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Penrose limits and co dimension 2 defects in supergravity theories

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ABSTRACT OF THE DISSERTATION

Penrose limits and co dimension 2 defects in supergravity theories

by

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In this dissertation, we will discuss two different holographic constructions. First, we construct a Penrose limit for the type IIB $AdS_6 \times S^2 \times \Sigma$ solutions of [1, 2, 3]. These solutions are dual to 5d SCFT which can often be described in terms of long quiver gauge theories. The resulting plane wave spacetime takes a universal form and the worldsheet action of the Green-Schwarz string is quadratic in the light cone gauge allowing us to obtain the spectrum of string excitation.

Next, we present a solution of $d = 7$ $N = 4$ gauged supergravity which is dual to co-dimension 2 defect living in a six-dimensional SCFT. Regularity conditions are satisfied by a one parameter family of solutions. We then uplift these to eleven dimensions where they can be understood as Lin-Lunin-Maldacena solutions. Using their electrostatic formulation, we find an infinite family of regular solutions describing holographic defects. We compute holographic observables in both the seven-dimensional and eleven-dimensional theories.

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M. Gutperle and N. Klein, "A note on co-dimension 2 defects in N=4,d=7 gauged supergravity," Nucl. Phys. B **984**, 115969 (2022) doi:10.1016/j.nuclphysb.2022.115969 [arXiv:2203.13839 [hep-th]].

M. Gutperle, N. Klein and D. Rathore, "Holographic 6d co-dimension 2 defect solutions in M-theory," [arXiv:2304.12899 [hep-th]]. Submitted to JHEP

Chapter 2 is based on [4] in collaboration with Michael Gutperle. Chapter 3 is based on [5] in collaboration with Michael Gutperle. Chapter 4 is based on [6] in collaboration with Michael Gutperle and Dikshant Rathore.

CHAPTER 1

Introduction: The AdS/CFT Correspondence

The AdS/CFT correspondence [7, 8, 9], the conjectured duality between certain theories of gravity in (d+1)-dimensional asymptotically anti-de Sitter spaces (AdS) and non-gravitational, conformal field theories (CFT) living on their d-dimensional boundaries, is one of the most important developments of modern theoretical physics and since being formulated, has been the impetus of much inquiry by (and an indispensable tool for) those looking to understand the physics on either side of the duality. It is in this context that the research being presented in this thesis - having to do with the computation of field theory observables using supergravity - was undertaken. While the computations in this thesis are restricted to the gravity side, we should mention that it is also possible to perform precise checks of these dualities with field theory calculations; the main tool here being supersymmetric localization [10] which provides us with results that can be reliably extended to strong coupling and compared against predictions from the gravity side.

In this introductory chapter, we will set the stage for the work described in the body of this thesis by presenting a overview of the correspondence as it was first arrived at through string theory. In the section 1.1, we will briefly review some aspects of perturbative string theory and in section 1.2, we discuss Dp-branes - the solitonic objects of the theory. An important aspect of these branes is that their low energy dynamics have two different descriptions; one which uses the effective dynamics of the theory living on their worldvolume and second which treats them as sources of the massless closed string fields in a low-energy limit. In the latter, these branes will have a geometric description given by a classical, supergravity solution of the low-energy field equations. It is precisely these two points of view which - after taking care to control corrections from the stringy dynamics - will provide us with each side of the duality between gauge theory and gravity. In sections 1.3 and 1.4, we will see the details of how this plays out in a few of the historically important cases in Type IIB and M theory. In section 1.5, we will explore a plane wave limit of the aforementioned duality in type IIB. This will give us computational tractability which so far is unavailable in the string σ model on the AdS background with RR fields. Finally, in section 1.6, we will explore holographic descriptions of field theories with conformal defects.

The rest of this thesis is comprised of three chapters, each of which is based on one of the following publications [4, 5, 6]. In chapter 2, we will review another class of supergravity models first proposed in [1, 2, 3] which are duals of 5 dimensional SCFTs. In these backgrounds, we will take a similar limit to the one in 1.5 which will allow us to quantize the superstring. In chapters 3 and 4, we will construct a new family of supergravity solutions in 7 dimensions which are the holographic duals to co-dimension 2 defects in 6 dimensional SCFTs. These solutions are related by analytic continuation to supersymmetric black hole solutions. We will compute holographically renormalized gravitational observables in both the original 7 dimensional supergravity as well as in the uplift to 11 dimensions where they can be understood using M theory.

1.1 Pertubative String Theory

We will start this overview by reviewing the basic string theory components that will show up in our effective descriptions in the next subsection.

A consistent quantization of gravity through string theory is obtained by replacing point particles by extended objects whose worldvolume can be parameterized by the variables τ, σ and are embedded in d dimensional spacetime with the coordinates $X^\mu(\tau,\sigma)$. We start with open strings, for which $0 < \sigma < \pi$. In the conformal gauge, the action that must quantized (though we will not reproduce the details here) is given by

$$\frac{1}{4\pi\alpha'}\int_{\mathcal{M}} d^2\sigma \partial_a X^\mu \partial_a X_\mu \tag{1.1}$$

which can be extended to superstrings by adding worldsheet fermions.

$$S = \frac{1}{4\pi} \int_{\mathcal{M}} d^2\sigma \left\{ \alpha'^{-1} \partial X^\mu \bar{\partial} X_\mu + \psi^\mu \bar{\partial} \psi_\mu + \tilde{\psi}^\mu \partial \tilde{\psi}_\mu \right\} \tag{1.2}$$

Enforcing the vanishing of the surface term in the resulting equations of motion allows for two possible types of boundary conditions of the worldsheet fermions:

$$\mathrm{R}: \quad \psi^\mu (0,\tau) = \tilde{\psi}^\mu (0,\tau) \quad \psi^\mu (\pi,\tau) = \tilde{\psi}^\mu (\pi,\tau) \tag{1.3}$$

$$\mathrm{NS}: \quad \psi^\mu (0,\tau) = -\tilde{\psi}^\mu (0,\tau) \quad \psi^\mu (\pi,\tau) = \tilde{\psi}^\mu (\pi,\tau) \tag{1.4}$$

giving us the so-called Ramond (R) and Neveu-Schwarz (NS) sectors of the theory.

It is a well known fact that in order for the superstring worldsheet Weyl anomaly to cancel, we must be at the the critical dimension $d = 10$. Massless states are therefore classified by their $SO(8)$ representations. The lowest lying states in the NS sector are the eight transverse polarizations of the open string photon, A^μ which form the vector representation of $SO(8)$ labeled $\mathbf{8}_v$. The Gliozzi-Scherk-Olive (GSO) projection [11] onto states with even fermion number, $(-1)^F = 1$, removes the open string tachyon which also appears in the bosonic string spectrum.

In the Ramond sector, there is a degenerate vacuum forming a representation of the Dirac algebra which can be decomposed into the $\mathbf{8}_s$ and $\mathbf{8}_c$ representations of $SO(8)$. The GSO projection picks out one of these two, however they are physically equivalent, differing by a spacetime parity redefinition. Together these two open string sectors, $\mathbf{8_v} \oplus \mathbf{8_s}$, give us the spectrum of a massless vector multiplet with $d = 10$, $\mathcal{N} = 1$ supersymmetry.

The closed string spectrum can be obtained using two copies of the open string spectrum with left and right moving levels matched. The two choices of GSO projection therefore lead to two choices of massless sectors:

$$\text{Type IIA:} \quad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_c}) \tag{1.5}$$

$$\text{Type IIB:} \quad (\mathbf{8_v} \oplus \mathbf{8_s}) \otimes (\mathbf{8_v} \oplus \mathbf{8_s}) \tag{1.6}$$

The NS-NS sectors of either choice are the same, giving us a dilaton ϕ, graviton $G_{\mu\nu}$ and antisymmetric tensor $B_{\mu\nu}$:

$$(\mathbf{8_v} \oplus \mathbf{8_s}) = \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \tag{1.7}$$

however the R-R sectors decompose differently

$$(\mathbf{8_s} \oplus \mathbf{8_c}) = [1] \oplus [3] \tag{1.8}$$

$$(\mathbf{8_s} \oplus \mathbf{8_s}) = [0] \oplus [2] \oplus [4]_+ \tag{1.9}$$

where $[n]$ denotes the n-times antisymmetrized representation of $SO(8)$ where $[4]_+$ is self dual. One can also show that the physical state conditions for these representations reduce to the Bianchi identity and field equation for an antisymmetric tensor field. Duals of the $[n]$ representation can then be understood as the Hodge dual of the associated rank-$(n+1)$ field strengths. Together these NS-NS and R-R fields form the bosonic components of $d = 10$ type IIA or type IIB supergravity. We can write down the effective action of the latter (as it will be most relevant to us throughout the rest of this thesis), which is fixed completely by supersymmetry. The self-duality constraint is imposed as a supplementary field equation.

$$\begin{aligned} S_{IIB} =& \frac{1}{4\kappa_B}\Big[\int d^{10}x\sqrt{g}e^{-2\phi}\left(2R + 8(\partial\phi)^2 - H^2_{(3))}\right) \\ & - \int d^{10}x\sqrt{g}\left(F^2_{(1)} + \tilde{F}^2_{(3)} + \frac{\tilde{F}^2_{(5)}}{2}\right) \\ & - \int A_{(4)} \wedge H_{(3)} \wedge F_{(3)}\Big] + \text{ fermions} \end{aligned} \tag{1.10}$$

$$*\tilde{F}_5 = \tilde{F}_5 \tag{1.11}$$

Here $H = dB$, $F_i = dC_{i-1}$ for the potentials C_0, C_2, C_4 and $\tilde{F}_{\mu_1\ldots\mu_{p+1}} = F_{\mu_1\ldots\mu_p+1} - B_{\{\mu_1\mu_2}F_{\ldots\mu_{p+1}\}}$.

There are two parameters here that will control the perturbative expansion. The first is α' which controls the masses of the higher string modes and the strength of higher derivative corrections and the other is string coupling constant $g_s = \langle e^{\phi}\rangle$ which controls stringy interactions and quantum corrections. In principle, all of these corrections can be calculated in string perturbation theory; at each order in the loop expansion, the Feynman graph - which can be calculated using the worldsheet CFT - is represented by a Riemann surface for which the genus is the number of loops.

1.2 D-Branes

The type II theories discussed above (both string theory and the supergravities) have extended charged objects called p-branes which generalize the 1 dimensional strings to p space-like dimensions. They interact with gravity and gauge fields through couplings of the form

$$S_p = T_p \int dx^{p+1} \sqrt{g} + \mu_p \int dx^{p+1} A_{p+1} \tag{1.12}$$

where, like for the string, we integrate over the worldvolume. The first term defines the tension τ of the brane and the second term is the coupling to the A_{p+1} antisymmetric field where q is the brane charge. These charges obey a Dirac quantization condition which can be derived by integrating the associated field strength over an S^{8-p} in the space transverse to the brane:

$$(2\pi)^{p-6}(\alpha')^{\frac{p-7}{2}} \int_{S^{8-p}} *F_{p+2} = 2\pi N \tag{1.13}$$

1.2.1 Geometry

There exist classical solutions of type II supergravity (Equation 1.10 for type IIB) with sources that correspond to these massive charged p-branes. They are called black p-branes as they generalize charged black hole solutions. As in the charged black hole situation, evasion of naked singularities leads to a bound on the mass/tension and charge which, when saturated, lead to so-called "extremal" p-branes. Such solutions are special since extremality is equivalent to preserving half of the original 32 supersymmetries of the theory.

The geometry of an extremal p-brane with flux N parallel to the first $p+1$ directions is given by:

$$ds^2 = H^{-1/2}(r) dx_\mu^2 + H^{1/2}(r) dy^2 \tag{1.14}$$

$$A_{0\cdots p} = H(r) \tag{1.15}$$

$$e^\phi = g_s H(r)^{(3-p)/4} \tag{1.16}$$

$$H(r) = 1 + \frac{c g_s N (\alpha')^{(7-p)/2}}{r^{7-p}} \tag{1.17}$$

where c is some numberical constant, y are the transverse directions and r is the radial distance from the brane. This solution can be generalized by replacing the harmonic function H with the more general

$$H(y_i) = 1 + cg_s(\alpha')^{(7-p)/2} \sum_{a=1}^{N} \frac{1}{|y - y_a|^{7-p}} \tag{1.18}$$

This solution is still extremal and has flux N however it corresponds to N branes of unit charge at various locations $\vec{y_a}$. For obvious reasons, it is called the multi-center solution. Extremality of this more general solution tells us that these p-branes can be moved around at no energy cost. This is due to the exact cancellation of repulsive and attractive forces resulting from the gauge and gravity interactions.

1.2.2 World Volume Theory

In perturbative string theory, the extremal p-branes of the previous section can be described in terms of planes where open strings can end. It is for this reason that they are referred to as D(irichlet)-branes after the boundary conditions with which the strings end. These D-branes should, like strings, interact with the background fields of the theory. We will briefly comment on the worldvolume theory that describes their dynamics.

Much like in the case of the Nambu-Goto action for the string, we can start by choosing a set of embedding functions $X^A(x^a)$ where $A = 1, ..., 10$ and $a = 1, ..., p+1$ which parameterize the fluctuations of the brane in transverse space. We can set $X^a = x^a$ for the first $p + 1$ directions along the worldvolume and let $X^i = \phi^i$ for the remaining $9-p$ transverse directions. This is called the "static gauge". The Nambu-Goto action becomes

$$\int dx^{p+1}\sqrt{g} = \int dx^{p+1}\sqrt{det_{ab}(g_{ab} + g_{ij}\partial_a\phi^i\partial_b\phi^j)} \approx \int dx^{p+1}\sqrt{g} + \frac{1}{2}\sqrt{g}g_{ij}\partial_a\phi^i\partial_b\phi^j + .. \tag{1.19}$$

for ϕ^i which vary slowly. We obtain the kinetic term for $9 - p$ transverse scalars living on the worldvolume.

There are of course additional fields living on the brane than just these scalars. We also have to take into account new couplings for the embedding coordinates (and the gauge fields $A_a(x)$ from strings ending on the brane) which result from other background fields such as $B_{\mu\nu}$ as well as the various R-R fields. Fortunately since D-branes are half-BPS, the worldvolume theory should preserve 16 supersymmetries and so we know in advance that the additional worldvolume fields must make up the rest of the vector supermultiplet; the only supermultiplet with the right number of supercharges.

By quantizing open strings ending on these branes, one can obtain massless excitations corresponding to exactly such a multiplet. The effective action for these fields can be determined by perturbation theory of open and closed strings:

$$\frac{1}{(\alpha')^{(p+1)/2}}\int dx^{p+1}\left(e^{-\phi}\sqrt{det(g+(2\pi\alpha' F+B_2))}+e^{1\pi\alpha' F+B_2}\wedge\sum_k C_k|_{p+1}\right) \tag{1.20}$$

where the first term, called the Dirac-Born-Infeld (DBI) action, includes the gauge fields and their interactions with the bulk field $B_{\mu\nu}$ and the second term, called the Wess-Zumino term, generalizes the coupling to A_p in equation 1.12.

A crucial element of the dualities we will encounter in the following sections is the non-abelian generalization of the above story. This arises from considering a set of N Dp-branes such as those described by the supergravity solutions 1.14 and 1.18. We have already seen that open strings ending on a single brane give rise to massless vector multiplets. However, open strings which connect separated branes give rise to *massive* vector multiplets with $m \sim |\vec{\phi}_1 - \vec{\phi}_2|/\alpha'$ since the string with tension $1/\alpha'$ has non-zero length. When the branes coincide, these masses vanish and we obtain the N^2 massless vector fields of an enhanced $U(N)$ symmetry. This can be thought of in the world volume field theory as a sort of inverse Higgs mechanism where there is a direct correspondence between the set of vacua of the field theory and the minimal energy configurations of the D branes in spacetime.

1.3 The D3-Brane System

We are now ready to combine all of the ingredients of the preceding section for the special case of N coincident D3 branes in IIB string theory [12]. As we saw, the open strings ending on such branes are described by the DBI part of the worldvolume theory given by 1.20 with gauge group $U(N)$.

We will now want to consider the low energy limit of this action such that only massless excitations are taken into account. As we saw previously, this amounts to an expansion in the parameter α'. The resulting action is

$$S_{DBI} = -\frac{1}{2\pi g_s}\mathrm{Tr}\int d^4x \left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{6}\partial^\mu\phi^i\partial_\mu\phi^i - \pi g_s \sum_{i,j=1}^{6}[\phi^i,\phi^j]^2\right) + \text{fermions} + \mathcal{O}(\alpha') \tag{1.21}$$

In the low energy limit $\alpha' \to 0$, open strings decouple from from any closed string modes in the full $d = 10$ spacetime. So what we are left with is the action 1.21 plus some free massless gravity excitations about flat space coming from the decoupled closed strings. Notice that this leaves us with the $\mathcal{N} = 4$ Super Yang-Mills theory for the massless vector supermultiplet as long as we identify $4\pi g_s = g_{YM}^2$. That is,

$$\lim_{\alpha'\to 0} S_{DBI} = S_{\mathcal{N}=4\ \mathrm{SYM}}|_{g_{YM}^2=4\pi g_s} \tag{1.22}$$

We must, however, be careful to perform this limit in a way that keeps field theory quantities finite. The Higgs phase masses of the gauge bosons are given by $\Delta r/\alpha'$ so we take $\alpha' \to 0$ while leaving the masses finite.

On the other hand, we may also view these N coincident D3 branes from the supergravity point of view, 1.14, when $g_sN \gg 1$:

$$ds^2 = H(r)^{-1/2}dx_\mu^2 + H(r)^{1/2}dy^2 = H(r)^{-1/2}dx_\mu^2 + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{1.23}$$

$$H(r) = 1 + \frac{4\pi g_s N\alpha'^2}{r^4} = 1 + \frac{g_{YM}^2 N\alpha'^2}{r^4} \tag{1.24}$$

We can define the new variable $U \equiv r/\alpha'$ and take the $\alpha' \to 0$ limit keeping U fixed. The resulting metric is

$$ds^2 = \alpha'\left(\frac{U^2}{\sqrt{4\pi g_s N}}dx_\mu^2 + \sqrt{4\pi g_s N}\frac{dU^2}{U^2} + \sqrt{4\pi g_s N}d\Omega_5^2\right) \tag{1.25}$$

which describes $AdS_5 \times S^5$ where the radius of the sphere and AdS both equal to $R^2 = L_{AdS}^2 = \alpha'\sqrt{4\pi g_s N}$.

By considering the massless modes in each of these two pictures we can formulate the correspondence - In both, there is free gravity in $d = 9+1$ but in the first, we have $d = 3+1$ $\mathcal{N} = 4$ SU(N) SYM and in the second, we have type IIB supergravity on $AdS_5 \times S^5$. The "strong" form of the correspondence proposes that this duality goes beyond the supergravity approximation presented here.

We also note that the symmetries of these two side agree; the supersymmetry group of $AdS_5 \times S^5$ is the same as the superconformal group in $3+1$ dimensions and the $SL(2, \mathbb{Z})$ symmetry of SYM matches that of type IIB string theory. Additionally, the correspondence can be seen as an example of a strong-weak duality in the sense that as the effective coupling $g_s N$ gets large, we can trust supergravity calculations but not perturbative SYM calculations.

1.4 AdS Solutions from M Theory

An important result [13, 14, 15] which helped kick off the Second Superstring Revolution was the realization that all of the superstring theories are dual to each other at strong coupling. This subsequently led people to consider the relationships between these string theories and supergravity in one dimension higher ($d = 11$) in search of the overarching theory within which they each appear. To see why, we can consider, for example, coincident D0 branes in IIA string theory. Here one encounters infinitely many particle states of evenly spaced mass $m = n/(\alpha'^{1/2} g_s)$ which depend on the number of D0 branes, n, making up the bound state. As we go to strong coupling, these states all become light. Such a spectrum is reminiscent of dimensionally reduced theories (say, on an S^1 of radius R) which acquire an infinite tower of Kaluza-Klein states with $m = n/R$ that become a continuum as $R \to \infty$. This argument motivates us to consider $d = 11$ supergravity as the strong coupling limit of the type IIA string. While we will not comment on the full UV theory, called "M Theory", presently we will just need the low-energy effective theory.

In 11 dimensional supergravity, we have a graviton, antisymmetric 3-form tensor field

$A_{(3)}$ and their superpartners whose physics is governed by the following action:

$$S_{11d} = \frac{1}{16\pi G_{11}} \int d^{11}x\sqrt{-G}\left(R - \frac{1}{48}F_{(4)}^2\right) - \frac{1}{96\pi G_{11}} \int A_{(3)} \wedge F_{(4)} \wedge F_{(4)}$$
$$+ \text{ fermions} \tag{1.26}$$

with the following supersymmetry transformations

$$\begin{aligned}
\delta e_\mu^a &= \frac{1}{2}\bar{\epsilon}\Gamma^a\Psi_\mu \\
\delta A_{\mu\nu\rho} &= -3\bar{\epsilon}\Gamma_{[\mu\nu}\Psi_{\rho]} \\
\delta\Psi_\mu &= (\partial_\mu - \frac{i}{2}\omega_{\mu ab}\Sigma^{ab})\epsilon + \frac{1}{288}(\Gamma_\mu^{\nu\rho\alpha\beta} - 8\delta_\mu^{[\nu}\Gamma^{\rho\alpha\beta]})\epsilon F_{\nu\rho\alpha\beta}
\end{aligned} \tag{1.27}$$

Just as in the other supergravities we have encountered so far, we can make extended objects which couple to the antisymmetric tensor fields of the theory. In the present case we have just the self dual $A_{(3)}$ so that gives us an electrically charged "M2-brane" whose solution is

$$\begin{aligned}
ds^2 &= f_3^{-2/3}dx_{M_3}^2 + f_3^{1/3}(dr^2 + r^2 d\Omega_7^2) \\
f_3 &= 1 + \frac{\pi N \ell_p^3}{r^3} \\
A_{(3)} &= f_3^{-1} dt \wedge dx^1 \wedge dx^2
\end{aligned} \tag{1.28}$$

and a magnetically charged "M5-brane"

$$\begin{aligned}
ds^2 &= f_5^{-2/3}dx_{M_6}^2 + f_5^{1/3}(dr^2 + r^2 d\Omega_4^2) \\
f_3 &= 1 + \frac{32\pi N \ell_p^6}{r^6} \\
A_{(6)} &= f_5^{-1} dt \wedge dx^1 \wedge ... \wedge dx^5
\end{aligned} \tag{1.29}$$

where $dx_{M_d}^2 = (-dt^2 + ... + dx_{d-1}^2)$ is the d-dimensional Minkowski metric.

Basically everything that was done in section 1.3 for the D3-brane can be reproduced for these branes in M-theory allowing us to conjecture a few more holographic dualities.

For the M5 brane, the decoupling limit is now obtained by sending the Planck length ℓ_p to zero while keeping $U \equiv r/\ell_p^3$ fixed. Applying this limit to the metric 1.29 we obtain

$$ds^2 = \ell_p^2\left(\frac{U^2}{(\pi N)^{1/3}}dx_{M_6}^2 + 4(\pi N)^{2/3}\frac{dU^2}{U^2} + (\pi N)^{2/3} d\Omega_4^2\right) \tag{1.30}$$

where now the radii of the sphere and AdS_7 are $R = L_{AdS}/2 = \ell_p(\pi N)^{1/3}$. The low energy worldvolume theory realized on N coincident M5 branes is an $\mathcal{N} = (2,0)$ theory the superconformal algebra of which has bosonic subgroup $SO(2,6) \times SO(5)$; precisely the isometries of the above $AdS_7 \times S^4$ near-horizon geometry.

For the M2 brane[1], we take the slightly different decoupling limit $\ell_p \to 0$ keeping $U \equiv r/\ell_p^{3/2}$ fixed. In this limit, we obtain $AdS_4 \times S^7$ with $R = 2L_{AdS} = \ell_p(32\pi N)^{1/6}$. As in the previous cases, we can identify the $SO(3,2) \times SO(8)$ isometry of the near horizon geometry with the $SO(3,2)$ conformal group and $SO(8)$ R-symmetry of the $\mathcal{N} = 8$, $d = 2+1$ worldvolume theory on N M2 branes.

1.5 A Penrose Limit for type IIB $AdS_5 \times S^5$

Having conjectured above that string theory on $AdS_5 \times S^5$ is dual to a particular CFT in $d = 3+1$, one may be tempted to find a dictionary between states by directly quantizing the superstring in this background. Unfortunately, quantization in the presence of general R-R flux is a very difficult problem; in the Ramond-Neveu-Schwarz (RNS) formalism [16, 17], the resulting fields become non-local and in the Green-Schwarz (GS) formalism [18], explicit quantization is only straightforward in the lightcone gauge and in general it is unknown how to formulate the appropriate sigma model that will allow one to do this. There is, however, a way to make some progress in this direction through a particular limit of this space which arises from considering particles with large angular momentum on the S^5.

In this section, we will summarize the developments - starting with the original paper of Berenstein, Maldacena and Natase [19] - of a particular concrete realization of an AdS/CFT-like duality relating type IIB superstrings in a maximally symmetric plane-wave background to double scaling limit of 4d $\mathcal{N} = 4$ SYM. One of the most important features of this plane wave background is that in the lightcone gauge, the string sigma model reduces to a free

[1] The resulting duality could also be obtained by considering D2 branes in type IIA (since they are just M2 branes transverse to X^{10}) and subsequently taking the strong coupling limit to get the resulting CFT. In that case we let $\alpha' \to 0$ keeping $g_{YM}^2 \sim g_s/\alpha'$ fixed.

massive theory and therefore, as in the flat background, we can easily find its spectrum. This is in sharp contrast to the seemingly intractable $AdS_5 \times S^5$ string which has yet to be quantized. Furthermore, unlike in the strong/weak couple duality of AdS/CFT, the plane wave/gauge theory duality is perturbatively accessible from both sides and so we can compare their respective spectra in a perturbative expansion from either point of view.

We will obtain these plane wave backgrounds by taking a so-called Penrose limit of the original $AdS_5 \times S^5$ background where we zoom in on a particular light-like geodesic which wraps the S^5. This limit corresponds, in the dual gauge theory language, to a "double-scaling" where we take not only the rank N of the gauge group to infinity but also only consider correlation functions of operators with diverging R-charge $J \sim \sqrt{N}$.

1.5.1 The Plane Wave Geometry as a Penrose Limit of $AdS_5 \times S^5$

In global coordinates the $AdS_5 \times S^5$ metric is given by

$$ds^2_{AdS_5 \times S^5} = R^2 \left[-dt^2 \cosh^2 \rho + d\rho^2 + \sinh^2 \rho d\Omega_3^2 + d\psi^2 \cos^2 \theta + d\theta^2 + \sin^2 \theta d\Omega_3'^2 \right] \quad (1.31)$$

To get the Penrose limit of this background, we will consider zooming in on a lightlike trajectory parameterized by λ along $\rho = 0, \theta = 0, t = t(\lambda), \psi = \psi(\lambda)$. A massless, relativistic particle moving along this trajectory is governed by the following action:

$$S = \frac{1}{2} \int d\lambda \left(e^{-1} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu - em^2 \right) = \frac{R^2}{2} \int d\lambda e^{-1} \left(-\dot{t}^2 + \dot{\psi}^2 \right) \quad (1.32)$$

If we introduce lightcone coordinates $\tilde{x}^\pm = \frac{1}{2}(t \pm \psi)$, we find that the equations of motion are solved by the lightlike trajectory $\tilde{x}^- = \lambda, \tilde{x}^+ = const$. To study the geometry near this trajectory, we can rescale these coordinates by R in a convenient way and take $R \to \infty$.

$$x^+ = \frac{\tilde{x}^+}{\mu}, \quad x^- = \mu R^2 \tilde{x}^-, \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R} \quad (1.33)$$

where μ is a new mass parameter that we introduce to get the right length dimensions. Rewritten in these new coordinates, the metric (1.31) becomes (to lowest order in R)

$$ds^2 = -4dx^+ dx^- - \left(\vec{r}^2 + \vec{y}^2 \right) \left(dx^+ \right)^2 + d\vec{y}^2 + d\vec{r}^2 \quad (1.34)$$

$$= -4dx^+ dx^- - \mu^2 \vec{z}^2 \left(dx^+ \right)^2 + d\vec{z}^2 \quad (1.35)$$

and likewise, the non-vanishing self-dual five-form becomes

$$F_{+1234} = F_{+5678} = 4\mu \tag{1.36}$$

We notice that this breaks the transverse $SO(8)$ invariance of the original metric to a $SO(4)\times SO(4)$ subgroup.

