aligned}$$

But s_i normalizes X_{s_i} and $w^{-1} X_i w = X_{w^{-1}(r_i)} \le X$.

Thus $\alpha_{s_i} \alpha_w h = q_{s_i w} \overline{X}(s_i)(w)h\overline{X}$.

If h is chosen so that $(s_i)(w)h = (s_i w)$ it follows that

$$\alpha_{s_i} \alpha_w h = \alpha_{s_i w} .$$

The other assertion of the lemma follows by induction on $\ell(w)$.

__6.6 LEMMA__ The elements $\alpha_w h$ $(w \in W,\ h \in H)$ have inverses in $\bar{X}kG\bar{X}$.

__Proof__ For each i $(1 \leq i \leq n)$

$$\begin{aligned}\alpha_{s_i}^2 &= q_i^2 \bar{X}(s_i)\bar{X}(s_i)\bar{X} \\ &= q_i^2 \bar{X}(s_i)\bar{X}_i(s_i)\bar{X} \\ &= q_i \bar{X}(s_i)^2\bar{X} + q_i \Sigma \bar{X}(s_i)x(s_i)\bar{X} \\ &= q_i (s_i)^2\bar{X} + q_i \Sigma \bar{X}h_i(x)(s_i)^3\bar{X}\end{aligned}$$

where the summation is over the non-identity elements of X_i, and $h_i(x)$ is as defined in 3.10. Note that $(s_i)^2 \in H$. It follows that

$$\alpha_{s_i}^2 (s_i)^{-2} = q_i \bar{X} + H_i \alpha_{s_i}$$

where $H_i = \Sigma h_i(x)$ $(x \in X_i,\ x \neq 1)$.

Now $\bar{X}$ is the identity element of $\bar{X}kG\bar{X}$, and so

$$q_i^{-1}(\alpha_{s_i}(s_i)^{-2} - H_i\bar{X})$$

is an inverse for α_{s_i} in $\bar{X}kG\bar{X}$. We now use induction on $\ell(w)$ to show that each $\alpha_w h$ has an inverse.

For $\ell(w) = 0$, $\alpha_w h = \bar{X}h$ and the inverse is $\bar{X}h^{-1}$.

For $\ell(w) \geq 1$ there exists i with $w = s_i v$ and $\ell(v) = \ell(w) - 1$. The inductive hypothesis permits the assumption that each $\alpha_v f$ $(f \in H)$ has an inverse, and then for appropriate choice of f, 6.5 gives

$$\alpha_w h = \alpha_{s_i}\alpha_v f h$$

which is a product of elements with inverses, and so has an inverse.

(Results like 6.5 and 6.6 are well known. See [21], for example).

It will be convenient to adopt the notation "$\alpha \sim \beta$" for elements $\alpha, \beta \in kG$ to mean that α is a nonzero scalar multiple of β.

<u>6.7 LEMMA</u> For all $w \in W_J$, $\alpha_w B_\lambda X_{\mu_J}^{w_J} \sim B_{\lambda^v} X_{\mu_J}^{w_J}$ (where $v = w^{-1}$) and $X_{\mu_J}^{w_J} B_\lambda \alpha_w \sim X_{\mu_J}^{w_J} B_{\lambda^w}$

<u>Proof</u> The idempotents B_λ and $X_{\mu_J}^{w_J}$ afford the characters λ^{G_J} and $\mu_J^{G_J}$ of G_J, and $(\lambda^{G_J}, \mu_J^{G_J}) = 1$ by 5.1. Hence $B_\lambda kG_J X_{\mu_J}^{w_J}$ is one-dimensional (by 6.3). Similarly $B_{\lambda^v} kG_J X_{\mu_J}^{w_J}$ is one-dimensional. Therefore

$$\alpha_w B_\lambda X_{\mu_J}^{w_J} = B_{\lambda^v} \alpha_w B_\lambda X_{\mu_J}^{w_J} \qquad \text{(by 6.2)}$$

must be a scalar multiple of $B_{\lambda^v} X_{\mu_J}^{w_J}$. The scalar must be nonzero since α_w has an inverse. The proof of the second part is similar.

<u>6.8 LEMMA</u> $B_\lambda X_{\mu_J}^{w_J} B_\lambda X_\mu^{w_o} B_\lambda \sim B_\lambda X_\mu^{w_o} B_\lambda$

<u>Proof</u> Let Y be the product of the positive root subgroups in X^{w_J} and Z the product of the negative ones. Let

$$\alpha = |Z|^{-1} \Sigma \mu_J (x^{-1}) x \qquad (x \in Z)$$

$$= |Z|^{-1} \Sigma \mu (x^{-1}) x \qquad (x \in Z)$$

Then $X_{\mu_J}^{w_J} = \overline{Y}\alpha$ and $\alpha X_\mu^{w_o} = X_\mu^{w_o}$.

Hence $B_\lambda X_{\mu_J}^{w_J} = B_\lambda \alpha$, and so $B_\lambda \alpha B_\lambda \alpha \sim B_\lambda \alpha$ (by 5.2)

(Note in passing that $B_\lambda \alpha B_\lambda \neq 0$).

Therefore $B_\lambda \alpha B_\lambda \alpha X_\mu^{w_o} \sim B_\lambda \alpha X_\mu^{w_o}$, and the result follows.

<u>6.9 COROLLARY</u>. Let χ_J and χ be the characters of G_J and G defined in chapter 5, and let θ_J and θ be the corresponding homomorphisms of $B_\lambda kG_J B_\lambda$ and $B_\lambda kGB_\lambda$ into k. Then θ_J is the restriction of θ.

<u>Proof</u> By the note in the proof of 6.8, $B_\lambda X_{\mu_J}^{w_J} B_\lambda$ is nonzero, and so is a nonzero multiple of e, the primitive idempotent in $B_\lambda kG_J B_\lambda$ affording χ_J (see 6.4). Similarly $B_\lambda X_\mu^{w_o} B_\lambda \sim f$, the primitive idempotent in $B_\lambda kGB_\lambda$ affording χ. By 6.8 $ef \sim f$, and so $\theta(e) = 1$ (since $e^2=e$). For $w \in W_J \cap S$,

$$\theta_J(\beta_w)ef = (\beta_w e)f$$

$$= \beta_w(ef)$$

$$= \beta_w f$$

$$= \theta(\beta_w) f$$

$$= \theta(\beta_w) ef$$

and so $\qquad \theta_J(\beta_w) = \theta(\beta_w)$

(6.4, 6.7, 6.8 and 6.9 will also be used with λ replaced by any of the characters λ^w $(w \in W)$).

CHAPTER 7

SOME MORE CHARACTERS

7.1 LEMMA (i) For a fixed $w \in W$ the elements $\{\alpha_{vw}B_\lambda \mid v \in W_J\}$ and $\{\alpha_v \alpha_w B_\lambda \mid v \in W_J\}$ span the same space, and similarly $\{B_\lambda \alpha_{wv} \mid v \in W_J\}$ and $\{B_\lambda \alpha_w \alpha_v \mid v \in W_J\}$ span the same space.

(ii) $kBwG_J = kBwW_J X$

Proof (ii) is a standard result about BN pairs. For (i) note first that since α_w has an inverse in $\overline{X}kG\overline{X}$ the two spaces have the same dimension, namely $|W_J|$. It remains to prove that each $\alpha_{vw}B_\lambda$ $(v \in W_J)$ can be written in the form $\alpha\alpha_w B_\lambda$ for some $\alpha \in \overline{X}kG_J\overline{X}$.

Use induction on $\ell(v)$. The case $\ell(v) = 0$ is trivial.

Suppose $v = s_i u$, $i \in J$, $\ell(v) = \ell(u)+1$.

Then $\alpha_{uw}B_\lambda = \gamma\alpha_w B_\lambda$ for some $\gamma \in \overline{X}kG_J\overline{X}$, by the inductive hypothesis.

If $\ell(vw) > \ell(uw)$ then
$$\alpha_{vw}B_\lambda = \alpha_{suw} \sim \alpha_s \alpha_{uw}B_\lambda$$
(where $s=s_i$), while if $\ell(vw) < \ell(uw)$ then $\alpha_{uw}B_\lambda \sim \alpha_s \alpha_{vw}B_\lambda$. Choosing α to be the appropriate scalar multiple of $\alpha_s\gamma$ or $\alpha_s^{-1}\gamma$ it follows that $\alpha \in \overline{X}kG_J\overline{X}$ and $\alpha_{vw}B_\lambda = \alpha\alpha_w B_\lambda$.

7.2 COROLLARY Let $w \in W$, $J \subseteq \{1,2,\ldots n\}$. Then
$$B_\lambda kG_J B_\lambda \alpha_w = B_\lambda k(W_J \cap S)wB_{\lambda w}$$
(i.e. the space spanned by $B_\lambda \alpha_{vw} B_{\lambda w}$ for $v \in W_J \cap S$). Similarly,
$$\alpha_w B_\lambda kG_J B_\lambda = B_{\lambda u}\, kw(W_J \cap S)B_\lambda \qquad \text{(where } u=w^{-1}\text{)}$$

Proof $\quad B_\lambda kG_J B_\lambda \alpha_w = B_\lambda \overline{X}kG_J \overline{X}\alpha_w B_{\lambda w}$ (by 6.2)

and this is the space spanned by $\{B_\lambda \alpha_v \alpha_w B_{\lambda w} \mid v\in W_J\}$.

This is the same as the span of $\{B_\lambda \alpha_{v\,w} B_{\lambda w} \mid v\in W_J\}$.

However $B_\lambda \alpha_{v\,w} B_{\lambda\,w} = 0$ unless $\lambda^{v\,w}=\lambda^w$; i.e. unless $v\in S$.

Thus the basis consists of $\alpha_{v\,w}B_{\lambda w}$, $v\in W_J \cap S$.

