Sufficient Conditions for the Unsolvability and Solvability of the Absolute Value Equation

Olvi L. Mangasarian *

Abstract

We give linear-inequalities-based sufficient conditions for the unsolvability and solvability of the NP-hard absolute value equation (AVE): $Ax - |x| = b$, where A is an $n \times n$ square matrix. The satisfaction of the linear inequalities is easily verified using a linear program.

Keywords: absolute value equation, unsolvability, solvability, linear inequalities

1 Introduction

We consider the absolute value equation (AVE):

$$Ax - |x| = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are given, and $|\cdot|$ denotes absolute value. As was shown in [14], the general NP-hard linear complementarity problem (LCP) [2, 3, 1], which subsumes many mathematical programming problems, can be formulated as an AVE (1.1). This implies that (1.1) is NP-hard in its general form. There have been few results on the unsolvability of the absolute value equation such as [14, Proposition 7] which can be derived as a special case of the nonexistence results presented here. However, there have been numerous results on absolute value equations and inequalities which we briefly cite below.

In [12] a primal-dual bilinear programming approach is proposed for solving the absolute value equation, while in [11] dual complementarity is utilized as a solution method. In [8] a more general concept of absolute value programming is introduced. In [7] absolute value equations are solved using concave minimization, and in [9] solution is achieved by a generalized Newton method. In [10] the knapsack feasibility problem is formulated as an absolute value equation and solved by successive linear programming. In [13] the unsupervised classification problem is formulated and solved via a convex inequalities formulation.

In Section 2 we outline the background behind our approach and in Section 3 we state our sufficient conditions for the nonexistence of AVE solutions. In Section 4 we give sufficient conditions for the existence of a solution to AVE. In Section 5 we briefly cite some numerical results. Section 6 concludes the paper.

We describe our notation now. All vectors will be column vectors unless transposed to a row vector by a prime $'$. For a vector $x \in \mathbb{R}^n$ the notation x_j will signify the j-th component. The scalar (inner) product of two vectors x and y in the n-dimensional real space $\mathbb{R}^n$ will be denoted by $x'y$. The notation $A \in \mathbb{R}^{m\times n}$ will signify a real $m \times n$ matrix. For such a matrix, A' will denote the transpose of A. A vector of ones in a real space of arbitrary dimension will be denoted by e, while I will denote the

*Computer Sciences Department, University of Wisconsin, Madison, WI 53706 and Department of Mathematics, University of California at San Diego, La Jolla, CA 92093. *olvi@cs.wisc.edu*.

identity matrix. Thus for $e \in \mathbb{R}^m$ and $y \in \mathbb{R}^m$ the notation $e'y$ will denote the sum of the components of y. A vector of zeros in a real space of arbitrary dimension will be denoted by 0. The abbreviation "s.t." stands for "subject to".

2 AVE Solution and Dual Linear Programming Complementarity

We begin with the linear program:

$$\min_{x,y} 0'x + 0'y \text{ s.t. } Ax - y = b,\ x + y \geq 0,\ -x + y \geq 0, \tag{2.2}$$

and its dual:

$$\max_{u,v,w} b'u + 0'v + 0'w \text{ s.t. } A'u + v - w = 0,\ -u + v + w = 0,\ v \geq 0,\ w \geq 0. \tag{2.3}$$

Remark 2.1. *We note immediately that if x solves AVE (1.1) then x solves the primal linear program (2.2) with $y = |x|$.*

We further note that the following proposition gives a sufficient condition, in terms of an optimal dual variable u of (2.3), which ensures that a solution of the primal linear program (2.2) solves AVE (1.1).

Proposition 2.2. *Let (x, y) be a solution of the primal problem (2.2) and (u, v, w) be a solution of the corresponding dual problem (2.3). Then:*

$$u > 0 \implies Ax - |x| = b. \tag{2.4}$$

Proof From the dual complementarity condition we have that:

$$v'(x + y) + w'(-x + y) = 0. \tag{2.5}$$

Hence, if $u > 0$, then $v + w = u > 0$, which by (2.5) implies that either $y_i = -x_i$, or $y_i = x_i$, for $i = 1, \ldots, n$. Consequently since $y \geq 0$, which follows by adding the last two constraints of (2.2), we have that $y = |x|$. Hence by the first constraint of (2.2) we have that $Ax - |x| = b$. □

Note that the proof of the above proposition is based on a fundamental idea first utilized in [5, Lemma 1] for solving a general linear complementarity problem by a single linear program.

We now make use of the above results to obtain nonexistence and existence of solution results for AVE (1.1) in terms of the variables (u, v, w) of the dual linear program (2.3).

3 Nonexistence of AVE Solution

The following nonexistence result is based on a sufficient condition for the constraints of the linear program (2.2) being empty, in which case AVE (1.1) has no solution. This is so because for any solution x of AVE (1.1) we have that x and $y = |x|$ satisfy the constraints of the linear program (2.2).

