Arbitrary Abstraction and Logicality

Ludovica Conti
ludovica.conti@iusspavia.it

School of Advanced Study IUSS, Pavia

In this talk, I will discuss a criterion (*Weak Objectual Invariance* – WOI) that has been recently suggested in order to argue for the logicality of abstraction operators, when they are understood as arbitrary expressions (cf. Boccuni Woods 2020). I aim at showing that an arbitrary interpretation of the abstractionist vocabulary, on the one side, allows us to actually recover Fregean purpose of logicality, but, on the other side, unavoidably presupposes a non-Fregean semantics for abstraction.

Abstractionist theories are systems composed by a logical theory augmented with one or more abstraction principles (AP) – of form: $f_R\alpha = f_R\beta \leftrightarrow R(\alpha, \beta)$ – that introduce, namely rule and implicitly define, a corresponding term-forming operators f_R. Usually, the interpretation of these principles follows (what we can call) a *canonical reading*, that consists in, at least, two assumptions: an "at face value" reading of the abstractionist vocabulary and an informal primitive notion of reference. Following the first one, we can read the abstraction operator (f_R) as a function symbol and the abstract terms as genuine complex singular terms. Following the second one, the abstraction operator denotes a unique, determinate and knowable set of ordered pairs and abstract terms are interpreted as always referential and denoting singular, knowable and standard objects – named abstract objects.

The logicality of the abstractionist theories plainly depends on the logicality of the abstraction principles and this last issue was originally raised into the seminal abstractionist program, Frege's Logicism, i.e. a theory equivalent to second-order logic augmented with Basic Law V (BLV), that has been proposed with the foundational purposes to derive arithmetical laws as logical theorems and to define arithmetical expressions by logical terms. The inconsistency of this project seemed to determine the inconsistency and, then (in a classical logic) the non-logicality of BLV and – *a fortiori* – of any other abstraction principle[1]. Recently, the issue of logicality has been resumed regarding consistent abstraction principles, in order to question such a failure and the limits of new versions of Logicist project (e.g. Neologicism and consistent revisions of *Grundgsetze*), in light of the intervening studies about logicality. Briefly, a standard account of logicality has been provided, in semantical terms, by means of the Tarskian notion of invariance under permutation and isomorphism (cfr. [8]). In order to apply these criteria to abstraction principles, we can specify at least three different subjects to be examined: the whole abstraction principle

[1] We will describe a relation between the abstraction principles based on the *finesse* of their equivalence relations. Cfr. [1].

[2], the abstraction relation[3] and the abstraction operator[4]. Different results has been already proved (cf. [8], [6], [1], [4], [2], [5]) but a new *Logicist Dilemma* appeared: given a semantical definition of logicality as invariance, we are able to prove that some abstraction principles (e.g. Hume's Principle) are logical ([4])[6] but their implicit *definienda* (e.g. cardinal operator) are not ([1])[7] – so preventing a full achievement of the main Logicist goal.

I preliminarily aim at showing that this unfortunate situation closely depends on the *canonical reading* of the APs and, particularly, on the (unjustified) adoption of a same notion of reference for all the expressions of a same syntactical category (e.g. singular terms as always referential and denoting singular, knowable and standard objects). On the contrary, a less demanding reading of the abstractionist vocabulary is available: by admitting a different evaluation of primitive and defined expressions, it is able to focus on the only information actually provided by the APs and could turn out to be preferable as more faithful to the theory. Thus, chosen this *undemanding reading* of the APs and, particularly, an arbitrary interpretation (cf. [3]) of the abstractionist vocabulary, my main aim will consist in inquiring its effects on the logicality of the abstractionist theories, both from a formal and from a philosophical point of view.

On the one side, given such an arbitrary interpretation of the APs, we can rephrase the main criterion of logicality for abstraction operators (*objectual invariance*, cf. [1]), obtaining a weaker one (*Weak Objectual Invariance*[8], WOI, cf. [9], [2]) and proving that it is satisfied not only by cardinal operator but also by many other second-order ones, including those implicitly defined by consistent weakenings of Fregean Basic Law V. So, we will note that, given (what I argued as) a preferable reading of the APs, both main strategies pursued in the last century to save Fregean project – Neologicism and consistent revisions of *Grundgesetze* – are able to achieve the desirable logicality objective. Further generalising, I will prove that the logicality criterion could be satisfied by a large range of APs and it is apparently liable to a triviality objection – e.g. it is not able to distinguish between HP and some of its Bad Companions (like

[2]Regarding the abstraction principle, the more informative criterion consists of *contextual invariance*: an abstraction principle AP is *contextually invariant* if and only if , for any abstraction function f_R: $D_2 \to D_1$ and permutation π, $\pi(f_R)$ satisfies AP whenever f_R does (cfr. [1]).