7.3 COROLLARY Let $v,w,u \in W$ and $J,K \subseteq \{1,2,\ldots n\}$.

Then $B_{\lambda^v} kG_J B_{\lambda^v} \alpha_w B_{\lambda^u} kG_K B_{\lambda^u} = B_{\lambda^v} k(W_J \cap S^v)w(W_K \cap S^u)B_{\lambda^u}$

Proof First observe that both sides are zero unless $\lambda^{v\,w}=\lambda^u$; that is, unless $w\in v^{-1}Su$. (Equivalently, $(W_J \cap S^v)w(W_K \cap S^u) \subseteq v^{-1}Su$). Assuming $w \in v^{-1}Su$, 7.2 with λ^v replacing λ gives

$$B_{\lambda^v} kG_J B_{\lambda^v} \alpha_w B_{\lambda^u} kG_K B_{\lambda^u} = B_{\lambda^u} k(W_J \cap S^v)wB_{\lambda^u} kG_K B_{\lambda^u}$$

and this is the sum of the spaces $\alpha_{t\,w}B_{\lambda^u} kG_K B_{\lambda^u}$ for $t\in W_J \cap S^v$. But $\alpha_{t\,w}B_{\lambda^u} kG_K B_{\lambda^u}$ equals $B_{\lambda^v} tw\,k(W_K \cap S^u)B_{\lambda^u}$, and the sum of these as t runs through elements of $W_J \cap S^v$ is

$$B_{\lambda^v} k(W_J \cap S^v)w(W_K \cap S^u)B_{\lambda^u}.$$

7.4 LEMMA Let $J,K \subseteq \{1,2,\ldots n\}$ and let $v,w\in W$. Let e be the primitive idempotent in $B_{\lambda^v} kG_J B_{\lambda^v}$ which affords the character $\chi_{J\,v}$, and f the primitive idempotent in $B_{\lambda^w} kG_K B_{\lambda^w}$ which affords χ_{Kw}. Let $t_1, t_2, \ldots t_m$ be representatives of the orbits of $v^{-1}Sw$ under premultiplication by elements of $W_J \cap S^v$ and postmultiplication by elements of $W_K \cap S^w$. Then $ekGf$ has basis $\{e\alpha_{t_i} f \mid i=1,2,\ldots m\}$.

Proof $B_{\lambda^v} kGB_{\lambda^w}$ has basis $\{B_{\lambda^v} \alpha_u B_{\lambda^w} \mid u\in v^{-1}Sw\}$

$$= \bigcup_{i=1}^{m} \{B_{\lambda^v} \alpha_u B_{\lambda^w} \mid u\in (W_J \cap S^v)t_i(W_K \cap S^w)\}$$

Hence $B_{\lambda^v} kGB_{\lambda^w}$ is the direct sum of the spaces

$$B_{\lambda^v} k(W_J \cap S^v)t_i(W_K \cap S^w)B_{\lambda^w} \qquad i=1,2,\ldots m.$$

Now by 7.3,

$B_{\lambda^v} k(W_J \cap S^v) t_i (W_K \cap S^w) B_{\lambda^w} = B_{\lambda^v} kG_J B_{\lambda^v} \alpha_{t_i} B_{\lambda^w} kG_K B_{\lambda^w}$
and this contains the element $e\alpha_{t_i} f$. Therefore the $e\alpha_{t_i} f$ are linearly independent if they are nonzero. But for any $t \in v^{-1}Sw$,

$$B_{\lambda^v} X_\mu^{w_o} B_{\lambda^v} e\alpha_t f B_{\lambda^w} X_\mu^{w_o} B_{\lambda^w} \alpha_{t^{-1}}$$

$$\sim (B_{\lambda^v} X_\mu^{w_o} B_{\lambda^v})(\alpha_t B_{\lambda^w} X_\mu^{w_o} B_{\lambda^w} \alpha_{t^{-1}}) \qquad \text{(by 6.8)}$$

$$\sim (B_{\lambda^v} X_\mu^{w_o} B_{\lambda^v})(B_{\lambda^v} X_\mu^{w_o} B_{\lambda^v}) \qquad \text{(by 6.7,}$$

making use of the fact that $\lambda^{wt^{-1}} = \lambda^v$)
and this is nonzero, by 5.2. Hence $e\alpha_t f \neq 0$.
It remains to prove that the $e\alpha_{t_i} f$ span $ekGf$.
It was shown above that $B_{\lambda^v} kGB_{\lambda^w}$ is the sum of the spaces $B_{\lambda^v} kG_J B_{\lambda^v} \alpha_{t_i} B_{\lambda^w} kG_K B_{\lambda^w}$, $i=1,2,\ldots m$. Therefore $ekGf$ $= eB_{\lambda^v} kGB_{\lambda^w} f$ is the sum of

$$\{eB_{\lambda^v} kG_J B_{\lambda^v} \alpha_{t_i} B_{\lambda^w} kG_K B_{\lambda^w} f \mid i=1,2,\ldots m\}$$

$$= \{ke\alpha_{t_i} f \mid i=1,2,\ldots m\}$$

since $eB_{\lambda^v} kG_J B_{\lambda^v} = ke$ and $B_{\lambda^w} kG_K B_{\lambda^w} f = kf$ (see 6.4)

<u>7.5 THEOREM</u> (i) Let $J,K \subseteq \{1,2,\ldots n\}$ and $v,w \in W$. Then $(\chi_{J^v}^G, \chi_{K^w}^G)$ equals the number of $(W_J^{v^{-1}} \cap S)\backslash S/(W_K^{w^{-1}} \cap S)$ double cosets.

(ii) For each subset J of $\{1,2,\ldots n\}$ and each $v \in W$ define $\delta(S,J,v)$ to be the character of S induced from the 1-character of $W_J^{v^{-1}} \cap S$. Then

$$\chi_{J^v}^G \mapsto \delta(S,J,v)$$

is an isometry between the inner product spaces generated by these characters.

<u>Proof</u> (i) With e,f and $t_1,t_2,\ldots t_m$ as in 7.4 the module kGe affords the character $\chi_{J^v}^G$ of G and kGf affords $\chi_{K^w}^G$. Hence 6.3 gives

$$(\chi_{J^v}^G, \chi_{K^w}^G) = \dim ekGf = m$$

But it is clear that $\{vt_i w^{-1} | i=1,2,\ldots m\}$ is a set of representatives of the $(W_J^{v^{-1}} \cap S)\backslash S/(W_K^{w^{-1}} \cap S)$ cosets, and so m also equals the number of these cosets.

(ii) This is merely a restatement of (i) since

$$m = (\delta(S,J,v), \delta(S,K,w))$$

7.6 DEFINITION Let $J \subseteq \{1,2,\ldots n\}$ and let V be a set of representatives of the $S\backslash W/W_J$ cosets. Define

$$\varphi_J = \varphi_J(\mu_1,\mu_2,\ldots\mu_n) = \Sigma\chi_{J\,v}^G \qquad (v \in V)$$

Note that the choice of coset representatives is immaterial. If $t \in S$ it is obvious that $\chi_{J\,v} = \chi_{J\,t\,v}$ (since $\lambda^v = \lambda^{t\,v}$). Furthermore if $w \in W_J$ and $u=w^{-1}$,

$$\alpha_u B_{\lambda^v} X_{\mu_J}^{w_J} \sim B_{\lambda^{v\,w}} X_{\mu_J}^{w_J} \qquad (6.7)$$

so that the right modules $B_{\lambda^v} X_{\mu_J}^{w_J} kG$ (which affords $\chi_{J\,v}^G$) and $B_{\lambda^{v\,w}} X_{\mu_J}^{w_J} kG$ (which affords $\chi_{J\,v\,w}^G$) are isomorphic.

7.7 LEMMA If $J,K \subseteq \{1,2,\ldots n\}$ and δ_J, δ_K are the characters of W induced from the 1-characters of W_J and W_K then $(\varphi_J, \varphi_K) = (\delta_J|_S, \delta_K|_S)$ (where $\delta_J|_S$ is the restriction of δ_J to S, etc.)

Proof This is immediate from 7.5 (ii) and 7.6 since by Mackey's theorem (with V as in 7.6)

$$\delta_J|_S = \Sigma\ \delta(S,J,v) \qquad v \in V.$$

We can now prove theorem C:

7.8 THEOREM $\Sigma(-1)^{|J|}\varphi_J$ is an irreducible character of G, and it occurs with multiplicity one in λ^G (the summation is over all subsets J of $\{1,2,\ldots n\}$).

Proof From 2.8, $\Sigma(-1)^{|J|}\delta_J|_S = \varepsilon|_S$ is an irreducible character of S, and so

$$1 = (\Sigma(-1)^{|J|}\delta_J\big|_S,\ \Sigma(-1)^{|K|}\delta_K\big|_S)$$

$$= (\Sigma(-1)^{|J|}\varphi_J,\ \Sigma(-1)^{|K|}\varphi_K) \qquad \text{(by 7.7)}$$

so that $\pm\Sigma(-1)^{|J|}\varphi_J$ is irreducible.

Let L be the empty subset of $\{1,2,\ldots n\}$. Then $G_L = B$ and λ is the only component of λ^{G_L}. Thus $\lambda = \chi_L$.

$$(\Sigma(-1)^{|J|}\varphi_J,\lambda^G) = \Sigma(-1)^{|J|}(\chi^G_{J\,v},\chi^G_L)$$

where J runs through subsets of $\{1,2,\ldots n\}$ and v through a set of representatives of the $S\backslash W/W_J$ cosets. But by 7.5

$$(\chi^G_{J\,v},\chi^G_L) = (\delta(S,J,v),\ \delta(S,L,1))$$

and since L is empty $\delta(S,L,1)$ is ρ, the character of the regular representation of S. Therefore

$$(\Sigma(-1)^{|J|}\varphi_J,\lambda^G) = \Sigma(-1)^{|J|}(\delta(S,J,v),\rho)$$
$$= (\Sigma(-1)^{|J|}\delta_J\big|_S,\rho)$$
$$= (\varepsilon\big|_S,\rho)$$
$$= 1 \text{ since } \varepsilon\big|_S \text{ is}$$

a linear character of S.

