ABSTRACT COMMENSURABILITY AND QUASI-ISOMETRY CLASSIFICATION OF HYPERBOLIC SURFACE GROUP AMALGAMS

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Abstract

Let $\mathcal{X}_S$ denote the class of spaces homeomorphic to two closed orientable surfaces of genus greater than one identified to each other along an essential simple closed curve in each surface. Let $\mathcal{C}_S$ denote the set of fundamental groups of spaces in $\mathcal{X}_S$. In this dissertation, we characterize the abstract commensurability classes in $\mathcal{C}_S$ in terms of the ratio of the Euler characteristic of the surfaces identified and the topological type of the curves identified. We characterize which abstract commensurability classes in $\mathcal{C}_S$ contain a maximal element in $\mathcal{C}_S$. We apply our abstract commensurability classification to prove each group in $\mathcal{C}_S$ is abstractly commensurable to a right-angled Coxeter group; in particular, we show that two subclasses of groups in $\mathcal{C}_S$ embed as finite-index subgroups in right-angled Coxeter groups. We characterize which groups in $\mathcal{C}_S$ are abstractly commensurable to the right-angled Coxeter groups studied by Crisp–Paoluzzi in [CP08], and we exhibit a maximal element within the class of right-angled Coxeter groups for certain abstract commensurability classes in $\mathcal{C}_S$. We prove that all groups in $\mathcal{C}_S$ are quasi-isometric by exhibiting a bilipschitz map between the universal covers of two spaces in $\mathcal{X}_S$. In particular, we prove that the universal covers of any two such spaces may be realized as isomorphic cell complexes with finitely many isometry types of hyperbolic polygons as cells.

Dedication: To my schoolmates, housemates, friends, and family

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Abstract commensurability and quasi-isometry classification of hyperbolic surface group amalgams

Chapter 1.

Introduction

Finitely generated infinite groups carry both an algebraic and a geometric structure, and to study such groups, one may study both algebraic and geometric classifications. Abstract commensurability defines an algebraic equivalence relation on the class of groups, where two groups are said to be *abstractly commensurable* if they contain isomorphic subgroups of finite-index. Finitely generated groups may also be viewed as geometric objects, since a finitely generated group has a natural word metric which is well-defined up to quasi-isometric equivalence. Gromov posed the program of classifying finitely generated groups up to quasi-isometry.

A finitely generated group is quasi-isometric to any finite-index subgroup, so, if two finitely generated groups are abstractly commensurable, then they are quasi-isometric. Two fundamental questions in geometric group theory are to classify the abstract commensurability and quasi-isometry classes within a class of finitely generated groups and to understand for which classes of groups the characterizations coincide.

A basic and motivating example is the class of groups isomorphic to the fundamental group of a closed orientable surface of genus greater than one. These groups act properly discontinuously and cocompactly by isometries on the hyperbolic plane, hence all such groups are quasi-isometric. In addition, every surface of genus greater than one finitely covers the genus two surface, so all groups in this class are abstractly commensurable. In particular, the quasi-isometry and abstract commensurability classifications coincide in this setting. Free groups, which may be realized as the fundamental group of surfaces with non-empty boundary, exhibit the same behavior; there is a unique quasi-isometry and abstract commensurability class among non-abelian free groups.

In this thesis, we present a complete solution to the quasi-isometry and abstract commensurability classification questions within the class $\mathcal{C}_S$ of groups isomorphic to the fundamental group of two closed orientable surfaces of genus greater than one identified along an essential simple closed curve in each. We prove that there is a single quasi-isometry class within $\mathcal{C}_S$ and infinitely many abstract commensurability classes.

1.1 Overview of main results

In Chapter 3, we characterize the abstract commensurability classes within $\mathcal{C}_S$. Our classification uses work of Lafont, who proved that spaces obtained by identifying hyperbolic surfaces with non-empty boundary along their boundary components are *topologically rigid*: any isomorphism between fundamental groups of these spaces is induced by a homeomorphism [Laf07] (see also [CP08]). As a consequence, groups in the class $\mathcal{C}_S$ are abstractly commensurable if and only if the corresponding spaces built by identifying two surfaces along an essential closed curve in each have homeomorphic finite-sheeted covering spaces. We use this fact to obtain topological obstructions to commensurability.

Before stating the full classification theorem, we present two corollaries: the abstract commensurability classification in the case that groups G_1 and G_2 are the fundamental groups of surfaces identified along separating curves, and the abstract commensurability classification in the case that groups G_1 and G_2 are the fundamental groups of surfaces identified along non-separating curves.

Corollary 3.3.5 *If S_1, S_2, S_3, S_4 and T_1, T_2, T_3, T_4 are orientable surfaces of genus greater than or equal to one and with one boundary component, the S_i are glued along their boundary to form X_1, and the T_i are glued along their boundary to form X_2, then $\pi_1(X_1)$ and $\pi_1(X_2)$ are abstractly commensurable if and only if, up to reindexing, the quadruples $(\chi(S_1), \ldots, \chi(S_4))$ and $(\chi(T_1), \ldots, \chi(T_4))$ are equal up to scale.*

Corollary 3.3.6 *If S_{g_i} and $S_{g_i'}$ are orientable surfaces of genus greater than one identified to each other along a non-separating curve in each to form the space X_i for $i = 1, 2$, then $\pi_1(X_1)$ and $\pi_1(X_2)$ are abstractly commensurable if and only if, up to reindexing, $\frac{\chi(S_{g_1})}{\chi(S_{g_1'})} = \frac{\chi(S_{g_2})}{\chi(S_{g_2'})}$.*

The additional condition in the full classification within $\mathcal{C}_S$ given in Theorem 3.3.3 is that a separating curve that divides the surface exactly in half may be replaced by a non-separating curve on the same surface without changing the abstract commensurability class. We use the following notation. If γ is an essential simple closed curve on a surface, the number $t(\gamma)$ is equal to one if γ is non-separating, and is equal to $\frac{\chi(S_{r,1})}{\chi(S_{s,1})}$ if γ separates the surface into two subsurfaces $S_{r,1}$ and $S_{s,1}$ and $\chi(S_{r,1}) \leq \chi(S_{s,1})$. Our full classification theorem is given as follows.

Theorem 3.3.3. *If $G_1, G_2 \in \mathcal{C}_S$, then G_1 and G_2 are abstractly commensurable if and only if, up to relabeling,$G_1 \cong \pi_1(S_{g_1}) *_{\langle a_1 \rangle} \pi_1(S_{g_1'})$ and $G_2 \cong \pi_1(S_{g_2}) *_{\langle a_2 \rangle} \pi_1(S_{g_2'})$, the amalgams are given by the monomorphisms $a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i})$ and $a_i \mapsto [\gamma_i'] \in \pi_1(S_{g_i'})$, and the following conditions hold:*

$$\text{(a)} \quad \frac{\chi(S_{g_1})}{\chi(S_{g_1'})} = \frac{\chi(S_{g_2})}{\chi(S_{g_2'})}, \qquad \text{(b)} \quad t(\gamma_1) = t(\gamma_2), \qquad \text{(c)} \quad t(\gamma_1') = t(\gamma_2').$$

Let $\mathcal{G} \subset \mathcal{C}_S$ be an abstract commensurability class within $\mathcal{C}_S$. A *maximal element* for $\mathcal{G}$ is a group G_0 that contains every group in $\mathcal{G}$ as a finite-index subgroup. In Chapter 3.4, we show that for abstract commensurability classes $\mathcal{G} \subset \mathcal{C}_S$, the existence of a maximal element $G_0 \in \mathcal{C}_S$ depends on whether the class contains the fundamental group of surfaces identified along non-separating curves. In Proposition 3.4.4, we prove that a maximal element in $\mathcal{C}_S$ exists if and only if the abstract commensurability class does not contain the fundamental group of two surfaces identified along a non-separating curve in either surface.

In Chapter 3.5, we show that if the abstract commensurability class $\mathcal{G} \subset \mathcal{C}_S$ contains the fundamental group of two surfaces identified along non-separating curves in both surfaces, then there exists a right-angled Coxeter group that is a maximal element for the class. In the remaining case, that the class contains the fundamental group of two surfaces identified along a non-separating curve in exactly one of the surfaces and does not contain the fundamental group of two surfaces identified along a non-separating curve in both, Proposition 3.4.4 shows there is no maximal element in $\mathcal{C}_S$, and the existence of a maximal element outside of $\mathcal{C}_S$ remains open.

The quasi-isometry classification within $\mathcal{C}_S$ stands in contrast to the abstract commensurability classification. A *quasi-isometry* is a map between metric spaces that distorts distances by uniformly bounded multiplicative and additive factors. Such maps do not capture local structure, but rather large-scale, coarse geometry. The geometry of a finitely generated group is defined up to quasi-isometry by a word metric on the group, and this geometry may also be studied via a *geometric* action of a group on a metric space. This point of view, that the geometry of a finitely generated group may be identified with the quasi-isometry type of a *model space*, is central to this thesis. Groups in the class $\mathcal{C}_S$ act geometrically on a piecewise hyperbolic CAT(-1) space built by identifying infinitely many copies of the hyperbolic plane along geodesic lines in a 'tree-like' fashion. The following theorem, proven in Chapter 4.3, states that all such spaces have the same large-scale geometry; the quasi-isometry classification follows as a consequence.

Theorem 4.3.1. *Let $\mathcal{X}_S$ denote the class of spaces homeomorphic to two closed orientable surfaces of genus greater than one identified along an essential simple closed curve in each. If $X_1, X_2 \in \mathcal{X}_S$ and $\widetilde{X}_1$ and $\widetilde{X}_2$ are their universal covers equipped with a CAT(-1) metric that is hyperbolic on each surface, then there exists a bilipschitz equivalence $\phi : \widetilde{X}_1 \to \widetilde{X}_2$.*

Corollary 4.3.2. *If $G_1, G_2 \in \mathcal{C}_S$, then G_1 and G_2 are quasi-isometric.*

Our approach in the proof of Theorem 4.3.1 is to realize $\widetilde{X}_1$ and $\widetilde{X}_2$ as isomorphic cell complexes with finitely many isometry types of convex hyperbolic polygons as cells. We show there is a bilipschitz equivalence between hyperbolic n-gons that restricts to dilation on each edge. Thus, there is a well-defined cellular homeomorphism $\widetilde{X}_1 \to \widetilde{X}_2$ that restricts to a bilipschitz map on each tile, and we prove this extends to a bilipschitz map $\widetilde{X}_1 \to \widetilde{X}_2$.

Groups in the class $\mathcal{C}_S$ also admit a CAT(0) geometry, and an alternative approach to the quasi-isometry classification was given by Malone [Mal10], who applied the work of Behrstock–Neumann on the bilipschitz equivalence of fattened trees used in the quasi-isometric classification of graph manifold groups [BN08].

1.2 Historical context and related results

Commensurability has early foundations in the work of Euclid: in the *Elements*, two line segments are said to be commensurable if there is a third segment c that, when copies are laid end-to-end, evenly covers both a and b. Viewing a and b as real numbers, a and b are commensurable if and only if $\frac{a}{b}$ is rational. Research has developed to generalize commensurability in both the topological and algebraic setting. For example, commensurability classes of hyperbolic 3-manifolds is an active area of study, and in this setting, two manifolds that are commensurable have volumes that are commensurable in the sense of Euclid. Recent surveys on notions of commensurability are given by Paoluzzi [Pao13] and Walsh [Wal11].

In Proposition 3.4.4, we characterize the abstract commensurability classes within $\mathcal{C}_S$ that contain a maximal element in $\mathcal{C}_S$. A classic result in the setting of hyperbolic 3-manifolds is that of Margulis [Mar75], who proved that if $H \leq PSL(2, \mathbb{C})$ is a discrete subgroup of finite covolume, then there exists a maximal element in the abstract commensurability class of H if and only if H is non-arithmetic. It follows that the commensurability class of a non-arithmetic finite-volume hyperbolic 3-manifold contains a *minimal element*: there exists an orbifold finitely covered by every other manifold in the commensurability class.

The abstract commensurability classes within $\mathcal{C}_S$ are finer than the quasi-isometry classes; there is a unique quasi-isometry class in $\mathcal{C}_S$ and there are infinitely many abstract commensurability classes. Whyte, in [Why99], proves a similar result for free products of hyperbolic surface groups.

Theorem 1.2.1. ([Why99], Theorem 1.6, 1.7) *Let Σ_g be the fundamental group of a surface of genus $g \geq 2$ and let $m, n \geq 2$. Let $\Gamma_1 \cong \Sigma_{a_1} * \Sigma_{a_2} * \ldots * \Sigma_{a_n}$ and $\Gamma_2 \cong \Sigma_{b_1} * \Sigma_{b_2} * \ldots * \Sigma_{b_m}$. Then Γ_1 and Γ_2 are quasi-isometric, and Γ_1 and Γ_2 are abstractly commensurable if and only if*

$$\frac{\chi(\Gamma_1)}{n-1} = \frac{\chi(\Gamma_2)}{m-1}.$$

On the other hand, there are many classes of groups for which the quasi-isometry and abstract commensurability classifications coincide. Such classes include non-trivial free products of finitely many finitely generated abelian groups excluding

$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ [BJN09], non-uniform lattices in the isometry group of a symmetric space of strictly negative sectional curvature other than the hyperbolic plane [Sch95], and fundamental groups of n-dimensional ($n \geq 3$) connected complete finite-volume hyperbolic manifolds with nonempty geodesic boundary (which must be compact in dimension three) [Fri06].

This dissertation concerns surfaces of negative Euler characteristic; Cashen in [Cas10] provides a quasi-isometry classification of the fundamental groups of a disjoint union of (Euclidean) tori glued together along annuli.

Hyperbolic surface groups are finite-index subgroups of right-angled Coxeter groups. We apply our abstract commensurability classification (Theorem 3.3.3) to prove, in Proposition 3.5.6, that each group in $\mathcal{C}_S$ is abstractly commensurable to a right-angled Coxeter group. In other words, each abstract commensurability class of a group in $\mathcal{C}_S$ contains a right-angled Coxeter group. In particular, in Chapter 3.5, we show the fundamental group of two surfaces identified along a separating curve in each and the fundamental group of two surfaces identified along curves of *topological type one* (See definition 3.2.1) are finite-index subgroups of a right-angled Coxeter group. It is an open question whether each group in $\mathcal{C}_S$ is a finite-index subgroup of a right-angled Coxeter group in the remaining case.

The result in Theorem 3.3.3 is related to the abstract commensurability classification of the right-angled Coxeter groups introduced by Crisp–Paoluzzi in [CP08] and further studied by Dani–Thomas in [DT14]. Let

$$W_{m,n} = W(\Gamma_{m,n}),$$

be the right-angled Coxeter group associated to the graph $\Gamma_{m,n}$, which consists of a circuit of length $m+4$ and a circuit of length $n+4$ which are identified along a common subpath of edge-length 2. For all m and n, the group $W_{m,n}$ is the orbifold fundamental group of a 2-dimensional reflection orbi-complex $\mathcal{O}_{m,n}$. We show in Lemma 3.5.7 that for all m and n, $\mathcal{O}_{m,n}$ is finitely covered by a space consisting of two hyperbolic surfaces identified along non-separating essential simple closed curves. Conversely, we prove all amalgams of surface groups over non-separating essential simple closed curves are finite index subgroups of $W_{m,n}$ for some m and n, dependent on the Euler characteristic of the two surfaces. Thus, our theorem extends their result.

