

PRESERVICE MATHEMATICS TEACHERS' UNDERSTANDINGS OF INTENSIVE
QUANTITIES AND FUNCTIONS INVOLVED IN INTRODUCTORY MATHEMATICAL
MODELS FOR GLOBAL WARMING

by

DARÍO ANDRÉS GONZÁLEZ MARTÍNEZ

(Under the Direction of Dorothy Y. White)

ABSTRACT

This dissertation examined preservice mathematics teachers' (PSTs) understandings of intensive quantities and functions involved in introductory mathematical models for global warming. The study had two parts. The first part explored PSTs' conceptions of three intensive quantities: *concentration*, *energy density*, and *energy flux density*. The second part examined PSTs' covariational reasoning regarding three functions: *the forcing by CO₂ function $F(C)$* , *the planetary energy imbalance function $N(t)$* , and *the mean surface temperature function $T(t)$* . The study followed an exploratory, multiple-case research design. Three PSTs enrolled in a mathematics education program at a large Southeastern university participated in the study, each completing six mathematical tasks designed for the study during four individual, task-based interviews. The study's first part revealed that PSTs' conceptions of the intensive quantities were shaped by their understandings of a quantity's: (a) *measurable attribute* (what is being measured) and (b) *measurement process* (how the attribute is being measured). To identify a measurable attribute, PSTs needed to conceptualize each quantity as a *constant multiplicative relationship between quantities that can vary*. Depending on the quantity, PSTs used division

differently to measure such attribute (measurement process): *measurement division* for concentration (Type 1) and *partitive division* for energy density and energy flux density (Type 2). The latter two quantities were challenging for PSTs because they conceived temperature and energy as equivalent quantities. The study's second part revealed that PSTs did not reason about co-variation in terms of the rate of change when making sense of the functions. Two PSTs were often unable to reason beyond the direction in which change was occurring. The other consistently reasoned in terms of amounts of change in y for changes in x . Additionally, PSTs' conceptions of intensive quantities and ability to reason about co-variation impacted their understanding of two central concepts regarding global warming: *the Earth's energy budget* and *the radiative equilibrium*. Also, there was an unexpected finding regarding PSTs' ability to conceive what I termed *monotonically asymptotic variation*, which appeared central to make sense of all three functions. I concluded with a discussion of the study's implications and suggestions for future research.

INDEX WORDS: Conceptions, Covariational Reasoning, Climate Change, Functions, Global Warming, Intensive Quantities, Preservice Teacher Education, Quantitative Reasoning, Rate of Change, Ratios, STEM Education

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DEDICATION

Where I am from, an accomplishment such as this is often the result of a whole family effort. I dedicate this dissertation to those people that provided the foundation upon which this dissertation was built and gave support without expecting anything in return. I dedicate this dissertation to my *Wife Rachel* whose endless love gave me the strength to complete this challenge, and whose remarkable intellectual insights gave me tools to improve my academic work. I dedicate this dissertation to my *Mother* and *Tía* who have always believed in me even when I didn't believe in myself. I dedicate this dissertation to my *Father* and *Grandmother Teresa* whose spirits I believe are always by my side. I dedicate this dissertation to my in-laws *Mitchell, Laura, and Jolene Combs* who accepted me as one of their own and gave me a family in a foreign land. I dedicate this dissertation to *The Moras* who also accepted me as one of their own and gave me a family in a foreign land. Finally, I dedicate this dissertation to my friend *Miguel* who helped me start this fantastic journey. ¡Les estaré eternamente agradecido y los amo mucho!

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CHAPTER 1

INTRODUCTION

In recent years, there have been several calls to include global warming in school and college instruction (McKeown & Hopkins, 2010; UNESCO, 2012). Yet, the public lacks important knowledge needed to understand global warming (Leiserowitz, Smith, & Marlon, 2010; Pruneau, Khattabi, & Demers, 2010). Even among college students, including preservice science teachers, there exist many misconceptions about important concepts related to this environmental phenomenon (Wachholz, Artz, & Chene, 2014). Teachers and students can have difficulties recalling or understanding these concepts without a certain level of scientific and mathematical literacy (Barwell, 2013a, 2013b; Lambert & Bleicher, 2013). By studying global warming, teachers and their students can experience how mathematics and science, together, can help them understand environmental changes that are reaching all corners of our planet (Intergovernmental Panel on Climate Change [IPCC], 2013). Therefore, there is a need for studies connecting mathematics education research and education about global warming.