The character defined in 7.8 will be called "$\zeta(\lambda)$" or "$\zeta(\lambda,\mu_1,\mu_2,\ldots\mu_n)$" since as well as depending on λ it also depends on the characters μ_i of $X_{\circ r_i}$ $(i=1,2,\ldots n)$ that have been fixed throughout. The dependence on the μ_i will not be investigated here, but note that if λ and λ' are two linear characters of B such that λ' is not of the form λ^w $(w\in W)$ then $\zeta(\lambda) \neq \zeta(\lambda')$, since $(\lambda^G,\lambda'^G) = 0$. It can also be seen that $\zeta(\lambda) = \chi(\lambda)$ if and only if ε is trivial on S, since

$$(\zeta(\lambda),\chi(\lambda)) = \Sigma(-1)^{|J|}(\varphi_J,\chi(\lambda)) \qquad J \subseteq \{1,2,\ldots n\}$$
$$= \Sigma(-1)^{|J|}(\delta_J\big|_S,1)$$
$$= (\varepsilon\big|_S,1).$$

(Thus $(\zeta(\lambda),\mu^G) = 0$ unless ε is trivial on S).

CHAPTER 8

ON THE DEGREES OF COMPONENTS OF 1_X^G

8.1 THEOREM If σ is any component of λ^G

$$(\alpha, \mu_J) = (\sigma, \varphi_J)$$

where α is the restriction of σ to X^{W_J}.

Proof If $SvW_J \neq SwW_J$ then $\lambda^v \neq (\lambda^w)^u$ for any $u \in W_J$ and so the characters of G_J induced from λ^v and λ^w have no common component. Hence $\chi_{J\,v} \neq \chi_{J\,w}$. Now if V is a set of representatives of the $S\backslash W/W_J$ cosets then the characters $\{\chi_{J\,v} | v \in V\}$ are distinct, and components of $\mu_J^{G_J}$ (by their definition: see 5.1). Hence

$$\mu_J^{G_J} - \Sigma\chi_{J\,v} \qquad (v \in V)$$

is a proper character (i.e. a positive integral combination of irreducible characters). Inducing to G and using 7.7

$$\mu_J^G - \varphi_J$$

is a proper character of G. Now

$$(\varphi_J, \lambda^G) = (\delta_J |_S, \rho)$$

where ρ is the character of the regular representation of S (c.f. proof of 7.8)

$$= (\delta_J, \delta)$$

where δ is the character of the regular representation of W.

Therefore $(\varphi_J, \lambda^G) = |W:W_J| = (\mu_J^G, \lambda^G)$ (by 5.1)

and so $(\lambda^G, \mu_J^G - \varphi_J) = 0$

Since $\mu_J^G - \varphi_J$ is a proper character it follows that for all components σ of λ^G,

$$(\sigma, \mu_J^G - \varphi_J) = 0$$

and the result now follows by Frobenius reciprocity.

(The proof given for 8.1 is an improvement of the author's original proof, given in [13], and is based on a method used in [11]).

8.2 THEOREM If σ is any component of 1_X^G other than one of the $\zeta(\lambda)$ (for some λ) then the degree of σ is divisible by p.

Proof The character μ_J, and hence the characters $\chi_{J\,w}$, depend on the values of μ_i for $i \in J$ and not on μ_i for $i \notin J$. Now for any choice of λ,

$$0 = (\sigma, \sum_{\mu_i} \zeta(\lambda, \mu_1, \mu_2 \ldots \mu_n))$$

(where the summation is over all possible choices for the characters $\mu_1, \mu_2, \ldots \mu_n$)

$$= (\sigma, \sum_{\mu_i} \sum_J (-1)^{|J|} \varphi\ (\mu_1, \mu_2 \ldots \mu_n))$$

$$= (\sigma, \sum_J \Sigma_1 \Sigma_2 (-1)^{|J|} \varphi_J\ (\mu_1, \mu_2 \ldots \mu_n))$$

(where Σ_1 is over μ_i for $i \in J$ and Σ_2 is for $i \notin J$)

$$\equiv (\sigma, \sum_J \Sigma_1 (-1)^{n-|J|} (-1)^{|J|} \varphi_J\ (\mu_1, \mu_2 \ldots \mu_n)) \pmod p$$

since the number of nontrivial linear characters of X_{-r_i} for $i \notin J$ is congruent to -1 modulo p, and all give the same value for φ_J. Therefore

$$0 \equiv \sum_J \Sigma_1 (\alpha, \mu_J (\mu_1, \mu_2, \ldots \mu_n)) \qquad \text{(as in 8.1)}$$

$$= \sum_J \Sigma\ (\alpha, \mu_J)$$

where the second summation is over all linear characters μ_J of X^{w_J} which are nontrivial on exactly those root subgroups X_{-r_i} for $i \in J$. But

$$\Sigma(\alpha, \mu_J) = \Sigma(\sigma|_X, \mu_J^{w_J})$$

where $\mu_J^{w_J}$ is the character of X defined by $\mu_J^{w_J}(x) = \mu_J(x^{w_J})$, and we see that $\mu_J^{w_J}$ runs through all linear characters of X

nontrivial on exactly those root subgroups X_{r_i} $i \in J$.

Thus $0 \equiv \sum_J \sum_{\mu_J} (\sigma|_X, \mu_J^{w_J})$ (mod p)

$= \sum_\mu (\sigma|_X, \mu)$

where μ runs through all linear characters of X. But if the linear character μ of X occurs with multiplicity m_μ in $\sigma|_X$ then the degree of σ is congruent to $\sum_\mu m_\mu$, since nonlinear characters have degree divisible by p. Therefore

$$\text{degree } \sigma \equiv \sum_\mu (\sigma|_X, \mu) \equiv 0 \pmod p.$$

8.1 and 8.2 are theorems D and E respectively. The next lemmas prepare for theorem H.

<u>8.3 LEMMA</u> Let $i,j \in \{1,2,...n\}$ and $w \in W$ such that $w(r_i) = r_j$. Then $\alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1} = \overline{X}(s_j)^{-1}\overline{X}(s_j)\overline{X}$

<u>Proof</u> $w(r_i) = r_j > 0$ and so $\overline{X}(w)\overline{X}(s_i)\overline{X} = \overline{X}(w)(s_i)\overline{X}$.

Further, by 3.6, $(w)X_{-r_i} = X_{-r_j}(w)$

Hence $\overline{X}(w)\overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}$

$= \overline{X}(w)(s_i)^{-1}\overline{X}_i(s_i)\overline{X}$

$= \overline{X}(s_j)^{-1}\overline{X}_j(s_j)(w)\overline{X}$

$= \overline{X}(s_j)^{-1}\overline{X}(s_j)\overline{X}(w)\overline{X}$

and so $\alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X} = \overline{X}(s_j)^{-1}\overline{X}(s_j)\overline{X}\alpha_w$

<u>8.4 LEMMA</u> Let θ be a linear representation of $B_\lambda kGB_\lambda$, and let r be any root. Let $r = w(r_i)$ for some $w \in W$ and $i \in \{1,2,...n\}$. Then

$$\theta_r = q_i\,\theta(B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda)$$

depends only on the root r and not on the choice of w and i.

<u>Proof</u> $q_i = |X_i| = |X_r|$ depends only on r. Suppose that $w(r_i) = v(r_j)$ $v,w \in W$ $1 \le i, j \le n$. Let $u = v^{-1}w$.

By 8.3,

$$\theta(B_\lambda \alpha_v \overline{X}(s_j)^{-1}\overline{X}(s_j)\overline{X}\alpha_v^{-1}B_\lambda)$$

$$= \theta(B_\lambda \alpha_v \alpha_u \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_u^{-1}\alpha^{-1} \; B_\lambda)$$

$$= \theta(B_\lambda \alpha_v \alpha_u \alpha_w^{-1}\alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}\alpha_w\alpha_u^{-1}\alpha_v^{-1}B_\lambda)$$

$$= \theta(B_\lambda \alpha_v \alpha_u \alpha_w^{-1}B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda \alpha_w\alpha_u^{-1}\alpha_v^{-1}B_\lambda)$$

(using the fact that $B_\lambda \alpha_v = \alpha_v B_{\lambda^v}$, etc.)

$$= \theta(\beta)\theta(B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda)\theta(\beta^{-1})$$

(where $\beta = B_\lambda \alpha_v \alpha_u \alpha_w^{-1} B_\lambda$)

$$= \theta(B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda)$$

<u>8.5 LEMMA</u> If r is any root and if $v \in S$ then $\theta_r = \theta_{v(r)}$.

<u>Proof</u> Let $w \in W$, $1 \le i \le n$ such that $w(r_i) = r$.

Then $\theta_r = q_i\, \theta(B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda)$

and $\theta_{v(r)} = q_i\, \theta(B_\lambda \alpha_{vw} \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_{vw}^{-1}B_\lambda)$

$= q_i\, \theta(B_\lambda \alpha_{vw}\alpha_w^{-1}B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda \alpha_w\alpha_{vw}^{-1}B_\lambda)$

$= \theta_r$

<u>8.6 LEMMA</u> Let $N(w)$ be the set of positive roots r such that $w(r)$ is negative. Define $\theta_w = \Pi\theta_r$, $r \in N(w)$.

Then $q_w^{-1}\theta(B_\lambda \hat{\alpha}_w \alpha_w B_\lambda) = \theta_w$.