Corollary 1.2.2. ([CP08] Theorem 1.1) *Let* $1 \leq m \leq n$ *and* $1 \leq k \leq \ell$. *Then* $W_{m,n}$ *and* $W_{k,\ell}$ *are abstractly commensurable if and only if* $\frac{m}{n} = \frac{k}{\ell}$.

Moreover, in Proposition 3.5.9, we apply our abstract commensurability classification to prove that if $G \in \mathcal{C}_S$, then G is abstractly commensurable to $W_{m,n}$ for some m and n if and only if G is the fundamental group of two surfaces identified to each other along curves of *topological type one* (see Definition 3.2.1).

1.3 Outline

In Chapter 2, we define the spaces $\mathcal{X}_S$ and the class of groups $\mathcal{C}_S$ examined in this dissertation. Chapter 3 contains the abstract commensurability classification, the characterization of maximal elements, and a description of the relation of groups in $\mathcal{C}_S$ to the class of right-angled Coxeter groups. In Chapter 4, we define a piecewise hyperbolic metric on spaces in $\mathcal{X}_S$, construct a bilipschitz equivalence between the universal covers of any such spaces, and conclude all groups in $\mathcal{C}_S$ are quasi-isometric.

Chapter 2.

Surfaces and the class of groups $\mathcal{C}_S$

We use $S_{g,b}$ to denote the orientable surface of genus g and b boundary components. The *Euler characteristic* of a surface $S_{g,b}$ is $\chi(S_{g,b}) = 2 - 2g - b$. Unless stated otherwise, we will say "surface" to mean a compact, connected, oriented surface. We will typically be interested in surfaces of negative Euler characteristic.

We say a surface S *admits a hyperbolic metric* if there exists a complete, finite-area Riemannian metric on S of constant curvature -1 and the boundary of S is totally geodesic: the geodesics in ∂S are geodesics in S. A surface S may be endowed with a hyperbolic metric via a free and properly discontinuous action by isometries of $\pi_1(S)$ on the hyperbolic plane $\mathbb{H}^2$.

Theorem 2.0.1. *If S is a surface with $\chi(S) < 0$, then S admits a hyperbolic metric.*

A *closed curve* in a surface S is a continuous map $S^1 \to S$, and we often identify a closed curve with its image in S. We use $[\gamma]$ to denote the homotopy class of a curve γ. A closed curve is *essential* if it is not homotopic to a point or boundary component. An essential closed curve γ is *primitive* if is not the case that $[\gamma] = [\rho^n]$ for some closed curve ρ. A closed curve is *simple* if it is embedded. A *homotopy class of simple closed curves* is a homotopy class in which there exists a simple closed curve representative. A *multicurve* in S is the union of a finite collection of disjoint simple closed curves in S.

If γ is a simple closed curve on a surface S, the surface obtained by *cutting* S along γ is a compact surface S_γ equipped with a homeomorphism h between these two boundary components of S_γ so that the quotient $S_\gamma/(x \sim h(x))$ is homeomorphic to S and the image of these distinguished boundary components under the quotient map is γ.

If X_1 and X_2 are topological spaces and $A_1 \subset X_1$, $A_2 \subset X_2$ so that $A_1 \cong A_2$, we say X is obtained by *identifying* X_1 and X_2 along A_1 and A_2 if $X = X_1 \sqcup X_2/(x \sim h(x))$ for some homeomorphism $h : A_1 \to A_2$ and all $x \in A_1$. If A is the image of A_1 and A_2 under the quotient map, we denote the space X as $X = X_1 \cup_A X_2$.

Let $\mathcal{X}$ denote the class of spaces homeomorphic to two hyperbolic surfaces identified along an essential closed curve in each. Let $\mathcal{X}_S \subset \mathcal{X}$ be the subclass in which the curves that are identified are simple. Let $\mathcal{C}$ be the class of groups isomorphic to the fundamental group of a space in $\mathcal{X}$, and let $\mathcal{C}_S \subset \mathcal{C}$ be the subclass of groups isomorphic to the fundamental group of a space in $\mathcal{X}_S$. If $G \in \mathcal{C}$ then $G \cong \pi_1(S_g) *_{\langle\gamma\rangle} \pi_1(S_h)$, the amalgamated free product of two hyperbolic surface groups over $\mathbb{Z}$. We suppress in our notation the monomorphisms $i_g : \langle\gamma\rangle \to \pi_1(S_g)$ and $i_h : \langle\gamma\rangle \to \pi_1(S_h)$ given by $i_g : \gamma \mapsto [\gamma_g]$, $i_h : \gamma \mapsto [\gamma_h]$, where $\gamma_g : S^1 \to S_g$ and $\gamma_h : S^1 \to S_h$. Note that if $X \in \mathcal{X}_S$ consists of two surfaces identified to each other along separating curves, $\pi_1(X)$ may be expressed as an amalgamated free product of surface groups in up to three ways.

Chapter 3.

Abstract commensurability classes within $\mathcal{C}_S$

There are many notions of commensurability in group theory and topology. The first step taken in our abstract commensurability classification is to translate this algebraic question into a topological one, as described in the following section.

3.1 Finite covers and topological rigidity

A description of the subgroup structure of an amalgamated free product is given in the following theorem of Scott and Wall.

Theorem 3.1.1. ([SW79], Theorem 3.7) *If $G \cong A *_C B$ and if $H \leq G$, then H is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of A or B and the edge groups are subgroups of conjugates of C.*

Any finite sheeted cover of the space $X = S_g \cup_\gamma S_h$, where γ is the image of $\gamma_g : S^1 \to S_g$ and $\gamma_h : S^1 \to S_h$ under identification, consists of a set of surfaces which cover S_g and a set of surfaces which cover S_h, identified along multicurves that are the preimages of γ_g and γ_h. These covers are examples of *simple, thick, 2-dimensional hyperbolic P-manifolds* (see [Laf07], Definition 2.3.) The following *topological rigidity* theorem of Lafont allows us to address the abstract commensurability classification for members in $\mathcal{C}_S$ from a topological point of view. Corollary 3.1.3 also follows from the proof of Proposition 3.1 in [CP08].

Theorem 3.1.2. ([Laf07], Theorem 1.2) *Let* X_1 *and* X_2 *be a pair of simple, thick, 2-dimensional hyperbolic* P*-manifolds, and assume that* $\phi : \pi_1(X_1) \to \pi_1(X_2)$ *is an isomorphism. Then there exists a homeomorphism* $\Phi : X_1 \to X_2$ *that induces* ϕ *on the level of fundamental groups.*

Corollary 3.1.3. *Let* $G, G' \in \mathcal{C}_S$ *with* $G \cong \pi_1(X)$*,* $G' \cong \pi_1(X')$ *and* $X, X' \in \mathcal{X}_{\mathcal{S}}$*. Then* G *and* G' *are abstractly commensurable if and only if* X *and* X' *have homeomorphic finite-sheeted covering spaces.*

We will make repeated use of the following lemma.

Lemma 3.1.4. *If* X *is a CW-complex and* X' *is a degree* n *cover of* X*, then* $\chi(X') = n\chi(X)$*, where* χ *denotes Euler characteristic.*

3.2 Statement of the classification and outline of the proof

The abstract commensurability classification in the class $\mathcal{C}_S$ is given in terms of the ratio of the Euler characteristic of the surfaces identified and the *topological type* of the curves identified, which is defined as follows. An essential simple closed curve γ on a surface S is *non-separating* if $S \backslash \gamma$ is connected and is *separating* if $S \backslash \gamma$ consists of two connected surfaces, $S_{r,1}$ and $S_{s,1}$, of lower genus and a single boundary component.

Definition 3.2.1. The *topological type* of an essential simple closed curve $\gamma : S^1 \to S$, denoted $t(\gamma)$, is equal to one if the curve is non-separating and equal to $\frac{\chi(S_{r,1})}{\chi(S_{s,1})}$ if the curve separates S into subsurfaces $S_{r,1}$ and $S_{s,1}$ and $\chi(S_{r,1}) \leq \chi(S_{s,1})$.

Theorem 3.3.3. (Abstract commensurability classification within $\mathcal{C}_S$.) *If* $G_1, G_2 \in \mathcal{C}_S$*, then* G_1 *and* G_2 *are abstractly commensurable if and only if, up* *to relabeling,* $G_1 \cong \pi_1(S_{g_1}) *_{\langle a_1\rangle} \pi_1(S_{g_1'})$ *and* $G_2 \cong \pi_1(S_{g_2}) *_{\langle a_2\rangle} \pi_1(S_{g_2'})$*, the amalgams are given by the monomorphisms* $a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i})$ *and* $a_i \mapsto [\gamma_i'] \in \pi_1(S_{g_i'})$*, and the following conditions hold.*

$$\text{(a)}\quad \frac{\chi(S_{g_1})}{\chi(S_{g_1'})} = \frac{\chi(S_{g_2})}{\chi(S_{g_2'})}, \qquad \text{(b)}\quad t(\gamma_1) = t(\gamma_2), \qquad \text{(c)}\quad t(\gamma_1') = t(\gamma_2').$$

One direction of the proof is constructive: if $G_1 \cong \pi_1(X_1)$ and $G_2 \cong \pi_1(X_2)$ satisfy the conditions of the theorem, we construct a common (regular) cover of the spaces X_1 and X_2. The other direction of the proof has three steps:

(1) Construct finite covers $p_i : Y_i \to X_i$ so that Y_i consists of four surfaces each with two boundary components, one colored red and one colored blue; all red boundary components are identified and all blue boundary components are identified to form the connected space Y_i with two singular curves; and, $\chi(Y_1) = \chi(Y_2)$. The existence of such covers is proven in Lemma 3.3.1, and an example of these covers is given in Figure 1.

(2) Apply Proposition 3.3.2, which generalizes [Mal10, Theorem 5.3], and proves that since G_1 and G_2 are abstractly commensurable, the finite covers Y_1 and Y_2 are homeomorphic.

(3) Use the covering maps p_1 and p_2 to label the surfaces in X_1 and X_2 so that G_1 and G_2 are expressed as in the theorem and the conditions (a), (b), and (c) hold.

3.3 Abstract commensurability classification

In this section we prove Theorem 3.3.3, characterizing the abstract commensurability classes in $\mathcal{C}_S$. To prove the conditions in the theorem are necessary, the first step, denoted (1) above, is to take covers of spaces $X_1, X_2 \in \mathcal{X}_S$ with abstractly commensurable fundamental groups so that the covers of X_1 and X_2 have equal Euler characteristic.

Lemma 3.3.1. *If $X_1, X_2 \in \mathcal{X}_S$, then there exist finite-sheeted covers $p_i : Y_i \to X_i$ so that Y_i consists of four surfaces each with two boundary components, one colored red and one colored blue; all red boundary components are identified and all blue boundary components are identified to form the connected space Y_i with two singular curves; and, $\chi(Y_1) = \chi(Y_2)$.*

Proof. Let $X_1, X_2 \in \mathcal{X}_S$. Let

$$L = -2 \cdot \ell cm(|\chi(X_1)|, |\chi(X_2)|)$$

and

$$d_i = \frac{L}{\chi(X_i)}.$$

Suppose $X_1 = S_{h_1} \cup_{c_1} S_{h'_1}$ and $X_2 = S_{h_2} \cup_{c_2} S_{h'_2}$ where c_i identifies the curves $\rho_i : S^1 \to S_{h_i}$ and $\rho'_i : S^1 \to S_{h'_i}$. To build the covers Y_i, first let $\widetilde{S_{h_i}}$ be a 2-fold cover of S_{h_i} so that ρ_i has two preimages in the cover: if ρ_i is non-separating, cut along ρ_i, take two copies of the resulting surface with boundary, and re-glue the boundary components in pairs; if ρ_i is separating, cut along a non-separating essential simple closed curve in each of the subsurfaces bounded by ρ_i, take two copies of the resulting surface with boundary, and re-glue the boundary components in pairs. An example of these degree two covers appears in Figure 1. Next, cut along a non-separating curve in the cover $\widetilde{S_{h_i}}$ that intersects each curve in the pre-image of ρ_i in exactly one point. Take $\frac{d_i}{2}$ copies of the resulting surface with two boundary components and reglue the boundary components in pairs to get a surface $\widehat{S_{h_i}}$ which forms a $\frac{d_i}{2}$-fold cyclic cover of $\widetilde{S_{h_i}}$ and so that ρ_i has two preimages in $\widehat{S_{h_i}}$, each of which covers ρ_i by degree $\frac{d_i}{2}$. Construct $\widehat{S_{h'_i}}$ in the same way. Identify the two components of the preimage of ρ_i in $\widehat{S_{h_i}}$ with the two components of the preimage of ρ'_i in $\widehat{S_{h'_i}}$ in pairs to form Y_i, a d_i-fold cover of X_i. An example of these covers is illustrated in Figure 1. By construction, $\chi(Y_1) = \chi(Y_2) = L$. ∎

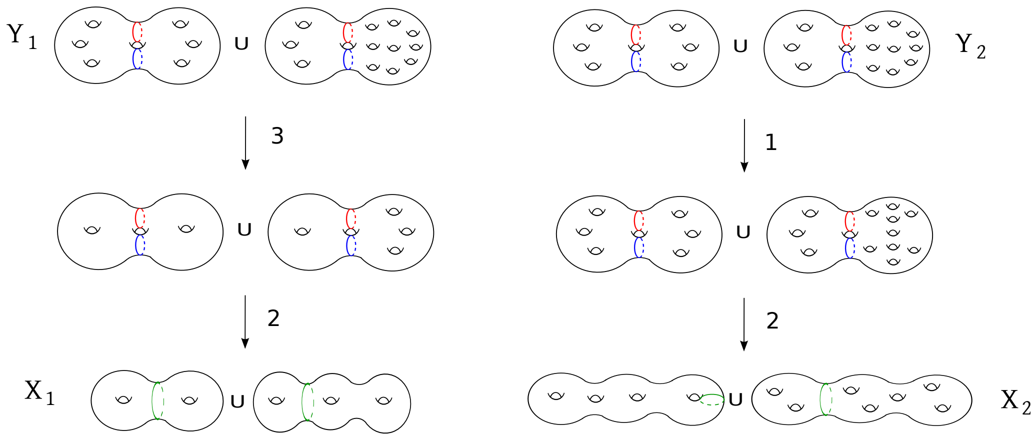

FIGURE 1. Above is an example of the covers $p_i : Y_i \to X_i$ constructed in Lemma 3.3.1. In each union, the two curves of the same color are glued together to form singular curves. In this example, $\pi_1(X_1)$ and $\pi_1(X_2)$ are abstractly commensurable; one can check that conditions (a), (b), and (c) hold.