We may now wonder how this $R \to \infty$ limit manifests in the dual gauge theory. To understand this, we will look at how the energy $E = i\partial_t$ and angular momentum $J = -i\partial_\psi$ relate to the new light-cone quantities and their conjugate momenta:

$$\begin{aligned} \mathcal{H}_{\text{lc}} := 2p^- = i\partial_{x^+} = \mu i\left(\partial_t + \partial_\psi\right) = \mu(E-J) \\ 2p^+ = i\partial_{x^-} = \frac{1}{\mu R^2} i\left(\partial_t - \partial_\psi\right) = \frac{E+J}{\mu R^2} \end{aligned} \tag{1.37}$$

In the $R \to \infty$ limit, we can see that generic excitations will have vanishing p^+ unless the angular momentum grows with R as $J \sim R^2$. In order to maintain a finite light-cone momentum for such states, we must also require that $J \sim E$. The nature of these restrictions can be understood in to the $\mathcal{N} = 4$ SYM language as follows: The energy E in global coordinates is identified with the scaling dimension Δ of a composite SYM operator. On the other hand, the angular momentum J is identified to the charge of a $U(1)$ subgroup of the $SO(6)$ R-symmetry. Therefore (1.37) can be rephrased as

$$\frac{\mathcal{H}_{lc}}{\mu} = \Delta - J \tag{1.38}$$

The BPS condition $\Delta \geq |J|$ will guarantee that $p^\pm$ are non-negative. Given the well-known AdS/CFT relation $R^4 = \alpha'^2 g_{YM}^2 N$, the limit $R \to \infty$ with $J \sim R^2$ translates to the gauge theory limit:

$$N \to \infty, \quad J \sim \sqrt{N}, \quad g_{YM} \text{ held fixed} \tag{1.39}$$

Note that g_{YM} being held fixed in this limit corresponds to a finite value of the string coupling $g_{string} \sim g_{YM}^2$ on the string theory side. Moreover, the finite light-cone energy tells us that in the gauge theory, only SYM operators with $\Delta \approx J$ will survive.

1.5.2 Light Cone Quantization of the Type IIB String in the Plane Wave

As we remarked above, one of the most important features of these plane wave NS (or RR) backgrounds is that the string action becomes dramatically simplified in the light-cone gauge. Letting $x^+ = \tau$ (the worldsheet time), the action for the 8 transverse directions becomes the action for eight massive bosons and the coupling to the RR background gives a mass to the 8 transverse fermions equal to the mass of the bosons. This is because the 16 supersymmetries realized in the light-cone gauge commute with $\mathcal{H}_{lc}$. Upon gauge-fixing, the action is

$$S = \frac{1}{2\pi\alpha'} \int dt \int_0^{2\pi\alpha' p^+} d\sigma \left[\frac{1}{2}\dot{z}^2 - \frac{1}{2}z'^2 - \frac{1}{2}\mu^2 z^2 + i\bar{S}(\not{\partial} + \mu\Gamma^{1234})S \right] \tag{1.40}$$

where S is a Majorana spinor on the worldsheet and a positive chirality $SO(8)$ spinor under transverse rotations.

We quantize by expanding fields in Fourier modes on the circle labelled by σ just as in the flat background and obtain bosonic and fermionic harmonic oscillators for each value of n. The light cone Hamiltonian is

$$2p^- = -p^+ = \mathcal{H}_{lc} = \sum_{n=-\infty}^{+\infty} N_n \sqrt{\mu^2 + \frac{n^2}{(\alpha' p^+)^2}} \tag{1.41}$$

As usual, $n > 0$ label left-movers, $n < 0$ label right-movers and N_n denotes the total occupation number of that mode. We also have the condition that the total momentum on the string vanishes

$$P = \sum_{n=-\infty}^{+\infty} n N_n = 0 \tag{1.42}$$

It is illuminating to rewrite the spectrum (1.41) in terms of the original $AdS_5 \times S^5$ variables. The contribution to $2p^- = \Delta - J$ for each oscillator is its frequency:

$$(\Delta - J)_n = \omega_n = \sqrt{1 + \frac{4\pi g N n^2}{J^2}} \tag{1.43}$$

Using the identifications in the previous section, we notice that N/J^2 remains fixed in the limit (1.39).

1.5.3 The BMN sector of $\mathcal{N} = 4$ SYM

The dual theory of the $AdS_5 \times S^5$ superstring is the maximally supersymmetric ($\mathcal{N} = 4$) SYM theory in 4 dimensions. The field content is comprised of a gluon field, six scalars and four Majorana gluoinos which can be written as a 16 component 10d Majorana-Weyl spinor. All fields are in the adjoint of $U(N)$. The action is uniquely determined be two parameters, the coupling g_{YM} and the rank N of the gauge group:

$$S = \frac{2}{g_{\mathrm{YM}}^2} \int d^4x \,\mathrm{Tr} \left\{ \frac{1}{4} (F_{\mu\nu})^2 + \frac{1}{2} (D_\mu \phi_i)^2 - \frac{1}{4} [\phi_i, \phi_j] [\phi_i, \phi_j] + \frac{1}{2} \bar{\chi} \, \not{D} \chi - \frac{i}{2} \bar{\chi} \Gamma_i [\phi_i, \chi] \right\} \tag{1.44}$$

This theory has a global $SO(6)$ R-symmetry acting as internal rotations on the six scalers and four spinors. Also, due to the large amount of supersymmetry, the conformal invariance of the classical field theory survives quantization (the β function for g_{YM} is believed to vanish to all orders in perturbation theory). Taking into account the 4d conformal group, the full bosonic symmetry group of the theory is $SO(2,4) \times SO(6)_R$ (precisely the isometry group of the $AdS_5 \times S^5$ geometry).

The observables of interest to us in this theory are local, composite, gauge invariant operators (i.e. traces of products of fundamental fields at a particular spacetime point). In particular, an important class of such operators for a conformal theory are the conformal primary operators with definite scaling dimension whose two-point functions are fixed by symmetry to be

$$\langle \mathcal{O}_A(x) \mathcal{O}_B(y) \rangle = \frac{\delta_{AB}}{(x-y)^{2\Delta_{\mathcal{O}_A}}} \tag{1.45}$$

In the quantum theory, the scaling dimensions will generally receive radiative corrections organized in an expansion in $\lambda = g_{YM}^2 N$

$$\Delta = \Delta_0 + \sum_{l=1}^{\infty} \lambda^l \sum_{g=0}^{\infty} \frac{1}{N^{2g}} \Delta_{l,g} \tag{1.46}$$

However, one feature of the $\mathcal{N} = 4$ theory is the existence of a class of operators (called "superconformal primaries") whose scaling dimensions do not receive any corrections.

	Z	$\bar{Z}$	$\phi_{i=1,2,3,4}$	A_μ	ψ_A	$\tilde{\psi}_{\dot{A}}$
Δ_0	1	1	1	1	3/2	3/2
J	1	-1	0	0	1/2	$-1/2$
$\Delta_0 - J$	0	2	1	1	1	2

In order to identify the BMN limit of this model, we need to identify the $U(1)$ charge J corresponding to the angular momentum along the equator of the S^5 on the string side. It is given by the charge associated to the combination of two scalars, say ϕ^5 and ϕ^6

$$Z = \frac{1}{\sqrt{2}}(\phi^5 + i\phi^6) \tag{1.47}$$

The classical scaling dimensions and J charges of the fundamental fields are summarized in the table above.

We are interested in the limit (1.39) which we stress is distict from the standart 't Hooft large N limit where one keeps λ fixed. This new limit, letting $\lambda \to \infty$ seems disastrous from the perturbation theory point of view however, we should remember that we are also restricting our attention to the set of operators whose scaling dimensions are of the order of J. These operators are made out of a long string of Z's. For the protected conformal primary operators, the strong coupling nature of our BMN limit is not visible. Examples of these operators include

$$Tr[Z^J] \quad \text{and} \quad Tr[\phi^i Z^J] \tag{1.48}$$

Another insight of [19] was to violate this protectedness by inserting a small number of operators with $\Delta - J = 1$

$$Tr[\phi^i Z...Z\phi^j Z...ZD_\mu Z...Z\psi_\alpha Z...Z] \tag{1.49}$$

The authors showed that the two and three-point functions of these types of operators receive quantum corrections through an effective loop counting parameter $\lambda' = g_{YM}^2 N/J^2$ which remains finite in the limit (1.39). Even though the scaling dimensions of generic $\mathcal{N} = 4$ operators diverge, there remains a perturbatively accessible "BMN sector". Note, however, that this effective weak coupling breaks down for higher point functions.

1.5.4 The Plane Wave String State/Gauge Theory Operator Dictionary

Guided by the value of $\Delta - J = E_{lc}/\mu$, let's identifying the gauge theory operators of the previous section with plane wave string states. For the ground state $|0, p^+\rangle$ with vanishing energy there is a unique single trace operator with $\Delta - J = 0$, namely

$$\left|0, p^+\right\rangle = \frac{1}{\sqrt{JN^J}} Tr[Z^J] \tag{1.50}$$

Since this operator is protected, the scaling dimension will equal J to all orders in the full quantum theory.

Moving on to $\Delta - J = 1$, we have the following set of operators

$$E_{lc} = \mu \quad \alpha_0^{\dagger i} \left|0, p^+\right\rangle = \frac{1}{\sqrt{N^J}} Tr[\phi^i Z^J] \tag{1.51}$$

$$E_{lc} = \mu \quad \alpha_0^{\dagger \mu} \left|0, p^+\right\rangle = \frac{1}{\sqrt{N^J}} Tr[D_\mu Z Z^{J-1}] \tag{1.52}$$

$$E_{lc} = \mu \quad \theta_{0A}^{\dagger} \left|0, p^+\right\rangle = \frac{1}{\sqrt{N^J}} Tr[\psi_A Z^J] \tag{1.53}$$

corresponding to the eight bosonic and eight fermionic excitations of the string. Note that while the first of these is a conformal primary, the second is a descendent of the groundstate operator (1.50). Similarly, the third state is a superconformal descendent of the groundstate. All three operators are again protected and have $\Delta - J = 1$ to all orders. Higher zero mode excitations are obtained by symmetrized insertions of ϕ^i, $D_\mu Z$ and ψ_A. For example,

$$E_{lc} = 2\mu \quad \alpha_0^{\dagger i} \alpha_0^{\dagger j} \left|0, p^+\right\rangle = \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}] \tag{1.54}$$

The challenge now lies in reproducing the string spectrum through the non-protected SYM operators introduced above. At the first level we are searching for a gauge theory object $\mathcal{O}_n^{ij}$ with

$$E_{lc} = 2\mu \sqrt{1 + \frac{n^2}{(\alpha' p^+ \mu)^2}} \quad \alpha_{-n}^i \tilde{\alpha}_{-n}^i \left|0, p^+\right\rangle = \mathcal{O}_n^{ij} \tag{1.55}$$

The operator should carry two ϕ's to reproduce the $SO(4)$ index structure and should reduce to (1.54) when $n \to 0$. It turns out that the correct answer is

$$\mathcal{O}_n^{ij} = \frac{1}{\sqrt{JN^J}} \sum_{l=0}^{J} Tr[\phi^i Z^l \phi^j Z^{J-l}] e^{2\pi i n l/J} \tag{1.56}$$

The scaling dimension of this operator to one loop is

$$\Delta_{\mathcal{O}_n^{ij}} = J + 2 + \frac{g_{YM}^2 N}{J^2} n^2 + ... \tag{1.57}$$

Note the appearance of the effective coupling λ'. Furthermore, after converting gauge theory parameters into string parameters, we can write

$$\lambda' = \frac{g_{YM}^2 N}{J^2} = \frac{1}{(\alpha' p^+ \mu)^2} \tag{1.58}$$

so that perturbation theory in λ' corresponds to a $\mu \to \infty$ expansion on the string side. In this regime, we precisely match the SYM result.

$$\frac{E_{lc}}{\mu} = 2\sqrt{1 + n^2 \lambda'} = 2 + \lambda' n^2 + ... \tag{1.59}$$

1.6 Holographic duals for conformal defects

In this section we will describe a slightly different type of holographic duality; one for which the supergravity describes the degrees of freedom living on a lower dimensional conformal defect. In such cases, these defects break the original conformal symmetry, say $SO(d-1,2)$ for a d-dimensional field theory, down to an $SO(d-n-1,2) \times SO(n)$ where n is the codimension of the defect. The subgroup factors represent the conformal symmetry along the defect and transverse rotations about the defect, respectively, and therefore can be realized in the holographic dual as the isometries of AdS_{d-n} and S^{n-1}.

Because of this, there are also local operators living on the defect itself which fuse according to their own OPE. Bulk operators, $\mathcal{O}(x)$, can excite defect operators when brought towards the defect and, in particular, can acquire a non-vanishing one-point function due to OPE with the defect identity operator. For a planar defect, this takes the simple form

$$\langle \mathcal{O}(x) \rangle = C_{\mathcal{O}} |x_\perp|^{-\Delta} \tag{1.60}$$

where $x_\perp$ are the directions perpendicular to the defect.

One approach to constructing such holographic duals in ten or eleven dimensional supergravity is to start with an ansatz metric which is a warped product of these spaces whose isometries realize the unbroken defect symmetries and solve the equations of motion and Killing spinor equations to find supersymmetric solutions. To illustrate this, we present the simpler non-supersymmetric case of [20] which deforms the $AdS_5 \times S^5$ solution of type IIB by allowing the dilaton to take a non-trivial profile over a warping coordinate.

To see how this works, we start by noticing that in a particular choice of coordinates, we can slice AdS_d by AdS_{d-1} spaces. That is, if we define AdS_d as the embedding of the hyperboloid

$$X_0^2 + X_d^2 - X_1^2 - ... - X_{d-1}^2 = 1 \tag{1.61}$$

in $\mathbb{R}^{2,d-1}$, we can then parameterize $X_{d-1} = z$ and the remaining X_i by any coordinate system of AdS_{d-1} with radius $\sqrt{1+z^2}$. The result is the following metric

$$ds^2_{AdS_d} = \frac{dz^2}{1+z^2} + (1+z^2) ds^2_{AdS_{d-1}} \tag{1.62}$$

We can then redefine $z = \tan \mu$, and rewrite (1.62) as

$$ds^2_{AdS_d} = f(\mu)(d\mu^2 + ds^2_{AdS_{d-1}}) \tag{1.63}$$

where $f(\mu) = 1/\cos^2 \mu$. Since z takes values over all of $\mathbb{R}$, $\mu \in [-\pi/2, \pi/2]$.

To understand the boundary in these coordinates, we start by writing the AdS_{d-1} in global coordinates

$$ds^2_{AdS_d} = \frac{1}{cos^2\mu \cos^2 \lambda}(-d\tau^2 + \cos^2 \lambda d\mu^2 + d\lambda^2 + \sin^2 \lambda d\Omega^2_{d-3}) \tag{1.64}$$

where $\lambda \in [0, \pi/2]$. At fixed time, the metric is conformal to half of S^{d-1} since μ only ranges from $[-\pi/2, \pi/2]$. The boundary of this space consists of two parts, $\mu = \pm\pi/2$, which are joined along the codimension one surface $\lambda = \pi/2$. These combine to make up an S^{d-2}.

We now consider a deformation of $AdS_5 \times S^5$ with non-trivial dilaton given by the following ansatz inspired by (1.63)

$$\begin{aligned} ds^2 &= f(\mu)(d\mu^2 + ds^2_{AdS_4}) + ds^2_{S^5} \\ \phi &= \phi(\mu) \\ F_5 &= 2 f(\mu)^{5/2} d\mu \wedge \omega_{AdS_4} + \omega_{S^5} \end{aligned} \tag{1.65}$$

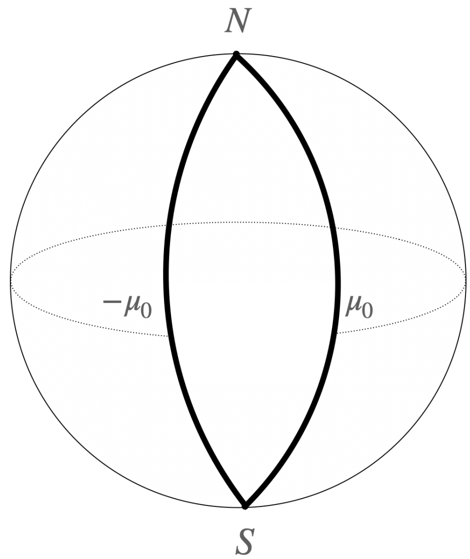

Figure 1.1: Constant time slice of the metric (1.65) for particular choice of c_0. Each point corresponds to an S^2 and the boundary (bold) is an S^3.

which is asymptotically AdS_5 and where ω_{AdS_4/S^5} are the unit volume forms on either space.

One can show that the dilaton equation of motion and Einstein equations can be combined to give the following simple, first order differential equation in terms of $f(\mu)$

$$f'f' = 4f^3 - 4f^2 + \frac{c_0}{6f} \tag{1.66}$$

for some constant c_0 ($c_0 = 0$ corresponding to $\phi(\mu) = const.$, $f(\mu) = 1/\cos^2\mu$). This equation can be solved numerically for different values of c_0. One finds that at each value, up to some critical value c_{critical}, μ is bounded by $[-\mu_0, \mu_0]$ where $\mu_0 > \pi/2$ and increases with c_0. When $c_0 = c_{\text{critical}}$, μ takes on values along the entire real line.

The boundary can be understood in global coordinates as we described above. As before, it consists of two parts; one at $\mu = -\mu_0$ and another at $\mu = \mu_0$ which are joined through the north and south poles (see Figure 1.1). The dilaton takes on two different values at either half of the boundary; $\phi_0^+ = \phi(\mu_0)$ and $\phi_0^- = \phi(-\mu_0)$. Since only the dilaton and metric vary non-trivially along the μ-direction, the field theory interpretation is simple; in the AdS/CFT dictionary, the asymptotic value of ϕ is associated with the Yang-Mills coupling of the 4d $N = 4$ theory on the boundary so therefore we interpret the solution (1.65) as describing a co-dimension one defect separating two half-spaces of different coupling. Solutions of this type are called "Janus" solutions, referring to the the Roman god with two faces. Since this non-supersymmetric example was first worked out, there has been subsequent work constructing supersymmetric Janus solutions [21, 22, 23, 24].

Briefly, we also note that it will also be useful for us to consider another slicing of AdS_d, this time by AdS spaces two dimensions lower. Similar to before, we can parameterize X_i such that

$$ds^2_{AdS_d} = du^2 + \cosh^2 u ds^2_{AdS_{d-2}} + \sinh^2 u d\phi^2 \tag{1.67}$$

where ϕ parameterizes an S^1. Inspired by the previous example, we may want to consider supergravity solutions which are are asymptotically of the form 1.67 but have non-trivial warping along u (or along other additional compact directions). Such solutions would now correspond to codimension two conformal defects. In chapters 3 and 4, we will see just such an example. There the solutions will describe four dimensional defects in six dimensional field theories.

CHAPTER 2

A Penrose limit for type IIB AdS_6 solutions

2.1 Introduction

The quantization of superstrings in the presence of Ramond-Ramond fields is a technically challenging problem and not solved in general. This constitutes a challenge to go beyond the semi-classical supergravity approximation in many examples of AdS/CFT. Instead of solving the general problem one approach is to consider limits or deformations of the supergravity solution which may lead to a simpler quantization problem. One such limit is given by taking the Penrose limit of an $AdS_p \times S^q$ background. The Penrose limit [?] (see also [25, 26, 27]) corresponds to zooming in on the close vicinity of a null geodesic in the spacetime and produces a plane wave geometry. In the case of the Penrose limit of the type IIB $AdS_5 \times S^5$ solution, where the null geodesic sits at the center of AdS_5 and on a great circle of S^5, one obtains a maximally supersymmetric plane wave [28, 29, 30]. It was proposed in [19] that on the field theory side this limit corresponds to singling out a special subset of CFT operators where both the conformal dimension Δ and a $U(1)$ R-charge J are taken to be of order $\sqrt{N}$, with the difference $\Delta - J$ finite, as $N \to \infty$. One important feature of this type IIB plane wave background is that the Green-Schwarz string can be quantized exactly [31, 32, 33]. The world sheet theory in the light cone gauge corresponds to the action of eight massive bosons and fermions. Furthermore the machinery of light-cone string field theory can be used to calculate string interactions and compare the results to field theory [34, 35, 36, 37]. For an incomplete list discussing Penrose limits of other supergravity backgrounds see [38, 39, 40, 41, 42].

The aim of this chapter is to study the Penrose limit for a different AdS solution, namely the type IIB solutions found in [1, 2], which are realized as warped products of $AdS_6 \times S^2$ over a Riemann surface Σ with boundary and provide duals of 5d SCFTs. These solutions are supported by R-R and NS-NS three form fluxes. Indeed the data characterizing the solution can be identified with the (p, q) five brane charges of semi-infinite five branes forming a five brane web.

Unlike the type IIB $AdS_5 \times S^5$ solution or the $AdS_{4|7} \times S^{7|4}$ solution of M-theory these backgrounds are warped product geometries and preserve only sixteen of the thirty-two supersymmetries of ten dimensional type IIB supergravity. This fact makes finding a suitable Penrose limit more challenging, since the radii of the AdS_6 and S^2 vary with the location on the Riemann surface Σ and therefore a general null geodesic will also have a nontrivial dependence on the Riemann surface. There is a special point on the Riemann surface, namely the critical point of a function G. At this point the metric factors of the AdS_6 and S^2 are extremized with respect to the coordinates on Σ. We choose the null-geodesic on which the Penrose limit is based to be localized at this critical point on Σ.

The structure of the chapter is as follows: In section 2.2 we review the type IIB supergravity solutions first found in [1, 2] which we use in the rest of the chapter. In section 2.3 we define the null geodesic on which the Penrose limit is based and present the resulting type IIB plane wave background, including all the other bosonic supergravity fields. In section 2.4 we discuss the quantization of the Green-Schwarz superstring action in the light cone gauge for this background. We discuss our results and possible directions for further research in section 2.5. Some detailed calculations and supplementary materials are relegated to the appendices after the discussion.

2.2 Type IIB AdS_6 solutions

In this section, we will explore beyond the canonical early examples of holographic duality mentioned in chapter 1 by introducing a family of supergravity solutions with AdS_6 factors. These solutions are candidates for duals of five dimensional superconformal field theories, or SCFTs. Holography is a particularly useful tool for understanding these theories since they are the strongly interacting UV fixed points of 5d supersymmetric Yang-Mills and do not possess a standard Lagrangian description. Furthermore, the Coulomb branch dynamics of the Yang-Mills theories - which are realized by webs of five branes in Type IIB string theory [43, 44] - do not give direct quantitative access to the SCFTs.