<u>Proof</u> The result is trivial for $\ell(w)=0$. Assuming that it holds for elements of length k, suppose that $\ell(w)=k+1$ and let $s=s_i$ be a fundamental reflection with $w=sv$, $\ell(v)=k$. Then

$$q_w^{-1}\theta(B_\lambda \hat{\alpha}_w\alpha_w B_\lambda) = q_w^{-1}\theta(B_\lambda \hat{\alpha}_v \hat{\alpha}_s \alpha_s \alpha_v B_\lambda)$$

(since if $(s)(v) = h(w)$ then $\alpha_s\alpha_v = h\alpha_w$ and $\hat{\alpha}_v\hat{\alpha}_s = \hat{\alpha}_w h^{-1}$)

$$= q_v^{-1}q_i^{-1}\theta(B_\lambda \hat{\alpha}_v \alpha_v B_\lambda \alpha_v^{-1}\hat{\alpha}_s \alpha_s \alpha_v B_\lambda)$$

$$= \theta_v\, q_i\, \theta(B_\lambda \alpha_v^{-1}\overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_v B_\lambda)$$

$= \theta_v\theta_a$ where $a = v^{-1}(r_i)$, since if we let $u=v^{-1}$ and $\beta = B_\lambda \alpha_v^{-1}\alpha_u^{-1}B_\lambda$ then

$$B_\lambda \alpha_v^{-1}\overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_v B_\lambda = \beta B_\lambda \alpha_u \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_u^{-1}B_\lambda\beta^{-1}.$$

However, $N(w) = N(v) \cup \{a\}$, and $\theta_v \theta_a = \theta_w$ as required.

8.7 COROLLARY Let ψ be a component of multiplicity 1 in λ^G and let θ be the restriction of ψ to $B_\lambda kGB_\lambda$. Then the degree of ψ is given by $\psi(1) = \sum_{w\in W} q_w / \sum_{w\in S} \theta_w$.

Proof Since $[G:B] = \Sigma q_w$ $(w\in W)$ this is immediate from the formula given in 6.4:

$$1 = \psi(1)[G:B]\Sigma q_w^{-1}\theta(\hat{\beta}_w)\theta(\beta_w)$$

and the fact that θ is a linear representation of $B_\lambda kGB_\lambda$.

8.8 LEMMA Let r be a root with corresponding reflection s, and choose any $w\in W$, $i \in \{1,2,\ldots n\}$ with $r = w(r_i)$. Then (i) There exists a nonnegative integer c_r, depending on λ but not θ, such that $\theta_r = p^{c_r}$ or p^{-c_r}. (By 8.5, $c_a = c_b$ if a and b are in the same S-orbit.)

(ii) $\theta_r = 1$ if and only if $\lambda^w(H_i) = 0$, where H_i is as defined in the proof of 6.6. This happens in particular if $s\notin S$.

Proof Since $\overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X} = q_i^{-1}\overline{X} + q_i^{-1}H_i\overline{X}(s_i)\overline{X}$ we have

$$\begin{aligned} & q_i B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1}B_\lambda - B_\lambda \\ &= q_i B_\lambda \alpha_w \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_w^{-1} - B_\lambda \quad \text{(by 6.2)} \\ &= (B_\lambda + B_\lambda \alpha_w H_i \overline{X}(s_i)\overline{X}\alpha_w^{-1}) - B_\lambda \\ &= \alpha_w \lambda^w(H_i) B_{\lambda^w}(s_i)\overline{X}\alpha_w^{-1} \\ &= \lambda^w(H_i) B_\lambda \alpha_w \overline{X}(s_i)\overline{X}\alpha_w^{-1} B_{\lambda^s} \quad \text{(by 6.2)} \\ &= \gamma \text{ say.} \end{aligned}$$

If $\lambda^w(H_i) \neq 0$ then by 6.2 and 6.6 there exists $\alpha\in\overline{X}kG\overline{X}$ with $\gamma\alpha = B_\lambda$. Hence $\gamma\neq 0$, and since $\gamma\in B_\lambda kGB_\lambda \cap B\, kGB_{\lambda^s}$ it follows that $\lambda^s = \lambda$. Furthermore, since γ has an inverse in $B_\lambda kGB_\lambda$ it follows that $\theta(\gamma) \neq 0$. Thus

$$\begin{aligned} \theta_r &= \theta(B_\lambda + \gamma) \\ &= \theta(B_\lambda) + \theta(\gamma) \\ &= 1 \text{ if and only if } \lambda^w(H_i) = 0. \end{aligned}$$

If $s \in S$, $\lambda^{ws_i} = \lambda^w$ and so if $P = B \cup Bs_i B$ then $(\lambda^w)^P$ has exactly two irreducible components, both of which occur with multiplicity one. If $\beta \in B_{\lambda w} kPB_{\lambda w}$ then $\alpha_w \beta \alpha_w^{-1} \in B_\lambda kGB_\lambda$ and so $\beta \mapsto \theta(\alpha_w \beta \alpha_w^{-1})$ is a linear representation of $B_{\lambda w} kPB_{\lambda w}$. Corresponding to this there is an irreducible component of $(\lambda^w)^P$ which has degree $d_1 = (1+q_i)/(1+\theta_r)$. Let d_2 be the degree of the other component, and let $d_1 = m_1 p^a$, $d_2 = m_2 p^b$, where m_1 and m_2 are not divisible by p. By theorem 3.1 of [7], m_1 and m_2 are both divisors of $1+q_i$, and since $d_1+d_2 = 1+q_i$ it follows that $m_1=m_2$. Now either $a=0$ or $b=0$, and so d_1 and d_2 are m and $p^c m$ where $m=m_1=m_2$ and $c=c_r=a+b$. If $d_1=m$ then $\theta_r = p^c$ and if $d_1=p^c m$ then $\theta_r = p^{-c}$.

Notice that the possibilities for c are limited by the requirement that $m = (1+q_i)(1+p^c)^{-1}$ is an integer. If the elements $h_i(x)$ $(x \in X_i, x \neq 1)$ form a group, all elements occurring with the same frequency, then $\lambda^w(H_i)=0$ or $q_i - 1$. If $\lambda^w(H_i)=0$ then $d_1=d_2=(1+q_i)/2$. If $\lambda^w(H_i)=q_i-1$, then setting $x = \theta(B_\lambda \alpha_w \overline{X}(s_i)\overline{X}\alpha_w^{-1} B_\lambda)$, $\theta_r - 1 = (q_i-1)x$ and so x is rational. But $q_i x^2 = \lambda((s)^2)\theta_r = \theta_r$ since θ_r and $q_i x^2$ are both positive rationals. Now

$$q_i x^2 - (q_i-1)x - 1 = 0$$

gives $x = -q_i^{-1}$ or 1. Thus $\theta_r = q_i$ or q_i^{-1}, and d_1 and d_2 are 1 and q_i. (So λ^w extends to a character of P). From the structural equations given in chapter 4 it can be seen that the above condition on the elements $h_i(x)$ is satisfied for all the Chevalley groups except for $G_2^1(q)$ and $A_{2\ell}^1(q)$ (for the root subgroup corresponding to S_ℓ).

In fact for $G_2^1(3)$, $q_i = 27$ and there exists a character λ for which $\lambda(H_i) = -6$, giving $\theta_r = 3$ or $\frac{1}{3}$, and d_1, d_2 equal to 7 and 21.

We now combine 8.7, 8.8 and 2.7 in a theorem to conclude chapter 8:

<u>8.9 THEOREM</u> Let ψ be an irreducible component of multiplicity 1 in λ^G, $S = \{w \in W | \lambda^w = \lambda\}$, θ the restriction of ψ to $B_\lambda kGB_\lambda$, θ_r as defined in 8.4, and W_S as defined in 2.7. Then the degree of ψ is

$$\psi(1) = [S:W_S]^{-1} W(\underset{\sim}{q})/W_S(\underset{\sim}{\theta})$$

where $W(\underset{\sim}{t})$ and $W_S(\underset{\sim}{u})$ are the Poincare polynomials of W and W_S (c.f.2.6), and the coordinates of the vector $\underset{\sim}{q}$ are given by the orders of corresponding root subgroups, and those of $\underset{\sim}{\theta}$ by corresponding θ_r. (In particular the coordinates are powers of p in both cases). (The proof of this is immediate.)

CHAPTER 9

AN AUTOMORPHISM OF $\overline{X}kG\overline{X}$.

In this chapter we prove theorem J, which is given by combining 9.1 and 9.4.

9.1 THEOREM Define $f:\overline{X}kG\overline{X} \to \overline{X}kG\overline{X}$ by setting

$$f(\alpha_w h) = (\hat{\alpha}_w)^{-1} h(-1)^{\ell(w)} q_w$$

and extending this linearly to the whole of $\overline{X}kG\overline{X}$. Then f is an automorphism.