We will apply the following proposition (with $r = 4$ and $n = 2$). The idea to restrict to the setting of spaces with equal Euler characteristic appears in [Mal10, Theorem 5.3], though the proof there has a small gap in the inductive step. In our proof, below, we complete Malone's proof and generalize his result.

Proposition 3.3.2. *Let $G_1 \cong \pi_1(X_1)$ and $G_2 \cong \pi_1(X_2)$ where*

$$X_1 = \bigcup_{i=1}^{r} S_i \quad \text{and} \quad X_2 = \bigcup_{i=1}^{r} T_i;$$

$r \geq 3$; S_i is a surface with n boundary components $\{\beta_{i1}, \ldots, \beta_{in}\}$; boundary components β_{ij} and β_{kj} are identified for all $1 \leq j \leq n$ and $1 \leq i \leq k \leq r$ so there are n singular curves in X_1; and X_2 is similar. Suppose that $\chi(S_1) \leq \ldots \leq \chi(S_r)$, $\chi(T_1) \leq \ldots \leq \chi(T_r)$, and $\chi(X_1) = \chi(X_2)$. Then G_1 and G_2 are abstractly commensurable if and only if $S_i \cong T_i$ for all $1 \leq i \leq r$.

Proof. Suppose G_1 and G_2 are abstractly commensurable. Then there exist finite covers $p_1 : \hat{X}_1 \to X_1$ and $p_2 : \hat{X}_2 \to X_2$ with $\pi_1(\hat{X}_1) \cong \pi_1(\hat{X}_2)$. Since $\chi(X_1) = \chi(X_2)$, the covering maps p_1 and p_2 have the same degree, d. By Theorem 3.1.2, there exists a homeomorphism $f : \hat{X}_1 \to \hat{X}_2$ inducing the isomorphism between $\pi_1(\hat{X}_1)$ and $\pi_1(\hat{X}_2)$.

Suppose

$$\chi(S_1) = \ldots = \chi(S_s) < \chi(S_{s+1}) \leq \ldots \leq \chi(S_r) \tag{1}$$

$$\chi(T_1) = \ldots = \chi(T_t) < \chi(T_{t+1}) \leq \ldots \leq \chi(T_r) \tag{2}$$

for some $s, t \leq r$. Without loss of generality, $\chi(S_1) \leq \chi(T_1)$ and if $\chi(S_1) = \chi(T_1)$, then $s \geq t$.

Consider the full preimage in $\hat{X}_1$ of the surfaces $S_1, \ldots, S_s$ of least Euler characteristic in X_1. Let

$$\mathcal{A}_i = p_1^{-1}(S_i).$$

The surface $\mathcal{A}_i$ may be disconnected; suppose $\mathcal{A}_i$ is the disjoint union of k_i connected surfaces,

$$\mathcal{A}_i = \bigsqcup_{j=1}^{k_i} A_{ij}.$$

Each component $f(A_{ij})$ of $f(\mathcal{A}_i)$ covers some surface $T_{ij} \in \{T_1, \ldots, T_r\} \subset X_2$ under the covering map p_2. Suppose $p_2 : f(A_{ij}) \to T_{ij}$ is a degree d_{ij} cover. For each i, the sum of the degrees d_{ij} is equal to d since the boundary of $f(\mathcal{A}_i)$ is the full preimage of the n singular curves in X_2 and no component of the preimage of the singular curves is incident to more than one component of $f(\mathcal{A}_i)$. Thus,

$$\begin{aligned} d \cdot \chi(S_1) &= \sum_{j=1}^{k_1} \chi(A_{1j}) \\ &= \sum_{j=1}^{k_1} \chi(f(A_{1j})) \\ &= \sum_{j=1}^{k_1} d_{1j} \cdot \chi(T_{1j}) \\ &\geq \chi(T_1) \cdot \sum_{j=1}^{k_1} d_{1j} \\ &= d \cdot \chi(T_1) \end{aligned}$$

Since $\chi(S_1) \leq \chi(T_1)$ by assumption, $\chi(S_1) = \chi(T_1)$. Each singular curve in $\hat{X}_2$ is incident to s surfaces in $f(\mathcal{A}_1) \cup \ldots \cup f(\mathcal{A}_s)$, so $p_2(f(\mathcal{A}_1) \cup \ldots \cup f(\mathcal{A}_s))$ must have in its image at least s surfaces in X_2, each of which must have Euler characteristic equal to $\chi(S_1)$ by the above argument. Thus, since $s \leq t$, we have $\chi(S_i) = \chi(T_i)$ for $1 \leq i \leq s = t$. Moreover, $p_1^{-1}\left(\bigcup_{i=1}^{s} S_i\right) = p_2^{-1}\left(\bigcup_{i=1}^{s} T_i\right)$, so the above argument can be repeated (at most finitely many times) with the remaining surfaces in X_1 and X_2 of strictly larger Euler characteristic, proving the claim.

The other direction of the statement is clear: if $a_i = b_i$ for $1 \leq i \leq r$, then $\pi_1(G_1) \cong \pi_1(G_2)$, so G_1 and G_2 are abstractly commensurable. □

Remark: The condition that $\chi(X_1) = \chi(X_2)$ can be omitted from the above proposition, and we get the conclusion that $\frac{\chi(S_i)}{\chi(T_i)} = c$ for some constant c and all $1 \leq i \leq r$. This generalization appears in upcoming joint work with Pallavi Dani and Anne Thomas on abstract commensurability classes of certain right-angled Coxeter groups.

Theorem 3.3.3. *If $G_1, G_2 \in \mathcal{C}_S$, then G_1 and G_2 are abstractly commensurable if and only if they may be expressed as $G_1 \cong \pi_1(S_{g_1}) *_{\langle a_1 \rangle} \pi_1(S_{g_1'})$ and $G_2 \cong$*

$\pi_1(S_{g_2}) *_{\langle a_2 \rangle} \pi_1(S_{g_2'})$, given by the monomorphisms $a_i \mapsto [\gamma_i] \in \pi_1(S_{g_i})$ and $a_i \mapsto [\gamma_i'] \in \pi_1(S_{g_i'})$, and the following conditions hold.

$$\text{(a)}\quad \frac{\chi(S_{g_1})}{\chi(S_{g_1'})} = \frac{\chi(S_{g_2})}{\chi(S_{g_2'})}, \qquad \text{(b)}\quad t(\gamma_1) = t(\gamma_2), \qquad \text{(c)}\quad t(\gamma_1') = t(\gamma_2').$$

Proof. Let $X_1, X_2 \in \mathcal{X}_S$. By Lemma 3.3.1, there exist covering spaces $p_1 : Y_1 \to X_1$ and $p_2 : Y_2 \to X_2$ so that $\chi(Y_1) = \chi(Y_2)$,

$$Y_1 = \bigcup_{i=1}^{4} S_i \quad \text{and} \quad Y_2 = \bigcup_{i=1}^{4} T_i;$$

the connected surfaces S_i in Y_1 have two boundary components, one colored red and one colored blue; all red boundary components are identified and all blue boundary components are identified; and likewise for Y_2.

Suppose $G_1 \cong \pi_1(X_1)$ and $G_2 \cong \pi_1(X_2)$ are abstractly commensurable, so $\pi_1(Y_1)$ and $\pi_1(Y_2)$ are abstractly commensurable. By Proposition 3.3.2, $S_i \cong T_i$ for $1 \leq i \leq 4$. The conditions of the theorem require a labeling of the surfaces and amalgamated curves in X_1 and X_2. Thus, it remains to assign S_{g_i}, $S_{g_i'}$, γ_i, and γ_i' for $i = 1, 2$ that satisfy conditions (a), (b), and (c). This assignment depends on whether the original curves ρ_i and ρ_i' are separating or non-separating. Let $p_1 : Y_1 \to X_1$ and $p_2 : Y_2 \to X_2$ be the covering maps constructed above.

If the curves ρ_i and ρ_i' are separating for $i = 1, 2$, suppose $\chi(S_i) \leq \chi(S_j)$ for $i \leq j$. Let

$$S_{g_1} = p_1(S_1) \cup_{\gamma_1} p_1(S_2) \quad \text{and} \quad S_{g_1'} = p_1(S_3) \cup_{\gamma_1'} p_1(S_4)$$

be the surfaces obtained by identifying $p_1(S_i)$ along their boundary curves and let γ_i and γ_i' be the images of the boundary curves. Similarly, let

$$S_{g_2} = p_2(T_1) \cup_{\gamma_2} p_2(T_2) \quad \text{and} \quad S_{g_2'} = p_2(T_3) \cup_{\gamma_2'} p_2(T_4).$$

One can easily check that the conditions of the theorem hold:

$$\begin{aligned} t(\gamma_1) &= \frac{\chi(p_1(S_1))}{\chi(p_1(S_2))} \\ &= \frac{\frac{\chi(S_1)}{d_1}}{\frac{\chi(S_2)}{d_1}} \\ &= \frac{\frac{\chi(S_1)}{d_2}}{\frac{\chi(S_2)}{d_2}} \\ &= \frac{\frac{\chi(T_1)}{d_2}}{\frac{\chi(T_2)}{d_2}} \\ &= \frac{\chi(p_2(T_1))}{\chi(p_2(T_2))} \\ &= t(\gamma_2), \end{aligned}$$

and an analogous calculation shows $t(\gamma_1') = t(\gamma_2')$, proving claims (b) and (c). Similarly,

$$\begin{aligned} \frac{\chi(S_{g_1})}{\chi(S_{g_1'})} &= \frac{\chi(p_1(S_1 \cup S_2))}{\chi(p_1(S_3 \cup S_4))} \\ &= \frac{\frac{\chi(S_1 \cup S_2)}{d_1}}{\frac{\chi(S_3 \cup S_4)}{d_1}} \\ &= \frac{\frac{\chi(S_1 \cup S_2)}{d_2}}{\frac{\chi(S_3 \cup S_4)}{d_2}} \\ &= \frac{\frac{\chi(T_1 \cup T_2)}{d_2}}{\frac{\chi(T_3 \cup T_4)}{d_2}} \\ &= \frac{\chi(p_2(T_1 \cup T_2))}{\chi(p_2(T_3 \cup T_4))} \\ &= \frac{\chi(S_{g_2})}{\chi(S_{g_2'})}, \end{aligned}$$

establishing (a) in this case.

Otherwise, at least one amalgamating curve ρ_i or ρ_i' is non-separating for $i = 1$ or $i = 2$. By the construction of the covers $p_i : Y_i \to X_i$, this situation implies $S_i \cong S_j$ for some $i \neq j$. Let k and ℓ denote the other indices. There are now three cases: among the S_i (and $T_i \cong S_i$) either two, three, or four of these connected surfaces with boundary are homeomorphic.

If neither S_k nor S_ℓ is homeomorphic to S_i, define

$$S_{g_1} = p_1(S_i) \cup_{\gamma_1} p_1(S_j),$$

$$S_{g_1'} = p_1(S_k) \cup_{\gamma_1'} p_1(S_\ell),$$

$$S_{g_2} = p_2(T_i) \cup_{\gamma_2} p_2(T_j),$$

$$S_{g_2'} = p_2(T_k) \cup_{\gamma_2'} p_2(T_\ell).$$

If, without loss of generality, $S_k \cong S_i$ and $S_\ell \neq S_i$, let S_{g_1} and S_{g_2} be the surfaces covered by two of $\{S_i, S_j, S_k\}$, and let $S_{g_1'}$ and $S_{g_2'}$ be covered by the remaining two subsurfaces. Let γ_i and γ_i' be the images of the boundary curves under the covering maps. Finally, if all four surfaces S_i are homeomorphic, define $(S_{g_i}, \gamma_i) = (S_{h_i}, \rho_i)$ and $(S_{g_i'}, \gamma_i') = (S_{h_i'}, \rho_i')$ to be the spaces given by the original labeling. In all three cases, conditions (a), (b), and (c) are verified in a manner similar to that above.

Suppose now that G_1 and G_2 are expressed as in the statement of the theorem and that conditions (a), (b), and (c) hold. Let $X_1 = S_{g_1} \cup_{c_1} S_{g_1'}$ and $X_2 = S_{g_2} \cup_{c_2} S_{g_2'}$ be the corresponding spaces where c_i identifies the essential simple closed curves $\gamma_i : S^1 \to S_{g_i}$ and $\gamma_i' : S^1 \to S_{g_i'}$. Construct finite covers $p_1 : Y_1 \to X_1$ of degree d_1 and $p_2 : Y_2 \to X_2$ of degree d_2 as in Lemma 3.3.1, with S_{g_i}, $S_{g_i'}$, γ_i, and γ_i' replacing S_{h_i}, $S_{h_i'}$, ρ_i, and ρ_i', respectively. We claim that Y_1 and Y_2 are homeomorphic. Let

$$S_1 \cup S_2 = p_1^{-1}(S_{g_1}),$$

$$S_3 \cup S_4 = p_1^{-1}(S_{g_1'}),$$

$$T_1 \cup T_2 = p_2^{-1}(S_{g_2}),$$

$$T_3 \cup T_4 = p_2^{-1}(S_{g_2'}).$$

Suppose $\chi(S_1) \leq \chi(S_2)$, $\chi(S_3) \leq \chi(S_4)$, $\chi(T_1) \leq \chi(T_2)$, and $\chi(T_3) \leq \chi(T_4)$; we use the conditions of the theorem to show $S_i \cong T_i$ for $1 \leq i \leq 4$. Since

$$\begin{aligned} d_1 \cdot \chi(S_{g_1}) &= \chi(S_1 \cup S_2), \\ d_1 \cdot \chi(S_{g_1'}) &= \chi(S_3 \cup S_4), \\ d_2 \cdot \chi(S_{g_2}) &= \chi(T_1 \cup T_2), \\ d_2 \cdot \chi(S_{g_2'}) &= \chi(T_3 \cup T_4), \end{aligned}$$

by condition (a),

$$\begin{aligned} \frac{\chi(S_1 \cup S_2)}{\chi(S_3 \cup S_4)} &= \frac{\chi(S_{g_1})}{\chi(S_{g_1'})} \\ &= \frac{\chi(S_{g_2})}{\chi(S_{g_2'})} \\ &= \frac{\chi(T_1 \cup T_2)}{\chi(T_3 \cup T_4)}. \end{aligned}$$

Since $\chi(Y_1) = \chi(Y_2) = L$,

$$\chi(S_1 \cup S_2) + \chi(S_3 \cup S_4) = \chi(T_1 \cup T_2) + \chi(T_3 \cup T_4),$$

hence

$$\chi(S_1 \cup S_2) = \chi(T_1 \cup T_2), \tag{3}$$

$$\chi(S_3 \cup S_4) = \chi(T_3 \cup T_4).$$

By condition (b), $t(\gamma_1) = t(\gamma_2)$. If $t(\gamma_i) = 1$, then by construction $\chi(S_1) = \chi(S_2) = \chi(T_1) = \chi(T_2)$. Otherwise,

$$\begin{aligned} \frac{\chi(S_1)}{\chi(S_2)} &= t(\gamma_1) \\ &= t(\gamma_2) \\ &= \frac{\chi(T_1)}{\chi(T_2)}, \end{aligned}$$