Since the unique superconformal algebra in five dimensions $F(4)$ with an $SO(2,5)$ factor has sixteen fermionic generators [45, 46, 47], the supergravity background preserves only 16 of the 32 supersymmetries of maximally symmetric type II or M-theory vacua. Furthermore the solutions are all realized as warped products of the AdS_6 over a base space. The first such solution was obtained in massive type IIA [48]. In this chapter we will focus on type IIB solutions first constructed in [1, 2], which is realized as a warped product of $AdS_6 \times S^2$ over a two dimensional Riemann surface Σ with boundary.

$$ds^2 = f_6^2 \, ds^2_{\mathrm{AdS_6}} + f_2^2 \, ds^2_{\mathrm{S^2}} + 4\hat{\rho}^2 \, |dw|^2 \, , \tag{2.1}$$

where w is a complex coordinate on Σ and $ds^2_{AdS_6}$ and $ds^2_{S^2}$ are the line elements for unit-radius AdS_6 and S^2, respectively[1]. One may, starting from this ansatz, solve the BPS equations for preserving 16 supersymmetries. The solutions are defined in terms of locally holomorphic functions $A_\pm$ on the Riemann surface Σ. The metric functions read

$$f_6^2 = \sqrt{6GT} \, , \qquad f_2^2 = \frac{1}{9}\sqrt{6G}\, T^{-\frac{3}{2}} \, , \qquad \hat{\rho}^2 = \frac{\kappa^2}{\sqrt{6G}} T^{\frac{1}{2}} \, . \tag{2.2}$$

and they are expressed in terms of the holomorphic functions $A_\pm$ as follows

$$\begin{aligned}
\kappa^2 &= -|\partial_w A_+|^2 + |\partial_w A_-|^2 \, , & \partial_w B &= A_+ \partial_w A_- - A_- \partial_w A_+ \, , \\
G &= |A_+|^2 - |A_-|^2 + B + \bar{B} \, , & T^2 &= \left(\frac{1+R}{1-R}\right)^2 = 1 + \frac{2|\partial_w G|^2}{3\kappa^2 \, G} \, .
\end{aligned} \tag{2.3}$$

The other non vanishing type IIB supergravity fields of the solution are the complex scalar B which is related to the axion dilaton $\tau = \chi + ie^{-\phi}$ via $B = (1+i\tau)/(1-i\tau)$ and

[1]We denote the metric factor of Σ by $\hat{\rho}$ to avoid confusion with ρ which radial direction of Ads_6 used later.

the complex two form antisymmetric tensor potential $C_{(2)}$

$$B = \frac{\partial_w A_+ \, \partial_{\bar{w}} G - R\, \partial_{\bar{w}} \bar{A}_- \partial_w G}{R\, \partial_{\bar{w}} \bar{A}_+ \partial_w G - \partial_w A_- \partial_{\bar{w}} G} \ , \qquad C_{(2)} \ = \frac{2i}{3} \left(\frac{\partial_{\bar{w}} G \partial_w A_+ + \partial_w G \partial_{\bar{w}} \bar{A}_-}{3\kappa^2 T^2} - \bar{\mathcal{A}}_- - \mathcal{A}_+ \right) vol_{S^2} \ . \tag{2.4}$$

As discussed in [2] the residues of $A_\pm$ can be identified with the charges of the (p,q) five brane web which realizes the dual SCFT via

$$Z_+^l = \frac{3}{4} \alpha' (p_l + i q_l), \quad l = 1, 2, \cdots, L \tag{2.5}$$

2.2.1 Solutions with and without D7 brane monodromy

In order to obtain regular and geodesically complete solutions one has to impose further conditions; namely that G vanishes along the boundary of the Riemann surface, which implies that the S^2 shrinks to zero size and the spacetime closes off. In [2] a large class of regular solutions were constructed by choosing Σ to be the upper half plane and the holomorphic functions such that $\partial_w A_\pm$ have L simple poles localized on the boundary which is the real line.

$$A_\pm = A_\pm^0 + \sum_{l=1}^{L} Z_\pm^l \ln(w - p_l), \quad \sum_{l=1}^{L} Z_\pm^l = 0, \quad Z_\pm^l = -\bar{Z}_\mp^l \tag{2.6}$$

The regularity condition translates into L conditions [2]

$$A_+^0 Z_-^k - A_-^0 Z_+^k + \sum_{l=1, l\neq k}^{L} (Z_+^l Z_-^k - Z_+^k Z_-^l) \ln |p_l - p_k| = 0, \qquad k = 1, 2, \cdots, L \tag{2.7}$$

These solutions have no monodromy but the procedure outlined above was later extended in [3] to include solutions with monodromy. Subsequent sections will only deal with the former solutions but for completeness, we will include the others here. The additional data for the case with D7 brane punctures are the locations $w_i \in \Sigma$ of the punctures and, for each puncture, a real number n_i and phase γ_i specifying the the orientation of the branch cut. The holomorphic functions are given by

$$A_{\pm} = A^0_{\pm} + \sum_{l=1}^{L} Z^l_{\pm} \ln(w - p_l) + \int_{\infty}^{w} dz f(z) \sum_{l=0}^{L} \frac{Y^l}{z - p_l} \tag{2.8}$$

where $Y^l \equiv Z^l_+ - Z^l_-$, the relationship between the constants is $A^0_+ = -\bar{A}^0_-$,

$$f(w) = \sum_{i=1}^{I} \frac{n_i^2}{4\pi} \ln\left(\gamma_i \frac{w - w_i}{w - \bar{w}_i}\right) \tag{2.9}$$

and the contour of the above integral is chosen to not cross any branch cuts. The regularity conditions outlined above constrain the parameters of the solution to satisty the following

$$0 = 2A^0_+ - 2A^0_- + \sum_{l=1}^{L} Y^l \ln|w_i - p_l|^2, \qquad i = 1, \cdots, I, \tag{2.10}$$

$$0 = 2A^0_+ \mathcal{Y}^k_- - 2A^0_- \mathcal{Y}^k_+ + \sum_{l \neq k} Z^{[l,k]} \ln|p_l - p_k|^2 + Y^k J_k, \qquad k = 1, \cdots, L. \tag{2.11}$$

where

$$J_k = \sum_{l=1}^{L} Y^l \left[\int_{\infty}^{p_k} dx f'(x) \ln|x - p_l|^2 + \sum_{i \in \mathcal{S}_k} \frac{i n_i^2}{2} \ln|w_i - p_l|^2\right] \tag{2.12}$$

$$\mathcal{Y}^l_{\pm} = Z^l_{\pm} + f(p_l) Y^l \tag{2.13}$$

where the $\mathcal{L}_{\pm}$'s are the residues of the poles of $\partial_w A_{\pm}$. The sum is taken over the set of branch points for which the associated cut intersects (p_k, ∞)

2.2.2 Behavior near the branes and charge quantization

We can now explore the metric 2.1 and supergravity fields around the poles p_l. (The residues will be denoted, as in the non-monodromy case, by $Z^l_{\pm}$ but the expressions are the same for the monodromy case making the replacement $Z \to \mathcal{Y}$). Near a pole p_l, we can express things in radial coordinates around the pole $w = p_l + re^{i\theta}$:

$$f_6^2 \approx 2 \cdot 3^{\frac{1}{4}} \kappa_l r^{\frac{1}{2}} |\ln r|^{\frac{3}{4}}, \quad \rho^2 \approx \frac{\kappa_l}{2 \cdot 3^{3/4}} r^{-\frac{3}{2}} |\ln r|^{-\frac{1}{4}}, \quad f_2^2 \approx 4r^2 \sin^2\theta \rho^2 \tag{2.14}$$

with κ_l a constant. The S^2 and θ-circle combine to form an S^3 around the pole. The three form field strength takes the form

$$F_3 = \frac{8}{3} Z_+^l \sin^2\theta d\theta \wedge vol_{S^2} \tag{2.15}$$

For solutions with monodromy, we can additionally zoom in on the region near a puncture w_i. The metric and axion-dilaton scalar τ are given in terms of a coordinate z with the puncture at $z = 0$:

$$ds^2 \approx ds^2_{AdS_6\times S^2} + Im(\mathcal{H})|dz|^2 \quad \tau \approx \mathcal{H} + \bar{\tau}_0 \quad \mathcal{H} = -\frac{in_i^2}{2\pi}\ln z \tag{2.16}$$

In the papers [2, 3], the poles were identified with (p,q) 5-branes and the punctures with 7-branes. We will now briefly summarize the relationship between the charges and quantities we have seen above.

The complex two form C_2 has real and imaginary parts given by the NS-NS two form B_2 and R-R two form C_2^{RR}. Charge quantization is then derived from the coupling of fundamental strings and D1-branes to B_2 and C_2^{RR} respectively. The quantization conditions are

$$\frac{1}{2\pi\alpha'}\int_{S^3} dB_2 = 2\pi N_{NS5} \tag{2.17}$$

$$\frac{1}{2\pi\alpha'}\int_{S^3} dC_2^{RR} = 2\pi N_{D5} \tag{2.18}$$

where we integrate over the S^3 surrounding the pole. Using the near-pole expressions for the field strength (2.15), we find that

$$Z_+^l = \frac{3}{4}\alpha'(N_{NS5} + iN_{D5}) \tag{2.19}$$

In the (p,q) notation, $p = N_{NS5}$ and $q = N_{D5}$.

For the solutions with monodromy, we can again replace $Z \to \mathcal{Y}$ in (2.19). We can also note that around $z = 0$ in (2.16), the monodromy of the scalar τ is $\tau \to \tau + n_i^2$ and so

$$N_{D7} = n_i^2 \tag{2.20}$$

since for a single D7 brane we have $\tau \to \tau + 1$.

2.2.3 Explicit examples

For definiteness we consider two explicit examples of the regular supergravity solution for which the dual SCFT are well studied, namely the T_N theory and the $+_{M,N}$ theory where the holomorphic functions $A_\pm$ are given by

$$\begin{aligned} A_\pm^{T_N} &= \frac{3N}{8\pi}\left[\pm\ln(w-1) + i\ln(2w) + (\mp 1 - i)\ln(w+1)\right] \ , \\ A_\pm^{+_{N,M}} &= \frac{3}{8\pi}\left[iN\left(\ln(2w-1) - \ln(w-1)\right) \pm M\left(\ln(3w-2) - \ln w\right)\right] \ , \end{aligned} \tag{2.21}$$

The T_N solution has three poles located at $w = \pm 1, 0$ whereas the $+_{M,N}$ has four poles located at $w = 0, \frac{1}{2}, \frac{2}{3}, 1$. The relevant brane webs for the two theories are given by the junction of N D5, N NS5 and N (1,1) 5-branes for the T_N theory and the intersection of N D5-branes and M NS5-branes for the $+_{M,N}$ theory.

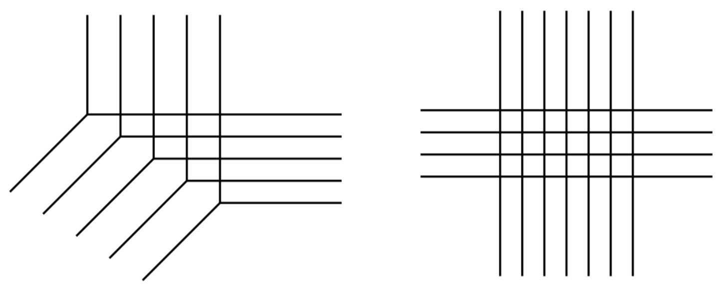

Figure 2.1: Left: brane web for the T_N theory. Right: brane web for the $+_{M,N}$ theory.

The quiver theories which at their conformal fixed points realize the SCFTs are given by long linear quivers with $SU(n)$ gauge nodes and bi-fundamental matter connecting them and a fundamental matter node at the end. For the T_N theories we have a linear quiver with $SU(k)$ nodes with increasing k along the quiver and fundamental representations attached to the end of the quiver

$$T_N: \quad [2] - (2) - (3) - \cdots - (N-1) - [N] \tag{2.22}$$

For the $+_{MN}$ theory we have $M-1$ $SU(N)$ nodes with matter in the fundamental representation of the end nodes attached at the end of the quiver.

$$+_{MN}: \quad [N] - (N) - (N) - \cdots - (N) - [N] \tag{2.23}$$

For the two example cases the function G defined in (2.3) takes the following compact form

$$
\begin{aligned}
G_{T_N} &= \frac{9}{8\pi^2} N^2 D\left(\frac{2w}{w+1}\right) \ , \\
G_{+_{N,M}} &= \frac{9}{8\pi^2} NM\left[D\left(\frac{3w-2}{w}\right) + D\left(\frac{w}{2-3w}\right)\right]
\end{aligned}
\tag{2.24}
$$

where D is the Bloch-Wigner function given by

$$
D(u) = Im\ [Li_2(u) + \ln(1-u)\ln|u|]\ . \tag{2.25}
$$

where Li_2 is the dilogarithm function.

2.3 Penrose limit and the plane wave background

The idea behind taking a Penrose limit of $AdS_5 \times S^5$ solution of type IIB is to consider the trajectory of a particle sitting at the center of AdS_5 and moving very fast along a great circle on S^5, such that in the limit the trajectory becomes a null geodesic. Zooming in on the vicinity of the null geodesic produces a pp-wave solution which preserves 32 supercharges.

For the $AdS_6 \times S^2$ solutions presented in section 4.2, the geometry is more complicated since the AdS_6 and S^2 are warped over the two dimensional surface Σ and in general a nice Penrose limit does not exist for geodesics through a generic point on Σ. Furthermore as discussed in appendix 2.B considering a geodesic with a nontrivial dependence on Σ looks daunting due to the complicated form of the geodesic equation.

Fortunately, we can find a special point on the Riemann surface Σ by considering critical points of the function G defined in (2.3), i.e $w_c \in \Sigma$ for which

$$
\partial_w G\ |_{w=w_c} = 0 \tag{2.26}
$$

It was observed in [49] that in all known examples, G has a unique critical point on the Riemann surface Σ. For our two examples the critical point on w_c on Σ is located at

$$
w_{c,T_N} = \frac{i}{\sqrt{3}}, \qquad w_{c,+_{M,N}} = \frac{3+i}{5} \tag{2.27}
$$

In [49] it was shown that the critical point w_c correspond to the location of BPS probe D3-branes that realize surface defects in the dual SCFT. As we shall see in the following, the existence of the critical point is essential for our construction of Penrose limit in the warped spacetime.

We can examine our solution near the critical point by expanding $w = w_c + \epsilon\zeta$. The most important expression, G, can be expanded in a power series in ϵ

$$G = G_0 + G_2\epsilon^2|\zeta|^2 + o(\epsilon^3) \tag{2.28}$$

Using (2.3) we find the following expansions for the other functions which appear in the metric factors

$$\begin{aligned}
\kappa^2 &= -G_2 + o(\epsilon) \\
T^2 &= 1 - \frac{2}{3}\frac{G_2}{G_0}\epsilon^2|\zeta|^2 + o(\epsilon^3) \\
R &= -\frac{G_2}{6G_0}|\zeta^2| + o(\epsilon)
\end{aligned} \tag{2.29}$$

Using the expressions for the metric factors (2.2) it is straightforward to show that the metric factors of the AdS_6 and S_2 are extremized at $w = w_c$.

For the T_N and $+_{N,M}$ examples we find the following expressions for the expansion coefficients G_0 and G_2

$$\begin{aligned}
T_N:\ G_0 &= Im(Li_2(e^{i\pi/3})), \quad G_2 = -\frac{81\sqrt{3}}{128\pi^2}N^2 \\
+_{N,M}:\ G_0 &= \frac{9CMN}{4\pi^2}, \qquad G_2 = -\frac{225MN}{16\pi^2}
\end{aligned} \tag{2.30}$$

where C is Catalan's constant.

2.3.1 Penrose limit

Using the expansions (2.28) and (2.29) the Einstein frame metric (2.1) can be expanded to second order which corresponds to the metric close to the critical point $w = w_c$.

$$ds^2 = \sqrt{6G_0}\Big\{(1+\epsilon^2\frac{G_2}{3G_0}|\zeta|^2)(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho ds^2_{S^4}) \\ + \frac{1}{9}(1+\epsilon^2\frac{G_2}{G_0}|\zeta|^2)(d\theta^2+\cos^2\theta d\phi^2) + \frac{2}{3}\epsilon^2\frac{|G_2|}{G_0}|d\zeta|^2\Big\} + o(\epsilon^3) \tag{2.31}$$

We can drop the expansion parameter ϵ in the following since it can be absorbed into a redefinition of the coordinate ζ. The null geodesic which gives the proper Penrose limit is defined by

$$\zeta = 0, \quad \rho = 0, \quad \theta = 0, \quad t - \frac{\phi}{3} = const \tag{2.32}$$

Note that $\zeta = 0$ implies that the null geodesic is localized at the critical point $w = w_c$ on the Riemann surface. We express the metric in terms of rescaled and relabelled coordinates

$$t = x^+ + \frac{x^-}{(6G_0)^{\frac{1}{2}}}, \qquad \phi = 3x^+ - \frac{3x^-}{(6G_0)^{\frac{1}{2}}} \\ \rho = \frac{1}{(6G_0)^{\frac{1}{4}}}r, \qquad \theta = \frac{3}{(6G_0)^{\frac{1}{4}}}x^6, \qquad \zeta = \epsilon\left(\frac{3G_0}{2|G_2|}\right)^{\frac{1}{2}}\frac{1}{(6G_0)^{\frac{1}{4}}}(x^7+ix^8) \tag{2.33}$$

The Penrose limit is obtained by taking $G_0 \to \infty$, which corresponds to taking $N \to \infty$ (for the T_N case) and $MN \to \infty$ (for the $+_{N,M}$ example) and keeping the finite terms in the metric (2.31). All expressions involving the ratio G_2/G_0 are finite in this limit. Note that the curvature radius of the AdS_6 and S^2 factors are controlled by G_0 and hence the limit of large G_0 is precisely the one taken for a Penrose limit in $AdS_p \times S^q$ spacetimes, i.e. zooming in on a region close to the null geodesic, which is much smaller than the curvature radius of the spacetime.

The resulting metric is given by

$$ds^2 = -4dx^+dx^- + (dx^+)^2\Big(-r^2 - 9x_6^2 - x_7^2 - x_8^2\Big) + dr^2 + r^2ds^2_{S^4} + dx_6^2 + dx_7^2 + dx_8^2 \tag{2.34}$$

where $r^2 = x_1^2 + x_2^3 + x_3^2 + x_4^2 + x_5^2$. The only difference between this metric and the one of the Penrose limit of $AdS_5 \times S^5$ that the coordinate x_6 is singled out and has a different normalization in the $(dx^+)^2$ term in the metric, which in the world sheet action leads to a different mass term for the field associated with x_6.

The energy in global AdS is associated with $i\partial_t$ whereas $-i\partial_\phi$ is related to a $U(1)$ generator of the $SU(2)_R$ symmetry, which is realized as the group of isometries of the S^2. Hence using the argument of [19] for the Penrose limit (2.33) we can identify the light cone momentum with the conformal dimension and the R-charge of a dual operator via

$$\begin{aligned} 2p^- &= i\partial_+ = i(\partial_t + 3\partial_\phi) = \Delta - 3J \\ 2p^+ &= i\partial_- = \frac{i}{R^2}(\partial_t - 3\partial_\phi) = \frac{1}{R^2}(\Delta + 3J) \end{aligned} \tag{2.35}$$

where we have identified the AdS radius with $R^2 = \sqrt{6G_0}$. Since we take R to be very large in the Penrose limit excitations with finite $p^\pm$ correspond to states with large energy and angular momentum which on the CFT side correspond to operators with large conformal dimension Δ and R-charge J, which are close to the BPS bound $\Delta = 3J$ [50, 51].

2.3.2 Other supergravity fields

In the Penrose limit, the complex scalar field B of type IIB supergravity becomes a constant

$$B = -\frac{\partial_w A_+}{\partial_w A_-}\,|_{w=w_c} + o(\epsilon) \tag{2.36}$$

Note that the standard RR axion χ and the dilaton ϕ is related to the complex scalar B via

$$B = \frac{1+i\tau}{1-i\tau}, \qquad \tau = \chi + ie^{-\Phi} \tag{2.37}$$

Consequently, the axion dilaton has the following expression in terms of the holomorphic functions at the critical point w_c

$$\tau = i\frac{\partial_w A_- + \partial_w A_+}{\partial_w A_- - \partial_w A_+}\,|_{w=w_c} \tag{2.38}$$

For the two examples we consider in this chapter one finds[2]

$$T_N: \ \chi = \frac{1}{2}, \ \ e^{-\phi} = \frac{\sqrt{3}}{2}, \qquad\qquad +_{N,M}: \ \ \chi = 0, \ \ e^{-\phi} = \frac{N}{M} \tag{2.39}$$

The complex rank two antisymmetric tensor potential $C^{(2)}$ is related to the NS-NS and R-R two form potentials by

$$\begin{aligned} C^{(2)} &= B^{NSNS} + iC^{RR} \\ &= \Big(c_0 - i\frac{8}{9}\Big(\partial_w A_+ \mid_{w=w_c} \epsilon\zeta + \partial_{\bar{w}}\bar{A}_- \mid_{w=w_c} \epsilon\bar{\zeta}\Big) + o(\epsilon^2)\Big) \cos\theta d\theta \wedge d\phi \end{aligned} \tag{2.40}$$

Note that the constant term c_0 is not important since in the bosonic part of the worldsheet action, a constant NS-NS anti-symmetric tensor potential will be a total derivative. In the fermionic worldsheet action only the field strength, which does not depend on c_0, appears. In the Penrose limit the finite part which survives (after using the rescaled variables)

$$C^{(2)} = -4i\frac{1}{|G_2|^{\frac{1}{2}}}\Big(\partial_w A_+ \mid_{w=w_c} (x^7 + ix^8) + \partial_{\bar{w}}\bar{A}_- \mid_{w=w_c} (x^7 - ix^8)\Big) dx^6 \wedge dx^+ \tag{2.41}$$

Note that in the scaling limit which produces the Penrose limit discussed below (2.33) the $\partial A_\pm$ and $|G_2|$ scale in the same way and hence the term above is the expression which survives the Penrose limit.

The NS-NS 3 form field strength H^3_{NS} and the RR field strength F^3_{RR} are given by

$$H^3_{NS} = Re\Big(dC_2\Big), \qquad F^3_{RR} = Im\Big(dC_2\Big), \tag{2.42}$$

For the examples we get

$$\begin{aligned} T_N: \quad H^3_{NS} &= -\frac{4\sqrt{2}}{3^{\frac{1}{4}}} dx^8 \wedge dx^6 \wedge dx^+, \\ F^3_{RR} &= -\frac{2\sqrt{2}}{3^{\frac{3}{4}}}\Big(3dx^7 + \sqrt{3}dx^8\Big) \wedge dx^6 \wedge dx^+ \end{aligned} \tag{2.43}$$

[2] In [1, 2] different normalization for the dilaton is used, we have translated the expression to the commonly used one.

and

$$+_{N,M}: \quad H^3_{NS} = -\frac{4}{5}\sqrt{\frac{M}{N}}(4dx^7 + 3dx^8) \wedge dx^6 \wedge dx^+,$$
$$F^3_{RR} = \frac{4}{5}\sqrt{\frac{N}{M}}(-3dx^7 + 4dx^8) \wedge dx^6 \wedge dx^+ \tag{2.44}$$

It is important to mention that the supergravity solutions of section 4.2 are given in the Einstein frame. In order to quantize the Green-Schwarz string in this background we have to transform to the string frame

$$G^{string}_{\mu\nu} = e^{\phi/2} G^{Einstein}_{\mu\nu} \tag{2.45}$$

Note that the particular combinations of the anti-symmetric tensor field strength which appear in the string frame supersymmetry transformations as well as the Green-Schwarz action are $H^{(3)}_{NS}$ and $e^{\phi}(F^{(3)}_R - \chi H^{(3)}_{NS})$.

In the Penrose limit, the transformation to the Einstein frame is just a constant rescaling of the metric. However, it is convenient for the string frame metric in to be the canonical plane wave form (2.34). This can be accomplished by a rescaling of all coordinates (except x^+) by a factor of $e^{-\phi/4}$. This has the effect of introducing a factor of $e^{-\phi/2}$ in each of the antisymmetric tensor fields. The end result is the following combination of fields which appears in the string worldsheet:

$$T_N: \quad H^3_{NS,string} = -4dx^8 \wedge dx^6 \wedge dx^+,$$
$$e^{\phi}(F^3_{RR,string} - \chi H^3_{NS,string}) = -4dx^7 \wedge dx^6 \wedge dx^+ \tag{2.46}$$
$$+_{N,M}: \quad H^3_{NS,string} = -\frac{4}{5}(4dx^7 + 3dx^8) \wedge dx^6 \wedge dx^+,$$
$$e^{\phi}(F^3_{RR,string} - \chi H^3_{NS,string}) = \frac{4}{5}(-3dx^7 + 4dx^8) \wedge dx^6 \wedge dx^+ \tag{2.47}$$

We will drop "string" from the subscripts moving forward since we will always be working in the string frame and there will be no confusion.

Note one interesting property of specific solutions (2.46) and (2.47): The orientation of the forms in the 7, 8 directions can be parameterized as $n_7 dx^7 + n_8 dx^8$, for both T_N and $+_{MN}$

examples the n_a associated with the $H^{(3)}_{NS}$ and $e^{\phi}(F^{(3)}_{R} - \chi H^{(3)}_{NS})$ tensors are orthogonal to each other and square to four. This means that we could rotate $x^{7,8}$ so that the fields which appear in the worldsheet action take exactly the same form in both T_N and $+_{MN}$ theories and therefore give the same string spectrum. In fact, this statement also holds for all other solutions [52, 53] that we have checked. We conjecture that all global solutions (2.1) - (2.4) share this property and in the quantization the follows, take our fields to have the form of the T_N example (2.46).

2.4 Light cone Green-Schwarz string spectrum

In this section we quantize the Green-Schwarz string the pp-wave spacetime obtained in the previous section. As shown in [32, 31, 33, 54] the fermionic part of the Green-Schwarz string becomes quadratic in fermionic fields for the pp-wave spacetimes in the light cone gauge, which makes the free string spectrum exactly solvable.

2.4.1 Bosonic Spectrum

We will start by examining the bosonic spectrum for the Green-Schwarz string in the plane wave background with NS-NS flux. The general action is given by

$$S_b = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{h} \left(h^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} + \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \right) \tag{2.48}$$

As in the $AdS_5 \times S^5$ case, this action simplifies considerably after light cone gauge-fixing, leaving us with a free theory which can be easily quantized following [55]. We set the target space coordinate $x^+ = \tau$ and worldsheet metric to $h^{\sigma\tau} = 0$, $h^{\sigma\sigma} = -h^{\tau\tau} = 1$ and after plugging in the background (2.34), (2.46), we are left with the following action for the eight remaining transverse scalar fields

$$S_b = \frac{1}{4\pi\alpha'}\int_0^{2\pi\alpha' p^+} d^2\sigma \left(\sum_{I=1}^{8}\left(\left(\partial_\tau X^I\right)^2 - \left(\partial_\sigma X^I\right)^2 - m_I^2\left(X^I\right)^2\right) + 4X^6\partial_\sigma X^7 - 4X^7\partial_\sigma X^6\right) \tag{2.49}$$

where $m_I^2 = 1$ for all $I \neq 6$ in which case $m_6^2 = 9$. The equations of motion deduced from this action are given by

$$\partial_\tau^2 X^I - \partial_\sigma^2 X^I + X^I = 0 \quad I = 1, ..., 5, 8 \tag{2.50}$$

$$\partial_\tau^2 X^6 - \partial_\sigma^2 X^6 + 9X^6 - 4\partial_\sigma X^7 = 0 \tag{2.51}$$

$$\partial_\tau^2 X^7 - \partial_\sigma^2 X^7 + X^7 + 4\partial_\sigma X^6 = 0 \tag{2.52}$$

Together with the periodicity condition $X^I(\sigma + 2\pi\alpha' p^+, \tau) = X^I(\sigma, \tau)$, the first set of equations (2.50) lead to the following familiar solutions for $I = 1, ..., 5, 8$

$$x^I(\sigma,\tau) = x_0^I \cos\tau + \frac{p_0^I}{p^+}\sin\tau + \sqrt{\frac{\alpha'}{2}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\omega_n^I}}[\alpha_n^I e^{-\frac{i}{\alpha' p^+}\left(\omega_n^I\tau + n\sigma\right)} + \tilde{\alpha}_n^I e^{-\frac{i}{\alpha' p^+}\left(\omega_n^I\tau - n\sigma\right)} + \alpha_n^{I\dagger} e^{\frac{i}{\alpha' p^+}\left(\omega_n^I\tau + n\sigma\right)} + \tilde{\alpha}_n^{I\dagger} e^{\frac{i}{\alpha' p^+}\left(\omega_n^I\tau - n\sigma\right)}] \tag{2.53}$$

where the frequencies will determine the energy of the string excitations

$$\omega_n^I = \sqrt{n^2 + (\alpha' p^+)^2} \tag{2.54}$$

To solve the remaining two equations of motion (2.51) and (2.52) for $I = 6, 7$ we use an ansatz of the following form

$$x^I(\sigma,\tau) = \sqrt{\frac{\alpha'}{2}}A_0(\tau) + \sqrt{\frac{\alpha'}{2}}\sum_{n>0}[A_n^I(\tau,\sigma) + \tilde{A}_n^I(\tau,\sigma)] \tag{2.55}$$

where A_n^I and $\tilde{A}_n^I$ for $I = 6, 7$ can be expressed in terms of eigenmodes

$$A_n^I(\tau,\sigma) = \sum_{J=6,7}\frac{1}{\sqrt{\omega_n^J}}\left\{(V_n)^I_J \alpha_n^J e^{-\frac{i}{\alpha' p^+}(\omega_n^J\tau + n\sigma)} + (\bar{V}_n)^I_J (\alpha^\dagger)^J_n e^{\frac{i}{\alpha' p^+}(\omega_n^J\tau + n\sigma)}\right\}$$

$$\tilde{A}_n^I(\tau,\sigma) = \sum_{J=6,7}\frac{1}{\sqrt{\omega_n^J}}\left\{(V_{-n})^I_J \alpha_n^J e^{-\frac{i}{\alpha' p^+}(\omega_n^J\tau - n\sigma)} + (\bar{V}_{-n})^I_J (\alpha^\dagger)^J_n e^{\frac{i}{\alpha' p^+}(\omega_n^J\tau - n\sigma)}\right\} \tag{2.56}$$

For the zero mode A_0 there only one eigenfunction which can be obtained from A_n^I in (2.56) by setting $n = 0$. The eigenfrequencies for $J = 6, 7$

$$\begin{aligned} \omega_n^6 &= \sqrt{n^2 + (\alpha' p^+)^2} + 2\alpha' p^+ \\ \omega_n^7 &= \left| \sqrt{n^2 + (\alpha' p^+)^2} - 2\alpha' p^+ \right| \end{aligned} \tag{2.57}$$

and $(V_n)^I{}_J$, $J = 6, 7$ are the orthonormal set of vectors which satisfy

$$\begin{pmatrix} \frac{-(\omega_n^J)^2 + n^2}{(\alpha' p^+)^2} + 9 & -\frac{4in}{\alpha' p^+} \\ +\frac{4in}{\alpha' p^+} & \frac{-(\omega_n^J)^2 + n^2}{(\alpha' p^+)^2} + 1 \end{pmatrix} \begin{pmatrix} (V_n)^6{}_J \\ (V_n)^7{}_J \end{pmatrix} = 0, \qquad J = 6, 7 \tag{2.58}$$

With this mode equation the the canonical quantization conditions

$$[x^I(\sigma, \tau), p^J(\sigma', \tau)] = i\delta^{IJ}\delta(\sigma - \sigma') \tag{2.59}$$

where $p^I = \frac{1}{2\pi\alpha'}\partial_\tau x^I$, yield the commutation relation for the creation and annihilation operators

$$[\alpha_n^I, \alpha_m^{J\dagger}] = [\tilde{\alpha}_n^I, \tilde{\alpha}_m^{J\dagger}] = \delta^{IJ}\delta_{mn} \tag{2.60}$$

where now we allow the indices I, J to run through all eight transverse oscillators $1, 2, \cdots, 8$.