Proof Let $s=s_i$ be any fundamental reflection. Then

$$f(\alpha_s^2) = f(q(s)^2\overline{X} + (s)^2 H_i \alpha_s) \qquad \text{(where } q=q_i\text{)}$$

$$= q(s)^2\overline{X} - (s)^2 H_i (\hat{\alpha}_s)^{-1} q$$

But, as in 6.6, $\hat{\alpha}_s^2 = q(s)^{-2}\overline{X} + q\Sigma(\overline{X}(s)^{-1}x(s)^{-1}\overline{X})$

(summation over non-identity elements of X_i)

$$= q(s)^{-2}\overline{X} + q(s)^{-2}\Sigma\overline{X}\, h_i(x)(s)\overline{X}$$

$$= q(s)^{-2}\overline{X} + H_i \hat{\alpha}_s$$

Therefore

$$(\hat{\alpha}_s)^{-2} q^2 = q(s)^2\overline{X} - (s)^2 H_i (\hat{\alpha}_s)^{-1} q$$

and so $\quad f(\alpha_s^2) = (\hat{\alpha}_s^{-2} q^2) = [f(\alpha_s)]^2.$

We now show that for all $w \in W$, $f(\alpha_s \alpha_w) = f(\alpha_s)f(\alpha_w)$

Firstly, if $\ell(sw) > \ell(w)$ and $h \in H$ is such that $(s)(w) = (sw)h$, then (as in 6.5)

$$\alpha_s \alpha_w = \alpha_{sw} h \quad \text{and} \quad \hat{\alpha}_w \hat{\alpha}_s = h^{-1} \hat{\alpha}_{sw}.$$

Thus $\quad f(\alpha_s \alpha_w) = f(\alpha_{sw} h)$

$$= (-1)^{\ell(sw)} (\hat{\alpha}_{sw})^{-1} h q_s^{-1} {}_w$$

$$= (-1)(-1)^{\ell(w)} (\hat{\alpha}_s)^{-1} (\hat{\alpha}_w)^{-1} q^{-1} q_w^{-1}$$

$$= f(\alpha_s) f(\alpha_w).$$

If $\ell(sw) < \ell(w)$ then by what we have just proved,

$$f(\alpha_s \alpha_{sw}) = f(\alpha_s) f(\alpha_{sw}).$$

Therefore $f(\alpha_s\alpha_w) = f(\alpha_s^2\alpha_{s\,w}h)$ (where $(s)(sw) = (w)h$)

$= f(q(s)^2\alpha_{s\,w}h + (s)^2H_i\,\alpha_s\alpha_{s\,w}h)$

$= f(q(s)^2\alpha_{s\,w}h) + f((s)^2H_i\,\alpha_s\alpha_{s\,w}h)$

$= f(q(s)^2)f(\alpha_{s\,w}h) + f((s)^2H_i\,\alpha_s)f(\alpha_{s\,w}h)$

$= f(\alpha_s^2)f(\alpha_{s\,w}h)$

$= f(\alpha_s)f(\alpha_s)f(\alpha_{s\,w}h)$

$= f(\alpha_s)f(\alpha_s\alpha_{s\,w}h)$

$= f(\alpha_s)f(\alpha_w)$

Now a simple induction completes the proof that $f(\alpha_v\alpha_w) = f(\alpha_v)f(\alpha_w)$ for all $v, w \in W$, and the rest is clear.

If e is a primitive idempotent in $\bar{X}kG\bar{X}$ affording an irreducible component ψ of 1_X^G then $f(e)$ is also a primitive idempotent, and the corresponding character will be called $f(\psi)$. It will be shown that for each λ, $f(\chi(\lambda)) = \zeta(\lambda)$.

<u>9.2 LEMMA</u> Let ψ be an irreducible character of G occurring with multiplicity 1 in λ^G. Then $\psi = f(\psi)$ if and only if ε is trivial on S.

<u>Proof</u> Let θ be the restriction of ψ to $B_\lambda kGB_\lambda$. Then $\psi = f(\psi)$ if and only if $\theta(\beta_w) = \theta(f(\beta_w))$ for all $w \in S$; i.e. if and only if $1 = \theta(f(\beta_w)^{-1}\beta_w)$

$= \theta((-1)^{\ell(w)}q_w^{-1}B_\lambda\hat{\alpha}_w\alpha_wB_\lambda)$ (see 9.1)

for all $w \in S$. If ε is trivial on S then by 8.8 (ii) $\theta_r = 1$ for all r, and so $\theta_w = 1$ for all w. Furthermore $(-1)^{\ell(w)} = 1$ for all $w \in S$, and so $(-1)^{\ell(w)}\theta_w = 1$ for all $w \in S$. Therefore $\psi = f(\psi)$.

Conversely, if $(-1)^{\ell(w)}\theta_w = 1$ for all $w \in S$ then since θ_w is a positive rational number it follows that $(-1)^{\ell(w)} = 1$ for all $w \in S$, and so ε is trivial on S.

9.3 LEMMA   Let $J \subseteq \{1,2,\ldots n\}$, $w \in W$. Then

$$(\chi_{J\,w}^{G}, f(\chi)) = (\delta(S,J,w), \varepsilon|_S)$$

Proof   By 7.5 (ii), $(\chi, \chi_{J\,w}^{G}) = (1, \delta(S,J,w)) = 1$ and it follows that $\chi_{J\,w}$ is the unique common component of $(\lambda^w)^{G_J}$ and $\chi|_{G_J}$. Therefore $f(\chi_{J\,w})$ is the unique common component of $(\lambda^w)^{G_J}$ and $f(\chi)|_{G_J}$. Therefore $(\chi_{J\,w}, f(\chi)|_{G_J})$ is zero if $\chi_{J\,w} \neq f(\chi_{J\,w})$ and one if $\chi_{J\,w} = f(\chi_{J\,w})$.

Therefore by 9.2,

$(\chi_{J\,w}^{G}, f(\chi)) = (1, \varepsilon)$ (where the inner product on the right hand side is taken over the group $W_J \cap S^w$)

$$= (\delta(S,J,w), \varepsilon|_S).$$

9.4 THEOREM $f(\chi(\lambda)) = \zeta(\lambda)$

Proof   By 9.3, $(\varphi_J, f(\chi)) = (\delta_J|_S, \varepsilon|_S)$

and so $(\zeta, f(\chi)) = \Sigma(-1)^{|J|}(\varphi_J, f(\chi))$

$$= \Sigma(-1)^{|J|}(\delta_J|_S, \varepsilon|_S)$$

(summation over all subsets J of $\{1,2,\ldots n\}$)

$$= (\varepsilon|_S, \varepsilon|_S)$$

$$= 1.$$

CHAPTER 10

THE STRUCTURE OF $B_\lambda kGB_\lambda$

Let D and W_S be as defined in 2.7 and let Γ be the root system of W_S. For simplicity assume that k is the complex field.

10.1 LEMMA Let $r=r_i$ be a fundamental root with corresponding reflection s, and let $v,w \in W$. If $w(r) \notin \Gamma$ then

$$B_{\lambda w}\, \alpha_s \alpha_v = \lambda^w((s)(v)(sv)^{-1}) \sqrt{q_s q_v q_{s v}^{-1}}\; B_{\lambda w}\, \alpha_{s v}$$

and $$\alpha_v \alpha_s B_{\lambda w} = \lambda^w((vs)^{-1}(v)(s)) \sqrt{q_v q_s q_{v s}^{-1}}\; \alpha_{v s} B_{\lambda w}$$

Proof We prove only the first of these, since the proof of the other is similar. Firstly, if $\ell(sv) > \ell(v)$ then $q_{s v} = q_s q_v$ and $\alpha_s \alpha_v = (s)(v)(sv)^{-1} \alpha_{s v}$, so that the result is trivial. If $\ell(sv) < \ell(v)$ then

$$\alpha_s \alpha_v = q_s (s)(v)(sv)^{-1} \alpha_{s v} + H_i (s)^2 \alpha_v .$$

By 8.8 $\lambda^w(H_i)=0$, and since $q_s q_{s v} = q_v$ the result follows.

10.2 LEMMA Let $v,w \in W$ and assume that $w^{-1}(r)$ is positive for all positive roots r such that $v(r)$ is a negative root of Γ. Then

$$B_\lambda \alpha_v \alpha_w = \lambda((v)(w)(vw)^{-1}) \sqrt{q_v q_w q_{v w}^{-1}}\; B_\lambda \alpha_{v w}$$

Proof Use induction on $\ell(v)$. The case $\ell(v)=0$ is trivial.

Assume $\ell(v) > 0$ and let $v=ts$ where $\ell(t)=\ell(v)-1$ and $s=s_i$ is a fundamental reflection. Let a be any positive root such that $t(a)$ is a negative root of Γ. Then a is not equal to r_i (since $t(r_i) > 0$), and so $s(a)$ is positive. Now $vs(a)$ is a negative root of Γ, and so $w^{-1}(s(a)) > 0$. Thus we have shown that $(sw)^{-1}(a)$ is positive for all positive roots a with $t(a)$ a negative root of Γ. Now

$$B_\lambda \alpha_v \alpha_w = \lambda((v)(s)^{-1}(t)^{-1}) B_\lambda \alpha_t \alpha_s \alpha_w \qquad \text{(by 6.6)}$$

$$= \lambda((v)(s)^{-1}(t)^{-1}) B_\lambda \alpha_t B_{\lambda^t} \alpha_s \alpha_w$$

$= \lambda^t((s)(w)(sw)^{-1})\sqrt{q_s q_w q_{sw}^{-1}}\ \lambda((v)(s)^{-1}(t)^{-1}) B_\lambda \alpha_t \alpha_{sw}$ (by 10.1 or 6.6 since either $w^{-1}(s) > 0$ or else $t(s) = -v(s) \notin \Gamma$)

$= \lambda((v)(w)(sw)^{-1}(t)^{-1})\sqrt{q_s q_w q_{sw}^{-1}}\ \lambda((t)(sw)(vw)^{-1})\sqrt{q_t q_{sw} q_{vw}^{-1}}\ B_\lambda \alpha_{vw}$

by the inductive hypothesis, and on cancellation we obtain the required formula. Using 10.2 we can prove theorems F and G:

<u>10.3 THEOREM</u> The character λ of H may be extended to a linear character of SH. (The extension will also be denoted by "λ").

<u>Proof</u> Let θ be the restriction to $B_\lambda kGB_\lambda$ of $\chi(\lambda)$. For $r \in \Gamma$ define $\eta_r = 1$ if θ_r is a positive power of p and $\eta_r = -1$ if θ_r is a negative power of p (see 8.8), and for $w \in S$ let $\eta(w) = \Pi\eta_r$ where the product is over positive roots of Γ negatived by w. It is clear from 8.5 that η is a character of S. Now for $w \in S$, $h \in H$ define $\lambda((w)h) = \eta(w)|\theta(\beta_w)|^{-1}\theta(\beta_w)\lambda(h)$.