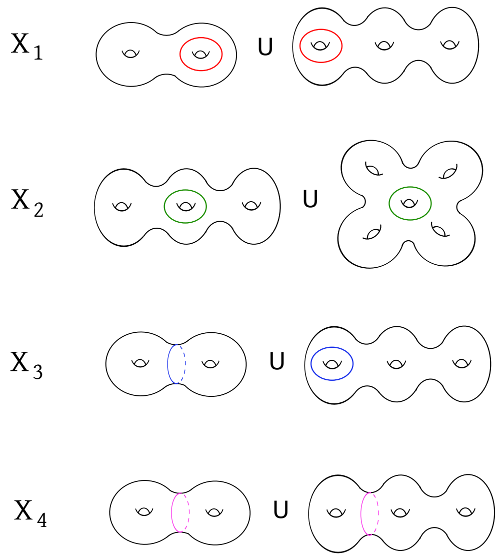

FIGURE 2. *Example:* The groups $\pi_1(X_1)$, $\pi_1(X_2)$, and $\pi_1(X_3)$ are abstractly commensurable, but are not abstractly commensurable with $\pi_1(X_4)$. All four groups are quasi-isometric by Theorem 4.3.1.

so by equation (3) above (and since Euler characteristic sums over these unions), we have $\chi(S_i) = \chi(T_i)$ for $i = 1, 2$. By condition (c) and an analogous calculation, we conclude $\chi(S_i) = \chi(T_i)$ for all $1 \leq i \leq 4$. Thus, $Y_1 \cong Y_2$, and therefore G_1 and G_2 are abstractly commensurable. $\square$

Corollary 3.3.4. *If $G_1, G_2 \in \mathcal{C}_S$ and G_1 and G_2 are abstractly commensurable, then there exist normal subgroups of finite index, $N_i \lhd G_i$ so that $N_1 \cong N_2$.*

Proof. In the proof of Theorem 3.3.3, the covers constructed are regular. $\square$

In the case that G_1 and G_2 are the fundamental groups of surfaces glued along separating curves, we have the following.

Corollary 3.3.5. *If S_1, S_2, S_3, S_4 and T_1, T_2, T_3, T_4 are orientable surfaces of genus greater than or equal to one and with one boundary component, the S_i are glued along their boundary to form X_1, and the T_i are glued along their boundary to form X_2, then $\pi_1(X_1)$ and $\pi_1(X_2)$ are abstractly commensurable if and only* *if, up to reindexing, the quadruples* $(\chi(S_1), \ldots, \chi(S_4))$ *and* $(\chi(T_1), \ldots, \chi(T_4))$ *are equal up to scale.*

If G_1 and G_2 are the fundamental groups of surfaces glued along non-separating curves, we have the following.

Corollary 3.3.6. *If* S_{g_i} *and* $S_{g_i'}$ *are orientable surfaces of genus greater than one identified to each other along a non-separating curve in each to form the space* X_i *for* $i = 1, 2$*, then* $\pi_1(X_1)$ *and* $\pi_1(X_2)$ *are abstractly commensurable if and only if, up to reindexing,* $\dfrac{\chi(S_{g_1})}{\chi(S_{g_1'})} = \dfrac{\chi(S_{g_2})}{\chi(S_{g_2'})}$.

3.4 Maximal elements in $\mathcal{C}_S$

Let $\mathcal{G} \subset \mathcal{C}_S$ be an abstract commensurability class. A *maximal element* for $\mathcal{G}$ is a group G_0 that contains every group in $\mathcal{G}$ as a finite-index subgroup. The existence of a maximal element that lies in $\mathcal{C}_S$ depends on whether the class contains the fundamental group of a surface identified along a non-separating curve. We define the following three subclasses that partition $\mathcal{X}_S$, $\mathcal{C}_S$. By Theorem 3.3.3, these subclasses partition the abstract commensurability classes within $\mathcal{C}_S$ as well.

Definition 3.4.1.

- Let $\mathcal{X}_0$ be the set of spaces $X \in \mathcal{X}_S$ for which the complement of the singular curve in X consists of four surfaces with one boundary component and unequal genus. Let $\mathcal{C}_0 \subset \mathcal{C}_S$ be the set of fundamental groups of spaces in $\mathcal{X}_0$.
- Let $\mathcal{X}_1$ be the set of spaces $X \in \mathcal{X}_S$ for which the complement of the singular curve in X contains either one surface with two boundary components and two surfaces with one boundary component and unequal

genus, or, four surfaces, exactly two of which have equal genus. Let $\mathcal{C}_1 \subset \mathcal{C}_S$ be the set of fundamental groups of spaces in $\mathcal{X}_1$.

- Let $\mathcal{X}_2$ be the set of spaces $X \in \mathcal{X}_S$ that can be realized as the union of two surfaces along curves of topological type one (see Definition 3.2.1). Let $\mathcal{C}_2 \subset \mathcal{C}_S$ be the set of fundamental groups of spaces in $\mathcal{X}_2$.

In this section, we prove that an abstract commensurability class $\mathcal{G} \subset \mathcal{C}_S$ contains a maximal element within $\mathcal{C}_S$ if and only if $\mathcal{G} \subset \mathcal{C}_0$. In the following section, in Corollary 3.5.10, we prove that if $\mathcal{G} \subset \mathcal{C}_2$, then there is a maximal element for $\mathcal{G}$ within the class of right-angled Coxeter groups. For $\mathcal{G} \subset \mathcal{C}_1$, it is not known whether there exists a maximal element for the abstract commensurability class $\mathcal{G}$.

To construct covers of surfaces glued along separating curves, we use the following lemma, which is a converse to Lemma 3.1.4 for hyperbolic surfaces with one boundary component.

Lemma 3.4.2. *For $g_i \geq 1$, if $\chi(S_{g_2,1}) = n\chi(S_{g_1,1})$, then $S_{g_2,1}$ n-fold covers $S_{g_1,1}$.*

Proof. Let

$$\pi_1(S_{g_1,1}) = \langle a_1, b_1, \ldots, a_{g_1}, b_{g_1} | \ \rangle \cong F_{2g_1}$$

be a presentation for the fundamental group of $S_{g_1,1}$. The homotopy class of the boundary element $\gamma_1 : S^1 \to S_{g_1,1}$ corresponds to the element $[a_1, b_1] \ldots [a_{g_1}, b_{g_1}] \in \pi_1(S_{g_1,1})$.

We exhibit $\pi_1(S_{g_2,1})$ as an index n subgroup of $\pi_1(S_{g_1,1})$ so that in the corresponding cover, γ_1 has preimage a single curve that n-fold covers γ_1.

Realize $\pi_1(S_{g_1,1})$ as the fundamental group of a wedge of $2g_1$ oriented circles labeled by the generating set. Construct an n-fold cover of this space as a

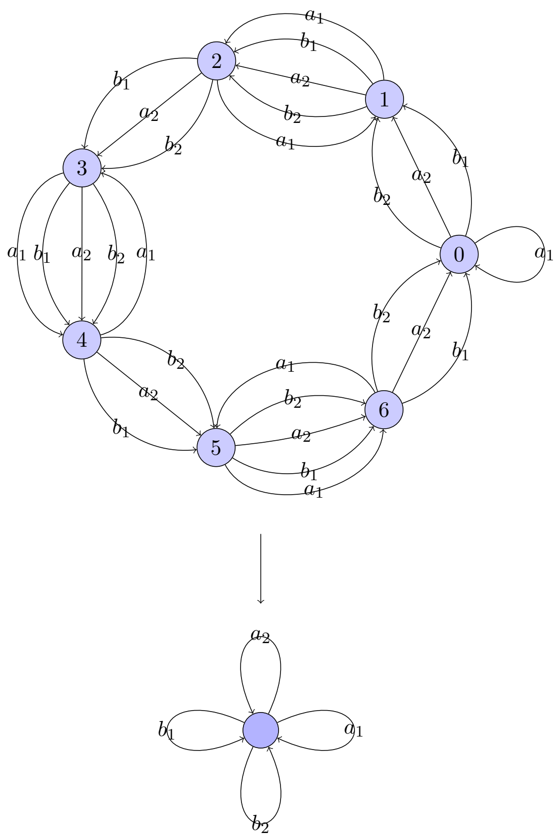

FIGURE 3. A covering map that realizes $S_{11,1}$ as a 7-fold cover of $S_{2,1}$.

graph, Γ, on n vertices labeled $\{0, \ldots, n-1\}$. For every generator besides a_1, construct an oriented n-cycle on the n vertices with each edge labeled by the generator. Since $\chi(S_{g_1,1})$ and $\chi(S_{g_2,1})$ are both odd, n must be odd as well by Lemma 3.1.4. Let $\{i, i+1\}$ and $\{i+1, 1\}$ be directed edges labeled by a_1 for $i < n$ and i odd. Construct a directed loop labeled a_1 at vertex $\{0\}$, as illustrated in Figure 3. By construction, Γ covers the wedge of circles given above.

To see that γ_1 has a preimage with one component, choose a vertex v in the graph Γ and consider the edge path p with edges labeled $([a_1, b_1] \ldots [a_{g_1}, b_{g_1}])^k$, which projects to γ_1 under the covering map. Then n is the smallest non-zero k for which p terminates at v. To see this, note that it suffices to consider the path $p' = [a_1, b_1]^k$ since every other segment $[a_j, b_j]$ returns to its initial vertex.

Starting at vertex $\{0\}$, observe that the path $[a_1, b_1]^k$ terminates at the vertex labeled

$$\begin{cases} 2k-1 & \text{if } 0 < k < \lfloor \frac{n}{2} \rfloor \bmod n \\ 2n-2k & \text{if } \lfloor \frac{n}{2} \rfloor \leq k < n \bmod n \\ 0 & \text{if } k = 0 \bmod n, \end{cases}$$

proving the claim. □

Lemma 3.4.2 may be restated in terms of the *Hurwitz realizability problem for branched coverings of surfaces*. In this language, Lemma 3.4.2 is a special case of [BB12, Lemma 7.1], proved first in [EKS84], [Hus62]. Lemma 3.4.2 is included since its proof is new and of independent interest.

In the proof of the characterization of the abstract commensurability classes that contain a maximal element, we will use the following definition.

Definition 3.4.3. If S_g and S_h are closed hyperbolic surfaces, γ is a multicurve on S_g and ρ is a multicurve on S_h, we say (S_g, γ) *covers* (S_h, ρ) if there exists a covering map $p : S_g \to S_h$ so that γ is the full preimage of ρ in S_g.

Proposition 3.4.4. *Let $\mathcal{G} \subset \mathcal{C}_S$ be an abstract commensurability class. There exists a maximal element in $\mathcal{C}_S$ for $\mathcal{G}$ if and only if $\mathcal{G} \subset \mathcal{C}_0$.*

Proof. Let $\mathcal{G} \subset \mathcal{C}_S$ be an abstract commensurability class. We first exhibit a maximal element when the conditions of the proposition hold. Suppose $G \cong \pi_1(S_g) *_{\langle \gamma \rangle} \pi_1(S_{g'}) \in \mathcal{G}$ where γ_g and $\gamma_{g'}$ are separating simple closed curves with $S_g = S_{r,1} \cup_{\gamma_g} S_{s,1}$ and $S_{g'} = S_{r',1} \cup_{\gamma_{g'}} S_{s',1}$ where $r \neq s$ and $r' \neq s'$. Let X_g denote the space obtained by identifying S_g and $S_{g'}$ along γ_g and $\gamma_{g'}$. We construct a space $X \in \mathcal{X}_S$ that X_g covers and prove that if $H \in \mathcal{G}$ with $H \cong \pi_1(X_h)$ and

$X_h \in \mathcal{X}_S$, then X_h covers X as well. Thus, we conclude, $\pi_1(X)$ is a maximal element in $\mathcal{G}$.

There exist relatively prime p and q and relatively prime p' and q' so that

$$\frac{\chi(S_{r,1})}{\chi(S_{s,1})} = \frac{p}{q} \quad \text{and} \quad \frac{\chi(S_{r',1})}{\chi(S_{s',1})} = \frac{p'}{q'}.$$

So, $\chi(S_{r,1}) = -dp$, $\chi(S_{s,1}) = -dq$, $\chi(S_{r',1}) = -d'p'$, and $\chi(S_{s',1}) = -d'q'$, for some odd integers d and d'. Let

- $S_{u,1}$ be the surface with Euler characteristic $-\dfrac{d}{gcd(d,d')}p$
- $S_{v,1}$ be the surface with Euler characteristic $-\dfrac{d}{gcd(d,d')}q$
- $S_{u',1}$ be the surface with Euler characteristic $-\dfrac{d'}{gcd(d,d')}p'$
- $S_{v',1}$ be the surface with Euler characteristic $-\dfrac{d'}{gcd(d,d')}q'$.

Let $S = S_{u,1} \cup_\gamma S_{v,1}$ be the surface obtained by identifying $S_{u,1}$ and $S_{v,1}$ along their boundary curves, let $S' = S_{u',1} \cup_{\gamma'} S_{v',1}$ be the surface obtained by identifying $S_{u',1}$ and $S_{v',1}$ along their boundary curves, and let X be the space obtained by identifying S and S' along these curves γ and γ'. Then, by Lemma 3.4.2, (S_g, γ_g) covers (S, γ) by degree $gcd(d, d')$ and $(S_{g'}, \gamma_{g'})$ covers (S', γ') by degree $gcd(d, d')$, so, X_g covers X by degree $gcd(d, d')$.

To see that $\pi_1(X)$ is a maximal element $\mathcal{G}$, let $H \cong \pi_1(S_h) *_{\langle\rho\rangle} \pi_1(S_{h'}) \in \mathcal{G}$ be given by the monomorphisms $\rho \mapsto [\gamma_h] \in \pi_1(S_h)$ and $\rho \mapsto [\gamma_{h'}] \in \pi_1(S_{h'})$. By Theorem 3.3.3 (b) and (c), γ_h and $\gamma_{h'}$ are separating simple closed curves so that $S_h = S_{m,1} \cup_{\gamma_h} S_{n,1}$ and $S_{h'} = S_{m',1} \cup_{\gamma_{h'}} S_{n',1}$, where

$$\frac{\chi(S_{m,1})}{\chi(S_{n,1})} = \frac{p}{q} \quad \text{and} \quad \frac{\chi(S_{m',1})}{\chi(S_{n',1})} = \frac{p'}{q'}.$$

Then $\chi(S_{m,1}) = -fp$, $\chi(S_{n,1}) = -fq$, $\chi(S_{m',1}) = -f'p'$, and $\chi(S_{n',1}) = -f'q'$ for some $f, f' \in \mathbb{N}$. By Theorem 3.3.3 (a), $\frac{\chi(S_g)}{\chi(S_{g'})} = \frac{\chi(S_h)}{\chi(S_{h'})}$, hence $\frac{-d(p+q)}{-d'(p'+q')} = \frac{-f(p+q)}{-f'(p'+q')}$, so $\frac{d}{d'} = \frac{f}{f'}$. So,

$$\frac{d}{gcd(d,d')} = \frac{f}{gcd(f,f')} \quad \text{and} \quad \frac{d'}{gcd(d,d')} = \frac{f'}{gcd(f,f')}.$$

Therefore, by Lemma 3.4.2, (S_h, γ_h) covers (S, γ) by degree $gcd(f, f')$ and $(S_{h'}, \gamma_{h'})$ covers (S', γ') by degree $gcd(f, f')$; thus, X_h covers X by degree $gcd(f, f')$ as desired.