The light-cone Hamiltonian can be expressed in terms of the creation and annihilation operators (2.53) and (2.55)

$$H_{l.c.}^{\mathrm{b}} = \frac{1}{\alpha' p^+} \sum_{I=0}^{8} \left[\omega_0^I \alpha_0^{I\dagger} \alpha_0^I + \sum_{n=0}^{\infty} \omega_n^I \left(\alpha_n^{I\dagger} \alpha_n^I + \tilde{\alpha}_n^{I\dagger} \tilde{\alpha}_n^I \right) \right] + \nu_{bos} \tag{2.61}$$

Here ν_{bos} denotes a normal ordering constant for the bosonic creation and annihilation operators. The zero mode modes x_0^I, p_0^I, $I = 1, 2, \cdots, 5, 8$ have been expressed in terms of creation and annihilation operators

$$\tilde{\alpha}_0^I = \frac{1}{\sqrt{2p^+}} p_0^I - i\sqrt{\frac{p^+}{2}} x_0^I \tag{2.62}$$

to maintain consistent normalization. It will be convenient to write ν_{bos} in terms of the oscillator frequencies:

$$\nu_{bos} = \frac{1}{2\alpha' p^+} \sum_{I=1}^{8} \left(\omega_0^I + 2\sum_n \omega_n^I \right) \tag{2.63}$$

2.4.2 Fermionic Spectrum

We can now turn our attention to the fermionic part of the Green-Schwarz action for type IIB strings, the term quadratic in the world sheet fermions is given by (see appendix 3.A for details on the notation).

$$S_f^{(2)} = \frac{i}{4\pi\alpha'}\int d^2\sigma\sqrt{h}\left(h^{ij}\delta_{ab} - \varepsilon^{ij}\left(\sigma_3\right)_{ab}\right)\partial_i x^M \bar{\theta}^a\Gamma_M\left(D_j\right)_c^b\theta^c \tag{2.64}$$

This Green-Schwarz action possesses a κ symmetry which is required to obtain spacetime supersymmetry for the on-shell string modes [31]. To obtain these physical fermionic modes, we can gauge fix (as in flat space) by choosing

$$\Gamma^+\theta^a = 0 \tag{2.65}$$

where $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)$. After enforcing the light cone gauge condition $x^+ = \tau$ and worldsheet metric to $h^{\sigma\tau} = 0$, $h^{\sigma\sigma} = -h^{\tau\tau} = 1$, the fermionic part of the Green-Schwarz action reduces to the quadratic one in the plane wave background [31, 32, 33] and takes the following form

$$S_f = \frac{i}{4\pi\alpha'}\int d^2\sigma\left(-\left(\theta^a\right)^T\Gamma^+\Gamma_+\left(D_\tau\right)_c^a\theta^c - \left(\theta^a\right)^T\Gamma^+\Gamma_+\sigma_{ab}^{(3)}\left(D_\sigma\right)_c^b\theta^c\right) \tag{2.66}$$

after inserting the antisymmetric tensor fields (2.46) for the T_N theory, the action becomes

$$\begin{aligned} S_f &= -\frac{i}{4\pi\alpha'}\int d^2\sigma(\theta^a)^T\Gamma^+\Gamma_+\Big(\delta_{ab}\partial_\tau + (\sigma_3)_{ab}\partial_\sigma - \frac{1}{4}H_{+67}\Gamma^{67}(\sigma_3)_{ab} + \frac{1}{4}\tilde{F}_{+68}\Gamma^{68}(\sigma_1)_{ab}\Big)\theta^b \\ &= -\frac{i}{4\pi\alpha'}\int d^2\sigma(\theta^a)^T\Gamma^+\Gamma_+\Big(\delta_{ab}\partial_\tau + (\sigma_3)_{ab}\partial_\sigma + \Gamma^{67}(\sigma_3)_{ab} - \Gamma^{68}\,(\sigma_1)_{ab}\Big)\theta^b \end{aligned} \tag{2.67}$$

where we have used the fact that $\Gamma^+\Gamma_+ = 1 + \Gamma^{09}$ and hence $(\Gamma^+\Gamma_+)^2 = 2\Gamma^+\Gamma_+$. The chirality condition $\Gamma^{11}\theta^a = \theta^a$ together with the light cone gauge condition (2.65) projects a general thirty-two component spinor onto an eight dimensional subspace. It is possible to find a representation of the Gamma matrices in this subspace and combine them with the σ matrices encoding the two chiral spinors such that the action can be written as

$$S_f = -\frac{i}{2\pi\alpha'}\int d^2\sigma\theta^T 1_4\otimes\Big(1_2\otimes 1_2\,\partial_\tau + 1_2\otimes\sigma_3\,\partial_\sigma + \sigma_3\otimes\sigma_3 - \sigma_1\otimes\sigma_1\Big)\theta \tag{2.68}$$

Due to the common 1_4 factor in all the terms it follows that there is a fourfold degeneracy and equation of motion for each degenerate spinor component is given by

$$\Big(1_2 \otimes 1_2\, \partial_\tau + 1_2 \otimes \sigma_3\, \partial_\sigma + i\sigma_3 \otimes \sigma_3 - i\sigma_1 \otimes \sigma_1\Big)\theta^\alpha = 0 \tag{2.69}$$

where $\alpha = 1, ..., 4$ labels four degenerate spinor components acting on the first 1_4 factor in the tensor product. We can proceed just as in the bosonic case with an ansatz of the form

$$\theta^\alpha(\sigma,\tau) = \frac{1}{\sqrt{2p^+}} \sum_{n=0}^{\infty} \Big(\theta_n^{\alpha,+}(\tau,\sigma) + \theta_n^{\alpha,-}(\tau,\sigma)\Big) \tag{2.70}$$

Where

$$\begin{aligned}\theta_n^{\alpha,\pm} = {} & V_n^\pm e^{-\frac{i}{\alpha' p^+}(\omega_n^\pm \tau + n\sigma)} \beta_n^{\alpha,\pm} + V_{-n}^\pm e^{\frac{i}{\alpha' p^+}(\omega_n^\pm \tau + n\sigma)} (\beta_n^{\alpha,\pm})^\dagger \\ & + V_{-n}^\pm e^{-\frac{i}{\alpha' p^+}(\omega_n^\pm \tau - n\sigma)} \tilde{\beta}_n^{\alpha,\pm} + V_n^\pm e^{\frac{i}{\alpha' p^+}(\omega_n^\pm \tau - n\sigma)} \tilde{(\beta_n^{\alpha,\pm})}^\dagger \end{aligned} \tag{2.71}$$

the positive eigenfrequencies are given by

$$\begin{aligned}\omega_n^+ &= \sqrt{(\alpha' p^+)^2 + n^2} + \alpha' p^+ \\ \omega_n^- &= \sqrt{(\alpha' p^+)^2 + n^2} - \alpha' p^+\end{aligned} \tag{2.72}$$

The vectors $V_n^\pm$ satisfy the following equation

$$\Big(-\omega_n^\pm\ 1_2 \otimes 1_2\ - n\ 1_2 \otimes \sigma_3 + \sigma_3 \otimes \sigma_3 - \sigma_1 \otimes \sigma_1\Big)V_n^\pm = 0 \tag{2.73}$$

ensuring that the mode expansion of θ (2.71) satisfies the equation of motion (2.69). For each $n > 0$ the four vectors $V_n^\pm, V_{-n}^\pm$ can be chosen to form a orthonormal basis. The canonical anti-commutation relations following from the action (2.68)

$$\{\theta^T(\tau,\sigma), \theta(\tau,\sigma')\} = i2\pi\alpha'\delta(\sigma-\sigma')1_{16\times16} \tag{2.74}$$

imply the following algebra for the fermionic creation and annihilation operators

$$\{\beta_n^{\alpha,\pm}, (\beta_m^{\gamma,\pm})^\dagger\} = \{\tilde{\beta}_n^{\alpha,\pm}, (\tilde{\beta}_m^{\gamma,\pm})^\dagger\} = \delta^{\alpha\gamma}\delta_{m,n} \tag{2.75}$$

with all other anti commutators vanishing.

Interestingly, we can see from (2.72) that our spectrum contains eight fermionic zero modes with $\omega_n^- = 0$ when $n = 0$. The zero modes are associated with θ in the original fermionic action (2.67) for which

$$\left(\Gamma^{67}\sigma^{(3)} - \Gamma^{68}\sigma^{(1)}\right)\theta = \Gamma^{67}\sigma^{(3)}\left(1 - i\Gamma^{78}\sigma^2\right)\theta = 0 \tag{2.76}$$

where we have dropped the index a for notational simplicity. Notice that $(1 - i\Gamma^{78}\sigma^2)$ is a projector and hence has an eight dimensional eigenspace with eigenvalue zero, matching precisely with the eight zero modes found above.

Lastly, we can write the fermionic light cone Hamiltonian

$$H^f_{l.c.} = \frac{1}{\alpha' p^+}\sum_\alpha \left[\omega_0^+ (\beta_0^{\alpha,+})^\dagger \beta_0^{\alpha,+} + \sum_{n=0}^{\infty}\sum_{\gamma=\pm}\omega_n^{\alpha,\gamma}\left((\beta_n^{\alpha,\gamma})^\dagger\beta_n^{\alpha+} + (\tilde{\beta}_n^{\alpha,\gamma})^\dagger\tilde{\beta}_n^{\alpha,\gamma}\right)\right] + \nu_f \tag{2.77}$$

Where ν_f is the constant obtained by normal ordering the fermionic creation and anilhilation operators.

$$\nu_f = -\frac{1}{2\alpha' p^+}\left(4\omega_0^+ + 8\sum_{n>0}(\omega_n^+ + \omega_n^-)\right)$$

2.4.3 Light cone spectrum

The Hamiltonian of the light cone string is the sum of the bosonic (2.61) and fermionic (2.77) parts

$$\begin{aligned} H_{l.c.} = \frac{1}{\alpha' p^+}&\left(\sum_{I=0}^{8}\left[\omega_0^I \alpha_0^{I\dagger}\alpha_0^I + \sum_{n=1}^{\infty}\omega_n^I\left(\alpha_n^{I\dagger}\alpha_n^I + \tilde{\alpha}_n^{I\dagger}\tilde{\alpha}_n^I\right)\right]\right. \\ &\left. + \sum_\alpha\left[\omega_0^+ (\beta_0^{\alpha,+})^\dagger\beta_0^{\alpha,+} + \sum_{n=1}^{\infty}\sum_{\gamma=\pm}\omega_n^{\alpha,\gamma}\left((\beta_n^{\alpha,\gamma})^\dagger\beta_n^{\alpha+} + (\tilde{\beta}_n^{\alpha,\gamma})^\dagger\tilde{\beta}_n^{\alpha,\gamma}\right)\right]\right) + \nu \end{aligned} \tag{2.78}$$

The normal ordering constant is the sum of the bosonic (2.63) and fermionic (2.78) contribution and the bosonic and fermionic normal ordering contributions cancel up to a finite number of terms

$$\begin{aligned}\nu &= \nu_b + \nu_f \\ &= \frac{1}{\alpha' p^+}\Big(2\alpha' p^+ + 4 \sum_{0<n<\sqrt{3}\alpha' p^+} \big(2\alpha' p^+ - \sqrt{n^2 + (\alpha' p^+)^2}\big)\Big)\end{aligned} \tag{2.79}$$

It is interesting to note that in the limit of vanishing $\alpha' p^+$ the finite normal ordering constant goes to zero, in agreement with the zero normal ordering constant for the light cone string in flat space. On the other hand for large $\alpha' p^+$, a large but finite number of modes contribute to the normal ordering constant[3].

In the light cone gauge the level matching constraint is obtained by considering the variation of the action with respect to $h^{\tau\sigma}$ in terms of the bosonic and fermionic modes one finds the constraint

$$\sum_{n=1}^{\infty} \omega_n^I \left(\alpha_n^{I\dagger}\alpha_n^I - \tilde{\alpha}_n^{I\dagger}\tilde{\alpha}_n^I\right) + \sum_{n=1}^{\infty}\sum_{\gamma=\pm} \omega_n^{\alpha,\gamma}\left((\beta_n^{\alpha,\gamma})^\dagger \beta_n^{\alpha+} - (\tilde{\beta}_n^{\alpha,\gamma})^\dagger \tilde{\beta}_n^{\alpha,\gamma}\right) = 0 \tag{2.80}$$

The spectrum of the light cone string has several features which distinguish it from the Penrose limit of $AdS_5 \times S^5$ discussed in [19]. First due to the absence of world sheet supersymmetry the bosonic and fermionic worldsheet energies ω_n^I and $\omega_n^{\alpha,\pm}$ are not the same. As discussed in appendix 2.A.1 this is a consequence of the fact that there are no "supernumerary" [56] supersymmetries in the pp-wave background we consider. A second difference is the presence of fermionic zero modes associated with the $\beta_0^{\alpha,-}, (\beta_0^{\alpha,-})^\dagger$ modes. These modes imply that even though the bosonic and fermionic oscillators have different energies that there are an equal number of bosonic and fermionic states at each energy level due to the 8 bosonic and 8 fermionic states created by $(\beta_0^{\alpha,-})^\dagger,\ \alpha = 1,2,3,4$. The third feature is the presence of a normal ordering constant and the dependence on $\alpha' p^+$. We discuss some possible reasons for the different properties of the lightcone string action in the next section. Presently, we have not been able to find an identification of the string spectrum with BMN like operators on the field theory side.

[3]The fact that the normal ordering constant is not divergent can be understood from the one loop ultraviolet finiteness of the Green-Schwarz string.

2.5 Discussion

In this chapter we have constructed a Penrose limit for type IIB warped $AdS_6 \times S^2$ solutions. We have constructed the limit by zooming in on a null geodesic at the center of the AdS_6 and along a great circle of the S^2. This construction only works if the geodesic is localized at the critical point $w = w_c$ of the function G on the two dimensional Riemann surface Σ. We constructed the pp-wave supergravity solution one obtains from taking the Penrose limit. After appropriate coordinate transformation the resulting pp-wave background is universal for all the cases we have considered. It would be interesting to consider a more general class of examples, such as the solutions including seven branes [3] or O7-branes [57] and confirm that the behavior of the pp-wave at the critical point is of the same form. Another observation is that all known regular solutions have a unique critical point w_c on the Riemann surface. This statement has not been proven but checked in all cases constructed in [58]. It is interesting that this critical point w_c is also relevant for supersymmetric embedding of D3-branes in the $AdS_6 \times S^2$ solutions which realizes BPS co-dimension two defects in the dual SCFT [49].

We presented the quadratic bosonic and fermionic world sheet actions of the Green-Schwarz string in this background and calculated the spectrum of bosonic and fermionic excitations of the string in the light cone gauge. The spectrum has some interesting features which are different from other cases such as the Penrose limit of $AdS_5 \times S^5$. Namely that the frequencies associated with the bosonic and fermionic creation and annihilation operators do not coincide, which is a consequence of the absence of any "extra" supersymmetries which would be associated with linearly realized supersymmetries. An additional new feature is the presence of fermionic zero modes, which makes sure that there are equal number of fermions and bosons at each level of the light cone hamiltonian.

One of the exciting features of the BMN correspondence was the identification of the light cone vacuum with a special class of BPS operators in the N=4 SYM theory, which can be viewed as a long spin chain. Furthermore, the action of creation operators on the string vacuum can be related to the insertion of impurities into the spin chain and diagonalizing the Hamiltonian. It is an interesting question whether a similar identification can be found for the light cone vacuum and excited states of the Green-Schwarz string in this plane wave background. This identification is more challenging for two reasons. While the Penrose limit identifies the light cone string vacuum with BPS states which satisfy $\Delta = 3J$, where J is a $U(1)$ inside the $SU(2)_R$ symmetry of the SCFT and therefore related to protected BPS multiplet, these theories are realized as conformal fixed points of long quiver theories and hence strongly coupled. For $d = 4, N = 4$ SYM the identification of these operators was possible in terms of the scalar fields of the SYM theory. For long quiver theories it is not clear how to construct these operators from the fundamental fields of the nodes of the quiver and the bi-fundamental matter. A class of such operators corresponding to "stringy" long meson and baryon operators was given in [52] which are constructed by taking products of the bi-fundamental hyper multiplets, from one end of the quiver to the other[4]. Since our null geodesic is at a fixed location on Σ is is natural to speculate that the BMN like operator would be associated with a single node in the quiver unlike the string like operators in [52] which stretch across the Riemann surface Σ. It would be interesting to investigate what mechanism on the field theory side singles out a specific node in the quiver theory. One could also investigate more general null geodesics which are not at a fixed point on Σ and consider the their Penrose limits. A very preliminary discussion of this issue can be found in appendix 2.B.

In [60, 61, 62] sphere partition functions and expectation values of BPS Wilson line operators for five dimensional SCFTs were calculated using localization techniques. For long quivers in the limit of large number of nodes and large gauge groups, the eigenvalues for the nodes where replaced by a continuous distribution and the saddle point of the path integral was reduced to an analogue electrostatic problem. It would be interesting to see whether these methods could be adapted to identify the BMN operators or whether the two dimensional electrostatic problem can be related to the light cone world sheet in some way and explain some of the curious properties of the light cone spectrum described in section 2.4.3.

[4]See [59] for a related construction in theories dual to six dimensional SCFTs.

2.A Supersymmetry of the plane wave background

The fermionic supersymmetry transformations in the string frame can be expressed in terms of the following two operators [63], that are also used to construct the part of the Green-Schwarz string action which is quadratic in space time fermions.

$$D^a_{\ b} = -\frac{1}{4\cdot 3!}H_{mnp}\Gamma^{mnp}(\sigma^3)^b_{\ a} - \frac{1}{4\cdot 3!}e^{\phi}(F_{mnp}-\chi H_{mnp})\Gamma^{mnp}(\sigma^1)^b_{\ a}$$
$$(D_M)^a_{\ b} = \partial_M + \frac{1}{4}\omega_{mn,M}\Gamma^{mn}\delta^b_a - \frac{1}{8}e^m_M H_{mnp}\Gamma^{np}(\sigma^3)^b_{\ a} + \frac{1}{8\cdot 3!}e^{\phi}(F_{npq}-\chi H_{npq})\Gamma^{npq}\Gamma_M(\sigma^1)^b_{\ a} \tag{2.81}$$

Here $\sigma^a, a = 1,3$ are the Pauli matrices which act on two component Majorna-Weyl spinors[5]. The supersymmetry transformations in the two component formalism are given by

$$\delta\lambda^a = D^a_b\epsilon^b, \qquad \delta\psi^a_M = (D_M)^a_b\epsilon^b \tag{2.82}$$

The susy transformations for the gravitino in the string frame are is different from the one in the Einstein frame given in [64], this is because the transformation $g_{MN} \to e^{\phi/2}g_{MN}$ induces a mixing of the dilatino supersymmetry transformation with the gravitino supersymmetry transformation [65]. The supersymmetry transformations (2.82) are given in the string frame.

The string world sheet action which is quadratic in fermions is given by

$$S^{(2)}_{RR} = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}\Big(h^{ij}\delta_{ab} - \epsilon^{ij}(\sigma^3)_{ab}\Big)\partial_i x^M\bar\theta^a\Gamma_M(D_j)^b_{\ c}\theta^c \tag{2.83}$$

where θ^a, $a = 1,2$ are the two ten dimensional Weyl-Majorana spinor world sheet fields with the same chirality as the supersymmetry transformation parameters ϵ^a in (2.82) . The operator $(D_j)^b_c$ is the pullback of the covariant derivative to the world sheet is related to

[5]The dictionary to go from the two component (with the matrices σ to formulation of using complex Weyl spinors spinor is given by $\epsilon = \epsilon^1 + i\epsilon^2$ and $\sigma^1\epsilon \to i\epsilon^*$, $\quad i\sigma^2 \to -i\epsilon$, $\quad \sigma^3\epsilon \to \epsilon^*$.

(2.81) by

$$(D_j)^b{}_c = \partial_j \delta^a_b + \partial_j x^M \Big(\frac{1}{4} \omega_{mn,M} \Gamma^{mn} \delta^b_a - \frac{1}{8} e^m_M H_{mnp} \Gamma^{np} (\sigma^3)^b{}_a + \frac{1}{8 \cdot 3!} e^\phi (F_{npq} - \chi H_{npq}) \Gamma^{npq} \Gamma_M (\sigma^1)^b{}_a \Big) \tag{2.84}$$

2.A.1 Integrability of supersymmetry transformations

To examine the supersymmetries of of the background (2.34), (2.46), we will first need to work out the spin connection.

$$X^\pm = \frac{1}{2}(T \pm X), \quad g_{+-} = g_{-+} = -2, \quad g^{+-} = g^{-+} = -\frac{1}{2} \tag{2.85}$$

The frame forms fields are given by

$$e^i_\mu dx^\mu = dx^i, \quad e^+_\mu dx^\mu = dx^+, \quad e^-_\mu dx^\mu = dx^- + \frac{1}{4} \sum_k m_k x_k^2 dx^+ \tag{2.86}$$

where $m_k = 1$ for all x^k except for $k = 6$ for which $m_6 = 3$. We can calculate the spin connection using the Cartan structure equations

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{2.87}$$

The the frame forms given above the only nontrivial one is

$$de^- = \frac{1}{2} \sum_k m_k x_k dx^k \wedge dx^+ \tag{2.88}$$

which implies that the connection 1-form is given by

$$\omega^-{}_k = \frac{1}{2} m_k x^k dx^+ \tag{2.89}$$

and we find that the only non vanishing components of the spin connection are given by

$$\omega_{+k,+} = -m_k x^k \tag{2.90}$$

Note that because (2.90) has two legs along the x^+ directions, the spin connection will not contribute to the fermion action (2.83), since the light-cone gauge $\Gamma^+ \theta^a = 0$ condition will make it vanish.

In the following we will present the explicit form of the supersymmetry transformations in the string frame two component formalism following from (2.82). For the antisymmetric tensor fields of the T_N example (2.46) the dilatino variation takes the form[6].

$$\delta\lambda^a = \Big(\Gamma^{67}(\sigma^3)^a_{\ b} + \Gamma^{68}(\sigma^1)^a_{\ b}\Big)\Gamma^+\epsilon^b \tag{2.91}$$

and the gravitino variations are given by

$$\begin{aligned}
\delta\psi^a_+ &= \partial_+\epsilon^a + \frac{1}{2}\sum_k m_k x^k\Gamma^{k+}\epsilon^a + \Gamma^{67}(\sigma^3)^a_{\ b}\epsilon^b - \frac{1}{2}\Gamma^{68}\Gamma^+\Gamma_+(\sigma^1)^a_{\ b}\epsilon^b \\
\delta\psi^a_i &= \partial_i\epsilon^a - \frac{1}{2}\Gamma^{68+}\Gamma_i(\sigma^1)^a_{\ b}\epsilon^b, \quad i = 1,2,3,4,5 \\
\delta\psi^a_6 &= \partial_6\epsilon^a + \Gamma^{7+}(\sigma^3)^a_{\ b}\epsilon^b - \frac{1}{2}\Gamma^{8+}(\sigma^1)^a_{\ b}\epsilon^b \\
\delta\psi^a_7 &= \partial_7\epsilon^a - \Gamma^{6+}(\sigma^3)^a_{\ b}\epsilon^b - \frac{1}{2}\Gamma^{678+}(\sigma^1)^a_{\ b}\epsilon^b \\
\delta\psi^a_8 &= \partial_8\epsilon^a + \frac{1}{2}\Gamma^{6+}(\sigma^1)^a_{\ b}\epsilon^b
\end{aligned} \tag{2.92}$$

It was pointed out in [56] that pp waves have $16+N_{sup}$ Killing spinors; 16 of which must occur in any background while the remaining $0 \le N_{sup} \le 16$ so-called "supernumerary" Killing spinors occur only in special backgrounds. After light-cone gauge fixing, only these extra spinors give rise to linearly-realized supersymmetries. Such linearly realized supersymmetries also act as two dimensional world sheet supersymmetries implying a degeneracy of the world sheet energies of bosonic and fermionic excitations. In our case, the sixteen ϵ which satisfy $\Gamma^+\epsilon = 0$ are the "automatic" supersymmetries. It's easy to check that for those the conditions (2.92) are integrable as only $\delta\psi_+$ is not automatically vanishing and can easily be integrated.

For the "supernumerary" Killing spinors the vanishing of the dilatino variation (2.91) imposes a projection condition on the supersymmetry transformation parameters ϵ. The integrability condition in the $i,+$ directions of $\delta\psi_M = 0$ becomes

$$\begin{aligned}
(\partial_+\partial_i - \partial_i\partial_+)\epsilon^a &= \frac{1}{2}\Gamma^{i+}\big(1 - \frac{1}{2}\Gamma_+\Gamma^+\big)\epsilon^a \\
&= \frac{1}{4}\Gamma^{i+}\Gamma^+\Gamma_+\epsilon^a \\
&= 0
\end{aligned} \tag{2.93}$$

[6] As discussed in section 4.3 any Penrose limit can be brought into the T_N form by a simple rotation in the 7-8 plane.

Where we used the identities

$$\Gamma^{+}\Gamma_{+} = 1 + \Gamma^{09}, \quad \Gamma_{+}\Gamma^{+} = 1 - \Gamma^{09} \tag{2.94}$$

Using the dilatino projection condition the $+, 7$ and $+, 8$ integrability conditions are also satisfied. However the $+, 6$ condition takes the following form

$$(\partial_{+}\partial_{6} - \partial_{6}\partial_{+})\epsilon^{a} = \frac{1}{2}\Gamma^{6+}\Big(\big(3 + \frac{1}{2}\Gamma_{+}\Gamma^{+}\big)\delta^{a}_{\ b} - 2i\Gamma^{78}\big(1 + \frac{1}{2}\Gamma_{+}\Gamma^{+}\big)(\sigma^{2})^{a}_{\ b}\Big)\epsilon^{b} \tag{2.95}$$

Where the factor in parenthesis is not a projector and consequently there are no "supernumerary" Killing spinors in our background. This is to be contrasted with the Penrose limit of $AdS_5 \times S^5$ where there are 16 additional supersymmetries [29].

2.B Null geodesics

In this appendix we consider null geodesics in the warped $AdS_6 \times S^2$ metric

$$ds^2 = f_6^2(-\cosh\rho^2 dt + d\rho^2 + \sinh\rho^2 ds^2_{S^4}) + f_2^2(d\theta^2 + \sin^2\theta d\phi^2) + 4\hat{\rho}^2\,|dw|^2\ , \tag{2.96}$$

We consider geodesics which are located at the center of AdS_6 at $\rho = 0$ and at the equator of the two sphere at $\theta = \frac{\pi}{2}$. These choices preserve the $U(1)$ of the two-sphere which we identify with a $U(1)$ inside the $SU(2)_R$ symmetry of the dual SCFT as well as the symmetries of the four-sphere of AdS_6, which means that the dual state has no angular momentum.

