Mechanical Theorem Proving in Tarski's Geometry.

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Outline

1. Interactive proof / Automated theorem proving
2. Tarski's axioms
3. Overview of the formalization
4. Degenerated cases
5. Comparison with related work

Interactive proof

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My goal is to merge the two approaches.

Related Work

Formalization of geometry

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Tarski's axioms

- Art Quaife (Otter)[Qua89]

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- They are simple.
- They have good meta-mathematical properties.
- They can be generalized to different dimensions and geometries.

The Coq proof assistant

- Interactive proof
- But some automation is available
- Intuitionist logic
- Proofs are performed using tactics

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- Your axioms

Tarski's axioms

Points (no lines, no planes).

Two predicates :

- equidistance $\equiv$
- betweeneness β

Axioms

1 Reflexivity of equidistance

$$AB \equiv BA$$

2 Pseudo-transitivity of equidistance

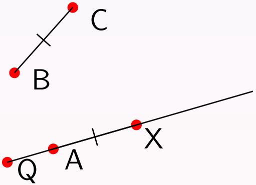
$$AB \equiv PQ \wedge AB \equiv RS \Rightarrow PQ \equiv RS$$

3 Identity of equidistance

$$AB \equiv CC \Rightarrow A = B$$

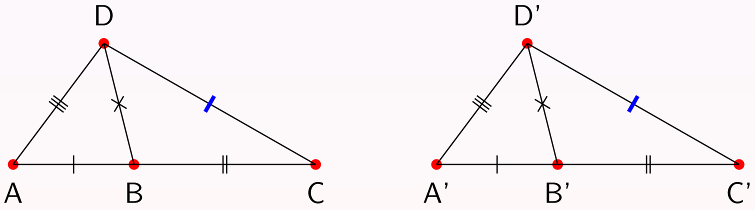
4 Segment construction

$$\exists X, \beta\ Q\,A\,X \wedge AX \equiv BC$$



5 Five segments

$$A \neq B \wedge \beta\ A\ B\ C \wedge \beta\ A'\ B'\ C' \wedge$$
$$AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \Rightarrow CD \equiv C'D'$$



5_1 Five segments (variant)

$$\left.\begin{array}{l} A \neq B \wedge B \neq C \wedge \beta\, A\, B\, C \wedge \beta\, A'\, B'\, C' \wedge \\ AB \equiv A'B' \wedge BC \equiv B'C' \wedge AD \equiv A'D' \wedge BD \equiv B'D' \end{array}\right. \Rightarrow CD \equiv C'D'$$

6 Identity of betweeness

$$\beta\ A\,B\,A \Rightarrow A = B$$

7 Pasch (inner)

$$\beta\ A\ P\ C \land \beta\ B\ Q\ C \Rightarrow \exists X, \beta\ P\ X\ B \land \beta\ Q\ X\ A$$

7_1 Pasch (outer)

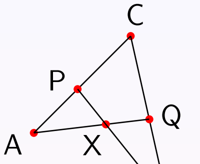
$$\beta\ A\ P\ C \land \beta\ Q\ C\ B \Rightarrow \exists X, \beta\ A\ X\ Q \land \beta\ B\ P\ X$$

7_2 Pasch (outer) (Variant)

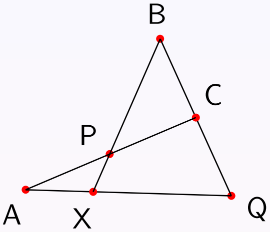
$$\beta\ A\ P\ C \land \beta\ Q\ C\ B \Rightarrow \exists X, \beta\ A\ X\ Q \land \beta\ X\ P\ B$$

7_3 Pasch weak

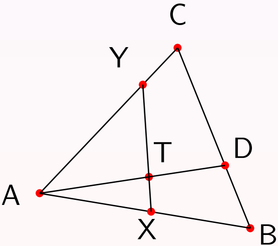
$$\beta\ A\ T\ D \land \beta\ B\ D\ C \Rightarrow \exists X, Y, \beta\ A\ X\ B \land \beta\ A\ Y\ C \land \beta\ Y\ T\ X$$