[3]Regarding the abstraction relation, we can distinguish, at least, four kind of invariance: *weak invariance*, *double invariance*, *internal invariance* and *double weak invariance* (cfr. [1], [4], [6].).

[4]Regarding the canonical reading of the abstraction operator, logicality is usually spelled out in terms of *objectual invariance*[5] (cf. [1]).

[6]More precisely, some abstraction principles (like Hume's Principle) satisfy the criterion of *contextual invariance* and their abstraction relations (e.g. equinumerosity) satisfy many logicality criteria, like *weak invariance*, *internal invariance*, *double internal invariance*. Cf. [1], [6], [4].

[7]More precisely, the corresponding abstraction operators (e.g. cardinal operators) do not satisfy the criterion of *objectual invariance*. Furthermore, such criterion fails precisely in case of operators related to internal (and,*a fortiori* double internal) invariant relation (cfr. [1]). So, operators fail to be *logical* though – just in case – they are implicitly defined by *logical* AP.

[8]An expression ϕ is *Weak Objectual Invariant* just in case, for all domains D, D' and bijections ι from D to D', the set of candidate denotations of ϕ on D (ϕ^{*D}) $= \{\gamma : \gamma$ is a candidate denotation for ϕ on $D\}$ is such that $\iota(\phi^{*D}) = \phi^{*D'} = \{\gamma : \gamma$ is a candidate denotation for ϕ on $D'\}$.

Nuisance Principle). I will answer to such a potential objection by showing that WOI however introduces interesting differences. More precisely, I will discuss the controversial case of Ordinal Abstraction and I will prove that WOI is not satisfied by any first-order abstraction principle (cf. [8], [9]). So, by comparing the respective schemas of first-order and second-order APs[9], we will note that logicality (in the chosen meaning) mirrors a relevant distinction between same-order and different-order abstraction principles.

On the other side, from the philosophical point of view, I will focus on the role of arbitrariness as a condition for the adoption of the abovementioned logicality criterion. Particularly, while this last one seems to testify the unexpected availability of the Logicist goal, the arbitrary interpretation of the vocabulary actually includes semantical insights that are radically alternative to Logicism. In order to argue for this latter consideration, I will suggest to precise the two main meanings of the informal notion of arbitrariness (i.e. the epistemicist meaning and the semantical one) in a model-theoretic perspective, by means of, respectively, a choice-like semantics and a modal semantics. Given these semantical frameworks, we will note that the arbitrary interpretation not only gives a structuralist nuance to the reading of the abstraction (by emphasising the role rather than the nature of the abstract entities), but its models are clearly more compatible with a nominalist account of the theoretical terms rather then with a Platonist one. Particularly, an analogy between the arbitrary interpretation of the APs and the semantics of some eliminative Structuralist reconstructions of the scientific theories ([7]) will be illustrated.

References

[1] Antonelli, G. A. (2010). Notions of invariance for abstraction principles. Philosophia Mathematica, 18(3), 276-292.

[2] Boccuni, F., Woods, J. (2020). Structuralist neologicism. Philosophia Mathematica, 28(3), 296-316.

[3] Breckenridge, W. Magidor, P. (2012), 'Arbitrary reference', Philosophical Studies 158(3):377–400.

[4] Cook, R. (2016).Abstraction and Four Kinds of Invariance (Or: What's So Logical About Counting), Philosophia Mathematica, 25,1, 3–25.

[5] Ebels-Duggan, S. C. (2019). Abstraction Principles and the Classification of Second-Order Equivalence Relations. Notre Dame Journal of Formal Logic, 60(1), 77-117.

[6] Fine, K. (2002). The limits of abstraction. Clarendon Press.

[9] More precisely, a schematic second-order abstraction principle – of form $\S(RF) = \S(RG) \leftrightarrow R(F,G)$, where § is a binary abstraction operator and E any isomorphism invariant equivalence relation – defines an abstraction function from $\wp(\wp(D) \times \wp(D)) \times \wp(D) \to D$ that satisfies the criterion of GWI and – differently from the specific unary operators – is total ([9]). On the other side, a schematic first-order abstraction principle – of form $\S(Ra) = \S(Rb) \leftrightarrow R(a,b)$, where § is a binary abstraction operator and E any first-order equivalence relation – defines an abstraction function from $\wp(D \times D) \times D \to D$ that is – differently from the corresponding unary operators – total, but however does not satisfy GWI.

[7] Gratzl, N. and Schiemer G. (2016). The epsilon-reconstruction of theories and scientific structuralism. Erkenntnis 81.2, 407–432.

[8] Tarski, A. (1956). The concept of truth in formalized languages. Logic, semantics, metamathematics, 2(152-278), 7.

[9] Woods, J. (2014). Logical indefinites. Logique et Analyse, 277-307.