TIME MEASUREMENTS, $1/F$ NOISE OF THE OSCILLATORS AND ALGEBRAIC NUMBERS

M. PLANAT
Laboratoire de Physique et Métrologie des Oscillateurs du CNRS
32 Avenue de l'Observatoire, 25044 Besançon Cedex, France
E-mail : planat@lpmo.edu

1. Introduction

Many complex systems from physics, biology, society... exhibit a $1/f$ power spectrum in their time variability so that it is tempting to regard $1/f$ noise as a unifying principle in the study of time. The principle may be useful in reconciling two opposite views of time, the cyclic and the linear one, the philosophic view of eternity as opposed to that of time and death. The temporal experience of such complex systems may only be obtained thanks to clocks which are continuously or occasionally slaved. Here time is discrete with a unit equal to the averaging time of each experience. Its structure is reflected into the measured arithmetical sequence. They are resets in the frequencies and couplings of the clocks, like in any human made calendar. The statistics of the resets shows about constant variability whatever the averaging time: this is characteristic of the flicker ($1/f$) noise. In a number of electronic experiments we related the variability in the oscillators to number theory, and time to prime numbers. In such a context, time (and $1/f$ noise) has to do with Riemann hypothesis that all zeros of the Riemann zeta function are located on the critical line, a mathematical conjecture still open after 150 years.

2. The Experience of Time from Electronic Oscillators

2.1. ASYNCHRONOUS OSCILLATORS: THE OPEN LOOP AND CONTINUED FRACTIONS

Low frequency noise of electronic oscillators is usefully interpreted in terms of arithmetic[1]: this is because the measurement of the frequency $f(t)$ of an oscillator under test is meaured versus the one f_0 of a reference oscillator thanks to a nonlinear mixing set-up and a filter. The beat frequency

$$f_B = |p_i f_0 - q_i f(t)| \,, \quad \text{with } p_i \text{ and } q_i \text{ integers,} \tag{1}$$

follows from the continued fraction expansion of the frequency ratio $\nu = \frac{f}{f_0} = [a_0; a_1, a_2, ...a_i, a, ...] = a_0 + 1/\{a_1 + 1/\{a_2 + ... + 1/\{a_i + 1/\{a...\}\}\}\} = \frac{p_i(a)}{q_i(a)} \simeq \frac{p_i}{q_i}$ of the input oscillators. Here $a_{\min} \leq a \leq a_{\max}$, with $a_{\min} = \lfloor \frac{f_0}{f_c q_i} \rfloor$, $a_{\max} = \lfloor \frac{f_0}{f_d q_i} \rfloor$ and f_c and f_d are the low and high frequency cut-off of the filter. Since $a \gg 1$ in typical measurements, the beat note is well approximated by the convergent p_i/q_i used in (1) which restricts to the partial quotient a_i in the expansion. Fig.1 shows a schematic of the resulting intermodulation spectrum.

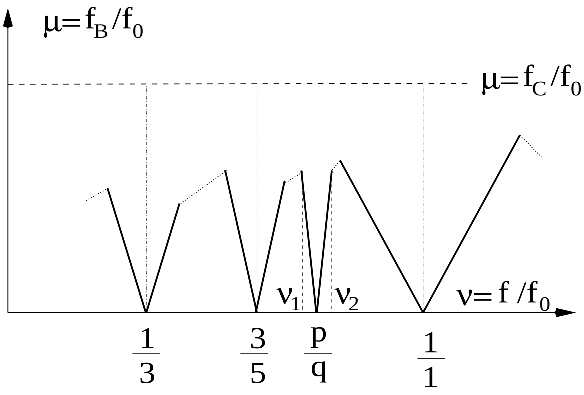

Figure 1. The intermodulation spectrum at the output of the mixer+filter set-up.

It may happen that the partial quotients after a don't play any role and frequency jumps occurs randomly at definite values of a leading to a large white frequency noise arising from the detection set-up instead of the oscillator under test[1].