My research investigated preservice mathematics teachers' (PSTs) understandings of intensive quantities and functions involved in introductory mathematical models for global warming—for which the mathematics can be accessible to high-school students. I consider PSTs an important group to be informed about global warming because they will educate the members and future leaders of this democratic society. PSTs may be reluctant to include global warming in their future classroom because they do not know much about the topic. I believe that helping PSTs see connections between mathematics and global warming can motivate them to include

these socially and ecologically complex phenomenon into their mathematics instruction. The mathematical models included in this study make use of intensive quantities to quantify variation in atmospheric gases, heat transfer, and surface temperature. Also, they make use of functions to express relationships between co-varying quantities and to link carbon dioxide pollution to global warming. Research in mathematics education, however, has demonstrated that students and future mathematics teachers can have persistent difficulties: (a) understanding intensive quantities and ratios in terms of quantities (Greer, 1992; Howe, Nunes, & Bryant, 2010; Nunes, Desli, & Bell, 2003; Thompson, 1994b) and (b) comprehending and mathematically expressing co-variation represented by functions (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012, 2015; Oehrtman, Carlson, Thompson, 2008; Thompson, 2011).

Background Information

Definition of Quantities

This study focused on three intensive quantities commonly used to model global warming: concentration, energy density, and energy flux density. *Concentration* is an intensive quantity defined by measuring the average abundance of a substance relative to the volume of a mixture containing the substance. Mathematically, concentration is the ratio between the amount of a substance, expressed in units of mass (e.g., pounds or kilograms) or volume (e.g., ounces or cubic centimeters), and the volume of the mixture. For instance, air is a mixture of different gases such as Nitrogen, Oxygen, Argon, and other trace gases (including Carbon Dioxide). Concentration emerges as a way of measuring the average abundance of a gas (substance) relative to a volume of air (mixture). Particularly, concentration emerges in its fractional form when defined as the ratio between the volume of a gas and the volume of air (dimensionless). When concentrations are relatively small, they are often measured in *parts per million by volume*

or *ppmv*. This unit is defined as the average volume of a substance per one million volume units of the mixture. For instance, today's atmospheric concentration of carbon dioxide (CO_2) is approximately 406 ppmv, which means that there is an average of 406 cm^3 of carbon dioxide per 1,000,000 cm^3 of air in the atmosphere. When modeling global warming, ppmv is the preferred unit to define concentration of greenhouse gases in the atmosphere.

Radiation is the emission of energy from one object (source) to another object (receiver). Broadly speaking, heat represents a type of radiation and is closely related to the concept of temperature. However, these two concepts are different: temperature is a measure of the average internal energy of an object, while heat is a measure of the magnitude of energy flowing from one object to another due to their temperature difference. For instance, the sun (source) radiates energy, in the form of heat, toward the Earth's surface (receiver). Following the previous definitions, the magnitude of solar energy incident to the planet's surface averaged over the planet's surface area is a measure of the average surface temperature of the planet. In this context, *energy density* can be defined as the magnitude of energy incident to an object's surface averaged over the object's surface area. Mathematically, energy density is a ratio between a magnitude of energy, measured in Joules (J), and a magnitude surface area, measured in square meters (m^2). *Energy flux density* is an intensive quantity defined by measuring the radiation rate of energy incident to an object's surface averaged over the object's surface area. Mathematically, energy flux density is the ratio between a magnitude of radiation rate, measured in Joules per second (J/s), and a magnitude of surface area, measured in square meters (m^2). For instance, the sun radiates energy toward the Earth's surface at a certain rate, let's called it solar radiation rate for convenience. The magnitude of the solar radiation rate incident to the planet's surface

averaged over the planet's surface area is a measure of the rate of change of the average surface temperature of the planet.