Inner



Outer



Weak

8(2) Dimension, lower bound 2

$$\exists ABC, \neg\beta\ A\ B\ C \land \neg\beta\ B\ C\ A \land \neg\beta\ C\ A\ B$$

8(n) Dimension, lower bound n

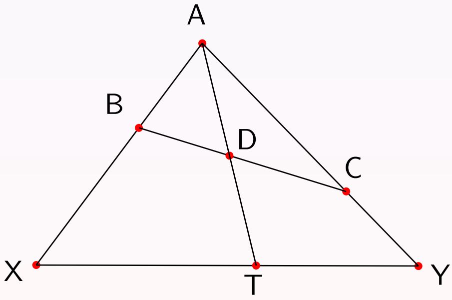
$$\exists ABCP_1P_2\ldots P_{n-1}, \begin{array}{l} \bigwedge_{1\leq i<j<n} p_i \neq p_j \land \\ \bigwedge_{i=2}^{n-1} AP_1 \equiv AP_i \land BP_1 \equiv BP_i \land CP_1 \equiv CP_i \land \\ \neg\beta\ A\ B\ C \land \neg\beta\ B\ C\ A \land \neg\beta\ C\ A\ B \end{array}$$

$9(n)$ Dimension, upper bound n

$$\begin{array}{l} \bigwedge_{1\leq i<j\leq n} p_i \neq p_j \wedge \\ \bigwedge_{i=2}^{n} \begin{array}{l} AP_1 \equiv AP_i \wedge \\ BP_1 \equiv BP_i \wedge \\ CP_1 \equiv CP_i \end{array} \end{array} \Rightarrow \beta\ A\ B\ C \vee \beta\ B\ C\ A \vee \beta\ C\ A\ B$$

10 Euclid's axiom

$$\beta\, A\, D\, T \wedge \beta\, B\, D\, C \wedge A \neq D \Rightarrow \exists X, Y \beta\, A\, B\, X \wedge \beta\, A\, C\, Y \wedge \beta\, X\, T\, Y$$



11 Continuity

$$\exists a, \forall xy, (x \in X \land y \in Y \Rightarrow \beta\ a\, x\, y) \Rightarrow \exists b, \forall xy, x \in X \land y \in Y \Rightarrow \beta\ x\, b\, y$$

Schema 11 Continuity (schema)

$$\exists a, \forall xy, (\alpha \land \beta \Rightarrow \beta\ a\, x\, y) \Rightarrow \exists b, \forall xy, \alpha \land \beta \Rightarrow \beta\ x\, b\, y$$

where α and β are first order formulas, such that a,b and y do not appear free in α and a,b and x do not appear free in β.

12 Reflexivity of β

$$\beta\ A\ B\ B$$

14 Symmetry of β

$$\beta\ A\ B\ C \Rightarrow \beta\ C\ B\ A$$

13 Compatibility with equality of β

$$A = B \Rightarrow \beta\ A\ B\ A$$

19 Compatibility with equality of $\equiv$

$$A = B \Rightarrow AC \equiv BC$$

15 Transitivity (inner) of β

$$\beta\ A\ B\ D \wedge \beta\ B\ C\ D \Rightarrow \beta\ A\ B\ C$$

16 Transitivity (outer) of β

$$\beta\ A\ B\ C \wedge \beta\ B\ C\ D \wedge B \neq C \Rightarrow \beta\ A\ B\ D$$

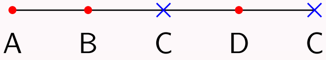
17 Pseudo-transitivity (inner) of β

$$\beta\ A\,B\,D \wedge \beta\ A\,C\,D \Rightarrow \beta\ A\,B\,C \vee \beta\ A\,C\,B$$

18 Pseudo-transitivity (outer) of β

$$\beta\ A\,B\,C \wedge \beta\ A\,B\,D \wedge A \neq B \Rightarrow \beta\ A\,C\,D \vee \beta\ A\,D\,C$$

Axiom 17



Axiom 18

20 Unicity of the triangle construction

$$\begin{array}{l} AC \equiv AC' \wedge BC \equiv BC' \wedge \\ \beta\ A\ D\ B \wedge \beta\ A\ D'\ B \wedge \beta\ C\ D\ X \wedge \\ \beta\ C'\ D'\ X \wedge D \neq X \wedge D' \neq X \end{array} \Rightarrow C = C'$$