This can be compared to the measurement of time from a moon-sun calendar. Early calendars have been devised from the motion of moon and sun as observed from the earth. The continued fraction expansion of the ratio ν between the sun year and the moon year is

$$\nu = \frac{365.242191}{29.530589} = [12; 2, 1, 2, 1, 1, 17, \ldots]. \tag{2}$$

The first approximation $\nu = 12$ (with 354 days) can be corrected by adding one month every two years, the second one (with 369 days) may be corrected by adding one month every three years and so on. Fluctuation of the integer

a in the frequency measurement set-up has the same aim to correct the measurement versus time.

2.2. SYNCHRONOUS OSCILLATORS: THE PHASE LOCKED LOOP AND THE PRIME NUMBERS

By controlling the frequency of the test oscillator from the error signal at the output of the detector, one gets phase locking at each harmonic p_i/q_i over a frequency window of width twice the open loop gain K_{q_i}. If one neglects harmonic interactions the phase difference between input oscillators is given from an Arnold map

$$\theta_{n+1} = \theta_n + 2\pi\Omega - c\ \sin\theta_n, \tag{3}$$

where $\Omega = f/f_0$ is the bare frequency ratio, $c = K/f_0$ and K is the open loop gain at the fundamental basin $p_i/q_i = 1/1$. Such a nonlinear map is studied by introducing the winding number $\nu = \lim_{n\to\infty}(\theta_n - \theta_0)/(2\pi n)$. The limit exists everywhere as long as $c < 1$, the curve ν versus Ω is a devil's staircase with steps attached to rational values of $\Omega = p_i/q_i$ and with width increasing with the coupling coefficient c. The phase locking zones may overlap if $c > 1$ leading to chaos from quasi-periodicity. In the experiments we used an open loop gain $K \ll f_0$ so that the variability of the beat signal, shown in Fig.2, is of a different origin.

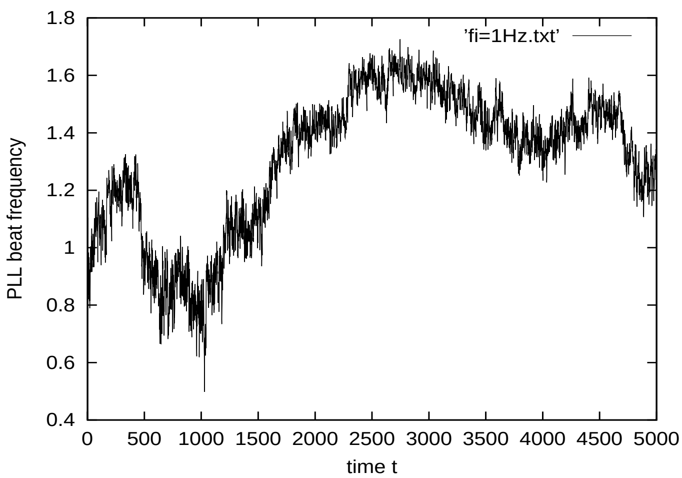

Figure 2. Beat frequency between two radiofrequency oscillators close to phase locking.

We looked at the Allan variance of beat frequency measurements, that is the mean squared value $\sigma^2(\tau)$ of the relative frequency deviations between adjacent samples in the time series, counted over an integration time τ. We found it was a constant (i.e. a $1/f$ noise), scaled by a nonlinear law, reflecting the dependance of the beat frequency versus the bare frequency deviation and the open loop gain[2]. A mechanism for the variability should come from the harmonic interactions that have been neglected so far in (3). To appreciate the impact of harmonics on the coupling coefficient one should observe that each harmonic of denominator q_i leads to the same fluctuating frequency $\delta f_B = q_i \delta f(t)$. There are $\phi(q_i)$ of them, where $\phi(q_i)$ is Euler totient function, that is the number of integers less or equal to q_i and prime to it; the average coupling coefficient is thus expected to be $1/\phi(q_i)$.