Modeling Global Warming

Lambert and Bleicher (2013) have suggested that future teachers need to learn about and work with two important notions in order to understand global warming: the Earth's energy budget and the link between carbon dioxide pollution and global warming. The sun powers the Earth's climate system and there is a continuous flow of energy between the sun, the planet's surface, and the atmosphere. This continuous flow of energy is known as the *Earth's energy budget* (Figure 1.1). The sun warms the planet's surface (S). As the surface warms up, it radiates (infrared) energy to the atmosphere (R), most of which is absorbed by *greenhouse gases* (GHG) such as water vapor (H₂O), carbon dioxide (CO₂), and methane (CH₄) (B). The atmosphere re-radiates the absorbed energy in both directions toward space and toward the surface (A). This continuous energy exchange between the surface and the atmosphere is known as the *greenhouse effect* and influences the planet's average surface temperature. Quantifying changes in the energy flows and the abundance of GHG in the atmosphere is required to accurately model climate and global warming. The energy flows S, R, B, L, and A are all measured in terms of energy flux density (J/m²/s), while the abundance of atmospheric GHG is measured in terms of concentration (ppmv). The parameter $0 < g < 1$ is related to the greenhouse effect. When the abundance of any GHG increases, the atmosphere absorbs more infrared energy from the planet's surface (B increases). Thus, the atmosphere then re-radiates more infrared energy towards the surface (A increases). Humans release large amounts of CO₂ into the atmosphere by burning fossil fuels such as oil and coal (IPCC, 2013). This has increased the abundance of CO₂ in the atmosphere

enhancing the greenhouse effect, which in turn influences the planet's average surface temperature. This is the connection between *CO₂ pollution* and *global warming*.

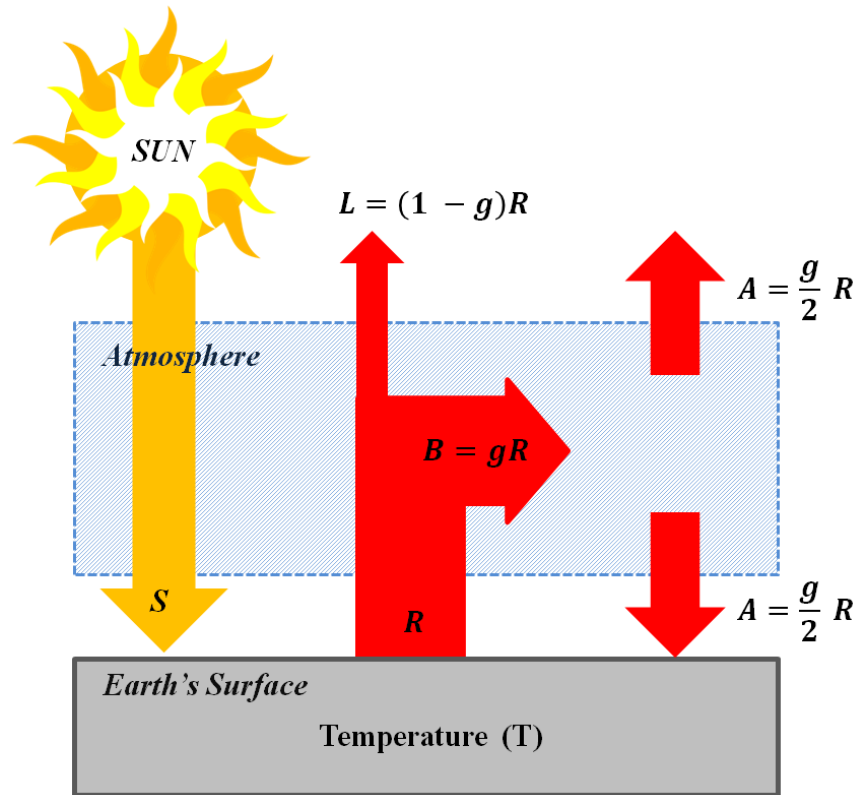


Figure 1.1. The Earth's energy budget, assuming a one-layered atmosphere

The *planetary energy imbalance* (N) is a measure of the energy imbalance in the Earth's energy budget. In particular, N can be defined as a difference between the combine magnitude of all downward energy flows and the combine magnitude of all upward energy flows at the planet's surface (Figure 1.1), or mathematically $N = (S + A) - R$. The Earth's energy budget is said to be in radiative equilibrium when $N = 0$, which implies that the average surface temperature of the planet remains constant. In contrast, if $N \neq 0$, then the budget experiences an energy imbalance. This imbalance results in a variation of the average surface temperature. $N > 0$ is of particular interest for my study. This results in a warming effect over the planet's surface (increase in temperature), which causes the surface's upward energy flow (R) to increase. This