Let $w \in S$ and s a fundamental reflection of W_S (i.e. the root r corresponding to s is in the base of Γ). Then by 10.2, if $w(r) > 0$

$$\lambda((w))\lambda((s)) = \lambda((w)(s)(ws)^{-1})\lambda((ws))$$
$$= \lambda((w)(s)).$$

If $w(r) < 0$ then $ws(r) > 0$ and so

$$\lambda((w))\lambda((s)) = \lambda((ws)(s))\lambda((s)^{-1}(ws)^{-1}(w))\lambda((s))$$
$$= \lambda((ws))\lambda((s))\lambda((s)^{-1}(ws)^{-1}(w))\lambda((s))$$
$$= \lambda((w)(s)^{-1})[\lambda((s))]^2$$

But $(\lambda((s)))^2 = |\theta(\beta_s)|^{-2}\theta(\beta_s)^2 = |\theta(\beta_s)|^{-2}\lambda((s)^2)\theta(\hat{\beta}_s\beta_s)$

$= \lambda((s)^2)$ since $\theta(\hat{\beta}_s\beta_s) = q_s\theta_s$ is real and positive. Thus $\lambda((w))\lambda((s)) = \lambda((w)(s))$ in this case also, and it is now clear that λ is a character of SH.

It is convenient at this point to introduce some new notation. If $r \in \Gamma$ define $p_r = p^{c_r}$ (see 8.8), and for $w \in S$ define $p_w = \Pi p_r$, product over positive roots $r \in \Gamma$ such that $w(r)$ is negative. Let

$$\gamma_w = \sqrt{p_w q_w^{-1}}\ \lambda((w)^{-1})\beta_w.$$

<u>10.4 THEOREM</u> Let $w \in S$, $v \in D$, and r a fundamental root of Γ with corresponding reflection s. Then

(i) $\gamma_v\gamma_w = \gamma_{vw}$ and $\gamma_w\gamma_v = \gamma_{wv}$

(ii) $\gamma_w\gamma_s = \gamma_{ws}$ if $w(r) > 0$

$\gamma_w\gamma_s = p_r\gamma_{ws} + (p_r-1)\gamma_w$ if $w(r) < 0$

(iii) $\gamma_s\gamma_w = \gamma_{sw}$ if $w^{-1}(r) > 0$

$\gamma_s\gamma_w = p_r\gamma_{sw} + (p_r-1)\gamma_w$ if $w^{-1}(r) < 0$.

<u>Proof</u> Elements of D permute the roots in Γ, leaving positive roots positive. Thus there are no positive roots r such that $v(r)$ is a negative root of Γ. Therefore by 10.2

$$\beta_v\beta_w = \lambda((v)(w)(vw)^{-1})\sqrt{q_v q_w q_{vw}^{-1}}\ \beta_{vw}$$

Furthermore $p_v = 1$ and $p_w = p_{vw}$, and so it follows that $\gamma_v\gamma_w = \gamma_{vw}$. The formulae for $\gamma_w\gamma_v$, $\gamma_w\gamma_s$ when $w(r) > 0$, and $\gamma_s\gamma_w$ when $w^{-1}(r) > 0$ also follow easily from 10.2, and we omit the proofs of these.

Let a be a fundamental root (i.e. fundamental in the root system of W) such that $s(a) < 0$, and let w_1 be the reflection corresponding to a. If $w_1 \neq s$ then $s(a) \neq -a$ and hence $w_1(s(a))$ is negative. That is, $(sw_1)^{-1}(a)$ is negative, and it follows that

$$\ell(w_1sw_1) = \ell(sw_1)-1 = \ell(s)-2.$$

Continuing in this way we can find a reduced expression for

s of the form $s = w_1 w_2 \ldots w_m s_i w_m \ldots w_2 w_1$. Let $v = w_1 w_2 \ldots w_m$ and $u = v^{-1}$. If b is a positive root such that $v(b)$ is a negative root in Γ then $-v(b)$ is a positive root in Γ which is negatived by u and hence by s also. Therefore $-v(b) = r$, and $b = -u(r) = -r_i$, contradicting the fact that b is positive. So no such b can exist, and we may apply 10.2 to conclude that

$$B_\lambda \alpha_v \alpha_u = q_v B_\lambda \lambda((v)(u)).$$

Therefore

$$\alpha_u B_\lambda \alpha_v = q_v B_{\lambda v} \lambda((v)(u)).$$

Now $\gamma_s^2 = p_s q_s^{-1} \lambda((s)^{-2}) \lambda((s)(u)^{-1}(s_i)^{-1}(v)^{-1})^2 (B_\lambda \alpha_v \alpha_{s_i} \alpha_u B_\lambda)^2$

$$= p_r q_s^{-1} \lambda((u)^{-1}(s_i)^{-1}(v)^{-1})^2 B_\lambda \alpha_v \alpha_{s_i}^2 \alpha_u B_\lambda q_v \lambda((v)(u))$$

$$= p_r q_s^{-1} q_v \lambda((u)^{-1}(v)^{-1}) B_\lambda \alpha_v (s_i)^{-2} \alpha_{s_i}^2 \alpha_u B_\lambda$$

But $(s_i)^{-2} \alpha_{s_i}^2 = q_i \overline{X} + H_i \alpha_{s_i}$, and so we have

$$\gamma_s^2 = p_r q_s^{-1} q_v^2 q_i B_\lambda + c\gamma_s \quad \text{for some scalar } c$$

$$= p_r B_\lambda + c\gamma_s$$

Let θ be the restriction to $B_\lambda kGB_\lambda$ of $\chi(\lambda)$. Then

$$\theta(\gamma_s) = \sqrt{p_r q_s^{-1}}\ |\theta(\beta_w)|\, \eta(w)$$

$$= \sqrt{p_r q_s^{-1}}\ \sqrt{q_s \theta_r}\ \eta(w)$$

$$= p_r \text{ or } -1$$

In either case $\theta(\gamma_s)^2 = p_r + c\theta(\gamma_s)$ gives $c = p_r - 1$.

Now if $w \in S$ such that $w(r) < 0$ then

$$\gamma_w \gamma_s = \gamma_{ws} \gamma_s^2 = \gamma_{ws}(p_r B_\lambda + (p_r - 1)\gamma_s)$$

$$= p_r \gamma_{ws} + (p_r - 1)\gamma_w$$

(and similarly $\gamma_s \gamma_w = p_r \gamma_{sw} + (p_r - 1)\gamma_w$ for w such that $w^{-1}(r) < 0$).

We now use 10.4 to determine representations of $B_\lambda kGB_\lambda$. In particular we have the following (from which theorem H follows):

10.5 THEOREM (i) For any linear representation ν of W_S there exists a linear representation θ of $B_\lambda kW_S B_\lambda$ such that if r is a fundamental root of Γ with corresponding reflection s, then $\theta(\gamma_s) = -1$ if $\nu(s) = -1$ and $\theta(\gamma_s) = p_r$ if $\nu(s) = 1$.

(ii) Let η be any irreducible character of D, and θ a linear representation of $B_\lambda kW_S B_\lambda$. Then $B_\lambda kGB_\lambda$ has an irreducible character κ such that $\kappa(\gamma_{vw}) = \eta(v)\theta(\gamma_w)$ for all $v \in D$, $w \in W_S$. Corresponding to κ there is an irreducible character of G which has multiplicity $\eta(1)$ in λ^G, and degree $\eta(1)|D|^{-1}W(\underset{\sim}{q})/W_S(\underset{\sim}{\theta})$.

(iii) D is an abelian p'-group.

(iv) There are precisely $|D|$ components of λ^G with degree obtained by setting $\eta(1) = 1$ and $\theta_r = p_r$ for all $r \in \Gamma$ in the formula given in (ii). (The θ_r are the coordinates of $\underset{\sim}{\theta}$). These are the only components with degree prime to p, unless W_S has an irreducible component W_1 of the form $[2m]$ or $[3^{n-2},4]$ (see chapter 2) and for all roots a of this component, $p_a|m$ or $p_a = 2$ (respectively). Then λ^G may have further components with degree prime to p such that $\theta_a\theta_b = 1$ when a and b are in different orbits in the root system of W_1.

Proof (i) For $r \in \Gamma$ with corresponding reflection s let $d_r = -1$ if $\nu(s) = -1$ and $d_r = p_r$ if $\nu(s) = 1$. Define $\theta(\gamma_w) = \Pi d_r$ (product over positive $r \in \Gamma$ negatived by w) and extend this linearly to the whole of $B_\lambda kW_S B_\lambda$. We use induction on the number of positive $r \in \Gamma$ negatived by w to show that $\theta(\alpha\gamma_w) = \theta(\alpha)\theta(\gamma_w)$ for all $w \in W_S$, from which it follows trivially that θ is a representation.

Firstly suppose $w=s$ is the reflection corresponding to a fundamental root r of Γ. Let $v \in W_S$. Then if $v(r) > 0$,

$$\theta(\gamma_v)\theta(\gamma_s) = (\Pi d_a)d_r$$ (where the product is over positive $a \in \Gamma$ negatived by v)

$$= (\Pi d_{s(a)})d_r$$ (since it is clear that $d_a = d_{u(a)}$ for any $a \in \Gamma$ and $u \in W_S$, in view of 8.8 (i))

$$= \Pi d_b$$ (product over positive $b \in \Gamma$ negatived by vs)

$$= \theta(\gamma_{vs})$$

$$= \theta(\gamma_v \gamma_s)$$

If $v(r) < 0$ then $\theta(\gamma_v)\theta(\gamma_s) = \theta(\gamma_{vs})\theta(\gamma_s)^2$

$$= \theta(\gamma_{vs})(p_r + (p_r - 1)\theta(\gamma_s))$$

$$= p_r\theta(\gamma_{vs}) + (p_r - 1)\theta(\gamma_{vs})\theta(\gamma_s)$$

$$= p_r\theta(\gamma_{vs}) + (p_r - 1)\theta(\gamma_v)$$

$$= \theta(\gamma_v \gamma_s)$$

Hence it follows that $\theta(\alpha)\theta(\gamma_s) = \theta(\alpha\gamma_s)$ for all $\alpha \in B_\lambda k W_S B_\lambda$.