If $G \in \mathcal{C}_S$ does not satisfy the conditions of the proposition, then there are two groups, H_1 and H_2, in the abstract commensurability class of G in $\mathcal{C}_S$, where $H_1 \cong \pi_1(S_{h_1}) *_{\langle\gamma\rangle} \pi_1(S_{h_1'})$ and $H_2 \cong \pi_1(S_{h_2}) *_{\langle\rho\rangle} \pi_1(S_{h_2'})$ and, up to relabeling, $\gamma \mapsto [\gamma_{h_1}] \in \pi_1(S_{h_1})$ and $\rho \mapsto [\gamma_{h_2}] \in \pi_1(S_{h_2})$, where γ_{h_1} is an essential non-separating simple closed curve and γ_{h_2} is a separating simple closed curve. Thus, (S_{h_1}, γ_{h_1}) and (S_{h_2}, γ_{h_2}) cannot cover the same pair (S, γ), so there is no maximal element in the abstract commensurability class of G in $\mathcal{C}_S$. □

3.5 Right-angled Coxeter groups and the Crisp–Paoluzzi examples

In this section, we discuss the relationship between groups in $\mathcal{C}_S$ and the class of right-angled Coxeter groups. We begin with the relevant background for this section.

Definition 3.5.1. Let Γ be a finite simplicial graph. The *right-angled Coxeter group with defining graph* Γ is

$$W(\Gamma) = \big\langle v \in V(\Gamma) \,|\, v^2 = 1 \text{ if } v \in V(\Gamma),\ [v,w] = 1 \text{ if } \{v,w\} \in E(\Gamma) \big\rangle.$$

For more on right-angled Coxeter groups, see [Dav08]. As shown in [Gre90], a right-angled Coxeter group is defined up to isomorphism by its defining graph; that is, $W(\Gamma) \cong W(\Gamma')$ if and only $\Gamma \cong \Gamma'$. Often, group theoretic properties of

$W(\Gamma)$ correspond to graph theoretic properties of Γ. Classic results relevant to our setting are recorded below.

Proposition 3.5.2. Let Γ be a simplicial graph.

(1) [Gro87, Pg. 123] The group $W(\Gamma)$ is word-hyperbolic if and only if every 4-cycle in Γ has a chord.

(2) [Dav08, Lemma 8.7.2] The group $W(\Gamma)$ is one-ended if and only if Γ is not a complete graph and there does not exist a complete subgraph K of Γ such that $\Gamma \backslash K$ is disconnected.

An *orbifold* is a topological space $\mathcal{O}$ in which each point has a neighborhood modeled on $\tilde{U}/G$, where $\tilde{U}$ is an open ball in $\mathbb{R}^n$ and G is a finite subgroup of $SO(n)$. Associated to each point in the orbifold is the finite group G called its *isotropy group*. A point is called a *ramification point* if its isotropy group is non-trivial. The set of all ramification points is called the *ramification locus* of the orbifold. The underlying topological space of an orbifold $\mathcal{O}$ is denoted $|\mathcal{O}|$. Background and a more formal definition of orbifolds can be found in [Kap09, Chapter 6] and [Rat06, Chapter 13]; recent applications can be found in the survey paper [Wal11].

A *homeomorphism* between orbifolds $\mathcal{O}$ and $\mathcal{R}$ is a homeomorphism $h : |\mathcal{O}| \to |\mathcal{R}|$ such that for each point $x \in \mathcal{O}$, $y = h(x) \in \mathcal{R}$, there are coordinate neighborhoods $U_x = \tilde{U}_x/G_x$ and $V_y = \tilde{V}_y/G_y$ such that h lifts to an equivariant homeomorphism $\tilde{h}_{xy} : \tilde{U}_x \to \tilde{V}_y$. An *orbi-complex* is a disjoint union of orbifolds identified to each other along homeomorphic suborbifolds.

An *orbifold covering* $p : \mathcal{O}' \to \mathcal{O}$ is a continuous map $|\mathcal{O}'| \to |\mathcal{O}|$ such that if $x \in \mathcal{O}$ is a ramification point with neighborhood given by $U = \tilde{U}/G$, then each component V_i of $f^{-1}(U)$ is isomorphic to $\tilde{U}/G_i$ where $G_i \leq G$ and $p|_{V_i} : V_i \to U$

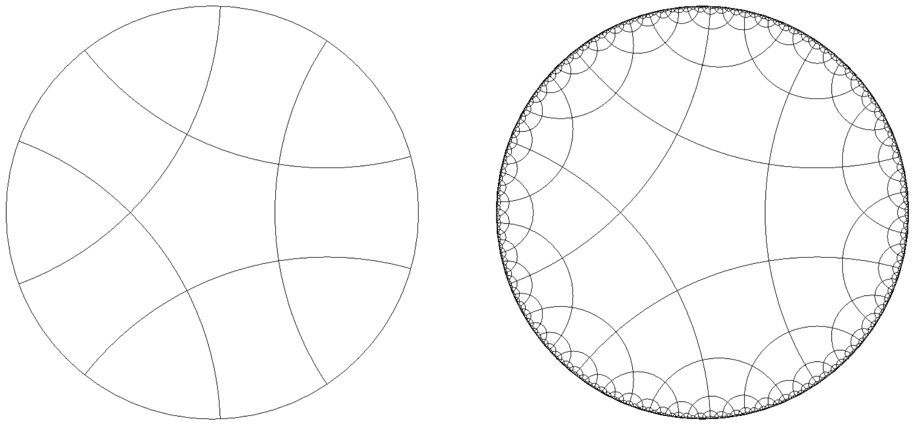

FIGURE 4. On the left are five geodesic lines in the disk model of the hyperbolic plane; on the right, is their orbit under the action of the right-angled Coxeter group W_5. Both figures were drawn with Curt McMullen's lim program [McM].

is $\tilde{U}/G_i \to \tilde{U}/G$. The *universal covering* $p : \tilde{\mathcal{O}} \to \mathcal{O}$ is a covering such that for any other covering $p' : \mathcal{O}' \to \mathcal{O}$ there exists a covering $\tilde{p} : \tilde{\mathcal{O}} \to \mathcal{O}'$ such that $p' \circ \tilde{p} = p$. The group of *deck transformations* of the orbifold covering $p : \mathcal{O}' \to \mathcal{O}$ is the group of self-diffeomorphisms $h : \mathcal{O}' \to \mathcal{O}'$ such that $p \circ h = p$. The *orbifold fundamental group*, $\pi_1^{orb}(\mathcal{O})$, is the group of deck transformations of its universal covering. Then $\mathcal{O} = \tilde{\mathcal{O}}/\pi_1^{orb}(\mathcal{O})$. The orbifold $\mathcal{O}$ is called a *reflection orbifold* if $\pi_1(\mathcal{O})$ is generated by reflections. The orbifold fundamental group can also be defined based on homotopy classes of loops in $\mathcal{O}$; this definition appears in [Rat06, Chapter 13]. A form of the Seifert-Van Kampen theorem allows one to compute the fundamental group of orbifolds; see Section 2 of [Sco83].

Let W_n be the right-angled Coxeter group with defining graph an n-cycle. If $n \geq 5$, W_n acts geometrically on the hyperbolic plane: W_n is isomorphic to the group generated by reflections about the geodesic lines through the n-sides of a right-angled hyperbolic n-gon. One such example is given in Figure 4. Let $\mathcal{O}_n$ denote the quotient of the hyperbolic plane under the action of W_n so $\pi_1^{orb}(\mathcal{O}_n) \cong W_n$. Every closed orientable surface of genus greater than one finitely covers $\mathcal{O}_5$ (for example, see [Sco78]), so $\pi_1(S_g)$ is a finite-index subgroup of W_5 for $g \geq 2$.

As orbifolds, $\mathcal{O}_n$ and $\mathcal{O}_m$ may be identified to each other along homeomorphic suborbifolds to form an orbi-complex. If the suborbifolds each have underlying space a geodesic segment that meets the boundary edges of the reflection orbifolds at right angles, then the orbi-complex obtained has orbifold fundamental group a right-angled Coxeter group. There are two homeomorphism types of such suborbifolds of $\mathcal{O}_n$: a reflection edge and the geodesic segment that connects the interior of reflection edges that are separated from each other by at least two reflection edges on either side.

The orbi-complex obtained by identifying $\mathcal{O}_n$ and $\mathcal{O}_m$ along a reflection edge in each is denoted $\mathcal{O}_{m,n}$. The orbifold fundamental group of $\mathcal{O}_{m,n}$ is the right-angled Coxeter group $W_{m,n}$ introduced by Crisp–Paoluzzi in [CP08], and is defined as follows.

Definition 3.5.3. [CP08] For $m, n \geq 5$, define $W_{m,n} = W(\Gamma_{m,n})$, where $\Gamma_{m,n}$ denotes the graph which consists of a circuit of length m and a circuit of length n identified along a common subpath of edge-length 2.

Our notation for $W_{m,n}$ varies slightly from that given in [CP08]; they define $\Gamma_{m,n}$ as the graph which consists of a circuit of length $m+4$ and a circuit of length $n+4$ identified along a common subpath of edge-length 2 and $m, n \geq 1$. One can easily translate between the two notations.

On the other hand, the orbi-complex obtained by identifying $\mathcal{O}_n$ and $\mathcal{O}_m$ along geodesics connecting reflection edges at distance greater than or equal to two from each other can also be viewed as the union of four right-angled reflection orbifolds with one boundary edge identified to each other along their boundary

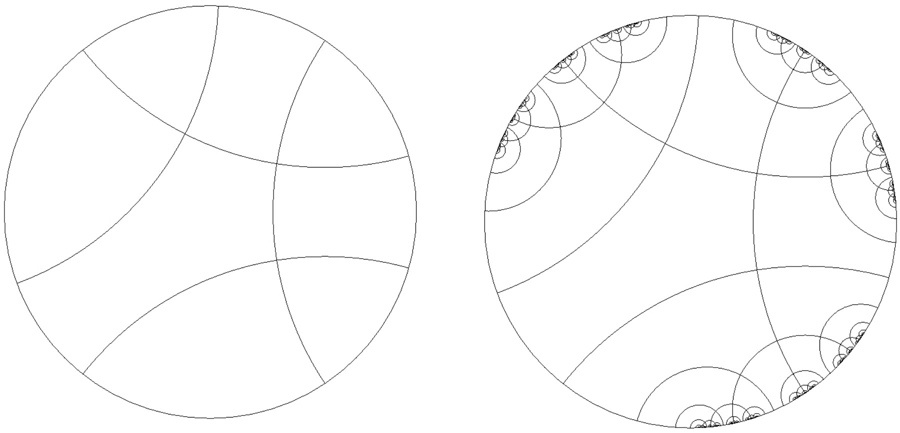

FIGURE 5. On the left are four geodesic lines in the disk model of the hyperbolic plane; on the right, is their orbit under the action of the right-angled Coxeter group with underlying graph a path of length four. Both figures were drawn with Curt McMullen's lim program [McM].

edges. The orbifold fundamental group of each component orbifold with boundary is P_n, the right-angled Coxeter group with underlying graph a path of length n for some $n \geq 4$. More specifically, for $n \geq 4$, P_n acts properly discontinuously by isometries on the hyperbolic plane by reflecting about n geodesic lines, whose intersection graph is a path of length n and so that the intersecting lines meet at right angles; an example is illustrated in Figure 5. The quotient of the hyperbolic plane under the group P_n is an open infinite-area right-angled hyperbolic reflection orbifold. Truncate this space along the unique geodesic in the homotopy class of the boundary to obtain the orbifold $\mathcal{O}_{n,1}$, a compact orbifold with boundary and $\pi_1^{orb}(\mathcal{O}_{n,1}) = P_n$.

For $n_i \geq 4$, the orbifolds $\mathcal{O}_{n_1,1}, \ldots, \mathcal{O}_{n_4,1}$ may be identified along their boundary curves to form an orbi-complex we denote $\mathcal{O}(n_1, \ldots, n_4)$. The orbifold fundamental group of the orbi-complex $\mathcal{O}(n_1, \ldots, n_4)$ is the right-angled Coxeter group with underlying graph denoted $\Theta(n_1, \ldots, n_4)$ that consists of four paths of length $n_i \geq 4$ glued to each other along their endpoints. The graphs $W_{m,n}$

and $\Theta(n_1, \ldots, n_4)$ are examples of *generalized* Θ*-graphs*, which were introduced by Dani–Thomas in [DT14], and which are defined more formally below.

Definition 3.5.4. Let $k \geq 3$, $n_1 \geq 3$ and $n_2, \ldots, n_k \geq 4$ be integers. Let Ψ_k be the graph with two vertices a and b and k edges $e_1, \ldots, e_k$ connecting the vertices a and b. The *generalized* Θ*-graph* $\Theta(n_1, \ldots, n_k)$ is obtained by subdividing the edge e_i of Ψ_k into $n_i - 1$ edges by inserting $n_i - 2$ new vertices along e_i for $1 \leq i \leq n$.

Remark. Each right-angled Coxeter group with defining graph a generalized Θ-graph is the orbifold fundamental group of a right-angled hyperbolic reflection orbi-complex of one of two types that generalize the orbi-complexes described above. That is, if $n_1 = 3$, the associated orbi-complex is similar to $\mathcal{O}_{m,n}$: it consists of $k-1$ right-angled hyperbolic reflection orbifolds identified to each other along a reflection edge in each. If $n_1 > 3$, the associated orbi-complex is similar to $\mathcal{O}(n_1, \ldots, n_4)$: it consists of k right-angled hyperbolic reflection orbifolds with boundary identified to each other along their boundary edges. In upcoming joint work with Pallavi Dani and Anne Thomas, we characterize the abstract commensurability classes in this setting.

In this section, we prove that the fundamental group of two surfaces identified along separating curves is a finite-index subgroup of a right-angled Coxeter group with defining graph $\Theta(n_1, \ldots, n_4)$ for $n_i \geq 4$. We prove the fundamental group of two surfaces identified along curves of topological type one (see Definition 3.2.1) is a finite-index subgroup of the right-angled Coxeter group $W_{m,n}$ with defining graph $\Theta(3, n_1, n_2)$ and $n_i \geq 4$. It remains open whether the fundamental group of the union of two surfaces obtained by gluing a non-separating curve to a curve that separates the surface into two subsurfaces of unequal genus is a finite-index subgroup of a right-angled Coxeter group.