flow will continue to increase until the magnitude of the upward energy flows equals that of the downward energy flows ($N = 0$) so that a new radiative equilibrium is reached. There are several factors, commonly known as *forcing agents* that can cause an energy imbalance. A change in the planetary energy imbalance (N) cause by a particular forcing agent is known as *radiative forcing* or simply *forcing*. The forcing by CO_2 (F) is of particular interest for my study because F is the change in N caused by a change in the atmospheric concentration of CO_2 (C). When C changes, the composition of the atmosphere changes, resulting in changes in the energy flows B and A . For instance, an increase in C results in an increase in the atmosphere's capacity to absorb infrared energy, or an increase in B . This causes the energy flow A to increase since $A = \frac{1}{2}B$ (Figure 1.1), which in turn causes $N = (S + A) - R$ to increase. This value of N is the magnitude of the forcing by CO_2 . In my study, I examined how PSTs' reasoning about co-variation while making sense of the forcing by CO_2 function $F(C)$. The magnitude of F , measured in $\text{J}/\text{m}^2/\text{s}$, grows logarithmically as C , measured in ppmv, increases (IPCC, 2013). Mathematically,

$$F(C) = \beta \ln\left(\frac{C}{C_0}\right)$$

where F is forcing by CO_2 , β is a positive constant, C is the atmospheric concentration of CO_2 , and C_0 is a reference concentration of CO_2 .

I also examined how PSTs' reasoning about co-variation regarding two other functions: the planetary energy imbalance with respect to time and the planet's mean surface temperature with respect to time, after the concentration of CO_2 is instantaneously increased at time zero (positive forcing). The planetary energy imbalance as a function of time can be defined by $N(t) = (S + A(t)) - R(t)$. The energy flow S is approximately constant and is known as the *solar constant*. When the concentration of CO_2 is instantaneously increased at $t = 0$, a positive forcing of magnitude $F > 0$ is created, which in turn implies that $N(0) = F$. This means that the

downward energy flow ($S + A$) exceed in magnitude the upward energy flow R , which produces a warming effect over the planet's surface, which in turn causes the surface's radiation R to increase. The surface's radiation R will continue to increase until it equals the downward energy flow ($S + A$) and a new radiative equilibrium is reached (i.e., $N = 0$). Mathematically, we have that $A = \frac{g}{2}R$ for $0 \leq g \leq 1$ (see Figure 1.1), which implies that

$$N(t) = (S + A(t)) - R(t) = S - \left(1 - \frac{g}{2}\right)R(t)$$

Therefore, as R increases, the magnitude of the energy imbalance decreases (i.e., if $t \rightarrow \infty$, then $R(t) \rightarrow (S + A(t))$ and $N(t) \rightarrow 0$). This function can be model by an exponential decay function such as

$$N(t) = F \cdot \exp\left(-\frac{t}{d}\right)$$

where F represents a positive forcing, t is time in years, and d is a constant expressing the mean lifetime. Finally, the planetary energy imbalance is directly related to the planet's average surface temperature. After a positive forcing has occurred, $N(t)$ decays exponentially but remains positive until a new radiative equilibrium has been reached. $N(t) > 0$ implies that the combine magnitude of the downward energy flows exceeds the combined magnitude of the upward energy flows. The content of energy in the planet's surface is then increasing, which means that the surface temperature is also increasing. In conclusion, the magnitude of $N(t)$ is a measure of how fast the surface temperature is increasing. Mathematically, the rate of change of the planet's average surface temperature with respect to time is proportional to the planetary energy imbalance (Lenton, 2000)

$$\frac{\partial T}{\partial t} = \alpha \cdot N(t)$$

where t is the time in years and α is a constant.

Statement of the Problem

As discussed above, global warming is a potentially engaging and motivating context to learn about important scientific and mathematical concepts. In particular, introductory mathematical models for global warming make use of intensive quantities such as concentration, energy density, and energy flux density. Intensive quantities are defined through the division of two quantities and are extensively used in physics and science (Howe, Nunes, & Bryant, 2010). Conceptualizing intensive quantities models can be more complex than conceptualizing extensive quantities such as distance, time, or mass because they involve proportional relationships between quantities (Nunes, Desli, & Bell, 2003). Additionally, introductory mathematical models for global warming used functions to express co-variation between quantities. Reasoning about quantities that vary together and functions is central for making sense of mathematical applications to physics and science (Carlson et al., 2002). Thus, it is important to explore how PSTs make sense of intensive quantities and reason about co-variation represented by functions involved in introductory mathematical models for global warming.