We consider however the possibility that the geodesic moves along the Riemann surface Σ, i.e. the coordinates t, ϕ, w and $\bar{w}$ depend on an affine parameter λ and the geodesic equation and null condition. With these choices the geodesic equation

$$\frac{d^2x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\ \nu\rho}\frac{dx^{\nu}}{d\lambda}\frac{dx^{\rho}}{d\lambda} = 0 \tag{2.97}$$

takes the following form

$$\frac{d}{d\lambda}\Big(\frac{dt}{d\lambda}f_6^2(w,\bar{w})\Big) = 0 \tag{2.98}$$

$$\frac{d}{d\lambda}\Big(\frac{d\phi}{d\lambda}f_2^2(w,\bar{w})\Big) = 0 \tag{2.99}$$

$$\frac{d^2w}{d\lambda^2} - \left(\frac{dt}{d\lambda}\right)^2\frac{\partial_{\bar{w}}f_6^2}{4\hat{\rho}^2} + \left(\frac{d\phi}{d\lambda}\right)^2\frac{\partial_{\bar{w}}f_2^2}{4\hat{\rho}^2} + \left(\frac{dw}{d\lambda}\right)^2\frac{\partial_w\hat{\rho}^2}{\hat{\rho}^2} = 0 \tag{2.100}$$

$$\frac{d^2\bar{w}}{d\lambda^2} - \left(\frac{dt}{d\lambda}\right)^2\frac{\partial_w f_6^2}{4\hat{\rho}^2} + \left(\frac{d\phi}{d\lambda}\right)^2\frac{\partial_w f_2^2}{4\hat{\rho}^2} + \left(\frac{d\bar{w}}{d\lambda}\right)^2\frac{\partial_{\bar{w}}\hat{\rho}^2}{\hat{\rho}^2} = 0 \tag{2.101}$$

Equations (2.98) and (2.99) can be integrated once

$$\frac{dt}{d\lambda} = \frac{c_6}{f_6^2}, \quad \frac{d\phi}{d\lambda} = \frac{c_2}{f_2^2} \tag{2.102}$$

The condition that the geodesic is null $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$, becomes

$$4\hat{\rho}^2\frac{dw}{d\lambda}\frac{d\bar{w}}{d\lambda} + \frac{c_2^2}{f_2^2} - \frac{c_6^2}{f_6^2} = 0 \tag{2.103}$$

As discussed in section 2.3.1 our goal is to describe dual operators close to the BPS bound $\Delta = 3J$ which leads to a condition on the coordinates t and ϕ

$$\frac{\partial\phi}{\partial t} = 3 \tag{2.104}$$

by identifying the generators of t and ϕ translations with the scaling and R symmetry generators. Using (2.102) this condition implies

$$3 = \frac{c_2}{c_6}\frac{f_6^2}{f_2^2} = 9\frac{c_2}{c_6}T^2 \tag{2.105}$$

Where (2.2) was used. It is easily confirmed that the choice used in the body of the chapter, where w is independent of λ and fixed at the critical point $w = w_c$, is indeed a null geodesic satisfying (2.104) for $c_6 = 3c_2$, due to the fact that $T|_{w=w_c} = 1$ and $\partial_w f_2|_{w=w_c} = \partial_w f_6|_{w=w_c} = 0$.

We briefly discuss the possibility of more general geodesics where $w, \bar{w}$ depend on the affine parameter λ. The condition (2.105) implies that T should be constant along such a geodesic. We have numerically searched for such geodesics for some examples and found that T varies along the null geodesics. This provides evidence that one has to drop either the condition (2.104), which may make the field theory identification of the Penrose limit challenging or consider more general trajectories on S^2. It would be very interesting to investigate the geodesics further and determine whether they are integrable. This is a quite challenging problem due to the complicated dependence of all the metric factors on the coordinates of Σ

2.C Plane wave limit for AdS_6 solution of massive type IIA

In [48] a solution of massive type IIA was found which is a warped product of AdS_6 over part of a four sphere S^4. This background was constructed by taking a near horizon limit of a D4-D8 semi-localized brane solution. In this appendix we briefly compare the Penrose limit for this supergravity background [66] to the one for the type IIB AdS_6 solutions described in the body of this chapter. The metric and four form antisymmetric tensor field strength of the solution are given by

$$\begin{aligned} ds^2 &= \frac{9}{2}W(\xi)\Big(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho ds^2_{S^4} + \frac{4}{9}\big(d\xi^2 + \sin^2\xi ds^2_{S_3}\big)\Big) \\ F_4 &= \frac{20\sqrt{2}}{3}(\cos\xi)^{\frac{1}{3}}\sin\xi^3 d\xi\wedge\omega_{S_3}, \qquad e^{\phi} = (\cos\xi)^{-\frac{5}{6}} \end{aligned} \tag{2.106}$$

Here the warp factor is given by $W(\xi) = \big(\cos\xi\big)^{\frac{1}{6}}$ in the string frame. The range of the warping coordinate $\xi\in[0,\pi/2]$ means that the three sphere warped over ξ produces only half of a four-sphere. The critical point of the warp factor of the three sphere is at $\xi=\pi/2$ and this corresponds to a Penrose limit considered in [66], which is analogous to the one discussed in this chapter. There is however one important difference since $\xi=\pi/2$ corresponds to the S^3 along the equator of the S^4 and therefore to the strong coupling region where the supergravity approximation breaks down. In contrast, the critical point in the type IIB AdS_6 is a regular point where the supergravity approximation is valid for large N. The same conclusion can be drawn from considering the T-dual of the Brandhuber-Oz solution [67] for which the holomorphic functions $A_\pm$ and G are [1]

$$A_\pm = \frac{a}{2}w^2 \mp bw, \qquad G = \frac{ab}{3}\big(1+(w+\bar{w})^3\big) \tag{2.107}$$

Here the condition $\partial_w G = 0$ is solved by $Im(w) = 0$ which under T-duality is mapped to $\xi = \pi/2$ on the type IIA side.

CHAPTER 3

Co-dimension 2 defect solutions in N=4, d=7 gauged supergravity

3.1 Introduction

The construction and study of extended conformal defects is an important subject in the investigation of superconformal field theories (SCFT). Defects are characterized by the broken and preserved symmetries. In a d-dimensional SCFT, a p-dimensional conformal defect preserves a $SO(p,2) \times SO(d-p)$ subgroup of the $SO(d,2)$ conformal group. The first factor is the conformal symmetry acting on the world volume of the defect and the second factor is the rotational symmetry in the transverse directions, which acts like a global symmetry on the degrees of freedom localized on the defect.

If the SCFT has a holographic dual it is interesting to look for the holographic description of such defects, which fall into two categories: First, a brane is placed in the bulk spacetime which ends on the boundary at the p dimensional defect [68, 69]. In a probe approximation the gravitational back reaction of such the brane is neglected, but the embedding is determined by solving the world volume equations of motion or the BPS-condition following from world volume kappa symmetry [70]. Second, a fully back reacted solution of the supergravity can be constructed using an ansatz of AdS and sphere factors warped over a base space (which can be a line or a Riemann surface with boundary). Solutions can either be constructed in lower dimensional gauged supergravities [71, 72] and in favorable circumstances be uplifted ten or eleven dimensions, or alternatively solutions can be constructed in ten or eleven dimensions [20, 22, 73, 74]. The former solutions are easier to obtain but the later are more general and in many cases give a top down understanding of the defects as backreacted solutions of intersecting brane systems, which allow us to identify the gauge theories, often of quiver type, which flow to the SCFTs.

In this note we consider the holographic description of $p = 4$ dimensional defects in $d = 6$ dimensional SCFTs. We construct solutions in a truncation of maximal $SO(5)$ gauged supergravity in seven dimensions with $U(1) \times U(1)$ gauge symmetry. These solutions are related by a double analytic continuation to supersymmetric black hole solutions. They are also closely related to compactifications of the seven dimensional theory on spindles - two dimensional compact surfaces with conical deficits which have been studied extensively in the past two years (see e.g.[75, 76, 77, 78, 79, 80]). Both constructions start with a ansatz $AdS_5 \times S^1$ warped over a real coordinate. Whereas the spindle solution the real coordinate takes values on a compact interval and the circle closes off at either end of the interval, in our case the real coordinate takes values on a real half-line and the geometry decompactifies to an asymptotic AdS_7 space. The solution therefore describes conformal a defect living inside a higher dimensional SCFT.

The structure of this note is as follows. In section 3.2 we describe the seven dimensional gauged supergravity and the relevant solutions which are obtained from double analytic continuation of black hole solutions. In section 3.3 we perform a regularity analysis based on the absence of conical singularities in the bulk and boundary and obtain a one parameter family of regular solutions, as well as solutions with conical singularities in the bulk related to spindles which have been actively investigated recently. In section 3.4 we perform some holographic calculations using the regular solutions, in particular we calculate the on-shell action of the solution, as well as the expectation value of the stress tensor and conserved R-symmetry currents. In section 3.5 we briefly discuss the uplift of the solution to eleven dimensions which is used to identify the R-symmetry currents of the six dimensional SCFT to which the seven dimensional gauge fields are dual. We close with a discussion of our results and leave some details of calculations to an appendix.

3.2 7-dim gauged supergravity

We consider a truncation of maximal $N = 4$, $SO(5)$ gauged supergravity in seven dimensions [81] with $U(1) \times U(1)$ gauge symmetry and two scalars [82, 83, 84]. There exists a consistent uplift of the seven dimensional solutions to eleven dimensional supergravity [82]. The solutions we consider are double analytic solutions of charged non-rotating black hole solutions [84, 83], where the S^5 factor is replaced by a AdS_5 factor and the time coordinate is replaced by a space-like compact circle coordinate. The black hole solution depends on a non-extremality parameter and two charges. The extremal solution preserves either half or a quarter of the thirty-two supersymmetries of the gauged supergravity theory for one or two nonzero charges respectively [84]. It was shown in [76] that the analytically continued extremal solutions also preserve the same amount of supersymmetry.

We follow the conventions of [76] to facilitate a comparison with their analysis. The action for the bosonic fields of $U(1) \times U(1)$ gauged supergravity in seven dimensions is given by

$$S = -\frac{1}{16\pi G_N} \int d^7x \sqrt{-g} \Big(R - g_c^2 V(\phi) - \frac{1}{2} \sum_{i=1}^{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{1}{4} e^{\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} F_1^2 \\ - \frac{1}{4} e^{-\sqrt{2}\phi_1 + \sqrt{\frac{2}{5}}\phi_2} F_2^2 \Big) \tag{3.1}$$

where $F_i = dA_i, i = 1, 2$ and the potential for the scalar fields is given by

$$V(\phi) = 2g_c^2 e^{-\sqrt{\frac{2}{5}}\phi_2} \Big(-8 + e^{\sqrt{10}\phi_2} - 8e^{\sqrt{\frac{5}{2}}\phi_2} \cosh \frac{\phi_1}{\sqrt{2}} \Big) \tag{3.2}$$

The solution given in [76] can be expressed in term of the following functions

$$\begin{aligned} h_i(y) &= y^2 + q_i, \quad i = 1, 2 \\ P(y) &= h_1(y) h_2(y) \\ Q(y) &= -y^3 - \mu y + g_c^2 h_1(y) h_2(y) \end{aligned} \tag{3.3}$$

and is given by

$$
\begin{aligned}
ds^2 &= \Big(yP(y)\Big)^{\frac{1}{5}}\Big\{ds^2_{AdS_5} + \frac{y}{4Q(y)}dy^2 + \frac{Q(y)}{P(y)}dz^2\Big\} \\
A_i &= \left(\frac{\sqrt{1-\frac{\mu}{q_i}}q_i}{h_i(y)}\right)dz, \quad i=1,2 \\
e^{\phi_1} &= \left(\frac{h_1(y)}{h_2(y)}\right)^{\frac{1}{\sqrt{2}}}, \quad e^{\phi_2} = \frac{(h_1(y)h_2(y))^{\frac{1}{\sqrt{10}}}}{y^{2\sqrt{\frac{2}{5}}}}
\end{aligned} \tag{3.4}
$$

It is easy to verify that the equations of motion following from the variation of the action (3.1) are satisfied for such a solution. Here q_1, q_2 are related to the charges and μ is a non-extremality parameter which we set to $\mu = 0$. This choice corresponds to a supersymmetric solution. We will also set $g_c = 1$ for simplicity. For these choices the solution with $q_1 = q_2 = 0$ corresponds to a unit radius AdS_7, using $AdS_5 \times S^1$ slicing coordinates.

3.3 Regularity analysis

In this section we present the conditions that regularity imposes on the solution. The analysis follows the general strategy employed in other cases of holographic description of defects. [85, 86, 87]. It is also closely related to the construction of holographic calculations of Renyi-entropies [88, 89], compactifications on spindles [76, 75, 77, 78] and related constructions [90, 91].

In [76] the solution presented in section 3.2 was used to construct a AdS_5 compactification of seven dimensional supergravity on a two dimensional compact space, a so-called spindle. A spindle is topologically a two sphere with two conical deficits at the north and south poles respectively. A spindle exists if the function $Q(y)$, defined in (3.3) has two real zeros and in between the zeros both $Q(y)$ and $P(y)$ are positive. The regularity, supersymmetry and the quantization of the deficit angle coming from a consistent interpretation of the uplift to eleven dimensions impose conditions on the parameters of the solution which were worked out in [76].

In our case the two dimensional space will be non-compact and we will look at the region from the largest positive zero of $Q(y)$ to infinity, which is a region where Q is positive. In the following we will investigate the regularity conditions imposed on the solution. For convenience we write out the functions which determined the regularity (recall we have set $\mu = 0$).

$$
\begin{aligned}
Q(y) &= -y^3 + (y^2+q_1)(y^2+q_2) = y^4 - y^3 + (q_1+q_2)y^2 + q_1 q_2 \\
P(y) &= (y^2+q_1)(y^2+q_2)
\end{aligned}
\tag{3.5}
$$

As $y \to \infty$ we approach an asymptotic AdS_7 region, with a six dimensional boundary. In this limit the metric takes the form

$$
\begin{aligned}
\lim_{y\to\infty} ds^2 &= y ds^2_{AdS_5} + y dz^2 + \frac{1}{4y^2} dy^2 + \cdots \\
&= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\Big(ds^2_{AdS_5} + dz^2 \Big) + \cdots
\end{aligned}
\tag{3.6}
$$

where we defined the Fefferman-Graham coordinate ζ as $y = 1/\rho$ and the dots denote sub-leading terms in y and ρ, which are determined in appendix 3.A. The metric is asymptotic to AdS_7, Since the z direction parameterizes a circle, the holographic boundary of the asymptotic AdS space is of the form $AdS_5 \times S^1$. The six dimensional metric on the boundary is given by

$$
\begin{aligned}
ds^2_6 &= \frac{dr^2 - dt^2 - \sum_{i=1}^3 dx_i^2}{r^2} + dz^2 \\
&= \frac{1}{r^2}\Big(d\zeta^2 - dt^2 - \sum_{i=1}^{3} dx_I^2 + r^2 dz^2 \Big)
\end{aligned}
\tag{3.7}
$$

which is conformal to $R^{1,5}$ if the coordinate z has periodicity 2π. For a different periodicity of z the boundary has a conical singularity at $r = 0$. In the standard formulation of AdS/CFT the boundary theory does not have dynamical gravity and hence a co-dimension two defect does not induce a conical deficit, as a cosmic string would in a gravitational theory. Consequently the condition of the absence of a conical deficit on the boundary fixes the periodicity of the S^1 coordinate z to be 2π.

We now seek conditions on q_1, q_2 such that there is at least one positive zero and that it is not a double zero. Once we have such a y_+, we can guarantee that in the range $[y_+, \infty)$

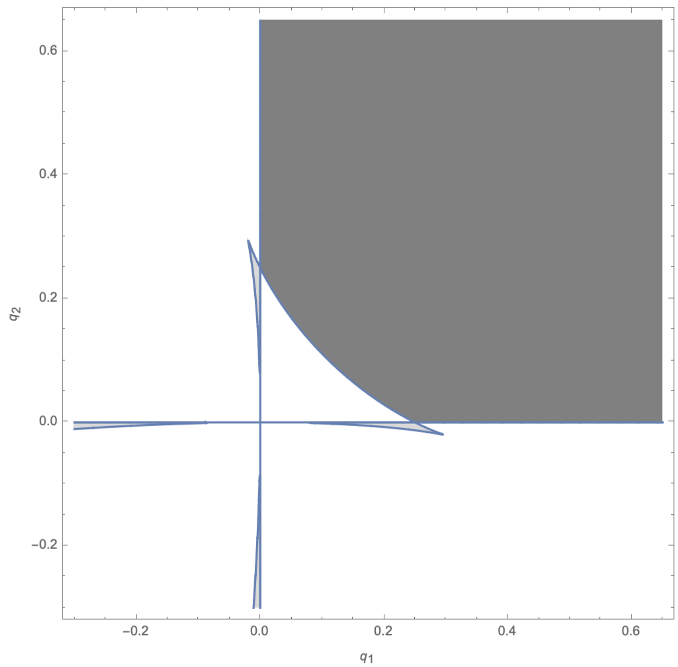

Figure 3.1: Sign of the discriminant (3.8) of the polynomial $Q(y)$ in the (q_1, q_2) plane

both metric functions $Q(y) > 0$ and $P(y) > y^3 > 0$ are positive and the metric is regular. An important quantity for the nature of the zeros of Q is the discriminant

$$D = q_1 q_2 \Big(16(q_1^4 + q_2^4) - 4(q_1^3 + q_2^3) - 64(q_1^3 q_2 + q_1 q_2^3) + 96 q_1^2 q_2^2 + 132(q_1^2 q_2 + q_1 q_2^2) - 27 q_1 q_2\Big) \tag{3.8}$$

Note that the vanishing of the discriminant implies the presence of a real double zero and for $D > 0$ we have either four or no real zeros whereas for $D < 0$ we have two real and two complex conjugate roots. We show a plot of the sign of the discriminant as a function of q_1, q_2 in figure 3.1, where locus of vanishing discriminant is represented by the blue curve and regions of positive discriminant are shaded grey.

We can use Descartes' rule of signs to show that in the region with either one or both q_1 and q_2 negative, we have two real roots in the (white) region where $D < 0$ and four real roots in the (grey) region where $D > 0$. In the region where both q_1, q_2 are positive we have two real zeros in the white region where $D < 0$ and no real zeros in the (dark grey) region, where $D > 0$. This implies that the dark grey region of charges is excluded since $Q(y)$ is never zero here and we will produce a naked singularity when y goes to zero and the Ricci scalar diverges.

Note that if $y = y_0$ is a double zero the metric will approach the following form near $y = y_0 + \rho$

$$ds^2 \sim (y_0 P(y_0)^{\frac{1}{5}} \Big(ds^2_{AdS_5} + \frac{y_0}{\gamma\rho^2} d\rho^2 + \frac{\gamma\rho^2}{P(y_0)} dz^2 \Big) \tag{3.9}$$

where $\gamma = \frac{1}{2} Q''(y) \mid_{y=y_0}$. This produces a singularity at $\rho = 0$. (We will see that we will never have to worry about this case for q_1, q_2 which satisfy the other regularity conditions)

Now we assume that we are in the allowed region of the q_1, q_2 plane and consider the $y \to y_+$ limit where y_+ is the largest positive zero of the function $Q(y)$. Letting $y = y_+ + \rho$, we have that

$$\begin{aligned} Q(y) &\approx Q'(y_+)\rho \\ P(y) &\approx P(y_+) = (Q(y_+) + y_+^3) = y_+^3 \end{aligned} \tag{3.10}$$

Plugging these into the metric (3.4) and defining the new radial coordinate $r = \rho^{\frac{1}{2}}$, we obtain

$$\Big(yP(y)\Big)^{\frac{1}{5}} \Big(\frac{y}{4Q(y)} dy^2 + \frac{Q(y)}{P(y)} dz^2 \Big) \sim \frac{y_+^{\frac{9}{5}}}{Q'(y_+)} \Big(dr^2 + \Big(\frac{Q'(y_+)}{y_+^2} \Big) r^2 dz^2 \Big) \tag{3.11}$$

As discussed above the absence of a conical deficit on the boundary fixes the periodicity of z to be 2π.

$$\frac{Q'(y_+)}{y_+^2} = \frac{1}{n} \tag{3.12}$$

gives us the metric on a half spindle which is regular everywhere except at $y = y_+$ where there is a conical deficit angle $2\pi(1 - \frac{1}{n})$.

Using the explict form of Q, we obtain the following constraint on the charges:

$$y_+ \Big(4y_+^2 - (3 + \frac{1}{n}) y_+ + 2(q_1 + q_2) \Big) = 0 \tag{3.13}$$

Note that the value of the largest root y_+ also depends on the charges q_1, q_2 and the resulting expression does not have a compact explicit expression. It is however clear that the condition will constrain the charges q_1, q_2 to lie on a on dimensional curve, which depends on the value of the conical deficit near the "half-spindle". In figure 3.2 we illustrate the curves of allowed

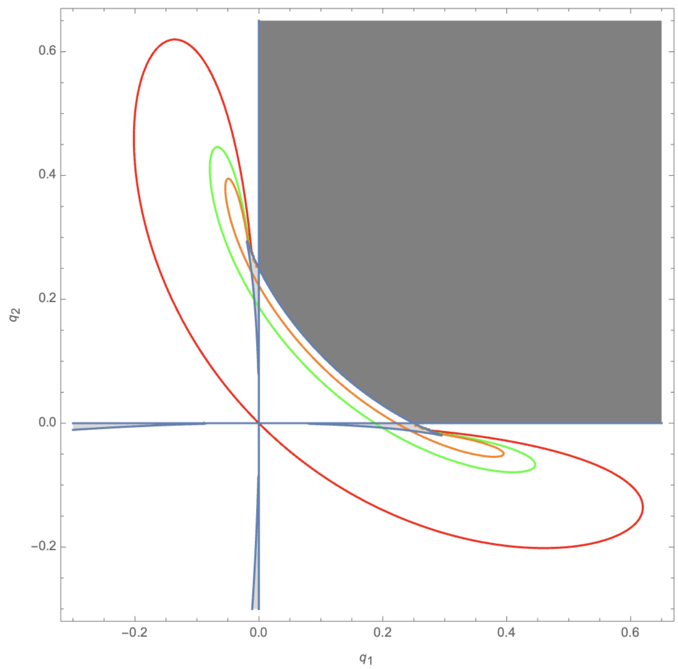

Figure 3.2: Allowed charges for different values of conical deficits: $n = 1$ (red) is the completely regular solution and two half-spindles with $n = 2$ (green) and $n = 3$ (orange)

charges for the case $n = 1$ which corresponds to a completely nonsingular spacetime, and $n = 2, 3$ which corresponds to spaces with conical deficits π and $\frac{2}{3}\pi$ respectively.

We note that there is no completely regular solution with one of the q_1 and q_2 charges set to zero. Hence all completely regular solutions preserve eight of the thirty two supersymmetries of the AdS_7 vacuum of the gauged supergravity. Consequently, the dual four dimensional defect preserves $N = 1, d = 4$ superconformal symmetry.

3.4 Holographic calculations

The solutions describe holographic co-dimension two defects in the six dimensional SCFT. In this section we calculate some holographic observables and discuss the implications for the defects imposed by regularity constraints. As discussed in section 3.3 the solution approaches AdS_7 asymptotically where the six dimensional boundary is $AdS_5 \times S^1$. While the boundary is conformal to $R^{1,5}$, it is simpler to work with the $AdS_5 \times S^1$ form of the boundary which is natural given the metric (3.4). All holographic calculations can be mapped to a flat boundary using the conformal mapping described in section 3.3.

3.4.1 On shell action

To evaluate the on shell action we have to add a Gibbons-Hawking term to the action (3.1) which is needed for a good variational principle. Using the trace of the Einstein equation the on-shell action can be expressed as

$$\begin{aligned} S_{on-shell} = &-\frac{1}{16\pi G_N}\int_M \sqrt{-g}\Big(\frac{2}{5}V - \frac{1}{10}e^{\sqrt{2}\phi_1+\sqrt{\frac{2}{5}}\phi_2}F_1^2 - \frac{1}{10}e^{-\sqrt{2}\phi_1+\sqrt{\frac{2}{5}}\phi_2}F_2^2\Big) \\ &+\frac{1}{8\pi G_N}\int_{\partial M}\sqrt{-h}\Theta \end{aligned} \tag{3.14}$$

The Gibbons-Hawking term is obtained from the trace of the second fundamental form

$$\Theta_{\mu\nu} = -\frac{1}{2}\Big(\nabla_\mu n_\nu + \nabla_\nu n_\mu\Big) \tag{3.15}$$

Here h_{ab} is the induced metric and n_μ is the outward pointing normal vector at the the cut-off surface. For the solution discussed in this chapter we choose the cutoff surface at large $y = y_c$. Furthermore since the spacetime closes off at the larges zero y_+ of $Q(y)$, the integral of the coordinate y in the action (3.14) is on $y \in [y_+, y_c]$. The on-shell action becomes

$$\begin{aligned} S_{on-shell} = &\frac{Vol_{AdS_5}}{16\pi G_N}\Big(-10y_c^3 + 10y_c^2 - 6(q_1+q_2)y_c - \frac{4}{5}(q_1+q_2) \\ &-\frac{2q_1q_2}{5y_+} - \frac{6(q_1+q_2)y_+}{5} - 2y_+^3 + \frac{4}{5}\frac{q_1^2}{q_1+y_+^2} + \frac{4}{5}\frac{q_2^2}{q_2+y_+^2}\Big) + o(y_c^{-1}) \end{aligned} \tag{3.16}$$

Here Vol_{AdS_5} is the regularized volume of AdS_5. The regularized on shell action is divergent in the limit $y_c \to \infty$ which removes the cutoff. In order to get a finite renormalized action we have to add covariant counter terms at the cutoff surface [92, 93, 94, 95]

$$\begin{aligned} S_{ct} &= \frac{1}{8\pi G_N}\int_{y=y_c}\sqrt{-h}\Big(W(\phi_1,\phi_2) + \frac{1}{8}R[h] + \frac{1}{64}(R[h]_{ab}R[h]^{ab} - \frac{3}{10}R[h]^2)\Big) \\ &= \frac{Vol_{AdS_5}}{16\pi G_N}\Big(10y_c^3 - 10y_c^2 + 6(q_1+q_2)y_c + \frac{5}{8}\Big) + o(y_c^{-1}) \end{aligned} \tag{3.17}$$

Here $R[h]_{ab}, R[h]$ are the Ricci tensor and scalar respectively calculated from the induced metric at the cutoff surface. $W(\phi)$ is the superpotential

$$W(\phi_1,\phi_2) = e^{2\sqrt{\frac{2}{5}}\phi_2} + 2e^{-\frac{1}{\sqrt{2}}\phi_1+\frac{1}{\sqrt{10}}\phi_2} + 2e^{-\frac{1}{\sqrt{2}}\phi_1-\frac{1}{\sqrt{10}}\phi_2} \tag{3.18}$$

Which is related to the scalar potential defined in (3.2) by

$$V = 2\sum_{i=1,2}\left(\frac{\partial W}{\partial \phi_i}\right)^2 - \frac{6}{5}W^2 \tag{3.19}$$

The renormalized action is the given by

$$\begin{aligned} S_{ren} &= \lim_{y_c\to\infty}\Big(S_{on-shell} + S_{ct}\Big) \\ &= \frac{Vol_{AdS_5}}{16\pi G_N}\Big(\frac{5}{8} - \frac{4}{5}(q_1+q_2) - \frac{2}{5}\frac{q_1q_2}{y_+} - \frac{6}{5}(q_1+q_2)y_+ - 2y_+^3 + \frac{4}{5}\frac{q_1^2}{q_1+y_+^2} + \frac{4}{5}\frac{q_2^2}{q_2+y_+^2}\Big) \end{aligned} \tag{3.20}$$

and when we include the relationship between the q_i's and y_+ implied by $Q(y_+) = 0$, we obtain a remarkably simple result:

$$S_{ren} = \frac{Vol_{AdS_5}}{16\pi G_N}\big(\frac{5}{8} - 2y_+^2\big) \tag{3.21}$$

As discussed above, our solutions describe holographic co-dimension 2 defects. In particular, when $q_1, q_2 = 0$ ($y_+ = 1$), we just obtain the AdS_7 vacuum which must be subtracted in order to identify the quantity above with the expectation value of the defect.

$$S_{ren} - S_{ren}|_{q_1,q_2=0} = \frac{Vol_{AdS_5}}{8\pi G_N}\big(1 - y_+^2\big) \tag{3.22}$$

Note that the volume of AdS_5 has to be regularized and will contain a scheme independent logarithmic divergent term. We interpret the coefficient (3.22) as the a central charge [96] associated with the four dimensional defect.