We developed a more refined model based on the properties of primes by defining the coupling coefficient as $c^* = c\Lambda(n; q_i, p_i)$ with

$$\Lambda(n; q_i, p_i) = \begin{cases} \ln b & \text{if } n = b^k, b \text{ a prime and } \mathrm{n \equiv p_i \ mod(q_i)}, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

This means a non zero coupling to harmonics at times $n = p_i + q_i\, l$, l integer whenever n is a power of a prime; the coupling at the fundamental mode is the so-called von Mangoldt function $\Lambda(n)$[3].

According to the generalized Riemann hypothesis, at large t, one gets the average[4]

$$c^*_{\mathrm{av}}/c = \frac{1}{t}\sum_{n=1}^{t} \Lambda(n; q_i, p_i) = \frac{1}{\phi(q_i)} + \epsilon(t), \quad (5)$$

with $\epsilon(t) = 0(t^{-1/2} ln^2(t))$ which is a good approximation as long as $q_i < \sqrt{t}$. A better estimate may also be obtained at larger q_i[4].

For $p_i/q_i = 1/1$ the fluctuating term may be expressed in terms of the zeros of the Riemann zeta function $\zeta(s)$ which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{b \text{ prime}} \frac{1}{1 - \frac{1}{b^s}} \quad \text{where } \Re(s) > 1. \quad (6)$$

Formula (5) is obtained by taking the logarithmic derivative $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n) n^{-s} = s\int_1^{\infty} t^{-s-1}\psi(t)dt$ and by inverting this latter Mellin integral[3]. The error term results as

$$\epsilon(t) = -\ln(2\pi) - \frac{1}{2}\ln(1 - t^{-2}) - \sum_{\rho} \frac{t^{\rho}}{\rho}. \quad (7)$$

The first term at the right hand side of (7) is due to the pole of $\zeta(s)$ at $s = 1$. The second term in $\epsilon(t)$ is due to the trivial zeros of $\zeta(s)$ which are located at $s = -2l$ (l a positive integer). The third term is due to the remaining zeros of $\zeta(s)$. Billions of them have been computed; all are found to be located on the line $s = \frac{1}{2}$. Riemann hypothesis is the (unsolved) conjecture that all non trivial zeros belong to the critical line. These zeros are very irregularly spaced and are responsible for the very irregular shape of the error term[1]. For arbitrary p_i/q_i, Riemann zeta function extends to a Dirichlet series and the generalized Riemann hypothesis holds[4].

The power spectral density of $\epsilon(t)$ roughly looks like that of a $1/f$ noise as shown in Fig.3 (lower curve). In the context of phase locking experiments[2], the $1/f$ noise of the coupling coefficient in (5) is responsible of the desynchronization of the oscillators and of the $1/f$ noise found in the beat frequency f_B.

3. Time, $1/f$ Noise, Ramanujan Sums and the Golden Ratio

From its definition $1/f$ noise comes from the use of the fast Fourier transform (FFT). But the FFT refers to the fast calculation of the discrete Fourier transform (DFT) with a finite period $q = 2^l$, l a positive integer. In the DFT one starts with all q^{th} roots of the unity $\exp(2i\pi p/q)$, $p = 1 \ldots q$ and the signal analysis of the arithmetical sequence $x(n)$ is performed by projecting onto the n^{th} powers (or characters of Z/qZ)with well known formulas[5].

$$e_p(n) = \exp(2i\pi \frac{p}{q} n), \tag{8}$$

The signal analysis based on the DFT is not well suited to aperiodic sequences with many resonances (by nature a resonance is a primitive root of the unity: $(p, q) = 1$), and the FFT may fail to discover the underlying structure in the spectrum. We recently introduced a new method based on Ramanujan sums[6],[8]

$$c_q(n) = \sum_{\substack{p=1 \\ (p,q)=1}}^{q} \exp(2i\pi \frac{p}{q} n), \tag{9}$$

which are n^{th} powers of the q^{th} primitive roots of the unity. The sums may be evaluated as[6]

$$c_q(n) = \mu\left(\frac{q}{(q,n)}\right) \frac{\phi(q)}{\phi\left(\frac{q}{(q,n)}\right)}, \tag{10}$$