Unfortunately, there is little research in mathematics education connecting mathematics learning and global warming. However, Barwell (2013a) have identified three questions connecting mathematics education research and global warming: (1) what mathematics are involved in modeling global warming?, (2) what mathematics related to global warming do people need to understand to take an informed stance about the issue?, and (3) what mathematics related to global warming is within the scope of school mathematics. In my dissertation, the mathematics of global warming that PSTs need to know and are within the scope of school mathematics include: the intensive quantities concentration, energy density, and energy flux density; the logarithmic function (forcing by CO₂ function); the exponential function (planetary

energy imbalance function); and the rate of change as a function (planetary energy imbalance function is proportional to the rate of change of the planet's mean surface temperature function).

Finally, the introductory mathematical models at the center of this dissertation combine distinct strands of mathematics education research that have not been combined within a single study (i.e. division, physical intensive quantities, reasoning about quantities that vary together). Literature in mathematics education research lacks studies investigating PSTs' conceptions of concentration, energy density, and energy flux density, and how PSTs conceive of co-variation between these intensive quantities. Because little is known about how PSTs explore and make sense of introductory mathematics models for global warming, I employed detailed case-studies to gain insights into how PSTs make sense of (1) intensive quantities commonly used to model global warming and (2) co-variation represented by functions linking carbon dioxide pollution to global warming.

Purpose of the Study and Research Questions

My dissertation study examined secondary preservice mathematics teachers' (PSTs) understandings of intensive quantities and functions involved in introductory mathematical models for global warming. The study was guided by two overarching research questions and four supporting questions:

- 1) What are PSTs' conceptions of intensive quantities commonly used to model global warming?
 - a. How do PSTs use division to define these intensive quantities?
 - b. What measurable attributes do PSTs identify for each intensive quantity?

- c. What differences can be observed in PSTs' work with Type 1 intensive quantities (concentration) versus Type 2 intensive quantities (energy density and energy flux density)?
- 2) How do PSTs reason about co-variation regarding three functions involved in introductory mathematical models for global warming?
 - a. What level of covariational reasoning do PSTs demonstrate when working with the functions involved in this study?
 - b. How do PSTs' understanding of the Earth's energy budget and radiative equilibrium relate to PSTs' understanding of the functions involved in this study?
 - 3) In what ways do PSTs' conceptions of intensive quantities relate to PSTs' understanding of the functions involved in this study?

The study followed an exploratory, multiple-case research design, which is a variation of *Case Study* design (Yin, 2013). The units of analysis included participants' responses to six mathematical tasks designed for this study. These tasks were divided into two sets, each set corresponding to one of the two overarching research questions. Set 1 included tasks describing situations in which PSTs were required to determine and work with concentration, energy density, and energy flux density to complete each task. Set 2 included tasks describing situations in which PSTs were required to conceptualize and make sense of three functions linking CO₂ pollution and global warming. Three PSTs enrolled in a mathematics education program at a large Southeastern university participated in my study. Each PST completed all mathematical tasks in the course of four individual, task-based interviews (Goldin, 2000). All interviews followed a semi-structured approach and the interview videos and transcripts were analyzed through the Framework Analysis (FA) method (Ward, Furber, Tierney, & Swallow, 2013). Initial

codes emerged from data and were also informed by the research literature. The initial codes were used to develop a working analytic framework that I applied back to the data; through this process, codes were refined and data was reduced.

Significance of the Research

By examining PSTs' understandings regarding intensive quantities and functions involved in introductory mathematical models for global warming, my dissertation study makes two fundamental contributions to education. First, it generates new knowledge about ways that mathematics and science can be studied together, which addresses recent calls to focus on integrated STEM (science, technology, engineering, and mathematics) education (Baldwin, 2009; McCright, O'Shea, Sweeder, Urquhart, & Zeleke, 2013). This dissertation involved two scientific ideas needed to understanding global warming: the *Earth's energy budget* and the connection between *carbon dioxide pollution* and *global warming* (Lambert & Bleicher, 2013). While exploring the Earth's energy budget, PSTs were able to determine how the variations in concentration of carbon dioxide and energy flows regulate the Earth's climate. They also were exposed to other related scientific concept such as radiation, heat, temperature, and the greenhouse effect. To quantify variations in concentration of CO₂ and energy flows, PSTs defined, worked with, and interpreted intensive quantities and their units. To make sense of intensive quantities, PSTs made use of division to model situations and reason about ratios in terms of quantities. Thus, exploring the Earth's energy budget is consistent with both science and mathematics education standards. Particularly, it relates to the Next Generation Science Standards HS-ESS 2 through 4 (National Research Council [NRC], 2013) and to the Common Core State Standard for Mathematics standards HS-NQ 1 through 3 (Common Core State Standards Initiative [CCSSI], 2010). Additionally, PSTs examined how changes in the