Proposition 3.1. *If there exist $(s, t) \in \mathbb{R}^{2n}$ such that:*

$$(A' - I)s + (A' + I)t = 0,\ \ b's + b't = 1.\ \ (s, t) \geq 0, \tag{3.6}$$

then the AVE (1.1) has no solution.

Proof As a consequence of (3.6) and by Gale's Theorem of the Alternative [6, Theorem 10,page 33], it follows that:

$$(A - I)x - b \geq 0, (A + I)x - b \geq 0, \textit{has no solution}. \tag{3.7}$$

This (3.7) consequence of (3.6) holding can also be seen directly from the ensuing contradiction if (3.7) had a solution x:

$$0 \leq s'(A-I)x - s'b + t'(A+I)x - t'b = -1 + x'((A'-I)s + (A'+I)t) = -1 < 0. \tag{3.8}$$

By setting $y = Ax - b$ in (3.7) we have that the following system has no solution (x, y):

$$Ax - y = b, \quad -x + y \geq 0, \quad x + y \geq 0. \tag{3.9}$$

This immediately implies that AVE (1.1) $Ax - |x| = b$ has no solution x, for if it did, then setting $y = |x|$, we have that $Ax - y = b$, $y = |x| \geq x \geq -|x| = -y$, thus contradicting (3.9) not having a solution (x, y). $\square$

We give an example now that implements the theorem above.

Example 3.2.

$$A = \begin{bmatrix} 2 & -5 & 4 \\ -3 & 3 & -2 \\ 6 & -7 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

By solving a linear program with a zero objective function and the linear inequalities of (3.6) as constraints, or more directly, we have that $s = 0$ and $t' = [1 \quad 3 \quad 1]$ satisfy (3.6). Thus we have that $t'(A+I) = [0 \ \ 0 \ \ 0]$, $t'b = 1$ and the assumptions of Proposition (3.1) are satisfied. Consequently for our Example (3.2), AVE (1.1) does not have a solution. To confirm that, we display the following contradiction if for this Example (3.2) AVE did have a solution x satisfying $Ax - |x| - b = 0$ and since by (3.6) $t'(A+I)x = 0$:

$$0 = t'(Ax - |x| - b) = t'(-x - |x| - b) \leq -t'b = -1. \tag{3.11}$$

Our nonexistence result Proposition (3.1) includes that of [14, Proposition 7] which can be derived as a special case of the nonexistence results presented here. Thus, if we add the assumption that $s > 0, \ \ t > 0$ to the inequalities of (3.6) of Proposition (3.1) we get that upon defining $r := s + t$:

$$(A'+I)r = r + s - t = 2s > 0, \quad (-A'+I)r = s + t - s + t = 2t > 0, \quad b'(s+t) = b'r > 0. \tag{3.12}$$

This then implies that:

$$r > A'r > -r, \quad b'r > 0, \tag{3.13}$$

which is the nonexistence sufficiency result of [14, Proposition 7].

We turn now to the existence of AVE solution.

4 Existence of AVE Solution

The existence of an AVE solution was already established in Proposition (2.2). We now give a more detailed existence result as follows.

PROPOSITION 4.1. *(AVE Solvability) The AVE (1.1) is solvable if there exist* $(x, y) \in \mathbb{R}^{2n}$ *that solve the linear program (2.2) and* $(u, v, w) \in \mathbb{R}^{3n}$ *that solve the dual program (2.3) such that:*

$$u > 0, \tag{4.14}$$

in which case x *solves AVE (1.1):* $Ax - |x| = b$ *with the following properties:*

$$x_i \geq 0, \quad for \;\; w_i > 0, \quad and \;\; x_i \leq 0, \quad for \;\; v_i > 0, \quad i = 1, \ldots, n. \tag{4.15}$$

Proof Note first that $y \geq 0$ follows from $y \geq x \geq -y$. Also, from the dual constraint (2.3) $-u+v+w = 0$ and the assumption here that $u > 0$, it follows that $v + w > 0$. Consequently from the complementarity conditions:

$$w'(-x + y) + v'(x + y) = 0, \;\; u'(Ax - y - b) = 0. \tag{4.16}$$

we have that:

$$Ax - y = b, \;\; w_i > 0 \implies x_i = y_i \geq 0, \;\; v_i > 0 \implies x_i = -y_i \leq 0, \;\; i = 1, \ldots, n. \tag{4.17}$$

Hence, $y = |x|$. Consequently $Ax - |x| = b$. □

We give now an example implementing the above theorem.