Now suppose that w negatives more than one positive root of Γ. Then there exists v which negatives one fewer and a fundamental reflection s of W_S such that $w=vs$. Then for any $\alpha \in B_\lambda k W_S B_\lambda$,

$$\theta(\alpha\gamma_w) = \theta(\alpha\gamma_v\gamma_s)$$

$$= \theta(\alpha\gamma_v)\theta(\gamma_s)$$

$$= \theta(\alpha)\theta(\gamma_v)\theta(\gamma_s)$$ by the inductive hypothesis

$$= \theta(\alpha)\theta(\gamma_w).$$

(ii) Let R be a matrix representation of D with character η. For $v \in D$, $w \in W_S$ define $T(\gamma_{vw}) = R(v)\theta(\gamma_w)$ and extend T linearly to the whole of $B_\lambda kGB_\lambda$. In particular, $T(\gamma_v \alpha) = R(v)\theta(\alpha)$

for all $v \in D$ and $\alpha \in B_\lambda k W_S B_\lambda$. Then T is clearly a representation, since if $v,w \in D$ and $\alpha,\beta \in B_\lambda k W_S B_\lambda$ then

$$T(\gamma_v \alpha \gamma_w \beta) = T(\gamma_{vw}(\gamma_w^{-1}\alpha\gamma_w\beta))$$

$$= R(vw)\theta(\gamma_w^{-1}\alpha\gamma_w\beta)$$

(since by 10.4 $\gamma_w^{-1}\alpha\gamma_w \in B_\lambda k W_S B_\lambda$)

$$= R(v)R(w)\theta(\alpha)\theta(\beta)$$

(since θ is a linear representation)

$$= T(\gamma_v \alpha)T(\gamma_w \beta).$$

The character κ of T obviously has the required property.

Now $\Sigma q_{vw}^{-1}\kappa(\hat{\beta}_{vw})\kappa(\beta_{vw}) = \Sigma q_{vw}^{-1}\eta(v)\eta(v^{-1})\theta(\hat{\beta}_{vw}\beta_{vw})$

$$= \Sigma\eta(v)\eta(v^{-1})\theta_{vw}$$

$$= \Sigma\eta(v)\eta(v^{-1})\theta_w \qquad \text{(see 2.7)}$$

where the summation is over all $v \in D$ and $w \in W_S$. Since $\Sigma\eta(v)\eta(v^{-1}) = |D|$, (ii) now follows from [7, theorem 2.4].

(iii) Choose the representation θ which corresponds to the 1-character of W_S, as in (i). Then for each $r \in \Gamma$, $\theta_r = p_r$. Hence $W_S(\underset{\sim}{\theta})$ is an integer. Using (ii) in the case $\eta=1$ it follows that $|D|$ divides $W(\underset{\sim}{q})$. Hence D is a p'-group. Apply (ii) again for the same θ and any non-linear irreducible character η of D. The character of G obtained has multiplicity greater than 1 in λ^G, and so by 7.8 is not of the form $\zeta(\lambda)$. Therefore by 8.2 it has degree divisible by p. But $W_S(\underset{\sim}{\theta})$ is an integer and $W(\underset{\sim}{q})$ is prime to p, and so p must divide $\eta(1)$. This contradicts the fact that D is a p'-group. Hence D has no nonlinear irreducible characters, and so is abelian.

(iv) By 8.2 and 7.8 any irreducible component ζ of λ^G with degree prime to p occurs with multiplicity 1. By

10.4 it is clear that the corresponding linear representation of $B_\lambda kGB_\lambda$ is composed of a representation η of D and a representation θ of $B_\lambda kW_S B_\lambda$ in the manner described in (ii). If $\theta_r = p_r^{-1}$ for any $r\in\Gamma$ then it can be seen from the factorizations given in [14] that (with the exceptions given in the theorem statement) $W_S(\underset{\sim}{\theta})$ has a factor of p in its denominator. (For example, if W_S has a component of the form [2m] then $W_S(\underset{\sim}{\theta})$ has factors $(1+\theta_a)$, $(1+\theta_b)$ and $(1+\theta_a\theta_b+\theta_a^2\theta_b^2+\ldots+\theta_a^{m-1}\theta_b^{m-1})$ where a and b are representatives of the two orbits of the root system [2m]. Since θ_a and θ_b are powers of p and not equal to 1, if $\theta_a = p_a^{-1}$ the only way to avoid a factor of p in the denominator is if $\theta_a\theta_b = 1$ and p_a divides $(1+\theta_a\theta_b+\ldots+\theta_a^{m-1}\theta_b^{m-1}) = m)$. If $W_S(\underset{\sim}{\theta})$ has a factor of p in its denominator then $p|\zeta(1)$, a contradiction. Thus (apart from the exceptional cases) $\theta_r = p_r$ for all $r\in\Gamma$, which means that $\theta(\gamma_s) = p_r$ for the corresponding reflection s (i.e. θ corresponds to the 1-character of W_S). The rest follows simply.

Our next aim is to develop the theme of 7.5 (ii) in more detail. In 10.5 (i) and (ii) an irreducible character of G is obtained from linear characters ν and η of W_S and D. Let us denote this character of G by "$\psi(\nu,\eta,\lambda)$". By 10.3 there is a linear character $\nu\eta\lambda$ of SH, and so inducing to N we obtain a character $\xi(\nu,\eta,\lambda)$ which is irreducible since for $w\notin S$ the restriction to H of $(\nu\eta\lambda)^w$ does not equal the restriction of $\nu\eta\lambda$. More generally, if J is any subset of $\{1,2,\ldots n\}$ the same process yields irreducible characters $\psi_J(\nu,\eta,\lambda)$ of G_J and $\xi_J(\nu,\eta,\lambda)$

of $N \cap G_J$.

Let J,K be two subsets of $\{1,2,...n\}$, and λ_1,λ_2 two linear characters of H. Let $S_1 = \{w \in W_J \mid \lambda_1^w = \lambda_1\}$ and $S_2 = \{w \in W_K \mid \lambda_2^w = \lambda_2\}$, and $S_1 = D_1W_1$, $S_2 = D_2W_2$ (as previously we had $S = DW_S$). For $w \in S_1$, $v \in S_2$ define γ_w and δ_v in the same way as previously we defined γ_w for $w \in S$. We now prove theorem I:

<u>10.6 THEOREM</u> If ν_1,ν_2 are linear characters of W_1,W_2, and η_1,η_2 linear characters of D_1, D_2 then

$$(\psi_1^G,\psi_2^G) = (\xi_1^N,\xi_2^N)$$

where $\psi_1 = \psi_J(\nu_1,\eta_1,\lambda_1)$, $\psi_2 = \psi_K(\nu_2,\eta_2,\lambda_2)$, $\xi_1 = \xi_J(\nu_1,\eta_1,\lambda_1)$ and $\xi_2 = \xi_K(\nu_2,\eta_2,\lambda_2)$.

First note that both inner products in the statement of the theorem are zero unless $\lambda_2 = \lambda_1^w$ for some $w \in W$. So assuming $\lambda_2 = \lambda_1^w = \lambda^w$, let θ be the restriction of $\chi(\lambda)$ to $B_\lambda kGB_\lambda$, and θ^w its restriction to $B_{\lambda^w} kGB_{\lambda^w}$. Then for any $\alpha \in B_\lambda kGB_\lambda$,

$$\theta(\alpha)B_\lambda X_\mu^{w_o} = \alpha B_\lambda X_\mu^{w_o} \qquad \text{(by 6.7)}$$
$$= \alpha_w^{-1}\alpha\alpha_w B_{\lambda^w} X_\mu^{w_o}$$

(since there is a scalar c with $\alpha_w B_{\lambda^w} X_\mu^{w_o} = cB_\lambda X_\mu^{w_o}$), and hence $\theta(\alpha) = \theta^w(\alpha_w^{-1}\alpha\alpha_w)$. Now let r be any root and suppose $r = v(r_i)$ for some fundamental r_i, and $v \in W$. Then

$$\theta_{w(r)} = q_i\,\theta(B_\lambda \alpha_{wv} \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_{wv}^{-1}B_\lambda)$$
$$= q_i\,\theta(B_\lambda \alpha_w\alpha_v \overline{X}(s_i)^{-1}\overline{X}(s_i)\overline{X}\alpha_v^{-1}\alpha_w^{-1}B_\lambda) \qquad \text{(cf. 8.5)}$$
$$= \theta_r^w$$

This obviously holds for any element of S_1wS_2.

The next step is to prove that S_1wS_2 contains an element t such that $t(r) > 0$ for all positive $r \in \Gamma_2$ (the root system of W_2) and $t^{-1}(r) > 0$ for all positive

$r \in \Gamma_1$ (root system of W_1). In fact let t be an element of $S_1 w S_2$ such that the number of positive $r \in \Gamma_2$ negatived by t plus the number of positive $r \in \Gamma_1$ negatived by t^{-1} is minimal. Suppose that $t^{-1}(r) < 0$ for some fundamental $r \in \Gamma_1$ with corresponding reflection s. Then $(st)^{-1}$ negatives one fewer positive root of Γ_1 than does t^{-1} (cf. proof of 2.4). Let a be a positive root in Γ_2 negatived by st but not by t. Then $t(a)$ is a positive root negatived by s, and since $s \in W_J$, $t(a)$ is in the root system of W_J. Since $\theta_a^t \neq 1$, $\theta_{t(a)} \neq 1$, and so $t(a) \in \Gamma_1$. But the only positive root in Γ_1 negatived by s is r, and so $t(a) = r$. But this contradicts $t^{-1}(r) < 0$. Thus there can be no such a, and it follows that the number of positive roots in Γ_2 negatived by st plus the number of positive roots in Γ_1 negatived by $(st)^{-1}$ is less than the same number for t, and this contradicts the definition of t. Therefore $t^{-1}(r) > 0$ for all positive $r \in \Gamma_1$, and similarly $t(r) > 0$ for all positive $r \in \Gamma_2$.