atmosphere due to human activity have increased CO₂ concentrations and thus affect climate. This allowed PSTs to see the link between carbon dioxide pollution and global warming. To establish a quantitative relationship, PSTs made use of functions to describe the co-variation between concentration of CO₂, energy flows, and surface temperature. Thus, exploring this link is consistent with Next Generation Science Standards HS-ESS 2 through 4 (NRC, 2013) and with Common Core State Standard for Mathematics standards HS-F-IF 2, HS-F-BF 1, and modeling with mathematics in general.

Second, this dissertation generated new knowledge regarding how *quantitative* and *covariational reasoning* can support or constrain PSTs' ability to make sense of mathematical models for real-world situations. Mathematical modeling is an essential cognitive tool to understand the world around us as well as an important skill to be learned according to national standards (NCTM, 2000; CCSSI, 2010). To construct a mathematical model, students must conceive of relevant quantities and their measurements, and reason about how these quantities vary together (co-variation) in the situation. To make sense of the introductory mathematical models for global warming involved in this dissertation, PSTs needed to make sense of the quantities and functions involved in such models. As a result, this dissertation combined distinct strands of mathematics education research that have not been combined within a single study: intensive quantities, co-variation, and mathematical modeling. Moreover, modeling real-world situations is an opportunity to explore and discuss important, motivating, and engaging phenomena regarding our society or environment. As a result, this dissertation sheds light on the role of mathematics learning and teaching in developing global warming awareness among PSTs.

CHAPTER 2

REVIEW OF THE LITERATURE

In this chapter, I review research literature relevant for the present study. I divided the study into two parts. The first part involved exploring PSTs' conceptions of intensive quantities commonly used to model global warming. The second part involved examining PSTs' ability to reason about co-variation when learning about introductory models for global warming. Thus, I have organized the chapter into four sections, two regarding the first part and two regarding the second part: (a) concerning intensive quantities and ratios, (b) conceptual framework: conceptions of intensive quantities, (c) concerning co-variation and covariational reasoning, and (d) the co-variation framework. In this chapter, I described the research and theory in which the present study is situated.

Concerning Intensive Quantities and Ratios

In this section, I reviewed and summarized research literature concerning topics related to the first part of my research: PSTs' conceptions of intensive quantities commonly used to model global warming. I have divided this literature review into four sections: quantitative reasoning, intensive quantities, division, and ratio and rate. I conclude this section with the discussion of the conceptual framework that I used to characterize PSTs' conceptions of concentration, energy density, and energy flux density.

Quantitative Reasoning

Thompson (2011) discussed the concept of *Quantitative Reasoning* (QR) to explain the process by which a person constructs a quantity. For Thompson, a quantity is a mental

construction and not an object in the *real world*. He explains that the construction of a quantity is an effortful and complex process in which the quantity's meaning emerges from the triad object-attribute-quantification. In other words, a person conceives an object to be quantified, identifies a measurable attribute of that object, and defines a unit and a measurement process to quantify that attribute. Thompson uses the quantity *torque* in order to illustrate how the triad object-attribute-quantification is enacted. The object a person may conceive when conceiving of torque is a system involving a force twisting something around an axis; the system behaves differently depending on how far away from the axis that turning force is applied. A measurable attribute of that object is "amount of twist." The conceptualization of this attribute involves the realization that the closer from the rotating axis a turning force is applied, the greater the turning force needs to be in order to produce the same "amount of twist." Finally, the person conceives of a unit to measure the attribute "amount of twist" and a measurement process in order to quantify torque. The quantification of torque must simultaneously take into account the distance at which the turning force is applied and the magnitude of that turning force; thus, a unit of torque is $(\text{distance}) \times (\text{force})$ such as Newton-meter or foot-pound (Figure 2.1). In addition to develop a unit of measure, the quantification of torque involves the understanding of the proportionality between the components of its unit of measure. For instance, if the force is applied at twice the distance, then we obtain twice the amount of torque (Figure 2.1). Through this illustration, Thompson emphasizes that the meaning for a quantity is inseparable from the quantification process of that quantity; they are defined through a dialectic relationship.