3.4.2 Stress tensor and currents

The expectation value of the renormalized holographic stress tensor was derived in [92, 94, 97] and can be obtained from the renormalized action

$$\langle T_{ab}\rangle_{ren} = \frac{2}{\sqrt{det(g_{(0)})}}\frac{\partial S_{ren}}{\partial g_{(0)}^{ab}} \tag{3.23}$$

Where $g_{(0)}$ is the asymptotic boundary metric in Fefferman-Graham coordinates.

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ab}(x,\rho)dx^a dx^b \tag{3.24}$$

with

$$g_{ab}(x,\rho) = g_{(0),ab} + \rho g_{(2),ab} + \rho^2 g_{(2),ab} + \rho^3 g_{(3),ab} + h_{(3),ab}\rho^3 \log\rho + \cdots \tag{3.25}$$

Here the asymptotic boundary is at $\rho = 0$. We defer the details of the calculation to the appendix 3.A but note one of the features of the expansion (3.25) is the absence of the logarithmic term, i.e. we find $h_{(3),ab}$ vanishes. The final result for the expectation value of the stress tensor is

$$\langle T_{ab}\rangle_{ren} dx^a dx^b = h_D ds_{AdS_5^2} - 5h_D ds^2_{S^1}, \qquad h_D = \Big(\frac{1}{18} - \frac{2}{15}(q_1+q_2)\Big) \tag{3.26}$$

which is traceless, indicating a vanishing six dimensional trace anomaly, which is in accordance with the absence of a logarithmic term in (3.25). The coefficient h_D can be called the defect's conformal dimension in analogy with other defects such as surface defects in four dimensions [98, 99, 100][1].

The gauge fields are dual to conserved currents and from the asymptotic behavior of A_i given in (3.4), we can read off the source and expectation value using the standard AdS/CFT dictionary.

$$\lim_{\rho\to 0} A_i = \Big(q_i\rho^4 + \cdots\Big)dz, \quad i = 1,2 \tag{3.27}$$

which implies that there is no source for the conserved currents and the expectation value of the currents is given by

$$\langle J_i\rangle = q_i dz \tag{3.28}$$

Since the currents are dual to the a $U(1)\times U(1)$ R-symmetry, we have a non-vanishing holonomy around the S^1. Recall that the regularity conditions derived in section 3.3 constrain the charges and hence the holonomies to a one parameter family.

Another holographic observable which can be calculated is the entanglement entropy in the presence of a defect (see e.g.[102, 103, 104, 105]). General arguments relate this quantity to the ones already calculated in this section. [106].

[1] See [101] for an in depth discussion of anomalies for co-dimension two conformal defects.

3.5 Uplift to 11 dimensions

The seven dimensional solutions presented in section 3.2 can be uplifted to solutions of eleven dimensional supergravity [82, 76]

$$ds_{11}^2 = \Omega^{\frac{1}{3}} ds_7^2 + \Omega^{-\frac{2}{3}} \Big(e^{-\sqrt{\frac{8}{5}}\phi_2} d\mu_0^2 + e^{\frac{\phi_1}{\sqrt{2}}+\frac{\phi_2}{\sqrt{10}}} (d\mu_1^2 + \mu_1^2 (d\phi_1 + A_1)^2) \\ + e^{-\frac{\phi_1}{\sqrt{2}}+\frac{\phi_2}{\sqrt{10}}} (d\mu_2^2 + \mu_2^2 (d\phi_2 + A_2)^2) \Big) \tag{3.29}$$

Where Ω is defined as

$$\Omega = e^{\sqrt{\frac{8}{5}}\phi_2} \mu_0^2 + e^{-\frac{\phi_1}{\sqrt{2}}-\frac{\phi_2}{\sqrt{10}}} \mu_1^2 + e^{\frac{\phi_1}{\sqrt{2}}-\frac{\phi_2}{\sqrt{10}}} \mu_2^2 \tag{3.30}$$

The coordinates $\phi_i, i = 1, 2$ are angular coordinates with periodicity 2π and the coordinates μ_i satisfy the constraint $\sum_{i=0}^{2} \mu_i^2 = 1$. The four form antisymmetric tensor flux is given by

$$*_{11} F_4 = \Big(2 \sum_{a=0}^{2} (X_a^2 \mu_a^2 - \Omega X_a) + \Omega X_a \Big) vol_7 + \frac{1}{2} \sum_{a=0}^{2} \frac{1}{X_a} (*_7 dX_a) \wedge d(\mu_a^2) \\ + \frac{1}{2} \sum_{a=1}^{2} \frac{1}{X_a^2} d(\mu_a^2) \wedge (d\phi_a + A_a) \wedge *_7 F_a \tag{3.31}$$

Here $*_{11}$ is the Hodge dual with respect to the eleven dimensional metric (4.19) whereas $*_7$ and vol_7 are the Hodge dual and volume with of to the seven dimensional metric (3.4) respectively. Note that the AdS_7 vacuum solution $q_1 = q_2 = 0$ gives the $AdS_7 \times S^4$ solution of eleven dimensional supergravity, dual to the vacuum of the six dimensional SCFT. Since the gauge fields $A_i, i = 1, 2$ twist the two angular coordinates ϕ_i in the metric (4.19) we can identify the gauge fields as dual to $U(1) \times U(1)$ R-symmetry currents inside the $SO(5)$ R-symmetry of the $N = (0, 2)$ six dimensional SCFT.

3.6 Discussion

In this note we constructed holographic solutions of $N = 4, d = 7$ gauged supergravity which describe four dimensional defects living inside a six-dimensional SCFT. The solutions are closely related to AdS_5 compactifications on spindles of the same theory [76]. The main difference lies in the fact that the two dimensional space transverse to the AdS_5 factor is compact in the spindle case, whereas in our case the space is noncompact and the solution has an asymptotic AdS_7 boundary. For the spindle [76] the two dimensional space is a sphere with two conical singularities at the north and south pole. The main result of the present chapter is that for the two charge extremal solutions it is possible to find completely regular solutions without any conical deficits in the bulk or on the asymptotic boundary. These solutions form a one parameter family in the space of extremal solutions. Another class of solutions are the "half-spindle" solutions of [90, 91] where the two dimensional space has the topology of the disk with one conical singularity in the center and M5-brane sources. It is possible to generalize our solutions to include a conical singularity in the bulk and in some sense this solution corresponds to a half-spindle on a plane instead of a disk since we have a non-compact space. It would be interesting to investigate whether a relation to the solutions [90, 91] exists. More generally speaking it would be interesting to see whether its possible to modify other holographic solutions of M-theory which describe AdS_5 compactifications, such as [107, 108, 109] to include a noncompact direction leading to an asymptotic AdS_7 boundary and hence describing a defect embedded in a higher dimensional theory.

The asymptotic boundary of the spacetime is $AdS_5 \times S^1$ which is conformal to $R^{1,5}$ under this map the circle parameterizes the angular direction of the transverse R^2. Since our solution have a non-vanishing expectation value of the $U(1) \times U(1)$ R-symmetry currents we can interpret the defect as a homolomy defect for the R-symmetry currents. Examples of such defects have been constructed for free field theories [110, 111, 112, 113, 114]. For surface defects in four dimensional $N = 4$ SYM such defects can be are related to probe brane and fully back reacted LLM geometries [115, 116, 117] and some observables were matched in [100]. It would be interesting to see whether such a relation exist for four dimensional defects in the six dimensional SCFT, in particular whether there is a field theory analogue of the regularity condition relating the two charges or holonomies that we found.

3.A Calculation of holographic stress tensor

In this section we calculate the expectation value of the holographic stress tensor following [94]. The metric (3.4) has the following large y expansion

$$ds^2 = \Big(\frac{1}{y^2} + \frac{1}{y^3} + \frac{5-4(q_1+q_2)}{5y^4} + \cdots\Big)\frac{dy^2}{4} + \Big(y + \frac{q_1+q_2}{5y} + \frac{-2q_1^2 - 2q_2^2 + q_1 q_2}{25y^3} + \cdots\Big)ds^2_{AdS_5}$$
$$+ \Big(y + \frac{q_1+q_2}{5y} + \frac{4q_1+4q_2}{5y^2} + \frac{-2q_1^2 - 2q_2^2 + q_1 q_2}{25y^3} + \cdots\Big)dz^2 \qquad (3.32)$$

where the dots denote terms which go faster to zero in the limit $y \to \infty$. The following coordinate transformation bring the metric into Fefferman-Graham form

$$y = \frac{1}{\rho} + \frac{1}{2} + \frac{5-16(q_1+q_2)}{80}\rho - \frac{q_1+q_2}{30}\rho^2 + o(\rho^3) \qquad (3.33)$$

Which takes the following form

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ab}(x,\rho)dx^a dx^b$$
$$g_{ab}(x,\rho) = g_{(0),ab} + \rho g_{(2),ab} + \rho^2 g_{(4),ab} + \rho^3 g_{(6),ab} + h_{(6),ab}\rho^3 \log\rho + \cdots \qquad (3.34)$$

The g_{ab} the takes the following form in Fefferman-Graham coordinates

$$g_{ab}(x,\rho)dx^a dx^b = \Big(1 + \frac{1}{2}\rho + \frac{1}{16}\rho^2 - \frac{2q_1+2q_2}{15}\rho^3 + \cdots\Big)ds^2_{AdS_5}$$
$$+ \Big(1 - \frac{1}{2}\rho + \frac{1}{16}\rho^2 + \frac{2(q_1+q_2)}{3}\rho^3 + \cdots\Big)dz^2 \qquad (3.35)$$

From which we can read off the $g_{(i),ab}, i = 0,2,4,5$. Note that there is no term logarithmic in ρ and hence $h_{(6),ab} = 0$ for the solution considered in this chapter. The expectation value of the holographic stress tensor is then given by

$$\langle T_{ab}\rangle = g_{(6),ab} - A_{(6),ab} + \frac{1}{24}S_{ab} \qquad (3.36)$$

Where A_6 and S are expressed in terms of $g_{(0)}, g_{(2)}, g_{(4)}$ and their derivatives. Explict expressions can be found in [94] and evaluating them for our background gives

$$\langle T_{ab}\rangle_{ren} dx^a dx^b = \Big(\frac{1}{18} - \frac{2}{15}(q_1+q_2)\Big)ds_{AdS_5^2} + \Big(-\frac{5}{18} + \frac{2}{3}(q_1+q_2)\Big)dz^2 \qquad (3.37)$$

CHAPTER 4

Holographic 6d co-dimension 2 defect solutions in M-theory

4.1 Introduction

In this chapter, we will discuss the uplift of solutions of seven dimensional gauged supergravity of [5] to eleven dimensions. These solutions describe holographic duals of co-dimension two defects in six dimensional SCFTs. The defects preserve four dimensional conformal symmetry as well as transverse rotational symmetry.

There are several approaches to constructing holographic duals of such defects. First, probe branes can be placed inside the AdS vacuum of the ten or eleven dimensional theory [68, 69]. The resulting embedding realizes the unbroken symmetries of the defect, which is localized at the intersection of the probe brane and the boundary of AdS. Second, one can construct solutions of the ten or eleven dimensional supergravity with the ansatz of a warped product of AdS and sphere factors which realize the defect symmetries and solve the supergravity Killing spinor equations to obtain a half-BPS solution. The second approach is generally quite involved and leads to “bubbling” solutions, see e.g. [115, 116, 118, 117, 73, 22, 23, 74, 119].

A more pedestrian approach is to consider a truncation of the ten or eleven dimensional theory to a lower dimensional gauged supergravity and construct solutions there. Generally, the ansatz and the BPS conditions following from the vanishing of the supersymmetry transformations are easier to solve in the lower dimensions than in higher dimensions. In many cases, such a lower dimensional solution can then be uplifted to the ten or eleven-dimensional supergravity and given a microscopic understanding by relating it to bubbling solutions.

In this chapter we will perform an uplift of the solutions found in [5] and embed it into a class of LLM solutions of M-theory [115, 108]. The seven dimensional solutions are constructed by warping $AdS_5 \times S^1$ over an interval with $U(1) \times U(1)$ gauge fields along the circle direction. They are related to hyperbolic (topological) black hole solutions by a double analytic continuation. These solutions have been used recently to construct spindle compactifications [75, 77, 76, 78, 79, 80, 120, 121, 122][1]. In this case, the warping coordinate takes values on a finite interval and the S^1 closes off at the ends of the interval. One ends up with a topological two sphere with two conical deficits $2\pi(1-\frac{1}{n_{n/s}})$, $n_{n/s} \in \mathbb{Z}$ at the north and south pole of the sphere. In our solution, the warping coordinate is a semi-infinite interval and the solution describes a co-dimension two defect in a six-dimensional SCFT. We note that the bulk gauge fields are dual to conserved currents in the CFT and the supergravity solution corresponds to turning on a source for these currents in the plane transverse to the defect. This means that these defects are twist/disorder defects where fields charged under these currents are picking up a phase when going around the defect. We list some examples of holographic co-dimension two defect solutions in supergravities in various dimensions [125, 86, 87].

The structure of the present chapter is as follows: In section 4.2, we review the defect solution of [5], in particular, the conditions for a completely non-singular solution with two non-vanishing gauge fields and a solution with a conical singularity in the bulk with only one gauge field turned on. In section 4.3 we use the formulas from [82] to lift the seven dimensional solution to eleven dimensions and investigate the nature of the conical singularity of the one charge solution. In section 4.4 we bring the uplifted one charge solution into canonical LLM form. Since our solution has an extra rotational symmetry the LLM solution can be described by an electrostatic potential by a change of variables and we determine the line charge distribution associated with the one charge solution. This allows us to identify the conical singularity with a "regular puncture" which was previously discussed in the context

[1] The hyperbolic black holes where also used to calculate charged Rényi entropies in holography, see e.g. [123, 124, 89].

of the LLM construction of duals of $d = 4,\ N = 2$ SCFTs by Gaiotto and Maldacena [108]. In addition, it allows us to construct generalized solutions with more complicated line charge distributions, some of which are completely regular. We calculate holographic observables namely the on-shell action and the vacuum-subtracted defect central charge. In an appendix 4.A, we construct a simple example for a co-dimension two defect in a $d = 6,\ N = (2, 0)$ SCFT using the six dimensional free tensor multiplet.

4.2 Seven dimensional solution

The seven dimensional supergravity theory is a truncation of the maximal $d = 7$ $SO(5)$ gauged supergravity, where we keep two scalars and two $U(1)$ gauge fields. The theory is defined by the Lagrangian [82]

$$S = \int d^7x\sqrt{-g}\Big(R - \frac{1}{2}\sum_{i=1}^{2}\partial_\mu\varphi_i\partial^\mu\varphi_i - g^2V - \frac{1}{4}\sum_{i=1}^{2}e^{\vec{a}_i\vec{\phi}}F_{(i)}^2\Big), \tag{4.1}$$

where we use

$$\vec{\alpha}_1 = (\sqrt{2}, \sqrt{\frac{2}{5}}), \quad \vec{\alpha}_2 = (-\sqrt{2}, \sqrt{\frac{2}{5}}) \tag{4.2}$$

to define

$$X_1 = e^{-\frac{1}{2}\vec{\alpha}_1\vec{\varphi}}, \quad X_2 = e^{-\frac{1}{2}\vec{\alpha}_2\vec{\varphi}}, \quad X_0 = (X_1X_2)^{-2}, \tag{4.3}$$

and the potential V can the be expressed as

$$V = -4X_1X_2 - 2X_0X_1 - 2X_0X_2 + \frac{1}{2}X_0^2. \tag{4.4}$$

We consider the following solution of the gauged supergravity which can be obtained by a double analytic continuation of charged black hole solutions [82, 126, 84]. These have been used to describe M5 branes wrapped on spindles [76], duals of $d = 4,\ N = 2$ Argyres-Douglass theories [90, 91], and co-dimension 2 defects [5] in this theory.

$$\begin{aligned} ds_7^2 &= \Big(yP(y)\Big)^{\frac{1}{5}}ds^2_{AdS_5} + \frac{y^{\frac{6}{5}}P(y)^{\frac{1}{5}}}{4Q(y)}dy^2 + \frac{y^{\frac{1}{5}}Q(y)}{P(y)^{\frac{4}{5}}}dz^2, \\ P(y) &= h_1(y)h_2(y), \quad Q(y) = -y^3 + \mu y^2 + \frac{1}{4}g^2h_1(y)h_2(y). \end{aligned} \tag{4.5}$$

The functions $h_i, i = 1, 2$ are given by

$$h_1 = y^2 + q_1, \quad h_2 = y^2 + q_2. \tag{4.6}$$

The scalar fields are expressed in terms of h_i as follows

$$X_1 = y^{\frac{2}{5}} \frac{h_2(y)^{\frac{2}{5}}}{h_1(y)^{\frac{3}{5}}}, \quad X_2 = y^{\frac{2}{5}} \frac{h_1(y)^{\frac{2}{5}}}{h_2(y)^{\frac{3}{5}}}, \tag{4.7}$$

and the two $U(1)$ gauge fields are given by

$$A_1 = \frac{\sqrt{1 - \frac{\mu}{q_1}} q_1}{h_1(y)} dz + a_1 dz, \quad A_2 = \frac{\sqrt{1 - \frac{\mu}{q_2}} q_2}{h_2(y)} dz + a_2 dz. \tag{4.8}$$

The constant μ is an extremality parameter and supersymmetric solutions are obtained by setting $\mu = 0$. A solution with both q_1, q_2 nonzero will preserve one-quarter of the supersymmetry and, as we shall review in the next section, completely nonsingular solutions are possible. Setting $q_2 = 0$ produces a solution that preserves half the supersymmetry of the seven dimensional gauged supergravity but such a solution suffers from conical singularities. For the gauge field to be non-singular at the location $y = y_+$, where the space closes off, we have to choose a_1 and a_2 such that

$$A_1(y_+) = A_2(y_+) = 0. \tag{4.9}$$

In the following we set the coupling $g = 2$. As discussed below, this implies that the asymptotic boundary $AdS_5 \times S^1$ is conformal to $\mathbb{R}^{1,5}$ without a conical deficit provided z has standard periodicity 2π.

4.2.1 Regular two charge solution

The case of completely regular solutions was analyzed in [5]. These solutions were constructed by allowing the warping coordinate y to take values in the semi-infinite interval $[y_+, \infty]$ where y_+ is the largest zero of $Q(y)$ defined in (4.5). The existence of such a positive y_+, which produces no double zero and a regular metric everywhere, is guaranteed as long as we place conditions (discussed in [5]) on the signs of the charges q_1, q_2 as well as the discriminant of the polynomial $Q(y)$.

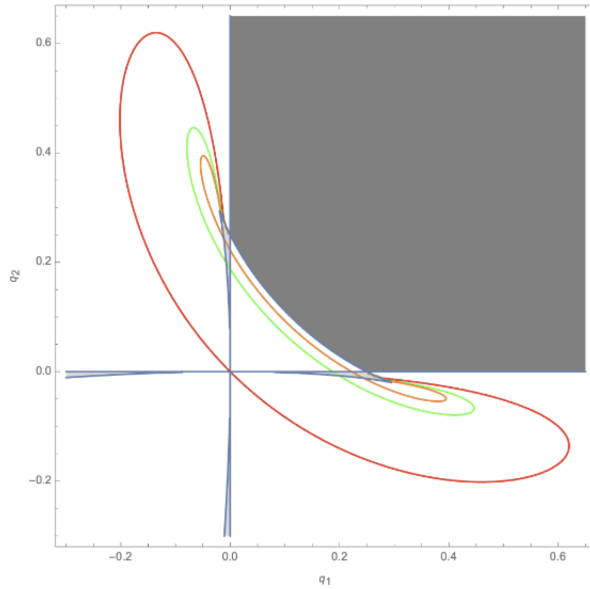

Figure 4.1: **Regular two charge solutions.** Allowed charges for different values of conical deficits: $n = 1$ (red) is completely regular. $n = 2$ (green), $n = 3$ (orange) correspond to the first two half-spindle solutions. The dark grey portion is the disallowed region where $Q(y)$ has no real zeros.

This interval produces a non-compact space and therefore, unlike in the spindle construction, we approach the asymptotic AdS_7 region as $y \to \infty$. In this limit the metric (4.5) takes the form

$$\begin{aligned}\lim_{y\to\infty} ds_7^2 &= y ds^2_{AdS_5} + y dz^2 + \frac{1}{4y^2} dy^2 + ... \\ &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(ds^2_{AdS_5} + dz^2\right) + ..., \end{aligned} \tag{4.10}$$

where we make the change of coordinates $y = 1/\rho$ and the dots denote subleading terms. Note that the boundary of this space is of the form $AdS_5 \times S^1$ which is conformal to $\mathbb{R}^{1,5}$ with no conical defect as long as the coordinate z parameterizing the S^1 has periodicity 2π.

Having fixed the periodicity of z, we can look at the metric in the region $y \to y_+$. Letting $y = y_+ + \rho$, we have that $Q(y) \approx Q'(y_+)\rho$ and $P(y) \approx P(y_+) = y_+^3$ so that the metric (4.5) takes the form

$$(yP(y))^{1/5}\left(\frac{y}{4Q(y)}dy^2+\frac{Q(y)}{P(y)}dz^2\right)\approx\frac{y^{9/5}}{Q'(y_+)}\left(dr^2+\left(\frac{Q'(y_+)}{y_+^2}\right)r^2dz^2\right), \tag{4.11}$$

where we define the new radial coordinate $r=\rho^{1/2}$. Notice that at $r=0$ ($y=y_+$) the z-circle shrinks to zero size and the space closes off. At this location, we may fix the values of q_1, q_2 such that we either have a regular solution or a $\mathbb{R}^2/\mathbb{Z}_k$ singularity by setting:

$$\frac{Q'(y_+)}{y_+^2}=\frac{1}{k}. \tag{4.12}$$

The values $k>1$ give the metric with deficit angle $2\pi(1-1/k)$ at $y=y_+$. Using the explicit form of the function $Q(y)$, we can express the constraint (4.12) as

$$y_+\left(4y_+^2-(3+1/k)y_+ +2(q_1+q_2)\right)=0. \tag{4.13}$$

Note that the root y_+ itself depends on the charges q_1, q_2 however we can clearly see that the above condition will constrain them to lie along a different one dimensional curve for each choice of k. In figure 4.1, we have plotted the first three of these families of solutions in the q_1, q_2-plane.

4.2.2 One charge solution

The solution with two nonzero charges is quarter BPS, i.e. preserves eight of the original thirty-two supersymmetries of the $d=7$ gauged supergravity. Our goal is to obtain solutions which fit into the LLM solutions in 11 dimensions, which preserve sixteen supersymmetries. We will have to set one of the two charges to zero in order to produce a half BPS solution. In the following we will set q_2 to zero. The metric components of (4.5) in the y and z direction become (recall that we have set $g=2$)

$$ds_7^2=\frac{(y^2+q_1)^{\frac{1}{5}}}{4y^{\frac{2}{5}}\left(y^2+q_1-y\right)}dy^2+\frac{y^{3/5}\left(y^2+q_1-y\right)}{(y^2+q_1)^{\frac{4}{5}}}dz^2+\cdots. \tag{4.14}$$

The larger zero of y is located at

$$y_c = \frac{1}{2}\big(1 + \sqrt{1-4q_1}\big). \tag{4.15}$$

With the following change of variable

$$y = y_c + \frac{1}{4}r^2, \tag{4.16}$$

the metric near $y \sim y_c$, i.e. $r \sim 0$ behaves as follows

$$ds^2 \sim \frac{1}{2^{\frac{9}{5}}\sqrt{1-4q_1}(1+\sqrt{1-4q_1})^{\frac{1}{5}}}\Big(dr^2 + (1-4q_1)r^2dz^2\Big) + \cdots\,. \tag{4.17}$$

Consequently, for nonzero q_1 there is a conical singularity in the bulk of the spacetime, whereas $q_1 = 0$ corresponds to the AdS_7 vacuum. For a $\mathbb{R}^2/\mathbb{Z}_k$ conical singularity with deficit $2\pi(1-\frac{1}{k})$, the charge q_1 is given by

$$\frac{1}{k} = \sqrt{1-4q_1}. \tag{4.18}$$

In seven dimensions a conical singularity in the bulk is problematic. In some cases uplifting a singular solution of lower dimensional supergravity to ten or eleven dimensions leads to a non-singular solution, in other cases the solution may have a well defined interpretation in terms of branes.

4.3 Uplift to eleven dimensions

A solution of seven dimensional gauged supergravity can be uplifted to eleven dimensional supergravity [82], the metric and the four-form antisymmetric tensor field strength take the following form

$$\begin{aligned}
ds_{11}^2 &= \Omega^{\frac{1}{3}}ds_7^2 + \frac{1}{g^2\Omega^{\frac{2}{3}}}\Big\{\frac{d\mu_0^2}{X_0} + \sum_{i=1}^{2}\frac{1}{X_i}\Big(d\mu_i^2 + \mu_i^2(d\phi_i + gA_i)^2\Big)\Big\},\\
*_{11}F_4 &= 2g\sum_{\alpha=0}^{2}\Big(X_\alpha^2\mu_\alpha^2 - \Omega X_\alpha\Big)\epsilon_7 + g\Omega X_0\epsilon_7 + \frac{1}{2g}\sum_{\alpha=0}^{2} *_7 d\ln X_\alpha \wedge d(\mu_\alpha^2)\\
&\quad + \frac{1}{2g^2}\sum_{i=1}^{2}\frac{1}{X_i^2}d(\mu_i^2)\wedge(d\phi_i + gA_i)\wedge *_7F_i,
\end{aligned} \tag{4.19}$$

where $F_i = dA_i$ and $*_7$ is the Hodge dual with respect to the seven dimensional metric (4.5) and $*_{11}$ the Hodge dual with respect to the eleven dimensional metric (4.19). $\phi_i, i = 1, 2$ are two angular coordinates with period 2π and the variables $\mu_\alpha, \alpha = 0, 1, 2$ parametrize a two sphere

$$\mu_0^2 + \mu_1^2 + \mu_2^2 = 1 \tag{4.20}$$

and the warp factor Ω is given by

$$\Omega = X_0\mu_0^2 + X_1\mu_1^2 + X_2\mu_2^2. \tag{4.21}$$

We will parameterize the μ_i in the following way

$$\mu_0 = \sin\alpha\cos\theta, \quad \mu_1 = \sin\theta, \quad \mu_2 = \cos\alpha\cos\theta. \tag{4.22}$$

4.3.1 Two charge solution

With our μ_i parameterization, the warp factor Ω becomes

$$\Omega = \frac{(y^2+q_1)^{\frac{2}{5}}(y^2+q_2\sin^2\alpha)\cos^2\theta}{y^{\frac{8}{5}}(y^2+q_2)^{\frac{3}{5}}} + \frac{y^{\frac{2}{5}}(y^2+q_2)^{\frac{2}{5}}\sin^2\theta}{(y^2+q_1)^{\frac{3}{5}}}. \tag{4.23}$$

As discussed in section 4.2.1, $y^2 + q_1 > 0$ and $y_2 + q_2 > 0$ for $y \geq y_c$ for the solutions which satisfy the regularity conditions. Hence, if the seven dimensional metric is regular then the eleven dimensional metric is also regular and describes a quarter-BPS co-dimension two defect in M-theory.

