History of Modern Mathematics

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ARTICLE 5: QUATERNIONS AND AUSDEHNUNGSLEHRE

Quaternions and Ausdehnungslehre1 are so closely related to complex quantity, and the latter to complex number, that the brief sketch of their development is introduced at this point. Caspar Wessel's contributions to the theory of complex quantity and quaternions remained unnoticed in the proceedings of the Copenhagen Academy. Argand's attempts to extend his method of complex numbers beyond the space of two dimensions failed. Servois (1813), however, almost trespassed on the quaternion field. Nevertheless there were fewer traces of the theory anterior to the labors of Hamilton than is usual in the case of great discoveries. Hamilton discovered the principle of quaternions in 1843, and the next year his first contribution to the theory appeared, thus extending the Argand idea to three-dimensional space. This step necessitated an expansion of the idea of r(cos □ + j sin □) such that while r should be a real number and □ a real angle, i, j , or k should be any directed unit line such that i2 = j2 = k2 = –1. It also necessitated a withdrawal of the commutative law of multiplication, the adherence to which obstructed earlier discovery. It was not until 1853 that Hamilton's Lectures on Quarternions appeared, followed (1866) by his Elements of Quaternions.

In the same year in which Hamilton published his discovery (1844), Grassmann gave to the world his famous work, Die lineale Ausdehnungslehre, although he seems to have been in possession of the theory as early as 1840. Differing from Hamilton's Quaternions in many features, there are several essential principles held in common which each writer discovered independently of the other.2

Following Hamilton, there have appeared in Great Britain numerous papers and works by Tait (1867), Kelland and Tait (1873), Sylvester, and McAulay (1893). On the Continent Hankel (1867), Hoüel (1874), and Laisant (1877, 1881) have written on the theory, but it has attracted relatively little attention. In America, Benjamin Peirce (1870) has been especially prominent in developing the quaternion theory, and Hardy (1881), Macfarlane, and Hathaway (1896) have contributed to the subject. The difficulties have been largely in the notation. In attempting to improve this symbolism Macfarlane has aimed at showing how a space analysis can be developed embracing algebra, trigonometry, complex numbers, Grassmann's method, and quaternions, and has considered the general principles of vector and versor analysis, the versor being circular, elliptic logarithmic, or hyperbolic. Other recent contributors to the algebra of vectors are Gibbs (from 1881) and Heaviside (from 1885).

The followers of Grassmann3 have not been much more numerous than those of Hamilton. Schlegel has been one of the chief contributors in Germany, and Peano in Italy. In America, Hyde (Directional Calculus, 1890) has made a plea for the Grassmann theory.4

Along lines analogous to those of Hamilton and Grassmann have been the contributions of Scheffler. While the two former sacrificed the commutative law, Scheffler (1846, 1851, 1880) sacrificed the distributive. This sacrifice of fundamental laws has led to an investigation of the field in which these laws are valid, an investigation to which Grassmann (1872), Cayley, Ellis, Boole, Schröder (1890-91), and Kraft (1893) have contributed. Another great contribution of Cayley's along similar lines is the theory of matrices (1858).

1 Tait, P. G., on Quaternions, Encyclopædia Britannica; Schlegel, V., Die Grassmann'sche Ausdehnungslehre, Schlömilch's Zeitschrift, Vol. XLI.

2 These are set forth in a paper by J. W. Gibbs, Nature, Vol. XLIV, p. 79.

3 For bibliography see Schlegel, V., Die Grassmann'sche Ausdehnungslehre, Schlömilch's Zeitschrift, Vol. XLI.

4 For Macfarlane's Digest of views of English and American writers, see Proceedings American Association for Advancement of Science, 1891.