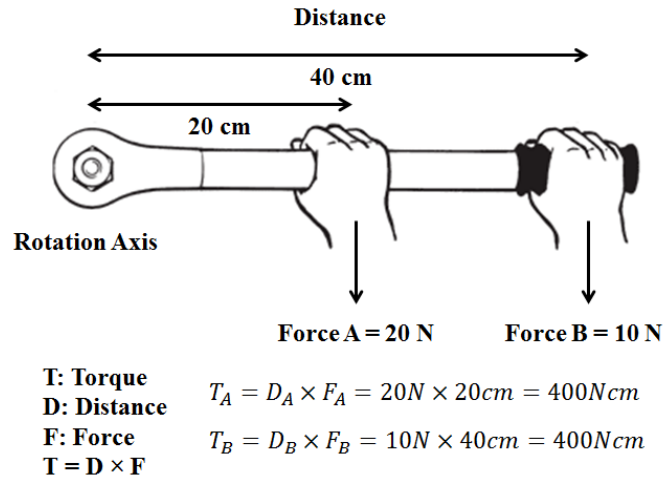


Figure 2.1. The quantity of torque

Intensive Quantities

Quantities can arise in either of two ways (Schwartz, 1988). The first way is when humans need to quantify a situation or phenomena either by counting (discrete quantities) or measuring (continuous quantities); these quantities are known as *extensive quantities*. Examples of extensive quantities include distance, mass, or surface area. The second way is when humans derive quantities by performing mathematical operations with already defined quantities. Examples of derived quantities include torque, correlation coefficient, or speed. Among the group of derived quantities, we find *intensive quantities* because they are the result of a ratio between two already defined quantities, which may or may not be extensive. Examples of intensive quantities include radians, density, or speed. Simon and Placa (2012) provided the following definition of intensive quantity, which reads

An intensive quantity is the relative size of the magnitudes (number of units) of measures from two different measure spaces, or of the measures of two different quantities from the same measure space, given particular units of measurement for each measure. (p. 39)

Nunes et al. (2003) distinguish between two types of intensive quantities: Type 1 includes intensive quantities whose constituent quantities form a part-whole relationship. For instance, in a mixture of nitrogen, oxygen, and carbon dioxide, the concentration of oxygen is the fraction of the mixture's volume that oxygen represents; here, oxygen and the mixture form a part-whole relationship. Intensive quantities of this type are often represented in terms of fractions, decimals, or percentages. Type 2 includes intensive quantities relating two quantities whose measures remain separate. For instance, energy density is a ratio between energy and surface area. Here, the measures of both constituent quantities remain separate because they belong to two different measure spaces. Intensive quantities of this type are often represented in terms of rate and not as fractions.

Simon and Placa (2012) explain that determining the magnitude of an intensive quantity is more complex than determining the magnitude of an extensive quantity. Extensive quantities such as length, area, or volume are measured in units defined within the same measure space (e.g., length is measured in units of length such as foot, meter, or mile). Thus, conceiving a unit of measure determines the magnitude of the extensive quantity. In contrast, intensive quantities are defined through division and express a constant multiplicative relationship between two quantities that may or may not involve units from the same measure space. For instance, a measure of radians involves a ratio between two measures from the same measure space; that is, distance per unit of distance such as feet per foot or centimeters per centimeter. We often refer to these intensive quantities as dimensionless since they involve a ratio between measures with the same units. In contrast, a measure of speed involves a ratio between two measures from different measure spaces, or distance per unit of time such as miles per hour or meter per second. Therefore, an intensive quantity's magnitude requires conceiving units for each of the constituent

quantities and multiplicatively relating these units so that a measure of quantity A is equivalent to a measure of quantity B (Simon & Placa, 2012). Because intensive quantities represent a multiplicative relationship between two quantities, they involve both *direct proportional relations* and *inverse proportional relations* with their constituent quantities (Nunes et al., 2003). For instance, energy density holds a direct proportional relation with energy – twice the energy implies twice the energy density – and an inverse proportional relation with surface area – half the surface area implies twice the energy density. Thus, constructing meaning for intensive quantities requires (a) to reason in terms of proportional relations and (b) to understand how the intensive quantity relates to each of its constituent quantities (Nunes et al., 2003).