We now investigate $t^{-1} S_1 t \cap S_2$. Let Ω be the set of roots $r \in \Gamma_2$ such that $t(r) \in \Gamma_1$ and let V be the subgroup generated by the corresponding reflections. For $r \in \Omega$, $r>0$, let $r = \Sigma \lambda_a a$ where each a is a fundamental root of Γ_2 and the λ_a are positive scalars. Then $t(r) = \Sigma \lambda_a t(a)$ and since $t(r)$ is in the root system of W_J so is each $t(a)$. Hence each a is in Ω. Therefore V is a parabolic subgroup of W_2 (generated by a set of fundamental reflections of Γ_2). If $v \in t^{-1} S_1 t \cap S_2$ and $v=dx$, $d \in D_2$, $x \in W_2$ then $r \mapsto t(r)$ maps the positive roots in

Γ_2 negatived by v onto the positive roots in Γ_1 negatived by tvt^{-1}. Each such r is in Ω, and so $x \in V$. Further, tdt^{-1} negatives no positive root of Γ_1, and so $tdt^{-1} \in D_1$. It can thus be seen that $t^{-1}S_1 t \cap S_2 = EV$ where $E = t^{-1}D_1 t \cap D_2$.

For $r \in \Gamma_1$, $v \in S_1$, p_r and p_v are defined as in the discussion preceding 10.4, and the character η of S_1 as in the proof of 10.3. (η is not related to η_1 or η_2). Define m_r, m_v for $r \in \Gamma_2$, $v \in S_2$ and a character κ of S_2 in a corresponding fashion. If $r \in \Omega$ with reflection s then $\theta_r = \theta^w_{t(r)}$ and so $m_r = p_{t(r)}$ and $\kappa(s) = \eta(tst^{-1})$. Further, if $v \in t^{-1}S_1 t \cap S_2$ then $m_v = p_u$, where $u = tvt^{-1}$, and $\eta(u) = \kappa(v)$. From 10.2,

$$B_\lambda \alpha_t \alpha_v = \sqrt{q_v q_u^{-1}}\,\lambda((t)(v)(t)^{-1}(u)^{-1}) B_\lambda \alpha_u \alpha_t \qquad (\sim B_\lambda \alpha_{ut})$$

(n.b. If $r>0$ it is impossible for $t(r)$ to be a negative root of Γ_1).

Thus
$$\theta^w(B_{\lambda w}\alpha_v) = \sqrt{q_v q_u^{-1}}\,\lambda((t)(v)(t)^{-1}(u)^{-1})\,\theta^w(\alpha_t^{-1}\beta_u\alpha_t)$$
$$= \sqrt{q_v q_u^{-1}}\,\lambda((t)(v)(t)^{-1}(u)^{-1})\,\theta(\beta_u)$$

and so $\lambda_2((v)) = \lambda_1((t)(v)(t)^{-1})$. Also

$$\delta_v = \sqrt{m_v q_v^{-1}}\,\lambda_2((v)^{-1}) B_{\lambda w}\alpha_v$$
$$= \sqrt{p_u q_v^{-1}}\,\lambda_1((t)(v)^{-1}(t)^{-1})\sqrt{q_v q_u^{-1}}\,\lambda((t)(v)(t)^{-1}(u)^{-1})\,\alpha_t^{-1}\beta_u\alpha_t$$
$$= \alpha_t^{-1}\gamma_u\alpha_t .$$

Let θ_1 be the restriction to $B_\lambda k G_J B_\lambda$ of ψ_1, θ_2 the restriction to $B_{\lambda w} k G_K B_{\lambda w}$ of ψ_2, and e_1, e_2 the corresponding idempotents. Suppose it is not true that $\eta_1\nu_1\lambda_1((t)(v)(t)^{-1}) = \eta_2\nu_2\lambda_2((v))$ whenever $v \in t^{-1}S_1 t \cap S_2$. Then either $\eta_1(tvt^{-1}) \neq \eta(v)$ for some $v \in E$ or else $\nu_1(tvt^{-1}) \neq \nu_2(v)$ for some fundamental reflection $v \in V$. In either case it is clear from 10.5 (i) and (ii) that $\theta_1(\gamma_u) \neq \theta_2(\delta_v)$ (where $u = tvt^{-1}$). However

$\theta_1(\gamma_u)e_1\alpha_t e_2 = e_1\gamma_u\alpha_t e_2 = e_1\alpha_t\delta_v e_2 = e_1\alpha_t e_2\theta_2(\delta_v)$ and so $e_1\alpha_t e_2 = 0$.

On the other hand, suppose

$$\eta_1\nu_1\lambda_1((t)(v)(t)^{-1}) = \eta_2\nu_2\lambda_2((v))$$

for all $v\in t^{-1}S_1t\cap S_2$. Then using 10.5 (i) and (ii) and the fact that $p_u=m_v$ it follows that $\theta_1(\gamma_u)=\theta_2(\delta_v)$ whenever v is a fundamental reflection of V or an element of E; hence it holds for all $v\in t^{-1}S_1t\cap S_2$ (with $u=tvt^{-1}$). Let A be a set of representatives of the $t^{-1}S_1t\cap S_2\backslash S_2$ cosets, such that for all $x\in A$, $x^{-1}(r) > 0$ for all positive $r\in\Omega$. Then $e_2\sim fe$, where $f = \Sigma m_v^{-1}\theta_2(\delta_{v^{-1}})\delta_v$ $(v\in t^{-1}S_1t\cap S_2)$ and $e = \Sigma m_x^{-1}\theta_2(\delta_{x^{-1}})\delta_x$ $(x\in A)$. Now

$$e_1\alpha_t e_2 \sim e_1(\alpha_t f\alpha_t^{-1})\alpha_t e \sim e_1\alpha_t e.$$

But for all $y\in S_1$, $x\in A$

$$B_\lambda\alpha_y\alpha_t\alpha_x \sim B_\lambda\alpha_{yt}\alpha_x \sim B_\lambda\alpha_{ytx} \qquad \text{(since } x^{-1}$$

negatives no positive root of $t^{-1}(\Gamma_1)$) and thus clearly $e_1\alpha_t e \neq 0$.

We can now complete the proof of 10.6. For any coset S_1wS_2 such that $\lambda_1^w=\lambda_2$ we have chosen a representative t with $t(r) > 0$ when r is a positive root of Γ_1 and $t^{-1}(r) > 0$ when r is a positive root of Γ_2; now let $t_1,t_2,\ldots t_m$ be all the representatives so obtained for the various cosets. Then as in the proof of 7.4, e_1kGe_2 has a basis consisting of those $e_1\alpha_{t_i}e_2$ which are nonzero. Hence the dimension equals the number of i such that $\eta_1\nu_1\lambda_1((t_i)(v)(t_i)^{-1})=\eta_2\nu_2\lambda_2((v))$ for all $v\in t_i^{-1}S_1t_i\cap S_2$, and this is just the number of cosets S_1HwS_2H $(w\in W)$ such that $\eta_1\nu_1\lambda_1((w)x(w)^{-1})=\eta_2\nu_2\lambda_2(x)$ for all $x \in w^{-1}S_1Hw \cap S_2H$ (obviously this can only occur when $\lambda_1^w = \lambda_2$).

Thus $(\psi_1^G, \psi_2^G) = \dim e_1kGe_2$

$=$ the number of $S_1H\backslash W/S_2H$ cosets with the above property

$= (\xi_1^N, \xi_2^N)$ by Mackey's theorem

and the proof of 10.6 is complete.

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CORRECTIONS

Summary, 2nd page, l.7	kG, not kg.
" l.b.9	"component ψ of λ^{G_J} with multiplicity 1 in λ^{G_J}".
p.3, l.1	Add "stabilized by D" to end of line 1.
p.11, Theorem 3.2 (3)	Add "where $\ell(w)$ is the minimum length for w as a product of the s_i (c.f. 2.1 & 2.2)".
p.12, l.7	Add in parenthesis "The involutions $s_1, s_2, \ldots s_n$ of definition 3.1 become fundamental reflections, & $r_1, r_2, \ldots r_n$ are the corresponding roots",
p. 16, et.seq.	The group U referred to in theorem 4.1 and its proof should be X.
p.30, end	(Versions of theorem B appear in the literature: see [11] or [19] as well as [9])"
p.32, l.b.8	$kG_J e$, not kGe.
p.49, l.b.5	$q_{s\,w}$, not $q_{s\,w}^{-1}$.
l.b.4	qq_w, not $q^{-1}q_w^{-1}$.
p.53, Footnote to 10.3	: "See also Kilmoyer, R., Notice 711-20-46, A.M.S. Notices 21."
p.56, Theorem 10.5	(i) Add "stabilized by D" to line 1 (ii) Add "stabilized by D" to end of 1st sentence.
p.60, l.b.11	Replace by $\theta(\alpha)B_{\lambda w}X_\mu^{w_0} = \alpha_w^{-1}\alpha\alpha_w B_{\lambda w}X_\mu^{w_0}$
p.59, l.b.3	Between the sentences insert "(Here for $w \in W_s$, $v \in D$, $\nu\eta(wv) = \nu(w)\eta(v)$. $\nu\eta$ is a character since ν is stabilized by D)".
p.62, l.11	Should be $\theta_{t(r)} = \theta_r^w$