4.3.2 One charge solution

The uplift of the $q_2 = 0$ solution given in section 4.2.2 and the eleven dimensional metric for the defect solution takes the following form

$$\begin{aligned} ds_{11}^2 = \kappa^{\frac{2}{3}}\Big\{ & y^{\frac{1}{3}}(y^2+q_1\cos^2\theta)^{\frac{1}{3}}ds^2_{AdS_5} + \frac{y^{\frac{4}{3}}\cos^2\theta}{4(y^2+q_1\cos^2\theta)^{\frac{2}{3}}}ds^2_{S_2} + \frac{(y^2+q_1\cos^2\theta)^{\frac{1}{3}}}{4y^{\frac{2}{3}}}d\theta^2 \\ & + \frac{(y^2+q_1\cos^2\theta)^{\frac{1}{3}}}{4y^{\frac{2}{3}}(y^2-y+q_1)}dy^2 + \frac{y^{\frac{1}{3}}(y^2+q_1\cos^2\theta)^{\frac{1}{3}}\left(y^2-y+q_1\right)}{(y^2+q_1)}dz^2 \\ & + \frac{(y^2+q_1)\sin^2\theta}{4y^{\frac{2}{3}}(y^2+q_1\cos^2\theta)^{\frac{2}{3}}}(d\phi_1 + \frac{2q_1}{y^2+q_1}dz + 2a_1dz)^2\Big\}, \end{aligned} \tag{4.24}$$

where we used the parameterization (4.22) for $\mu_\alpha, \alpha = 0, 1, 2$. The coordinates α and ϕ_2 will parameterize the round two sphere

$$ds^2_{S_2} = d\alpha + \sin^2\alpha \, d\phi_2^2. \tag{4.25}$$

The uplifted metric therefore geometrically realizes an $SU(2)$ symmetry, which will be interpreted as an R-symmetry from the perspective of the four dimensional $N = 2$ defect theory. Using the uplift formula (4.19), one obtains for the four form

$$F_4 = \kappa \Big\{ vol(S_2) \wedge (f_{\phi_1} d\phi_1 + f_z dz) \wedge d\theta + vol(S_2) \wedge (g_{\phi_1} d\phi_1 + g_z dz) \wedge dy \Big\} \tag{4.26}$$

with

$$\begin{aligned} f_{\phi_1} &= \frac{(y^2+q_1)(3y^2+q_1\cos^2\theta)\cos^2\theta\sin\theta}{8(y^2+q_1\cos^2\theta)^2}, \\ f_z &= \frac{\big(q_1 + a_1(y^2+q_1)\big)(3y^2+q_1\cos^2\theta)\cos^2\theta\sin\theta}{4(y^2+q_1\cos^2\theta)^2}, \\ g_{\phi_1} &= \frac{q_1 y \cos^3\theta\sin^2\theta}{4(y^2+q_1\cos^2\theta)^2}, \\ g_z &= \frac{q_1 y(1+a_1\sin^2\theta)\cos^3\theta}{2(y^2+q_1\cos^2\theta)^2}. \end{aligned} \tag{4.27}$$

Note that in contrast to solutions where y takes values on a compact interval, in our case the region $y \to \infty$ is part of the spacetime and corresponds to the asymptotic $AdS_7 \times S^4$ region. In this limit, the metric and the four form behave as follows

$$\begin{aligned} ds^2 &\sim \kappa^{\frac{2}{3}} \Big(y ds_{AdS_5} + \frac{1}{4}\cos^2\theta ds^2_{S^2} + \frac{1}{4}d\theta^2 + \frac{1}{4y^2}dy^2 + y dz^2 + \frac{1}{4}\sin^2\theta(d\phi_1 + 2a_1 dz)^2 + \mathcal{O}(1/y), \\ F_4 &\sim \kappa \, \frac{3}{8}\cos^2\theta\sin\theta \; vol(S^2) \wedge (d\phi_1 + 2a_1 dz) \wedge d\theta + \mathcal{O}(1/y). \end{aligned} \tag{4.28}$$

The angular coordinates z, ϕ_1 have period 2π. We can define a new angular coordinate $\tilde{\phi} = \phi_1 + 2a_1 z$, which has standard period 2π for $a_1 = k/2, k \in \mathbb{Z}$. The flux of the four form on the S^4 is given by

$$\begin{aligned} \int F_4 &= \kappa \frac{3}{8} \int_{S^2} vol(S^2) \int_0^\pi d\theta \cos^2\theta \sin\theta \int_0^{2\pi} d\tilde{\phi} \\ &= 2\pi^2\kappa = \frac{16}{g^3}\pi^2\kappa \end{aligned} \tag{4.29}$$

where we restored the gauge coupling g. The condition for charge quantization for the four form F_4 in M-theory is given by

$$\frac{1}{(2\pi)^3\ell_p^3}\int F_4 = N, \quad N \in \mathbb{Z}, \tag{4.30}$$

where N can be interpreted as the number of fivebranes leading to the $AdS_7 \times S^4$ vacuum in the near horizon limit and hence, the constant κ in the uplift formula is

$$\kappa = \frac{\pi}{2} g^3 N\, \ell_p^3. \tag{4.31}$$

Recall that the seven dimensional metric for the one charge solution has a conical singularity at $y = y_c$ (4.15). Defining $y = y_c + r^2$ and expanding around $r = 0$, the eleven dimensional metric takes the following form

$$\begin{aligned} ds^2 \sim \frac{(y_c^2 + q_1\cos^2\theta)^{\frac{1}{3}}}{y_c^{\frac{2}{3}}} \Big\{ & y_c ds^2_{AdS_5} + \frac{y_c^2\cos^2\theta}{4(y_c^2+q_1\cos^2\theta)} ds^2_{S^2} + \frac{d\theta^2}{4} + \frac{dr^2}{\sqrt{1-4q_1}} \\ & + \sqrt{1-4q_1}\, r^2 dz^2 + \frac{\sqrt{1-4q_1}\sin^2\theta}{1+\sqrt{1-4q_1}-2q_1\sin^2\theta}\Big(d\phi_1 + (1-\sqrt{1-4q_1}+2a_1)dz\Big)^2 \Big\} + \mathcal{O}(r^2) \end{aligned} \tag{4.32}$$

There are three potential conical singularities in the θ, z, r, ϕ_1 part of the metric. At $\theta = \pi/2$ the two sphere shrinks to zero size in a smooth way, and at $r = 0$ there is a $\mathbb{R}^2/\mathbb{Z}_k$ conical singularity if $1/k = \sqrt{1-4q_1}$ which is inherited from the seven dimensional metric. At $\theta = 0$ we can define a new angular variable

$$\hat{\phi} = \phi_1 + \big(1 + 2a_1 - \frac{1}{k}\big) z. \tag{4.33}$$

As argued above from the regularity in the asymptotic $AdS_7 \times S^4$ limit, $2a_1$ is an integer and both ϕ_1 and z have period 2π. Hence the new angular variable $\hat{\phi}$ has period $2\pi/n$ and the metric displays a $\mathbb{R}^4/\mathbb{Z}_k$ singularity near the point $r = 0, \theta = 0$.

4.4 Lin-Lunin-Maldacena solutions

The M-theory LLM solutions [115] are examples of "bubbling" supergravity solutions which holographically are the deformation of the $d = 6,\ N = 2$ SCFT by half-BPS states of dimension $\Delta \sim N^2$. In the same paper a double analytic continuation related these solutions to a general solution of eleven dimensional supergravity with $SO(2,4) \times SU(2) \times U(1)$ symmetry. These solutions have been used to find holographic duals [108] of a large class of $d = 4,\ N = 2$ SCFTs constructed in [127]. The goal of the present section is to show that our uplifted solution can be written in the LLM form. We briefly review the salient features of the LLM solution [108]. The metric is given by an $AdS_5 \times S^2$ warped over a four dimensional space, which is a $U(1)$ fibration over a three dimensional base space spanned by coordinates ξ, x_1, x_2

$$\begin{aligned} ds^2_{11,LLM} = \kappa_{11}^{\frac{2}{3}} e^{2\lambda} \Big\{ & 4 ds^2_{AdS_5} + \xi^2 e^{-6\lambda} ds^2_{S^2} + \frac{4}{1 - \xi \partial_\xi D} (d\chi - \frac{1}{2} v_i dx^i)^2 \\ & - \frac{\partial_\xi D}{\xi} \big(d\xi^2 + e^D (dx_1^2 + dx_2^2) \big) \Big\}. \end{aligned} \tag{4.34}$$

The four form field strength takes the following form

$$F_4 = 2\kappa_{11} vol(S_2) \wedge \Big((d\chi + v) \wedge d(\xi^3 e^{-6\lambda}) + (\xi - \xi^3 e^{-6\lambda}) dv - \frac{1}{2} \partial_\xi e^D dx_1 \wedge dx_2 \Big). \tag{4.35}$$

The dimensionful quantity $\kappa_{11} = \frac{\pi}{2} \ell_p^3$ is the standard choice, note that our κ has both N and g in it, this way we have to absorb the charges to D which makes the comparison easier to [108]. Therefore, we identify $\kappa = g^3 N \kappa_{11}$.

The solution is completely determined in terms of a single function $D(\xi, x_1, x_2)$

$$e^{-6\lambda} = \frac{-\partial_\xi D}{\xi(1 - \xi \partial_\xi D)}, \qquad dv = \sum_i v_i dx^i, \quad v_1 = -\partial_{x_2} D, \quad v_2 = \partial_{x_1} D. \tag{4.36}$$

The function $D(\xi, x_1, x_2)$ satisfies the partial differential equation of Toda type

$$\big(\partial_{x_1}^2 + \partial_{x_2}^2 \big) D + \partial_\xi^2 e^D = 0. \tag{4.37}$$

Our goal is to find the LLM form of our uplifted solution (4.24). We note that our solution has an additional rotational symmetry in the x_1, x_2 plane which allows us to write the metric as

$$\begin{aligned} ds^2_{11,LLM} = \kappa_{11}^{\frac{2}{3}} e^{2\lambda} \Big\{ & 4 ds^2_{AdS_5} + \xi^2 e^{-6\lambda} ds^2_{S^2} + \frac{4}{1 - \xi \partial_\xi D} (d\chi - \frac{\rho}{2} \partial_\rho D \, d\beta)^2 \\ & - \partial_\xi D \xi \big(d\xi^2 + e^D (d\rho^2 + \rho^2 d\beta^2) \big) \Big\}. \end{aligned} \tag{4.38}$$

As we will review in section 4.4.2, this additional symmetry allows for a reformulation in terms of an electrostatic problem [108, 128, 129, 130, 131] which replaces the Toda equation with a linear Laplace equation.

4.4.1 Map to LLM

In order to find the map of the metric (4.24) to an LLM form (4.38), we note that the metric (4.24) depends on the two coordinates y, θ while the LLM metric with the additional $U(1)$ isometry also depends on two coordinates ξ, ρ. In addition there are two angular coordinates ϕ, z which have to be related to χ, β.

By comparing the AdS_5 and S^2 parts of the two metrics, we can determine the radial coordinate ξ in terms of y, θ as well as an expression for λ in (4.38)

$$\xi = Ny\cos\theta, \qquad e^{6\lambda} = N^2\, y(y^2 + q_1\cos^2\theta), \tag{4.39}$$

and we can choose an ansatz for the second radial coordinate ρ

$$\rho = \sin\theta\; g(y) \tag{4.40}$$

for some function $g(y)$. Using these relations, the $g_{\xi\xi}, g_{\rho\rho}$ and the $g_{\xi\rho}$ components of (4.38) can be expressed in terms of the y, θ coordinates and be matched to the uplifted metric (4.24). This gives us a differential equation for the function $g(y)$

$$\frac{d}{dy}\ln g(y) = \frac{y}{y^2 - y + q_1}, \tag{4.41}$$

which can be integrated to obtain

$$g(y) = \left(y - \frac{1}{2}(1 + \sqrt{1-4q_1})\right)^{\frac{1}{2}\left(1+\frac{1}{\sqrt{1-4q_1}}\right)} \left(y - \frac{1}{2}(1 - \sqrt{1-4q_1})\right)^{\frac{1}{2}\left(1-\frac{1}{\sqrt{1-4q_1}}\right)} \tag{4.42}$$

as well as an expression for the function D expressed as a function of y

$$e^D = N^2\frac{(y^2 - y + q_1)}{g(y)^2}. \tag{4.43}$$

The function D satisfies the Toda equation (4.37), which can be verified using the mapping (4.39). The mapping is complete by finding the identification of angular variables

$$z = c_1\chi + c_2\beta, \qquad \phi = c_3\chi + c_4\beta. \tag{4.44}$$

Matching the angular components of the metric gives the following relations for $c_i, i = 1, \cdots, 4$

$$c_1 = \pm 1, \quad c_2 = 0, \quad c_3 = \mp 2(1 + a_1), \quad c_4 = \mp 1. \tag{4.45}$$

To match the metric components, both signs in (4.45) are possible, however, matching the four form components (4.26) and (4.35) selects the upper signs.

For the choice of the upper signs in (4.45), the relations for the angular variables become

$$z = \chi, \quad \phi = -\beta - 2(1 + a_1)\chi, \tag{4.46}$$

which means that the periodicity of both sets of angular variables is 2π.

4.4.2 $U(1)$ symmetric solutions

The LLM metric (4.38) has an additional $U(1)$ symmetry associated with shifts of the angle β. For such geometries, it is possible to find an implicit change of variables that turns the nonlinear Toda equation (4.37) into a linear Laplace equation. This idea goes back to the paper by Ward [132] and has been applied to the LLM solution in [108, 128, 129, 130, **?**]. Note that in some of these papers the $U(1)$ circle is compactified to obtain a type IIA solution from the M-theory one.

We map the LLM coordinates ξ, ρ to the new ones r, η and relate the function D to an electrostatic potential

$$\rho^2 e^{D(\xi,\rho)} = r^2, \quad \xi = r\partial_r V \equiv \dot{V}, \quad \ln \rho = \partial_\eta V \equiv V'. \tag{4.47}$$

The function $V(r, \eta)$ satisfies the Laplace equation in cylindrical coordinates

$$\frac{1}{r}\partial_r(r\partial r V) + \partial_\eta^2 V = 0. \tag{4.48}$$

The four dimensional metric and the three form potential are given by

$$\begin{aligned} ds_{11}^2 &= \kappa_{11}^{\frac{2}{3}} \left(\frac{\dot{V}\Delta}{2V''} \right)^{\frac{1}{3}} \Big\{ 4ds_{AdS_5}^2 + \frac{2V''\dot{V}}{\Delta} ds_{S_2}^2 + \frac{2V''}{\dot{V}} \big(dr^2 + \frac{2\dot{V}}{2\dot{V} - \ddot{V}} r^2 d\chi^2 + d\eta^2 \big) \\ &\quad + \frac{2(2\dot{V} - \ddot{V})}{\dot{V}\Delta} \big(d\beta + \frac{2\dot{V}\dot{V}'}{2\dot{V} - \ddot{V}} d\chi \big)^2 \Big\}, \\ C_3 &= 2\kappa_{11} \left(-2\frac{\dot{V}^2 V''}{\Delta} d\chi + \big(\frac{\dot{V}\dot{V}'}{\Delta} - \eta \big) d\beta \right) \wedge d\Omega_{S^2} \end{aligned} \tag{4.49}$$

where $d\Omega_{S^2}$ is the volume form on S^2 and Δ is defined as

$$\Delta = (2\dot{V} - \ddot{V})V'' + (\dot{V}')^2. \tag{4.50}$$

To determine the mapping to electrostatic coordinates we are following appendix C in [131]. The relation (4.47) gives $r = r(\xi, \rho)$ and the expression for the other variable $\eta = \eta(\xi, \rho)$ implies the exact differential

$$d\eta = \frac{\partial \eta}{\partial \xi} d\xi + \frac{\partial \eta}{\partial \rho} d\rho = \frac{\rho}{r} \partial_\rho r d\xi - \frac{r}{\rho} \partial_\xi r d\rho. \tag{4.51}$$

The electrostatic potential can be obtained from the exact differential

$$dV = \left(-\frac{r}{\rho} \partial_\xi r \ln \rho + \frac{\xi}{r} \partial_\rho r \right) d\rho + \left(\frac{\xi}{r} \partial_\xi r + \frac{\rho}{r} \partial_\rho r \, \ln \rho \right) d\xi. \tag{4.52}$$

The boundary condition that the sphere closes at $\xi = 0$ implies

$$\partial_r V \mid_{\eta=0} = 0. \tag{4.53}$$

The rotational symmetric solution corresponds to a conducting disk at $\eta = 0$, which is equivalent to (4.53) since $\partial_r V$ is the electrical field in the r direction which vanishes for a conductor at $\eta = 0$.

The potential V is determined by a line charge $\lambda(\eta)$ localized at $r = 0$

$$\lambda(\eta) = r\partial_r V \mid_{r=0} = \xi(r = 0, \eta). \tag{4.54}$$

Hence determining the change of variables gives the line charge density. The potential can then be obtained via the Green's function

$$V = -\frac{1}{2} \int d\eta' G(r, \eta, \eta') \lambda(\eta') \tag{4.55}$$

where the Green's function can be obtained by the method of images (adding a line charge at negative η)

$$G(r,\eta,\eta') = \frac{1}{\sqrt{r^2+(\eta-\eta')^2}} - \frac{1}{\sqrt{r^2+(\eta+\eta')^2}}. \tag{4.56}$$

A set of rules for the charge distributions $\lambda(\eta)$ which leads to regular solutions (or those with only A_k singularities) was found in [108]. The line charges must be piecewise linear and convex with integer slopes. Furthermore, the slopes can only change at integer values of η. We will say more about these conditions later, but a final point that we want to explore in this subsection is the relationship between the intercepts of these line segments and the flux of the four form field strength F_4.

To do this, we first note that at $r = 0$ the χ circle shrinks to zero size and at $\eta = 0$ the S^2 shrinks. This means that we can form a closed four-cycle by considering the χ circle, the S^2 and an arc in the r, η-plane which intercepts the η-axis near a region of constant slope (see Figure 4.2). Note that at this point, $\dot{V}'$ is the constant slope of this segment and the C_3 field (4.49) takes the following form:

$$C_3 \approx 2\kappa_{11}\left[(-\dot{V}+\eta\dot{V}')d\chi + \left(\frac{\dot{V}\dot{V}'}{\tilde{\Delta}} - \eta\right)(d\beta + \dot{V}'d\chi)\right] \wedge d\Omega_{S_2}. \tag{4.57}$$

We may now find the flux of F_4 on this cycle by using (4.57) to calculate the difference between C_3 at the two endpoints of the arc. If $\lambda(\eta)$ takes the form $s_i\eta + \lambda_i$ along the segment under consideration, we find that $Q_4 = 2\lambda_i$. We can therefore interpret these intercepts as counting the number of fivebranes at each location where the slope changes.

4.4.3 Electrostatic solution for uplifted solution

Using the map of our original coordinates y, θ to LLM coordinates ξ, ρ, we can express the electrostatic variables in terms of y, θ. The first relation in (4.47) gives

$$r = N\sqrt{y^2 - y + q_1}\sin\theta. \tag{4.58}$$

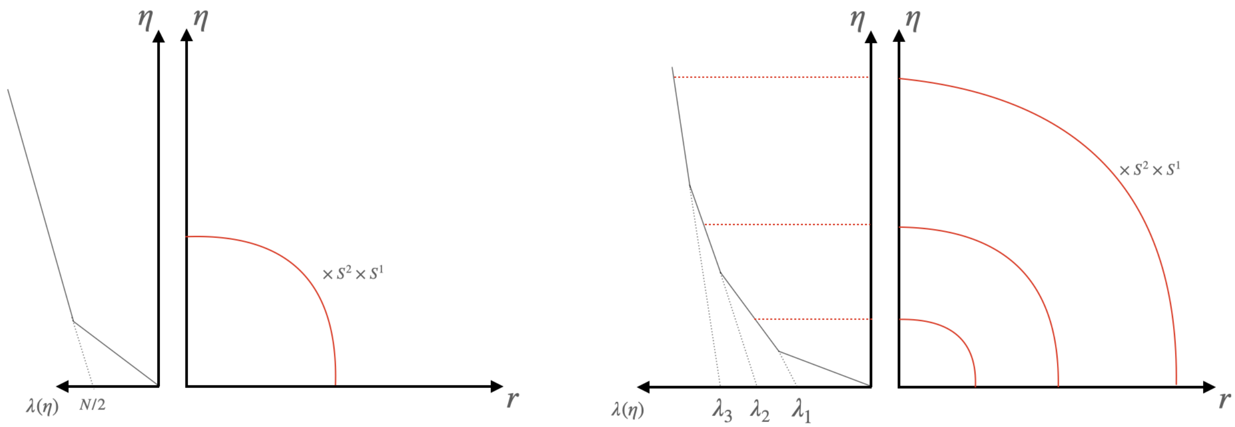

Figure 4.2: **Left:** An arc in the r, η-plane that can be combined with S^1_χ, S^2 to form a four cycle which measures flux N in the uplifted solution. **Right:** A generic solution with many kinks in the line charge. There are more choices of four cycles that can be used to count the number of fivebranes creating each kink.

The exact differential $d\eta$ (4.51) expressed in terms of the y, θ variables is given by

$$d\eta = N(\frac{1}{2} - y)\sin\theta d\theta + N\cos\theta dy \tag{4.59}$$

which can be integrated to give the map from y, θ to η, ξ

$$\eta = N(y - \frac{1}{2})\cos\theta, \qquad \xi = Ny\cos\theta. \tag{4.60}$$

It follows from (4.58) that $r = 0$ corresponds to either $y = y_c$ or $\theta = 0$. Plugging this relation into (4.54) determines the line charge

$$\lambda(\eta) = \begin{cases} \frac{y_c}{y_c - \frac{1}{2}}\eta & 0 < \eta < N(y_c - \frac{1}{2}) \\ \eta + \frac{N}{2} & \eta > N(y_c - \frac{1}{2}). \end{cases} \tag{4.61}$$

Using the relation of the charge q_1 (4.18) and y_c (4.15) for a $\mathbb{R}^4/\mathbb{Z}_k$ conical singularity with $n = 2, 3, \cdots$ then gives

$$\lambda(\eta) = \begin{cases} (k+1)\eta & 0 < \eta < \frac{N}{2k} \\ \eta + \frac{N}{2} & \eta > \frac{N}{2k}. \end{cases} \tag{4.62}$$

We note that $k = 1$ corresponds to $q_1 = 0$ and hence the $AdS_7 \times S^4$ vacuum. We have $\lambda(\eta = 0) = N/2$ which corresponds to a four form flux of N. Note that at $y = y_c = N/(2k)$ the slope of the line charge density $\lambda(\eta)$ changes from 1 to $k + 1$.

4.4.4 Generalization of electrostatic solution

We showed in the previous section that the uplifted defect solution corresponds to a specific line charge in electrostatic formulation. In [108] general conditions on the line charge distribution which are imposed by charge conservation and regularity, which we will briefly review.

First, we previously remarked upon the relationship between the F_4 flux and the intercepts of the line charge. Imposing charge quantization therefore quantizes these intercepts. Next, in order to find constraints on the slopes, we zoom into a region of constant charge density near $r = 0$ where (4.49) takes the form

$$\begin{aligned} ds^2 &\approx \kappa_{11}^{2/3}\left(\frac{\dot{V}\tilde{\Delta}}{2V''}\right)^{1/3}\left(4d^2_{AdS_5} + \frac{2V''\dot{V}}{\tilde{\Delta}}ds^2_{S^2} + \frac{2V''}{\dot{V}}(dr^2 + r^2 d\chi^2 + d\eta^2) + \frac{4}{\tilde{\Delta}}(d\beta + \dot{V}' d\chi)^2\right), \\ \tilde{\Delta} &\approx 2\dot{V}V'' + (\dot{V}')^2. \end{aligned} \tag{4.63}$$

As we mentioned previously, at $r = 0$ the χ-circle is shrinking however the circle $\beta + \dot{V}'\chi$ is not and so we can use it to define a new periodic coordinate provided that $\dot{V}'$ takes integer values there. Since $\dot{V}'(r = 0, \eta)$ is just the slope of the constant line segment, we find that regularity imposes our next quantization condition on $\lambda(\eta)$.

There are further constraints on the changes in slope which we can deduce by zooming in on the region $\eta = \eta_i$ where two slopes meet. Here V'' has a delta function source which means that

$$V'' \approx \frac{k}{2}\frac{1}{\sqrt{r^2 + (\eta - \eta_i)^2}} \tag{4.64}$$

where k is the change in slope. When we insert this into the metric (4.63), we find that the r, η and circle directions give us a space that is locally $\mathbb{R}^4/\mathbb{Z}_k$. Imposing regularity, therefore, quantizes the change in slope so that it takes on (positive) integer values. It can be shown that these A_{k-1} ($k > 1$) singularities give rise to non-abelian gauge fields in AdS_5 corresponding to global symmetries [108].

Finally, we can consider the geometry of our solution along the η-axis between two points η_i, η_{i+1} at which the slope of λ changes. Along any of these segments, we can form a closed four cycle by considering the segment $[\eta_i, \eta_{i+1}]$, the S^2, and the circle $\beta + \dot{V}'\chi$. Notice that at either endpoint, V'' and hence $\tilde{\Delta}$ blows up causing the circle to shrink. One can then use (4.57) to find that the flux of F_4 on this cycle is $\eta_{i+1} - \eta_i$. This can also be done for the first segment $[0, \eta_1]$ since the S^2 (but not the circle) shrinks at $\eta = 0$. Flux quantization thus constrains all η_i's to take on integer values.