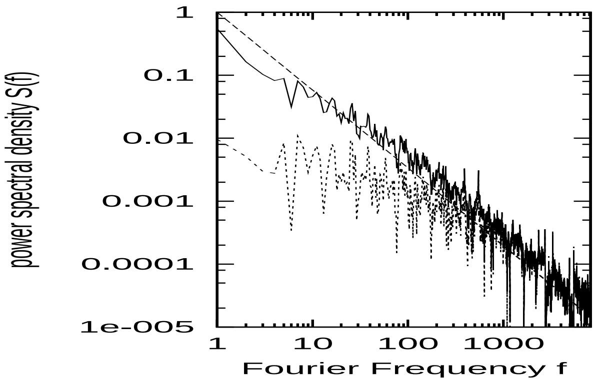

Figure 3. Power spectral density (FFT) of the error term of Mangoldt function $\Lambda(n)$(lower curve), of the error term in new Mangoldt function $b(n)$(upper curve) and the comparison to the power law $1/f^{2\alpha}$, with $\alpha = (\sqrt(5) - 1)/2$ the golden mean.

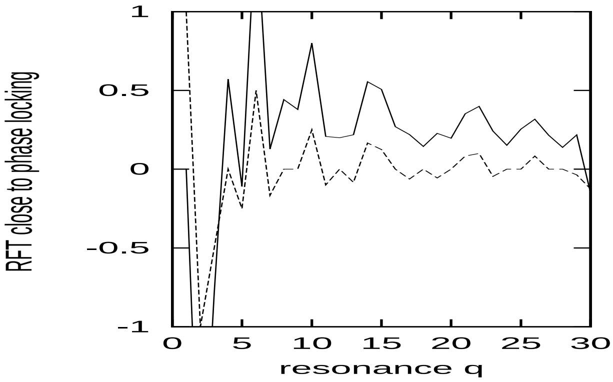

Figure 4. Ramanujan-Fourier transform (RFT) of the error term (upper curve) of new Mangoldt function $b(n)$ in comparison to the function $\mu(q)/\phi(q)$(lower curve).

where (q, n) means the greatest common divisor of q and n and the Möbius function follows from the unique prime number decomposition $n = \prod_k n_k^{\beta_k}$, ($n_k$ prime) of the integer n

$$\mu(n) = \begin{cases} 0 \text{ if } n \text{ contains a square } \beta_k > 1, \\ 1 \text{ if } n = 1, \\ (-1)^k \text{ if } n \text{ is the product} \\ \qquad \text{of } k \text{ distinct primes.} \end{cases} \tag{11}$$

The sums are quasiperiodic versus the time n ($c_1 = \overline{1}$; $c_2 = \overline{-1, 1}$; $c_3 =$

$\overline{-1,-1,2}$, where the bar indicate the period; for example $c_3(4) = -1$) and aperiodic versus the order q of the resonance. In particular $c_q(n) = \mu(q)$ whenever $(n, q) = 1$.

Möbius function can be considered as a coding sequence for prime numbers, as it is the case of Mangoldt function. Well known formulas are the inverse zeta function $1/\zeta(s) = \sum_{n\geq 1} \mu(n)/n^s$ and a formulation of Riemann hypothesis as $M(t) = \sum_{n=1}^{t} \mu(n) = O(t^{1/2+\epsilon})$, whatever ϵ. Mangoldt function is related to Möbius function thanks to the Ramanujan sums expansion found by Hardy[7]

$$b(n) = \frac{\phi(n)}{n}\Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n). \tag{12}$$