20_1 Unicity of the triangle construction (variant)

$$\begin{array}{l} A \neq B \wedge \\ AC \equiv AC' \wedge BC \equiv BC' \wedge \\ \beta\ B\ D\ C' \wedge (\beta\ A\ D\ C \vee \beta\ A\ C\ D) \end{array} \Rightarrow C = C'$$

21 Existence of the triangle construction

$$AB \equiv A'B' \Rightarrow \exists CX, \begin{array}{l} AC \equiv A'C' \wedge BC \equiv B'C' \wedge \\ \beta\ C\ X\ P \wedge (\beta\ A\ B\ X \vee \beta\ B\ X\ A \vee \beta\ X\ A\ B) \end{array}$$

History

1940 [Tar67]	1951 [Tar51]	1959 [Tar59]	1965 [Gup65]	1983 [SST83]
1	1	1	1	1
2	2	2	2	2
3	3	3	3	3
4	4	4	4	4
5_1	5_1	$\rightarrow 5$	5	5
6	6	6		6
7_2	7_2	$\rightarrow 7_1$	7_1	$\rightarrow 7$
8(2)	8(2)	8(2)	8(2)	8(2)
$9_1(2)$	$9_1(2)$	$\rightarrow 9(2)$	9(2)	9(2)
10	10	$\rightarrow 10_1$	10_1	$\rightarrow 10$
11	11	11	11	11
12	12			
13				
14	14			
15	15	15	15	
16	16			
17	17			
18	18	18		
19				
20	$\rightarrow 20_1$			
21	21			
20 + 1 schema	18 + 1 schema	12 + 1 schema	10 + 1 schema	10 + 1 schema

Formalization

W. Schwabhäuser
W. Szmielew
A. Tarski

Metamathematische Methoden in der Geometrie

Springer-Verlag 1983

Overview I

About 200 lemmas and 6000 lines of proofs and definitions.

The first chapter contains the axioms.

The second chapter contains some basic properties of equidistance (noted `Cong`).

The third chapter contains some basic properties of the betweeness predicate (noted `Bet`). In particular, it contains the proofs of the axioms 12, 14 and 16.

The fourth chapters provides properties about `Cong`, `Col` and `Bet`.

The fifth chapter contains the proof of the transitivity of `Bet` and the definition of a length comparison predicate. It contains the proof of the axioms 17 and 18.

The sixth chapter defines the `out` predicate which says that a point is not on a line, it is used to prove transitivity properties for `Col`.

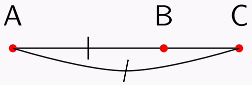
Overview II

The seventh chapter defines the midpoint and the symmetric point and prove some properties.

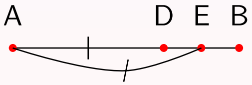
The eighth chapter contains the definition of the predicate "perpendicular", and finally proves the existence of the midpoint.

Two crucial lemmas

$$\forall ABC, \beta\ A\ C\ B \land AC \equiv AB \Rightarrow C = B$$



$$\forall ABDE, \beta\ A\ D\ B \land \beta\ A\ E\ B \land AD \equiv AE \Rightarrow D = E.$$



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- It is simple but effective !
- Still, the axiom system is important.

Comparison with other formalizations

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- ☺ It has good meta-mathematical properties.
- ☺ Generalization to other dimensions is easy.
- ☹ Lemma scheduling is more complicated.
- ☹ It is not well adapted to teaching.

Comparison with ATP

- We can not use a decision procedure specialized in geometry.
- Problems which can be solved by at least one general purpose ATP AND appear in my formalization have short proofs.

Examples

Lemma	Coq proof	Otter	Vampire
symmetry of betweeness	6 lines	0s	0s
reflexivity of equidistance	2 lines	0s	0s
transitivity of equidistance	2 lines	0s	0s
existence of the midpoint	6000 lines	timeout	timeout

Future work

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Alfred Tarski.
The completeness of elementary algebra and geometry, 1967.

An example.

Gupta

$$A \neq B \wedge \beta\ A\ B\ C \wedge \beta\ A\ B\ D \Rightarrow \beta\ A\ C\ D \vee \beta\ A\ D\ C$$