In summary, we find that charge distributions give rise to regular (or with A_k singularities) solutions provided that they are piecewise linear, have (decreasing) integer slopes and half-integer intercepts, and change slope only at integer values of η. Putting these together, we can write a multi-kink generalization of the uplifted flux N solution:

$$\lambda(\eta) = \begin{cases} s_1\eta & \eta \in [0, \eta_1] \\ s_2\eta + \lambda_2 & \eta \in [\eta_1, \eta_2] \\ s_3\eta + \lambda_3 & \eta \in [\eta_2, \eta_3] \\ \ldots & \ldots \\ \eta + N/2 & \eta \in [\eta_{n_{\text{kink}}}, \infty). \end{cases} \tag{4.65}$$

Note that the continuity of $\lambda(\eta)$ alone is enough to determine the η_i's in terms of the slope and intercept data. That is, $\eta_i = (\lambda_{i+1} - \lambda_i)/(s_i - s_{i+1})$ which can be written in terms of the slope changes $k_i \in \mathbb{Z}$ and the number of fivebranes creating the punctures N_i to give $\eta_i = N_i/2k_i \in \mathbb{Z}$. Substituting these into (4.65) gives

$$\lambda(\eta) = \begin{cases} (1 + \sum_{i=1}^{n_{\text{kink}}} k_i)\,\eta & \eta \in [0, N_1/2k_1] \\ (1 + \sum_{i=2}^{n_{\text{kink}}} k_i)\,\eta + N_1/2 & \eta \in [N_1/2k_1, N_2/2k_2] \\ (1 + \sum_{i=3}^{n_{\text{kink}}} k_i)\,\eta + (N_1 + N_2)/2 & \eta \in [N_2/2k_2, N_3/2k_3] \\ \ldots & \ldots \\ \eta + N/2 & \eta \in [N_{n_{\text{kink}}}/2k_{n_{\text{kink}}}, \infty). \end{cases} \tag{4.66}$$

where $N = \sum_{i=1}^{n_{\text{kinks}}} N_i$ is the total F_4 flux. One can plug this general solution into (4.49) and find that it produces the same asymptotic $AdS_7 \times S^4$ region (4.28) as the original uplifted solution.

4.5 Holographic observables

The supergravity solutions presented in the previous section can be used to calculate holographic observables. Examples of such observables are the entanglement entropy of a surface around the defect and the on-shell action. Due to the infinite volume of the asymptotic $AdS_7 \times S^4$ region, the holographic observables are divergent and have to be regularized. We can define a general cutoff surface

$$\eta(\epsilon, \theta) = y_c(\epsilon, \theta) \sin\theta, \quad r(\epsilon, \theta) = y_c(\epsilon, \theta) \cos\theta, \tag{4.67}$$

where

$$y_c(\epsilon, \theta) = \frac{1}{\epsilon} + f_0(\theta) + f_1(\theta)\epsilon + f_2(\theta)\epsilon^2 \tag{4.68}$$

and $f_i(\theta)$ are arbitrary bounded functions of the angle $\theta \in [0, \frac{\pi}{2}]$. The observables which we will consider here turn out to be integrals of total derivatives and become integrals over the boundary of the integration regions which is given by the integral along the η and r axis as well as the cutoff surface at large y_c. The cutoff of the integral along the η and r axis is given by setting $\theta = \pi/2$ and $\theta = 0$ in (4.67) respectively. The simplest choice for a cutoff surface would be given by setting all $f_i = 0$ which corresponds to a circular quarter arc in the η, r plane whose radius will go to infinity as $\epsilon \to 0$.

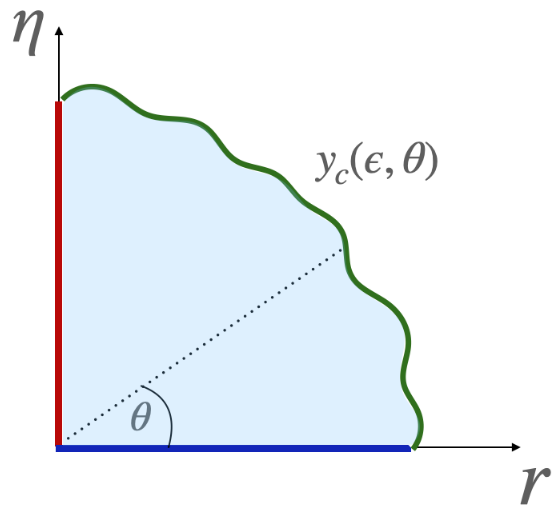

Figure 4.3: Integration region in the η, r-plane. We consider observables which reduce to integrals over the boundary comprised of the η-axis, r-axis, and a generic θ-dependent cutoff surface.

In order to obtain finite results we use vacuum subtraction, i.e. we subtract the regularized result by the result for the $AdS_7 \times S^4$ vacuum using the same cutoff surface. We use this prescription since a full set of covariant counterterms is not known for the eleven-dimensional supergravity and the standard method of holographic renormalization [94, 97] which can be used for AdS solutions of gauged supergravities in lower dimensions is not available.

The contributions from the cutoff surface can all be expressed in terms of moments of the large y_c expansion of derivatives of the potential $\dot{V}, V''$ (4.55)

$$\begin{aligned}\dot{V} &= y_c \sin\theta + m_1 \sin\theta - m_3 \frac{\cos^2\theta \sin\theta}{2y_c^2} + \mathcal{O}\left(\frac{1}{y_c^4}\right), \\ V'' &= m_1 \frac{\sin\theta}{y_c^2} - m_3 \frac{\sin\theta(1+5\cos 2\theta)}{4y_c^4} + \mathcal{O}\left(\frac{1}{y_c^6}\right).\end{aligned} \tag{4.69}$$

The moments m_1 and m_3 can be expressed in terms of line charge (4.66)

$$m_1 = \sum_{i=1}^{n_{\text{kinks}}} (s_i - s_{i+1})\eta_i = \frac{1}{2}\sum_{i=1}^{n_{\text{kinks}}} N_i = \frac{N}{2}, \tag{4.70}$$

$$m_3 = \sum_{i=1}^{n_{\text{kinks}}} (s_i - s_{i+1})\eta_i^3 = \frac{1}{8}\sum_{i=1}^{n_{\text{kinks}}} \frac{N_i^3}{k_i^2}, \tag{4.71}$$

where $s_{n_{\text{kinks}+1}} = 1$. When it is unambiguous, we will just write m_i but when we refer to a particular or multiple solutions at once (as in the case of vacuum subtraction), we will denote the moments with a superscript, e.g. $m_i^{(n_{\text{kinks}})}$ or $m_i^{(vac)}$.

4.5.1 Central charge

It was argued in [133, ?] that the holographic dual of the a central charge of a $d = 4$ SCFT coming from the 11 dimensional metric

$$ds_{11}^2 = \kappa_{11}^{2/3}\left(\frac{\dot{V}\Delta}{2V''}\right)^{1/3}\left[4ds^2_{\text{AdS}_5} + ds^2_{M_6}\right] \tag{4.72}$$

is give by the following expression

$$a = \frac{2^5\pi^3\kappa_{11}^3}{(2\pi\ell_p)^9}\int_{M_6}\left(\frac{\dot{V}\Delta}{2V''}\right)^{3/2} d\Omega_{M_6}, \tag{4.73}$$

where ℓ_p is the 11 dimensional Planck length and $d\Omega_{M_6}$ is the volume form of $ds^2_{M_6}$. For holographic duals of $d = 4$, $N = 2$ SCFTs the six dimensional space is compact and one obtains a finite result for the integral. As discussed above, for the defect solutions the integral will be taken over a non-compact space and will be divergent.

$$d\Omega_{M_6} = \frac{8\sqrt{2}r(V'')^{5/2}}{\dot{V}^{1/2}\Delta^{3/2}} d\Omega_{S_2}\wedge d\eta\wedge dr\wedge d\chi\wedge d\beta. \tag{4.74}$$

The central charge is therefore equal to

$$a = \frac{2^7\pi^3\kappa_{11}^3}{(2\pi\ell_p)^9}\int r\dot{V}V'' d\Omega_{S_2}\wedge d\eta\wedge dr\wedge d\chi\wedge d\beta. \tag{4.75}$$

We can now use the cylindrical Laplace equation (4.48) to write $r\dot{V}V'' = -\partial_r(\dot{V}^2)/2$ and the fact that χ and β are 2π periodic, as well as $\kappa_{11} = \frac{\pi}{2}\ell_p^3$ to write the central charge as

$$\begin{aligned} a &= \frac{1}{4}\int -\partial_r(\dot{V}^2) dr \wedge d\eta \\ &= \frac{1}{4}\int_0^{y_c(\epsilon,\pi/2)} d\eta\ \lambda(\eta)^2 - \frac{1}{4}\int_{\theta=0}^{\theta=\pi/2} (\dot{V})^2 d\big(y_c(\epsilon,\theta)\sin\theta\big) \\ &= \frac{1}{4}\int_0^{\eta_{n_{\text{kink}}}} d\eta\ \lambda(\eta)^2 + \frac{1}{4}\int_{\eta_{n_{\text{kink}}}}^{y_c(\epsilon,\pi/2)} d\eta\ (\eta+m_1)^2 - \frac{1}{4}\int_{\theta=0}^{\theta=\pi/2} (\dot{V})^2 d\big(y_c(\epsilon,\theta)\sin\theta\big), \end{aligned} \tag{4.76}$$

where we obtain the final line by noticing that $\lambda(\eta)$ has a universal form in the region beyond the final kink $\eta_{n_{\text{kink}}}$. Notice above that the first integral in the third line is finite. Inserting the generic cutoff surface (4.68) into this expression and integrating over θ gives us following:

$$\begin{aligned} a = &\frac{m_1/3 + f_0(\pi/2)}{4\epsilon^2} + \frac{2m_1^2/3 + 2m_1 f_0(\pi/2) + f_0(\pi/2)^2 + f_1(\pi/2)}{4\epsilon} \\ &+ \frac{1}{4}\int_0^{\eta_{n_{\text{kink}}}} d\eta\ \lambda(\eta)^2 + \frac{1}{60}\Big[2m_3 + 15m_1^2(-\eta_{n_{\text{kink}}} + f_0(\pi/2)) + 15m_1(-\eta_{n_{\text{kink}}}^2 + f_0(\pi/2)^2) \\ &+ 5(-\eta_{n_{\text{kink}}}^3 + f_0(\pi/2)^3) + 30(m_1 + f_0(\pi/2))f_1(\pi/2) + 15 f_2(\pi/2)\Big] \\ &+ \int_0^{\pi/2} I_{m_1,f_i}(\epsilon,\theta) d\theta + \mathcal{O}(\epsilon), \end{aligned} \tag{4.77}$$

where in the final line, $I_{m_1,f_i}(\epsilon,\theta)$ is an expression that depends on the cutoff surface functions f_i but only m_1 and therefore, since this is the same for all solutions, it will be eliminated by subtracting the contribution from the $AdS_7 \times S^4$ vacuum solution:

$$\begin{aligned} a_{(vac)} = &\frac{m_1/3 + f_0(\pi/2)}{4\epsilon^2} + \frac{2m_1^2/3 + 2m_1 f_0(\pi/2) + f_0(\pi/2)^2 + f_1(\pi/2)}{4\epsilon} \\ &+ \frac{1}{60}\Big[-13m_1^3 + 15m_1^2 f_0(\pi/2) + 15m_1 f_0(\pi/2)^2 \\ &+ 5f_0(\pi/2)^3 + 30(m_1 + f_0(\pi/2))f_1(\pi/2) + 15f_2(\pi/2)\Big] \\ &+ \int_0^{\pi/2} I_{m_1,f_i}(\epsilon,\theta) d\theta + \mathcal{O}(\epsilon). \end{aligned} \tag{4.78}$$

This expression can be obtained from (4.77) by noticing that $m_3^{(vac)} = m_1^3$ and $\eta_{n_{\text{kink}}}^{(vac)} = \eta_1^{(vac)} = m_1$. All of the divergent terms depend only on m_1. Furthermore, the cutoff surface functions, f_i, only appear in the finite term with m_1 (and no higher moments) so after vacuum subtraction we will be left with something finite and independent of the choice of cutoff:

$$a - a_{(vac)} = \frac{1}{4}\int_0^{\eta_{n_{\rm kink}}} d\eta\ \lambda(\eta)^2 + \frac{1}{60}(13m_1^3 + 2m_3 - 15m_1^2\eta_{n_{\rm kink}} - 15m_1\eta_{n_{\rm kink}}^2 - 5\eta_{n_{\rm kink}}^3). \tag{4.79}$$

It is useful to rewrite these expressions in terms of the more physical parameters k_i and N_i. For one and two kinks these become

$$\begin{aligned} a_{(2)} - a_{(vac)} = & \frac{(-3 + k_1(-10 + 13k_1) - 5k_2)N_1^3}{480k_1^2} + \frac{39N_1^2N_2}{480} \\ & + \frac{3(-5 + 13k_2)N_1N_2^2}{480k_2} + \frac{(-1 + k_2)(3 + 13k_2)N_2^3}{480k_2^2} \end{aligned} \tag{4.80}$$

and

$$a_{(1)} - a_{(vac)} = \frac{(k-1)(3+13k)N^3}{480k^2}. \tag{4.81}$$

4.5.2 On-shell action

For holographic defect solutions, among the simplest observables is the vacuum subtracted on-shell action which gives the defect partition function in the semi-classical approximation. Other observables, which we will not discuss here, include one-point functions of bulk operators in the presence of the defect or the entanglement entropy in the presence of the defect.

The action of eleven dimensional supergravity is given by

$$S = \frac{1}{2k_{11}^2}\int_M \sqrt{-g}\Big(R - \frac{1}{48}F_{\mu\nu\rho\lambda}F^{\mu\nu\rho\lambda}\Big) + \frac{1}{2k_{11}^2}\int_{\partial M}\sqrt{h}\ 2K + S_{CS}. \tag{4.82}$$

Here S_{CS} is the Chern-Simons term which vanishes for the LLM solutions and is dropped in the following. The second term is the Gibbons-Hawking term which is needed for a good variational principle for spacetimes with boundary. Here h_{ab} is the induced metric on the boundary and K is the trace of the second fundamental form $K_{\mu\nu} = -\frac{1}{2}(\nabla_\mu n_\nu + \nabla_\nu n_\mu)$ where n_μ is the outward pointing normal vector to the boundary ∂M. Using the equations of motion for the metric and the three form potential, it is easy to show that the bulk part of the on-shell action is a total derivative and the total action is given by a boundary term

$$S_{on\ shell} = \frac{1}{2k_{11}^2}\int_{\partial M}\left(-\frac{1}{3}\right)C_3 \wedge *F_4 + \frac{1}{2k_{11}^2}\int_{\partial M}\sqrt{h}\,2K. \tag{4.83}$$

Presently, we will compute this for the simple cutoff ($f_i = 0$ for all i) and later comment on generic cutoff-dependence. To start, we notice that the boundary region $\eta = 0$ gives no contribution since here the S^2 shrinks to zero volume. The contribution coming from the cutoff surface has a universal form for all solutions in terms of moments m_1 and m_3:

$$S_{bulk,cutoff} = \frac{Vol(AdS_5)Vol(S^2)}{2k_{11}^2}\left(-\frac{64(-2m_1+5m_3)}{15m_1\epsilon} - \frac{128(m_1^3+2m_3)}{15}\right) \tag{4.84}$$

and

$$S_{GH,cutoff} = \frac{Vol(AdS_5)Vol(S^2)}{2k_{11}^2}\left(\frac{128}{\epsilon^3} + \frac{512m_1}{3\epsilon^2} + \frac{512m_1^2}{15\epsilon} + \frac{128(m_1^3-3m_3)}{15}\right). \tag{4.85}$$

This is not unexpected since we take this boundary to be at a distance far away from the region where the slopes of $\lambda(\eta)$ are changing ($y_c(\epsilon,\theta) \gg \eta_{n_{kink}}$).

The final contribution comes from the region along the η-axis. Since this involves an integral over $0 < \eta < y_c(\epsilon, \pi/2)$, it will be sensitive to line charge data beyond just the moments. These integrals quickly become unwieldy for more complicated $\lambda(\eta)$ so in lieu of a generic expression, we can write down the answer for $n_{kink} = 2$ from which the $n_{kink} = 1$ case can be easily derived by setting $N_1 \to 0$ and $N_2 \to N$. We have that

$$S^{(2)}_{bulk,r=0} = \frac{Vol(AdS_5)Vol(S^2)}{2k_{11}^2}\left(-\frac{64}{3\epsilon^3} - \frac{64m_1}{\epsilon^2} - \frac{64(4m_1^3 - m_3^{(2)})}{3m_1\epsilon} + S^{(2),finite}_{bulk,r=0}\right), \tag{4.86}$$

$$S^{(2),finite}_{bulk,r=0} = \frac{64}{3}\left[(1+4s_1-2s_2)(s_1-s_2)\eta_1^3 + 6(s_1-s_2)(s_2-1)\eta_1\eta_2^2 + (s_2-1)(4s_2-1)\eta_2^2\right],$$

$$S^{(2)}_{GH,r=0} = \frac{Vol(AdS_5)Vol(S^2)}{2k_{11}^2}\left(\frac{128}{2\epsilon^3} + \frac{128m_1}{\epsilon^2} + \frac{128m_1^2}{\epsilon} + S^{(2),finite}_{GH,r=0}\right), \tag{4.87}$$

$$S^{(2),finite}_{GH,r=0} = \frac{-128}{3}\left[m_3^{(2)}(2s_1-s_2) + (s_2-1)(3m_1 - 2(s_1-1))\eta_2^2\right].$$

One can quickly inspect that the ϵ^{-3} and ϵ^{-2} divergences only depend on m_1 and so will cancel once we subtract the vacuum contribution. There are m_3's which appear in the ϵ^{-1} divergent term, however they cancel between (4.84) and (4.86). Combining all of the terms, we obtain the following

$$S^{(2)}_{\text{on shell}} = \frac{2\pi Vol(AdS_5)}{k_{11}^2}\left(\frac{448}{3\epsilon^3} + \frac{704m_1}{3\epsilon^2} + \frac{256m_1^2}{3\epsilon} - \frac{64}{3}m_3^{(2)}\right) \tag{4.88}$$

and after subtracting the $AdS_7 \times S^4$ vacuum, we are left with

$$\begin{aligned} S^{(2)}_{\text{on shell}} - S^{(vac)}_{\text{on shell}} &= \frac{-2\pi Vol(AdS_5)}{k_{11}^2}\frac{64}{3}(m_3^{(2)} - m_3^{(vac)}) \\ &= \frac{16\pi Vol(AdS_5)}{3k_{11}^2}\left((N_1+N_2)^3 - \frac{N_1^3}{k_1^2} + \frac{N_2^3}{k_2^2}\right). \end{aligned} \tag{4.89}$$

From this, we can set $N_1 = 0$, $N_2 = N$ and $k_2 = k$ to obtain the expression for one kink:

$$\begin{aligned} S^{(1)}_{\text{on shell}} - S^{(vac)}_{\text{on shell}} &= \frac{16 Vol(AdS_5)}{3k_{11}^2}N^3\left(1 - \frac{1}{k^2}\right) \\ &= \frac{-2\pi Vol(AdS_5)}{k_{11}^2}\frac{64}{3}(m_3^{(1)} - m_3^{(vac)}). \end{aligned} \tag{4.90}$$

The terms with $\frac{1}{\epsilon^{2n}}$ divergences cancel out of the vacuum subtracted on shell action. However, the result still has a divergence due to the infinite volume of AdS_5. For a more complete treatment one should introduce a Fefferman-Graham like cutoff which regularizes all divergences, see e.g. [103, 134] for discussions of such cutoffs in other holographic defect theories.

Another possible related feature of the vacuum subtracted on shell action is that the detailed form of finite terms depend on the choice of the cutoff surface. This is analogous to the possibility of finite counter terms in a covariant regularization procedure in lower dimensional supergravity. Such ambiguities can often be fixed by demanding the finite counter terms preserve supersymmetry, but how this implemented in the vacuum subtraction is not clear to us at this moment. While the results for a simple cutoff we have presented in this section are compellingly simple, it is not clear at the moment whether they are unambiguous.

4.6 Discussion

In this chapter we constructed solutions of eleven dimensional supergravity, which are holographic duals of co-dimension two defects in six dimensional SCFTs. The solutions preserve sixteen of the thirty two supersymmetries. In the classification given in [135], the solution preserves the $SU(2,2|2)$ superconformal algebra of the original $OSp(8|2)$ of the $AdS_7 \times S^4$ vacuum.

While it is possible to construct completely regular quarter-BPS solutions which carry two nonzero charges, the seven dimensional half-BPS solution with only one nonzero charge turned on suffers from a conical singularity in the bulk. Upon uplifting to eleven dimensions we showed that the singularity is also present in eleven dimensions. The uplift allows us to identify this type of singularity with a regular puncture which is locally $\mathbb{R}^4/\mathbb{Z}_k$ and was discussed already in the original paper of Gaiotto and Maldacena [108] that constructs holographic duals of $d = 4,\ N = 2$ SCFTs.

One of the main results in the present chapter is to use the electrostatic formulation of the LLM solution to construct new defect solutions based on more general linear charge densities. It is possible to obey all the conditions that charge quantization and periodicity of the angular coordinates impose. The generic solutions have singularities corresponding to a finite number of regular punctures, associated with the kinks in the linear charge density. It is however possible to construct solutions which can be completely regular. It is interesting to contrast these holographic solutions with the ones used to describe $d = 4,\ N = 2$ SCFTs [108, 129, 130, 131] as well as more recent ones constructing duals of Argyres-Douglas theories [90, 91, 136]. In the former, the η, r is compact and will be related to Maldacena-Nunez [107] solutions and class S $N = 2, d = 4$ theory [127] coming from compactifying a $d = 6,\ N = (2, 0)$ theory on a Riemann surface with (regular) punctures. In the latter, one considers a disk in the η, r plane with 5-brane source smeared on the boundary of the disk. This behavior is to be contrasted to our solutions where the η, r space is non-compact and the solutions are asymptotically $AdS_7 \times S^4$ in the limit where η, r go to infinity. Hence the supergravity solutions are holographically dual to co-dimension 2 defects in $d = 6,\ N = (2, 0)$ SCFTs. Since the superconformal symmetry preserved by the defect is the same as the one of $d = 4,\ N = 2$ SCFTs it is natural that these SCFTs describe the defect degrees of freedom. As discussed in appendix 4.A the defects can be interpreted as disorder type defects. It would be interesting to determine the exact identification of the defect theory, the calculation of some holographic observables given in this chapter is a first step in this direction. It may be possible to check the identification by matching holographic calculation with calculations on the field theory side using localization.

We note that the electromagnetic formulation involves an approximation where we consider a rotationally symmetric distribution of sources for the Toda equation and smear them. It would be interesting to consider solutions of the Toda equation corresponding to co-dimension two defect solutions. This would involve placing line sources in the three dimensional half space spanned by ξ, x_1, x_2. The holographic defects would correspond to solutions where this space is non-compact and the large ξ, x_i limit corresponds to the asymptotic $AdS_7 \times S^4$ region. The nonlinear nature of the Toda equation which determines the solution makes the construction of such solutions very challenging, however. It would also be interesting to find generalizations of the uplifts of the quarter-BPS defect solutions which are completely regular already in 7 dimensions however no general "bubbling" solution à la LLM is known for eight instead of sixteen preserved supersymmetries.

4.A Defects for the $d = 6,\ N = (2,0)$ tensor multiplet

In this appendix we construct a conformal co-dimension two defect for the free $d = 6,\ N = (2,0)$ tensor multiplet. The field content of the multiplet is a rank 2 antisymmetric tensor field $B_{\mu\nu}$ with self-dual field strength $H_{\mu\nu\rho}$, five scalars $\Phi_i, i = 1, \cdots, 5$ which transform as a 5 under the $SO(5)$ R-symmetry and four symplectic Majorana-Weyl spinors $\psi^a, a = 1, \cdots, 4$ which transform as 4 under the $USp(4) \equiv SO(5)$.

The super(conformal) symmetry transformations are given by [137]

$$\begin{aligned}
\delta\psi &= \frac{1}{2}\gamma^\mu \partial_\mu \phi_i \Gamma^i \epsilon - \frac{1}{6} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + 2\phi_i \Gamma^i \eta_0, \\
\delta\phi^i &= -2\bar{\epsilon}(\Gamma^i)\psi, \\
\delta B_{\mu\nu} &= -2\bar{\epsilon}\gamma_{\mu\nu}\lambda.
\end{aligned} \tag{4.91}$$

Here $\Gamma^i, i = 1, 2, \cdots, 5$ are $SO(5)$ gamma matrices and γ^μ are six dimensions gamma-matrices. The spinors are contracted using the symplectic metric Ω_{ab}. The supersymmetry transformation parameter ϵ is given by

$$\epsilon = \epsilon_0 + \gamma_\mu x^\mu \eta_0, \tag{4.92}$$

where ϵ_0 is a left handed constant symplectic Majorana spinor parameterizing the Poincare supersymmetries annd η_0 is a constant right handed symplectic Majorana spinor, parameterizing the superconformal transformations. We are constructing a co-dimension two defect in this six dimensional theory, which preserves some part of the superconformal symmetry. The simplest set-up is to consider a flat defect with a four dimensional world-volume directions, on which all fields do not depend. The two directions transverse to the defect are spanned by x^1, x^2 and we choose the defect to be located at $x^1 = x^2 = 0$. From the symmetries we can deduce that the antisymmetric tensor field is vanishing and hence only the scalars are turned on. It is useful to introduce complex coordinates $z = x^1 + ix^2$ and gamma matrices

$$\gamma^z = \frac{1}{\sqrt{2}}\Big(\gamma^1 + i\gamma^2\Big), \qquad \gamma^{\bar{z}} = \frac{1}{\sqrt{2}}\Big(\gamma^1 - i\gamma^2\Big). \tag{4.93}$$

From the supergravity solutions it follows that for a defect that preserves half the supersymmetries the $SO(5)$ R-symmetry is broken to $SU(2)$, hence we make the following ansatz for the scalar fields. The following complex combination of the scalar fields is nontrivial

$$\phi_\omega = \frac{1}{\sqrt{2}}\Big(\phi^1 + i\phi^2\Big) = \frac{\alpha + i\beta}{z}. \tag{4.94}$$

Unbroken supersymmetries satisfy $\delta\psi = 0$, it is easy to see that the supersymmetry transformation rules (4.91) lead to the condition on the Poincare supersymmetry

$$\gamma^a \Gamma^\omega \epsilon_0 = 0 \quad \Leftrightarrow \quad \gamma^{12}\Gamma^{12}\epsilon_0 = \epsilon_0. \tag{4.95}$$

The second condition is a projection which implies that half the Poincare supersymmetries are preserved. It is also easy to verify that for an η_0 satisfying the same projection condition (4.95) and the z dependence of the scalar (4.94) half of the superconformal symmetries are preserved and hence the defect is half BPS.

For a defect preserving a quarter of the supersymmetry we have a nontrivial profile for four scalars, breaking the $SO(5)$ R-symmetry to $U(1) \times U(1)$.

$$\phi_{\omega_1} = \frac{1}{\sqrt{2}}\Big(\phi^1 + i\phi^2\Big) = \frac{\alpha_1 + i\beta_1}{z}, \qquad \phi_{\omega_2} = \frac{1}{\sqrt{2}}\Big(\phi^3 + i\phi^4\Big) = \frac{\alpha_2 + i\beta_2}{z}, \tag{4.96}$$

which leads to two projectors

$$\gamma^{12}\Gamma^{12}\epsilon_0 = \epsilon_0, \qquad \gamma^{12}\Gamma^{34}\epsilon_0 = \epsilon_0. \tag{4.97}$$

Hence a quarter of the supersymmetries are preserved (as well as a quarter of the superconformal symmetries). The free tensor multiplet can be used to construct the $N = (2,0)$ superconformal current multiplet which contains the $SO(5)$ R-symmetry current and the stress tensor [138]. The free tensor multiplet corresponds to the "center of mass" degrees of freedom and the construction of the defect solution for the strongly coupled interacting $d = 6,\ N = (2,0)$ theory is beyond the scope of this appendix.

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