We call such a type of Fourier expansion a Ramanujan-Fourier transform (RFT). General formulas are given in our recent publication[8] and in the paper by Gadiyar[7]. This author also reports on a stimulating conjecture relating the autocorrelation function of $b(n)$ and the problem of prime pairs. In the special case (12), it is clear that $\mu(q)/\phi(q)$ is the RFT of the modified Mangoldt sequence $b(n)$. It is thus of high interest to compare the results of the RFT and of the FFT on $b(n)$. Let consider the summatory function

$$B(t) = \sum_{n=1}^{t} \Lambda(n)\phi(n)/n = t(1 + \epsilon_B(t)). \tag{13}$$

The fast Fourier transform $S_B(t)$ of the error term $\epsilon_B(t)$ is shown on Fig.3 (upper curve) and follows the approximate power law expansion

$$S_B(t) \sim f^{-2\alpha}, \tag{14}$$

with $\alpha = (\sqrt(5) - 1)/2 = 1/(1 + 1/(1 + 1/[1 + ...)))$. This spectrum shows a possible connexion between the golden ration α and $\mu(q)$ and thus a possible relationship between the theory of diophantine approximations for quadratic irrational numbers such as α and prime number theory. The RFT of $\epsilon_B(t)$ looks similar to the one $\mu(q)/\phi(q)$ of the new Mangoldt function $b(n)$ as shown in Fig.4.

Conclusion

In our description, time and its structure is revealed from the (non linear) interaction of two oscillators. The strength of this interaction fluctuates from universal principles, which also depend on the observer and its observation prism. Counting is the trace of observables, so that one should not be too much surprised that number theory, the Queen of Mathematics, takes part in the play.

The measured time durations over the integration time τ shows Allan deviation $\sigma(\tau)$ very similar to the Weber constant $K(\tau)$ of physiological perception[9]. The best perception occurs over a window $(\tau_{\min}, \tau_{\max})$ of constant variability, also called flicker$(1/f)$ noise. The flicker floor of time discrimination also depends on the sharpness of time measurements. In quartz oscillators, the sharpness is measured as the relative width $1/Q$ of the resonant line , with $Q = 10^6$ the quality factor of the acoustic resonator. It is far higher that the one $Q = 10^2$ of a conventional LC electronic resonator. Inversely microwave cavities may have wonderful sharp lines used in the design of atomic clocks and the physical determination of the second. May be the price to pay for consciousness is a very large relative bandwidth $1/Q \geq 1$ of our brain, the case $Q < 1$ meaning a meditation experience, and $Q = 0$ being the signature of death, the ultimate time experience of the Heidegger philosophy.

Time also lies at the boundary between order and chaos leading to the Golden ratio α, the most irrational number. A measure of time is the chambered snail Nautilus[10], that nature designed as a logarithmic spiral, each chamber being an up-scaled replica of the preceding chamber, with a constant scale factor equal to α. According to the extended relativity in $C-$spaces[11], the average dimension of our world $4 + \alpha^3 \simeq 4.236$ would also be related to α. And there is our newly proposed tautology that time $\equiv$ prime numbers $\equiv\ \alpha\ \equiv$ the Ramanujan-Fourier spectrum $\mu(q)$ and the Riemann hypothesis.

References

1. Planat, M. (2001), $1/f$ noise, the measurement of time and number theory, *Fluctuation and Noise Letters* **1**,R65.
2. Planat, M. and Henry, E. (2002), Arithmetic of $1/f$ noise in a phase locked loop, *Appl. Phys. Lett.* **80**(13), 2413–2415.
3. Edwards, H. M. (1974) *Riemann's Zeta Function*, Academic Press, N.Y.
4. Davenport, H. (1980) *Multiplicative Number Theory*, Springer Verlag, N.Y.
5. Terras, A. (1999) *Fourier Analysis on Finite Groups and Applications*, Cambridge University Press, Cambridge.
6. Hardy, G. H and Wright, E. M. (1979) *An Introduction to the Theory of Numbers*, fifth edition, Oxford University Press, Oxford.
7. Gadiyar, H. G. and Padma, R. (1999), Ramanujan-Fourier series, the Wiener-Khintchine formula and the distribution of prime pairs, *Physica* **A 269**, 503-510.
8. Planat, M. (to be published) Ramanujan sums for signal processing of low frequency noise.
9. Grondin, S. (this volume) Using Weber's law for studying psychological time.
10. Schroeder, M. (1991) *Fractals, Chaos, Power Laws*, W. H. Freeman and Co, New York.
11. Castro, C. (this volume) The program of the extended relativity theory in C-spaces.