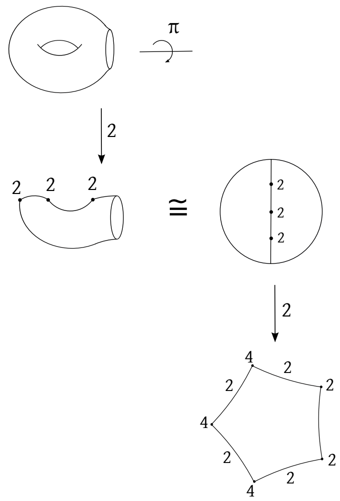

FIGURE 6. Illustrated above is a 4-fold cover of the orbifold $\mathcal{O}_{4,1}$ by the surface with boundary $S_{1,1}$.

Using the following lemma, we prove that in every abstract commensurability class of a group in $\mathcal{C}_S$ there is a group that is a finite-index subgroup of a right-angled Coxeter group with underlying graph $\Theta(n_1, \ldots, n_4)$ and $n_i \geq 4$.

Lemma 3.5.5. *If $S_1, \ldots, S_k$ are orientable hyperbolic surfaces with one boundary component, identified to each other along their boundary components to form the space X, then $\pi_1(X)$ is a finite-index subgroup of a right-angled Coxeter group.*

Proof. We prove X four-fold covers the reflection orbi-complex $\mathcal{O}(n_1, \ldots, n_k)$ for some $n_i \geq 4$ whose orbifold fundamental group is a right-angled Coxeter group with underlying graph the generalized Θ-graph $\Theta(n_1, \ldots, n_k)$.

The surface with boundary $S_i \subset X$ four-fold covers $\mathcal{O}_{n_i,1}$ for some $n_i \geq 4$ such that the boundary of S_i four-fold covers the boundary edge of $\mathcal{O}_{n_i,1}$ as illustrated in Figure 6. To see this, skewer S_i through its boundary component so that $2g_i + 1$ points on the surface intersect the skewer, and rotate by π. The quotient is homeomorphic to a disk with $2g_i + 1$ cone points of order two, which may be arranged on the diameter of the disk. Reflection across the diameter gives the desired covering map $S_i \to \mathcal{O}_{n_i,1}$. Thus, the union of these surfaces

S_i glued along their boundary curves four-fold covers the union of the orbifolds along their boundary lines concluding the proof. □

Corollary 3.5.6. *If $G \in \mathcal{C}_S$, then G is abstractly commensurable to a right-angled Coxeter group.*

Proof. Let $G \in \mathcal{C}_S$. By the abstract commensurability classification within $\mathcal{C}_S$ given in Theorem 3.3.3, there exists $Y \in \mathcal{X}_S$ whose fundamental group is abstractly commensurable to G and so that Y has one singular curve that identifies the boundary components of four surfaces each with one boundary component. The group $\pi_1(Y)$ is a finite-index subgroup of a right-angled Coxeter group by Lemma 3.5.5, so, G is abstractly commensurable to a right-angled Coxeter group. □

For the remainder of the section, we restrict attention to the relationship between the groups in $\mathcal{C}_S$ and the groups $W_{m,n}$ studied by Crisp–Paoluzzi in [CP08]. Recall, $\mathcal{X}_2 \subset \mathcal{X}_S$ is defined to be the set of spaces $X \in \mathcal{X}_S$ that can be realized as the union of two surfaces along curves of topological type one. The groups $\mathcal{C}_2 \subset \mathcal{C}_S$ are the fundamental groups of spaces in $\mathcal{X}_2$ (see Definition 3.4.1).

Lemma 3.5.7. *If $X = S_g \cup_\gamma S_h \in \mathcal{X}_2$, then X 8-fold covers $\mathcal{O}_{g+3,h+3}$. Conversely, if $m, n \geq 5$, then $\mathcal{O}_{m,n}$ is 8-fold covered by $S_{m-3} \cup_\gamma S_{n-3} \in \mathcal{X}_2$.*

Proof. We show that if $\gamma_g : S^1 \to S_g$ is an essential simple closed curve of topological type one, then there exists an 8-fold orbifold covering map $S_g \to \mathcal{O}_{g+3}$ so that γ_g orbifold covers a reflection edge by degree 8, as illustrated in Figure 7. Thus, if $X = S_g \cup_\gamma S_h$, where γ identifies two curves of topological type one, then $S_g \cup_\gamma S_h$ 8-fold orbifold covers $\mathcal{O}_{g+3,h+3}$.

First suppose $\gamma_g : S^1 \to S_g$ is non-separating. Skewer S_g so that $2g+2$ points on the surface intersect the skewer, and rotate by π. The quotient under this action

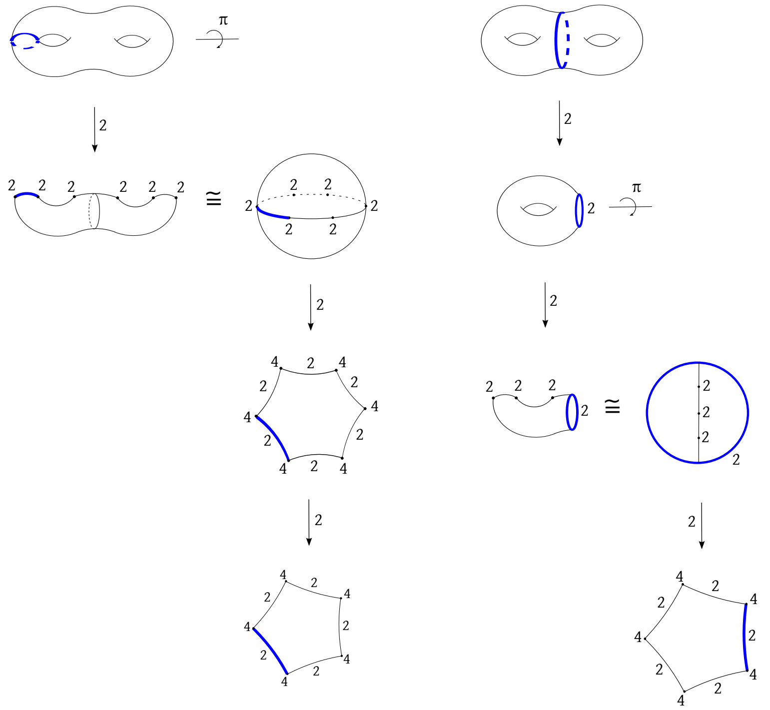

FIGURE 7. Shown above are orbifold covering maps $S_2 \to \mathcal{O}_5$ described in Lemma 3.5.7 and constructed so that the highlighted curves of topological type one cover a reflection edge in the orbifold $\mathcal{O}_5$. In particular, the union of these surfaces over the highlighted curves finitely covers the union of the orbifolds along the reflection edges.

is $S^2(2, \ldots, 2)$, the 2-sphere with $2g+2$ cone points of order two. This map $p_1 : S_g \to S^2(2, \ldots, 2)$ is an orbifold covering map: each ramification point in the sphere has a neighborhood in which the cover is given by rotation by π, and all other points have a neighborhood with preimage two homeomorphic copies of the neighborhood. The six cone points may be arranged along the equator of the sphere. Reflection through the equatorial plane has a quotient $\mathcal{O}_6$. Finally, $\mathcal{O}_6$ 2-fold orbifold covers $\mathcal{O}_5$ by reflection, which can be seen by unfolding $\mathcal{O}_5$ along a reflection edge. It is clear that this covering, illustrated in Figure 7 can be arranged so that γ_g 8-fold covers a reflection edge.

Now suppose $\gamma_g : S^1 \to S_g$ is separating. Reflecting S_g across the curve γ_g yields a 2-fold orbifold cover of an orbifold with orbifold boundary and underlying space $S_{\frac{g}{2},1}$. Skewer this orbifold along $g+1$ points and rotate by π yielding an orbifold with underlying space a disk, $g+1$ cone points or order two, and so that the boundary consists solely of reflection points. Finally arrange the cone points along a diameter of the disk and reflect about this line. These covering maps are illustrated in Figure 7. As in the non-separating case, one can easily verify each of these maps is an orbifold covering map. □

We immediately obtain the following corollary.

Corollary 3.5.8. *If $G \in \mathcal{C}_2$, then G embeds as a finite-index subgroup in the right-angled Coxeter group $W_{m,n}$ for some m and n.*

Remark: An alternative covering map $S_2 \to \mathcal{O}_5$ appears in [Sco78]. Under this covering map, illustrated in Figure 8, the curves of topological type one can also be chosen to cover a reflection edge in the pentagon orbifold.

Proposition 3.5.9. *If $G \in \mathcal{C}_S$, then G is abstractly commensurable to $W_{m,n}$ for some m and n if and only if $G \in \mathcal{C}_2$.*

Proof. Suppose $G \in \mathcal{C}_2$ so $G \cong \pi_1(X)$ with $X \in \mathcal{X}_2$. By Lemma 3.5.7, X finitely covers $\mathcal{O}_{m,n}$ for some m, n. Hence G is abstractly commensurable to $W_{m,n}$ for some m and n. Conversely, suppose $G \in \mathcal{C}_S$ and G is abstractly commensurable to $W_{m,n}$ for some m and n. By Lemma 3.5.7, $W_{m,n}$ is abstractly commensurable to G' for some $G' \in \mathcal{C}_2$. Since abstract commensurability is an equivalence relation, G is abstractly commensurable to G' so $G \in \mathcal{C}_2$ by Theorem 3.3.3. □

Finally, we may use the analysis of this section to produce a maximal element in the class of right-angled Coxeter groups for abstract commensurability classes within $\mathcal{C}_2$.

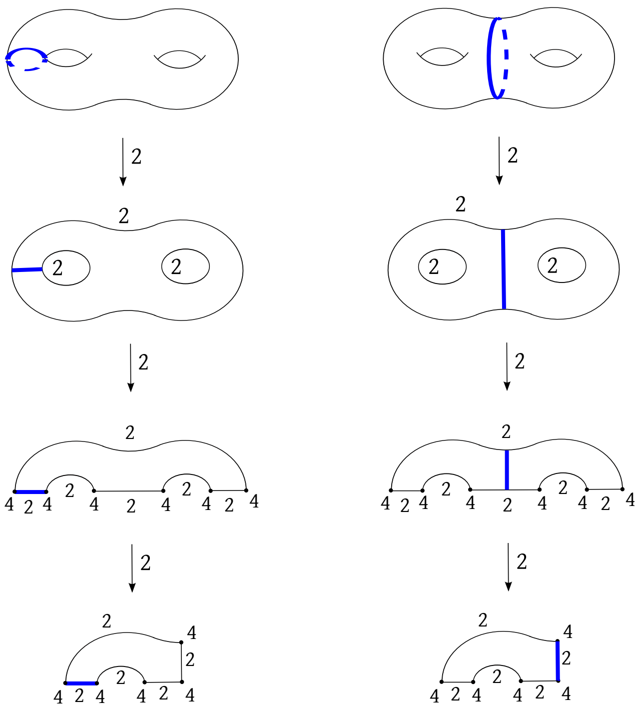

FIGURE 8. Pictured above are orbifold covering maps that appear in [Sco78]. Each map can be realized by embedding the surface in $\mathbb{R}^3$ and reflecting about a plane cutting through the surface. For our purposes, it is important to note that both curves of topological type one cover a reflection edge by degree eight.

Corollary 3.5.10. *If $G \in \mathcal{C}_2$, then there is a right-angled Coxeter group G_0 so that every group in $\mathcal{C}_S$ in the abstract commensurability class of G is a finite-index subgroup of G_0.*

Proof. Let $G \in \mathcal{C}_2$ and let $\mathcal{G} \subset \mathcal{C}_S$ denote the abstract commensurability class of G in $\mathcal{C}_S$. By Lemma 3.5.7, G is a finite-index subgroup of $W_{m,n}$ for some m and n, and, if $G' \in \mathcal{G}$, then G' is a finite-index subgroup of $W_{k,\ell}$ for some k and ℓ. By [CP08, Theorem 1.1], $W_{m,n}$ and $W_{k,\ell}$ are abstractly commensurable if and only if $\frac{m-4}{n-4} = \frac{k-4}{\ell-4}$. Furthermore, $\mathcal{O}_{m,n}$ finitely covers $\mathcal{O}_{p,q}$ whenever $\frac{p-4}{q-4} = \frac{m-4}{n-4}$ and $\mathtt{g}cd(p-4, q-4) = 1$. Thus, G' is a finite-index subgroup of $W_{p,q}$, and $W_{p,q}$ is a maximal element for $\mathcal{G}$ within the class of right-angled Coxeter groups. $\square$

Chapter 4.

Quasi-isometry classification within $\mathcal{C}_S$

Let G be a group in the class $\mathcal{C}_S$ so that $G \cong \pi_1(X)$, where X is a space in the class $\mathcal{X}_S$. Suppose $X = S_g \cup_\gamma S_h$ where S_g and S_h are closed orientable surfaces of negative Euler characteristic and γ denotes the image of the essential simple closed curves $\gamma_g : S^1 \to S_g$ identified to $\gamma_h : S^1 \to S_h$ in X. There are many metrics on X through which the geometry of the group G may be studied.

4.1 A CAT(-1) metric on $\widetilde{X}$

Let M^n_κ denote the complete, simply connected, Riemannian n-manifold of constant sectional curvature $\kappa \in \mathbb{R}$. As described in [BH99, Chapter I.2], depending on whether κ is positive, negative, or zero, M^n_κ can be obtained from one of $\mathbb{S}^n$, $\mathbb{H}^n$, or $\mathbb{E}^n$, respectively, by scaling the metric.

Definition 4.1.1 (see Chapter II.1 of [BH99])**.** Let $\Delta(p, q, r)$ be a geodesic triangle in a metric space X, which consists of three vertices p, q, and r, and three geodesic segments $[p, q]$, $[q, r]$, and $[r, p]$. A triangle $\bar{\Delta}(\bar{p}, \bar{q}, \bar{r}) \subset M^2_\kappa$ is called a *comparison triangle* for $\Delta(p, q, r)$ if $d(\bar{p}, \bar{q}) = d(p, q)$, $d(\bar{q}, \bar{r}) = d(q, r)$, and $d(\bar{r}, \bar{p}) = d(r, p)$. A point $\bar{x} \in [\bar{q}, \bar{r}]$ is called a *comparison point* for $x \in [q, r]$ if $d(q, x) = d(\bar{q}, \bar{x})$.