EXAMPLE 4.2.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ -1 & -3 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

For this example there is a dual optimal variable as follows:

$$u' = [1\ 1\ 1], \;\; v' = [0\ 1\ 0], \;\; w' = [1\ 0\ 1]. \tag{4.19}$$

Hence there is an AVE solution for this problem such that $x_1 \geq 0, \; x_2 \leq 0, \; x_3 \geq 0$. Consequently AVE $Ax - |x| = b$ can be rewritten as a linear matrix equation $Ax - Dx = b$, where D is a diagonal matrix defined according to the signs of x as follows: $D = diag([1 \;\; -1 \;\; 1])$. Thus $Ax - Dx = b$ is satisfied by $x' = [1 \quad \zeta \quad (-1 - 2\zeta)]$, $\zeta \in \mathbb{R}$, $\zeta \leq -1/2$ as a solution to $Ax - |x| = b$.

5 Numerical Results

Numerical implementation of our sufficiency results were carried out on ten solvable absolute value equations each with $n = 100$. We utilized the CPLEX linear programming code [4] within MATLAB [15]. The solvable absolute equations were easily generated by picking a random 100×100 matrix A, a random solution $x \in R^{100}$ and then defining $b = Ax - |x|$. We note that because it is not easy to generate large dimensional unsolvable absolute value equations, only solvable test equations were utlized.

When Proposition 3.1 was applied to each of the ten solvable absolute value equations, it gave the result that there was no feasible point. Hence the sufficient unsolvabilty conditions do not apply, as should be the case because each of the ten absolute value equations are solvable.

When Proposition 4.1 was applied to each of the ten solvable absolute value equations, it did not give a positive multiplier: $u > 0$. Hence we could not independently conclude that each of the ten absolute value equations is solvable.

6 Conclusion and Outlook

We have given sufficient conditions for the unsolvability and solvability of the NP-hard absolute value equation $Ax - |x| = b$. These conditions consist of linear inequalities that can be easily checked by using any linear programming software package such as CPLEX [4]. It would be a very significant contribution if the sufficient conditions can be modified so as to become necessary and sufficient conditions. This may be an extremely difficult task given that AVE is NP-hard.

Acknowledgments The research described here, based on Data Mining Institute Report 16-01, October 2016, was supported by Microsoft and ExxonMobil.

References

[1] S.-J. Chung. NP-completeness of the linear complementarity problem. *Journal of Optimization Theory and Applications*, 60:393–399, 1989.

[2] R. W. Cottle and G. Dantzig. Complementary pivot theory of mathematical programming. *Linear Algebra and its Applications*, 1:103–125, 1968.

[3] R. W. Cottle, J.-S. Pang, and R. E. Stone. *The Linear Complementarity Problem.* Academic Press, New York, 1992.

[4] ILOG CPLEX Division, 889 Alder Avenue, Incline Village, Nevada. *CPLEX Optimizer.* `http://www.cplex.com/`.

[5] O. L. Mangasarian. Linear complementarity problems solvable by a single linear program. *Mathematical Programming*, 10:263–270, 1976.

[6] O. L. Mangasarian. *Nonlinear Programming.* SIAM, Philadelphia, PA, 1994.

[7] O. L. Mangasarian. Absolute value equation solution via concave minimization. *Optimization Letters*, 1(1):3–8, 2007. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/06-02.pdf.

[8] O. L. Mangasarian. Absolute value programming. *Computational Optimization and Applications*, 36(1):43–53, 2007. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/05-04.ps.

[9] O. L. Mangasarian. A generalized Newton method for absolute value equations. Technical Report 08-01, Data Mining Institute, Computer Sciences Department, University of Wisconsin, May 2008. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/08-01.pdf. Optimization Letters 3(1), January 2009, 101-108. Online version: http://www.springerlink.com/content/c076875254r7tn38/.

[10] O. L. Mangasarian. Knapsack feasibility as an absolute value equation solvable by successive linear programming. *Optimization Letters*, 3:161–170, 2009. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/08-03.pdf. Optimization Letters online version: http://www.springerlink.com/content/7011287432285747/.

[11] O. L. Mangasarian. Absolute value equation solution via dual complementarity. Technical report, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, September 2011. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/11-03.pdf. Optimization Letters 7(4), 2013, 625-630.

[12] O. L. Mangasarian. Primal-dual bilinear programming solution of the absolute value equation. Technical report, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, February 2011. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/11-01.pdf, Optimization Letters 6(7), 1527-1533, 2012.

[13] O. L. Mangasarian. Unsupervised classification via convex absolute value inequalities. Technical Report 14-01, Data Mining Institute, Computer Sciences Department, University of Wisconsin, Madison, Wisconsin, March 2014. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/14-01.pdf. Optimization 64(1) 2015, 81-86.

[14] O. L. Mangasarian and R. R. Meyer. Absolute value equations. *Linear Algebra and Its Applications*, 419:359–367, 2006. ftp://ftp.cs.wisc.edu/pub/dmi/tech-reports/05-06.pdf.

[15] MATLAB. *User's Guide*. The MathWorks, Inc., Natick, MA 01760, 1994-2006. http://www.mathworks.com.