Definition 4.1.2 (see Definition II.1.1 of [BH99])**.** Let X be a metric space and let $\kappa \in \mathbb{R}$. Let Δ be a geodesic triangle in X with perimeter less than twice the diameter of M^2_κ. Let $\bar{\Delta} \subset M^2_\kappa$ be a comparison triangle for Δ. Then Δ satisfies the *CAT*(κ) *inequality* if for all $x, y \in \Delta$ and comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$,

$d(x, y) \leq d(\bar{x}, \bar{y})$. If $\kappa \leq 0$, then X is called a *$CAT(\kappa)$ space* if X is a geodesic space all of whose triangles satisfy the CAT(κ) inequality.

In [Mal10], Malone proves all groups in $\mathcal{C}_S$ are quasi-isometric by examining a CAT(0) geometry on X and applying the techniques of Behrstock–Neumann on the bilipschitz equivalence of fattened trees [BN08]. The bilipschitz equivalence constructed by Behrstock–Neumann relies on the Euclidean structure of fattened trees; their map is piecewise-linear. In this thesis, we study a CAT(-1) metric on X that is piecewise hyperbolic, and we define a bilipschitz equivalence with respect to this hyperbolic structure. The piecewise hyperbolic metric on $X \in \mathcal{X}_S$ can be constructed as follows.

One can choose hyperbolic metrics on S_g and S_h so that the length of the geodesic representatives of $[\gamma_g]$ and $[\gamma_h]$ is equal (see Chapter 10 of [FM12]). Gluing by an isometry yields a piecewise hyperbolic complex X. We call such a metric *hyperbolic on each surface.* The universal cover $\widetilde{X}$ consists of copies of $\mathbb{H}^2$ that are the lifts of the hyperbolic surfaces, identified along geodesic lines that are the lifts of the curve γ. The following proposition implies that $\widetilde{X}$ is a CAT(-1) metric space.

Proposition 4.1.3. [BH99, Proposition II.11.6] *Let X_1 and X_2 be metric spaces of curvature $\leq \kappa$ and let $A_1 \subset X_1$ and $A_2 \subset X_2$ be closed subspaces that are locally convex and complete. If $j : A_1 \to A_2$ is a bijective local isometry, then the quotient of the disjoint union $X = X_1 \bigsqcup X_2$ by the equivalence relation generated by $[a_1 \sim j(a_1)$ for all $a_1 \in A_1]$ has curvature $\leq \kappa$.*

For details on metric gluing constructions, see the work of Bridson–Haefliger ([BH99], Section II.11).

4.2 Bilipschitz maps and polygonal tilings

The bilipschitz equivalence between the universal covers of two spaces X_1 and X_2 in $\mathcal{X}_S$ is constructed by realizing $\widetilde{X}_1$ and $\widetilde{X}_2$ as isomorphic cell complexes with finitely many isometry types of hyperbolic polygons as cells. We will use the following definitions.

Definition 4.2.1. A map $f : (X, d_X) \to (Y, d_Y)$ is *K-bilipschitz* if there exists $K \geq 1$ so that for all $x_1, x_2 \in X$,

$$\frac{1}{K} d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2),$$

and f is a *K-bilipschitz equivalence* if, in addition, f is a homeomorphism. A map is said to be a *bilipschitz equivalence* if it is a K-bilipschitz equivalence for some K. Two spaces X and Y are *bilipschitz equivalent* if there exists a bilipschitz equivalence from X to Y.

Example 4.2.2. The map $f : [0, D] \to [0, D']$ given by $x \mapsto \frac{D'}{D} x$ is called *dilation*, and is a bilipschitz equivalence with bilipschitz constant $\frac{D'}{D}$.

Definition 4.2.3. A *convex hyperbolic polygon* is the convex hull of a finite set of points in the hyperbolic plane.

Lemma 4.2.4. *Let $\Delta_1, \Delta_2 \subset \mathbb{H}^2$ be hyperbolic triangles. Then there exists a bilipschitz equivalence $\phi : \Delta_1 \to \Delta_2$ that is dilation when restricted to each edge of Δ_1.*

Proof. It follows from [BB04, Lemma 5, Lemma 6] that there is a bilipschitz equivalence between a hyperbolic triangle and its Euclidean comparison triangle that restricts to an isometry on each of the edges. Then, composing with a linear map between Euclidean triangles gives the desired result. $\square$

Corollary 4.2.5. *If P and Q are convex hyperbolic n-gons, then there exists a bilipschitz equivalence $\phi : P \to Q$ that is dilation when restricted to each edge of P.*

For a more formal and general definition of polyhedral complexes and their metric, see [BH99, Chapter 1.7].

Lemma 4.2.6. *If $\widetilde{X}_1$ and $\widetilde{X}_2$ are geodesic metric spaces realized as isomorphic cell complexes with finitely many isometry types of hyperbolic polygons as cells, then $\widetilde{X}_1$ and $\widetilde{X}_2$ are bilipschitz equivalent.*

Proof. Suppose geodesic metric spaces $\widetilde{X}_1$ and $\widetilde{X}_2$ are realized as isomorphic cell complexes with polygonal cells $\{V_i\}_{i\in I}$ and $\{W_i\}_{i\in I}$, respectively. Suppose the cell complex isomorphism maps V_i to W_i for all $i \in I$. By Corollary 4.2.5 and since there are finitely many isometry types of hyperbolic polygons in the cell complexes, we may take this map $\phi_i : V_i \to W_i$ to be a K-bilipschitz equivalence for some $K \in \mathbb{R}$ that restricts to dilation on each of the edges of V_i. These maps agree along the intersection of two polygons, thus, there is a well-defined cellular homeomorphism $\Phi : \widetilde{X}_1 \to \widetilde{X}_2$ that restricts to the K-bilipschitz equivalence ϕ_i on each cell.

Let $x, y \in \widetilde{X}_1$, and let p be the geodesic path from x to y. Since the cell complex contains finitely many isometry types of convex hyperbolic polygons, the path p can be decomposed into a finite union of geodesic segments $\{[x_i, x_{i+1}]\}_{i=0}^{n-1}$, with $x_0 = x$ and $x_n = y$, and so that each subpath $[x_i, x_{i+1}]$ is contained entirely in a 2-cell V_i. Since $\Phi(p)$ is a path connecting $\Phi(x)$ and $\Phi(y)$,

$$\begin{aligned} d(\Phi(x), \Phi(y)) &\leq \sum_{i=0}^{n-1} d(\phi_i(x_i), \phi_i(x_{i+1})) \\ &\leq \sum_{i=0}^{n-1} K d(x_i, x_{i+1}) \\ &= K d(x, y). \end{aligned}$$

The other inequality follows similarly. Namely, suppose q is a geodesic path from $\Phi(x)$ to $\Phi(y)$. The path q can be decomposed into a union of geodesic segments $\{[w_i, w_{i+1}]_{i=0}^{m-1}\}$ where $w_0 = \Phi(x)$, $w_m = \Phi(y)$ and the interior of $[w_i, w_{i+1}]$ is contained entirely in a 2-cell W_i. Then, since $\Phi^{-1}(q)$ is a path from x to y and ϕ_i is a K-bilipschitz equivalence for all i,

$$\begin{aligned} d(\Phi(x), \Phi(y)) &= \sum_{i=0}^{m-1} d(w_i, w_{i+1}) \\ &\geq \sum_{i=0}^{m-1} \frac{1}{K} d(\phi_i^{-1}(w_i), \phi_i^{-1}(w_{i+1})) \\ &\geq \frac{1}{K} d(x, y). \end{aligned}$$

Thus, $\frac{1}{K} d(x,y) \leq d(\Phi(x), \Phi(y)) \leq K d(x,y)$, so Φ is a K-bilipschitz equivalence.

□

In the construction of the bilipschitz equivalence, we find it useful to restrict to a specific metric on a space $X \in \mathcal{X}_S$, and we will use the following lemma.

Lemma 4.2.7. *If $X_1, X_2 \in \mathcal{X}_S$ and $\pi_1(X_1)$ and $\pi_1(X_2)$ are abstractly commensurable, then $\widetilde{X}_1$ and $\widetilde{X}_2$ are bilipschitz equivalent with respect to any CAT(-1) metric on X_1 and X_2 that is hyperbolic on each surface.*

Proof. Let $X_1, X_2 \in \mathcal{X}_S$, and suppose $\pi_1(X_1)$ and $\pi_1(X_2)$ are abstractly commensurable. By Theorem 3.1.2, there exist finite-sheeted covers $Y_i \to X_i$ that are homeomorphic. Choose a locally CAT(-1) metric on X_1 and X_2 that is hyperbolic on each surface. This piecewise hyperbolic metric on X_i lifts to a piecewise hyperbolic metric on Y_i. Since Y_1 and Y_2 are homeomorphic, we may realize Y_1 and Y_2 as finite simplicial complexes with isomorphic 1-skeleta. After subdividing if necessary, we may assume each triangle in Y_i is isometric to a hyperbolic triangle. So, $\widetilde{Y_1} \equiv \widetilde{X}_1$ and $\widetilde{Y_2} \equiv \widetilde{X}_2$ may be realized as simplicial complexes with isomorphic 1-skeleta and each built from finitely many isometry types of hyperbolic triangles. By Lemma 4.2.6, $\widetilde{Y_1} \equiv \widetilde{X}_1$ and $\widetilde{Y_2} \equiv \widetilde{X}_2$ are bilipschitz equivalent. □

4.3 Construction of the cellular isomorphism

Theorem 4.3.1. *If $X_1, X_2 \in \mathcal{X}_S$ and $\widetilde{X}_1$ and $\widetilde{X}_2$ are their universal covers equipped with a CAT(-1) metric that is hyperbolic on each surface, then there exists a bilipschitz equivalence $\widetilde{X}_1 \to \widetilde{X}_2$.*

Proof. Let $X_1, X_2, \in \mathcal{X}_S$. If $X \in \mathcal{X}_S$, then by the abstract commensurability classification within $\mathcal{C}_S$ given in Theorem 3.3.3, there exists $Y \in \mathcal{X}_S$ so that Y consists of four surfaces of genus at least two and one boundary component, identified to each other along their boundary components and so that $\pi_1(X)$ and $\pi_1(Y)$ are abstractly commensurable. So, by Lemma 4.2.7, it suffices to consider the case where

$$X_1 = \bigcup_{i=1}^{4} S_i,$$
$$X_2 = \bigcup_{i=1}^{4} T_i,$$

where S_i is a surface of genus greater than two and one boundary component for $1 \leq i \leq 4$, and the union identifies the boundary components of the S_i; the space X_2 is similar. Choose locally CAT(-1) metrics on X_1 and X_2 that are hyperbolic on each surface, and let $\widetilde{X}_i$ denote the universal cover of X_i equipped with this metric.

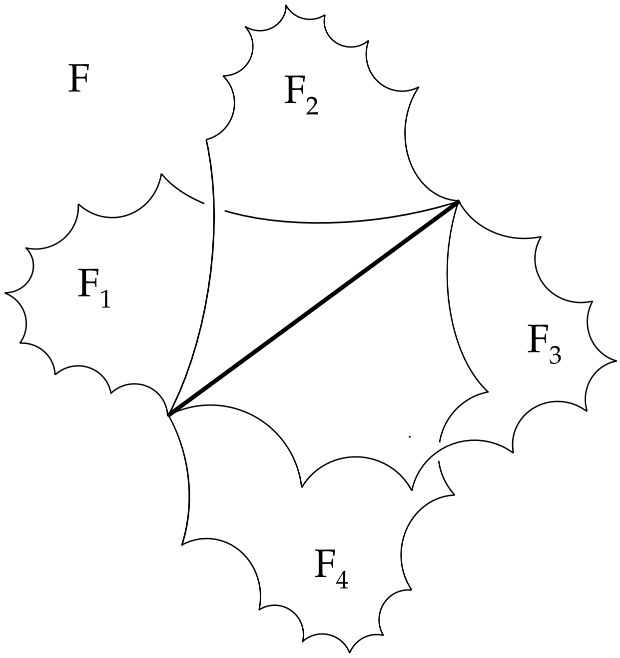

FIGURE 9. An illustration of the fundamental domain F for the action of $\pi_1(X_1)$ on $\widetilde{X}_1$. The fundamental domain is built from four convex hyperbolic polygons F_i. The darkened edge is referred to as the *branching edge*.

Let γ_i denote the singular curve in X_i and let $\widetilde{\gamma}_i$ represent the component of the preimage of γ_i in $\widetilde{X}_i$ stabilized by $\langle[\gamma_i]\rangle$. Let $\mathcal{L}_i = \{g \cdot \widetilde{\gamma}_i \,|\, g \in \pi_1(X_i)\}$. Let H_1, H_2, H_3, H_4 be the four components of $\widetilde{X}_1 \backslash \mathcal{L}_1$ incident to $\widetilde{\gamma}_1$ so that $\pi_1(S_i)$ stabilizes H_i, and let J_1, J_2, J_3, J_4 be the four components of $\widetilde{X}_2 \backslash \mathcal{L}_2$ incident to $\widetilde{\gamma}_2$ so that $\pi_1(T_i)$ stabilizes J_i.

Let $F = \displaystyle\bigcup_{i=1}^{4} F_i$ be a connected fundamental domain for the action of $\pi_1(X_1)$ on $\widetilde{X}_1$ that comes from a cell division of X_1 with a single vertex and so that

- $F_i \subset H_i$ is a fundamental domain for the action of $\pi_1(S_i)$ on H_i,
- F_i is a convex hyperbolic polygon with at least nine sides so that exactly one edge of F_i lies in $\widetilde{\gamma}_1$. We refer to this distinguished edge as the *branching edge* of F_i. The remaining vertices of F_i lie on $g\widetilde{\gamma}_1$ for distinct $g \in \pi_1(X_1)$,

- the branching edges F_i are identified via an isometry to form the connected fundamental domain F.

An example is given in Figure 9. Let $D = \bigcup_{i=1}^{4} D_i$ be a connected fundamental domain for the action of $\pi_1(X_2)$ on $\widetilde{X}_2$ constructed similarly. Note that F and D are not *strict fundamental domains* (see [BH99, Definition II.12.7]); in particular, F and D contain many vertices.

Isometry types of cells used in the cell decompositions:

Let x and y be one endpoint of the branching edges in F and D, respectively. We will show that each polygon in the cell complexes constructed lies in the finite set of polygons $\mathcal{P}$ that satisfy the following three conditions.

- The vertex sets are

$$\mathcal{V}_1 = \{g \cdot x \,|\, g \in \pi_1(X_1)\} \quad \text{and} \quad \mathcal{V}_2 = \{g \cdot y \,|\, g \in \pi_1(X_2)\},$$

  respectively, the same vertices that appear in the tilings by fundamental domains.

- Each edge is isometric to a geodesic segment connecting two vertices of F or D.

- The number of sides of each polygon is bounded above by $M \in \mathbb{N}$, where M is two times the maximum number of sides in F or D times the maximum valance x or y.

Construction of the first cell in H_1 and J_1:

Let V be the vertices in the fundamental domain F_1 and let W be the vertices in the fundamental domain D_1. If V and W have the same size, the fundamental domains themselves are the first cells used in the cell decomposition of H_1 and

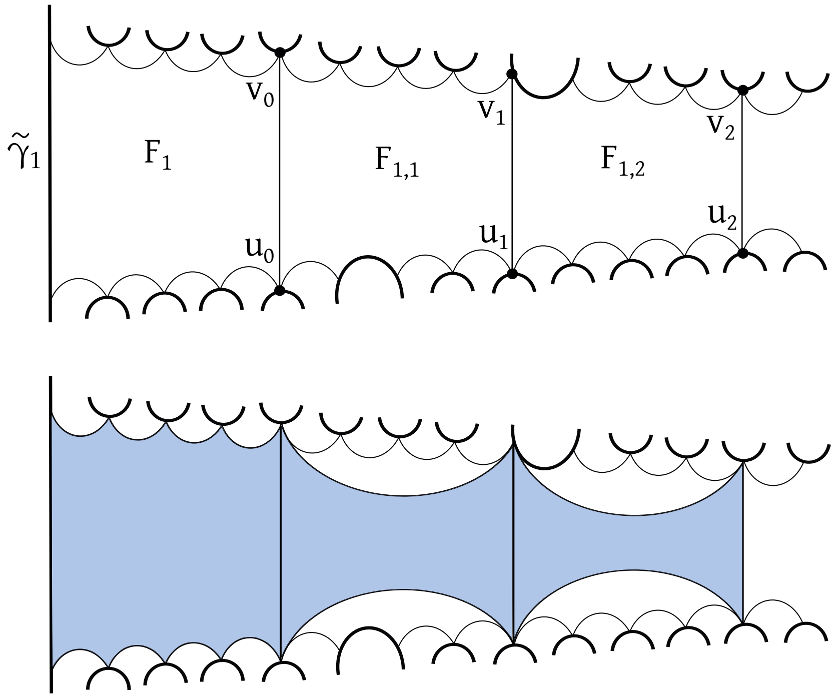

FIGURE 10. In the top image are translates $F_{1,i}$ of the fundamental domain F_1 in H_1. The dark lines are translates of $\widetilde{\gamma}_1$, which are boundary lines of the region H_1. The vertices u_i and v_i are selected as in the proof of the theorem. Shaded below is the first tile in H_1 in the setting where the fundamental domain D_1 has more sides than the fundamental domain F_1.

J_1; continue to the definition of the map. Otherwise, without loss of generality, $|W| - |V| = k > 0$. We will enlarge V until $|V| = |W|$.

Suppose $k = 2n + m$ for some $n \geq 0$ and $m \in \{0, 1\}$. By the choice of the fundamental domain, there is a non-branching edge $\{u_0, v_0\}$ of F_1 that is disjoint from the branching edge of F_1 and its two adjacent edges. The edge $\{u_0, v_0\}$ lies in a second translate of the fundamental domain $F_{1,1} \subset H_1$. There is a non-branching edge $\{u_1, v_1\}$ in $F_{1,1}$ disjoint from $\{u_0, v_0\}$ and its two adjacent edges. Similarly, there are edges $\{u_i, v_i\}$ for $1 \leq i \leq n + 1$, where $\{u_i, v_i\}$ and $\{u_{i+1}, v_{i+1}\}$ lie in the same fundamental domain $F_{1,i+1}$ for $1 \leq i \leq n$, and $\{u_i, v_i\}$

is disjoint from $\{u_j, v_j\}$ and its two adjacent edges for $i \neq j$, as illustrated in Figure 10.

To construct the cycle boundary of P_V, the first cell in H_1, start with the cycle boundary of F_1. Remove the edge $\{u_0, v_0\}$. Add geodesic segments $\{u_i, u_{i+1}\}$ and $\{v_i, v_{i+1}\}$ for $1 \leq i \leq n-1$. Up to relabeling the u_i and v_i, we may assume $\{u_i, u_{i+1}\}$ and $\{v_i, v_{i+1}\}$ do not intersect. If k is even, add $\{u_n, v_n\}$ to complete the cycle boundary of the polygon. If k is odd, add $\{u_n, u_{n+1}\}$ and $\{v_n, u_{n+1}\}$ to complete the cycle. Attach a 2-cell to this boundary cycle to form the first cell P_V in H_1. Let P_W be the fundamental domain D_1, the first cell in J_1.

Map P_V to P_W by a cellular homeomorphism ϕ, sending the branching edge of P_V to the branching edge of P_W, and dilating along each edge of the tile. After extending the fundamental domain F_1 to the tile P_V, it is possible that P_V is not convex. If this is the case, subdivide P_V and P_W isomorphically into convex polygons so the configurations have isomorphic 1-skeleta. Observe that the number of edges in any polygon is bounded above by the size of the largest fundamental domain, and each edge connects vertices that lie in a common translate of the fundamental domain. Thus, $P_V, P_W \in \mathcal{P}$.

Constructing the remaining cells in H_1 and J_1:

Extend the cell decompositions to all of H_1 and J_1 recursively. Along each new edge of a polygon built during the preceding stage, build one new polygon in H_1 and a corresponding new polygon in J_1. Each new polygon is constructed in a manner similar to the first polygons. Begin by constructing one new polygon along each edge of P_V and P_W that lies in the interior of H_1 and J_1 as follows.

Let $\{a, a_0\}$ be an edge of P_V that lies in the interior of H_1 and let $\{b, b_0\} = \phi(\{a, a_0\})$. By construction, the edge $\{a, a_0\}$ connects two vertices in a translate of the fundamental domain, and the interior of this geodesic segment either lies on a non-branching edge of a translate of the fundamental domain or in the interior of a translate of the fundamental domain. This distinction does not affect the construction of the new cells. The vertices a and a_0 lie in distinct translates of $\widetilde{\gamma}_1$ that are boundary lines of H_1. Let $\{a, a'\}$ and $\{a_0, a_0'\}$ be the branching edges on these translate of $\widetilde{\gamma}_1$ that lie in the component of $H_1 \backslash \{a, a_0\}$ that does not contain P_V. Let $\{b, b'\}$ and $\{b_0, b_0'\}$ be the analogous edges in J_1. We form cycles C_A in H_1 and C_B in J_1 that contain the paths $\{a', a, a_0, a_0'\}$ and $\{b', b, b_0, b_0'\}$, respectively, and will serve as the boundary cycles of the new cells constructed. The branching edges of the tiling by fundamental domains are distinguished; so, to ensure C_A can be mapped to C_B, we extend these paths $\{a', a, a_0, a_0'\}$ and $\{b', b, b_0, b_0'\}$ to cycles that contain no other branching edges.

Let $A_1, \ldots, A_m$ be the (non-empty) set of translates of the fundamental domain F_1 in H_1 that intersect a or a_0 and the component of $H_1 \backslash \{a, a_0\}$ that does not contain P_V. Note that if the edge $\{a, a_0\}$ lies in the interior of a fundamental domain, then A_i may only be part of a fundamental domain for some i. Suppose the A_i are labeled so that A_1 contains $\{a, a'\}$, A_n contains $\{a_0, a_0'\}$, and A_i and A_{i+1} intersect in an edge $\{\epsilon, a_i\}$ of the tiling by fundamental domains where $\epsilon \in \{a, a_0\}$ and $1 \leq i \leq m-1$, as illustrated in Figure 11. Let $B_1, \ldots, B_n$ and $b_1, \ldots, b_{n-1}$ be similar. Form an embedded cycle

$$C_A = \{a, a', p_1, a_1, p_2, a_2, \ldots, a_{m-1}, p_m, a_0', a_0\},$$

where p_i is an embedded path in A_i containing the remaining vertices of ∂A_i, but choosing only one vertex from a branching edge of A_i. Let

$$C_B = \{b, b', q_1, b_1, q_2, b_2, \ldots, b_{n-1}, q_n, b_0', b_0\}$$

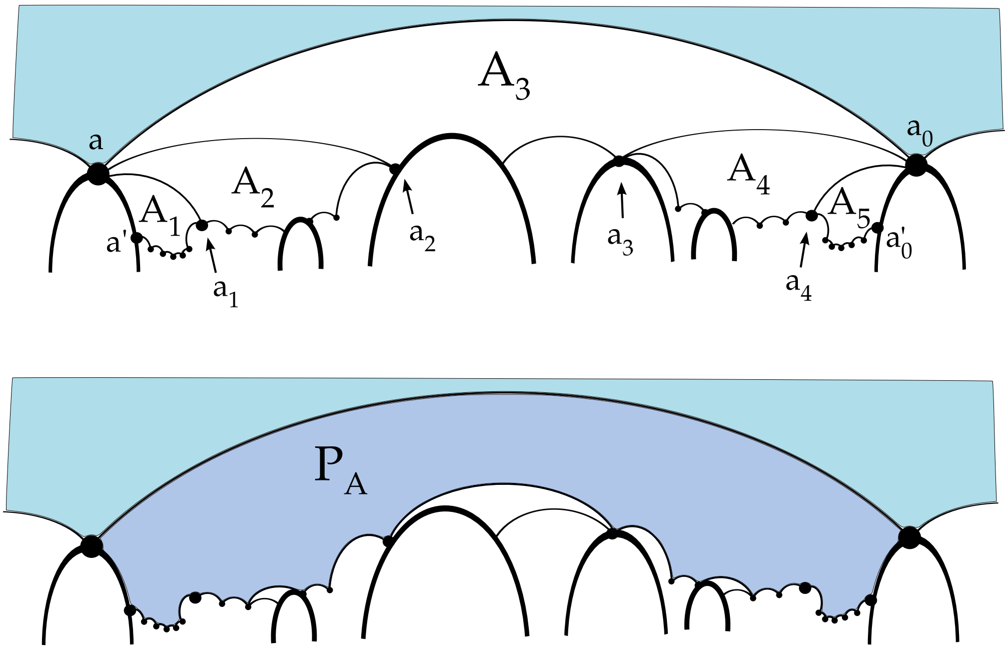

FIGURE 11. The figures illustrate how to extend the tiling recursively along an edge $\{a, a_0\}$ of a (shaded) tile previously constructed. The A_i are translates of the fundamental domain that intersect a or a_0, and the a_i are their points of intersection. The dark lines are translates of $\widetilde{\gamma}_1$ that bound H_1. Below, the new tile P_A is drawn; its cycle boundary contains all of the a_i as well as paths $p_i \subset A_i$ that include the other vertices of A_i, except that only one vertex is chosen from a boundary geodesic. Then, the only edges of P_A that lie on the boundary geodesics are $\{a, a'\}$ and $\{a_0, a_0'\}$.

be similar. If $|C_A| = |C_B|$, continue to the cell and map definitions. Otherwise, suppose without loss of generality, $|C_B| > |C_A|$. By the choice of fundamental domains, there is a non-branching edge of a fundamental domain in the cycle C_A disjoint from $\{a', a, a_0, a_0'\}$ and its adjacent edges, which can be used to extend the cycle C_A as with the first cell. After extending the cycle if necessary, attach 2-cells to these boundary cycles to form polygons P_A and P_B.

Map P_A to P_B by a cellular homeomorphism, sending $\{a', a, a_0, a_0'\}$ to $\{b', b, b_0, b_0'\}$, and dilating along each edge of the tile. As before, if P_A or P_B is not convex, subdivide P_A and P_B isomorphically into convex polygons so the configurations have isomorphic 1-skeleta. The map $P_A \to P_B$ extends the map $P_V \to P_W$ and the cellular isomorphism.

By construction, $P_A, P_B \in \mathcal{P}$. Continue construction in this way along each edge of each polygon constructed. The cell complexes built in the regions H_1 and J_1 are exhaustive since the tiling of these regions by the fundamental domains F_1 and D_1, respectively, is exhaustive. That is, in our cell decomposition of H_1, the first polygon contains the fundamental domain F_1, the next round of polygons contain all of the translates of the fundamental domain F_1 that are adjacent to F_1, the following round of polygons contain all of the translates of F_1 adjacent to these fundamental domains, and so on; the cell decomposition of J_1 is similar.

Extending the cell decomposition to the entire universal covers:

First, realize H_i and J_i as isomorphic cell complexes for $2 \leq i \leq 4$ in the same manner as with H_1 and J_1. Let

$$\phi_i : H_i \to J_i$$

be the cellular homeomorphism constructed, which is dilation when restricted to each boundary geodesic of H_i. So, the maps $\phi_i : H_i \to J_i$ and $\phi_j : H_j \to J_j$ agree when restricted to their intersection. We will use the action of the group to extend these maps and hence these cell decompositions to all of $\widetilde{X}_1$ and $\widetilde{X}_2$.

Recall, $\mathcal{L}_i = \{g \cdot \widetilde{\gamma}_i \,|\, g \in \pi_1(X_i)\}$ is the set of branching geodesics in $\widetilde{X}_i$. We define a cellular homeomorphism

$$\Phi : \widetilde{X}_1 \to \widetilde{X}_2$$

recursively, mapping components of $\mathcal{C}_1 = \widetilde{X}_1 \backslash \mathcal{L}_1$ to components of $\mathcal{C}_2 = \widetilde{X}_2 \backslash \mathcal{L}_2$.

Let

$$\Phi : \bigcup_{i=1}^{4} H_i \to \bigcup_{i=1}^{4} J_i$$

be defined by the maps above: $\Phi(H_i) = \phi_i(H_i)$.

Extend the map Φ along each unmapped branching geodesic of a component mapped during the preceding stage as follows. To begin, let $g\widetilde{\gamma}$ be a branching geodesic of H_1 for some nontrivial $g \in \pi_1(X_1)$. Suppose R_2, R_3, and R_4 are components of $\mathcal{C}_1$ that intersect the boundary of H_1 in the branching geodesic $g\widetilde{\gamma}_1$. Without loss of generality, $g^{-1}(R_i) = H_i$. The isometry $g : H_i \to R_i$ induces a cell decomposition of R_i isomorphic to the cell decomposition of H_i. Suppose $\Phi(g\widetilde{\gamma}_1) = h\widetilde{\gamma}_2 \in J_1$ for some $h \in \pi_1(X_2)$. Let S_2, S_3, and S_4 be the other components of $\mathcal{C}_2$ incident to $h\widetilde{\gamma}_2$ so that $h^{-1}(S_i) = J_i$. Then, $h : J_i \mapsto S_i$ induces a tiling of S_i isomorphic to the cell decompositions of J_i, H_i, and R_i. Map R_i to S_i by the cellular homeomorphism $h \circ \Phi_i \circ g^{-1}$ for $2 \leq i \leq 4$.

Repeat this procedure along each unmapped branching geodesic of the regions H_i and J_i, then along each unmapped branching geodesic of the regions incident to H_i and J_i, and so on to define Φ, an exhaustive cellular homeomorphism $\widetilde{X}_1 \to \widetilde{X}_2$. By Lemma 4.2.6, $\widetilde{X}_1$ and $\widetilde{X}_2$ are bilipschitz equivalent. □

Corollary 4.3.2. *If $G, G' \in \mathcal{C}_S$, then G and G' are quasi-isometric.*

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