Division

Division is an important arithmetic operation that can be used to model different real-world situations. Since intensive quantities are formed through division, how division is used and interpreted may influence one's conceptions of these quantities. Beckmann (2014) explains that there are two different ways of interpreting division: as a *partitive division* and as a *quotitive division*. Greer (1992) provides the following definition for these two interpretations

Dividing the total by the number of groups to find the number in each group is called partitive division, which correspond to the familiar practice of equal sharing (with social implications of equity). Dividing the total by the number in each group to find the number of groups is called quotitive division (sometimes termed measurement division, reflecting its conceptual links with the operations of measurement). (p. 276)

Regardless of how division may be interpreted, situations involving multiplication and division can present more difficulties for students than those situations involving addition and subtraction. Unlike addition and subtraction which are unidimensional, multiplication and

division involve dimensional complexity (Greer, 1992). For instance, an intensive quantity such as speed involves two dimensions: a dimension of distance and a dimension of time. Division can be applied in different ways depending on how a student conceives a particular situation. As a result, distinctions between different types of situations involving division can be made. Greer (1992) identifies and offers a classification of different situations modeled by multiplication and division. There are three ways of applying division to situations that are of particular interest for my study: a *part-whole relation*, a *multiplicative comparison*, and a *rate*. Each way of applying division includes two different types of situations depending on whether division is interpreted as a partitive or quotitive (Table 2.1).

Division as a *part-whole relation* involves applying division to determine what fraction of a total (whole) a certain quantity (part) represents. This application often includes situations in which two (or more) quantities can be added together to form a total; hence, these quantities can be measured in units from the same measure space. For instance, concentration is a ratio between the amount of a substance mixed in a solution and the total amount of solution, which is formed by several substances. The amount of each substance mixed in the solution is often measured in volume units such as ounces, liters, or cubic centimeters. The amount of solution is also measured in volume units since it represents is the sum of all volumes. Thus, concentration defines a part-whole relationship between amount of a substance and amount of solution.

Division as a *multiplicative comparison* involves applying division to define (and represent) a many-to-one correspondence between two quantities. Here, division defines an association between an amount of quantity A and one unit of quantity B. Unlike division as a part-whole relation, the quantities involved in the situation can be measured in units from different measure spaces. For example, energy density is a ratio between the magnitude of energy incident to an

object's surface and the object's surface area. By its definition, energy density defines a many-to-one correspondence between energy and surface area, such as 350 Joules correspond to 1 square meter (350 J/m^2). Finally, division as a *rate* involves applying division to define the relative magnitude of an intensive quantity. The relative magnitude represents a magnitude of a quantity measured in terms of another quantity, and thus it defines a single value representing constant multiplicative relationship between those quantities. Similar to division as a multiplicative comparison, division as a rate may or may not involve a ratio between units from the same measure space. For instance, using division as a rate to define energy density involves conceiving the resulting single value as a magnitude of energy averaged over a surface.

Table 2.1

Situations Involving Division according to Application and Interpretation

Class	Use of Division	
	Partitive	Measurement or Quotitive
<i>Part-whole Relation</i>	A college passed the top 3/5 of its students in an exam. If 48 passed, how many students sat the exam?	A college passed the top 48 out of 80 students who sat an exam. What fraction of the students passed?
<i>Multiplicative Comparison</i>	Iron is 0.88 times as heavy as cooper. If a piece of iron weighs 3.7 kg, how much does a piece of cooper the same size weigh?	If equally sized pieces of iron and copper weigh 3.7 kg and 4.2 kg respectively, how heavy is iron relative to copper?
<i>Rate</i>	A boat moves 13.9 meters in 3.3seconds. What is the average speed in meters per second?	How long does it take a boat to move 13.9 meters at a speed of 4.2 meters per second?

Note. Adapted from “Multiplication and division as models of situations” by B. Greer, 1992, In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York, NY: Macmillan, p. 280

Ratio and Rate

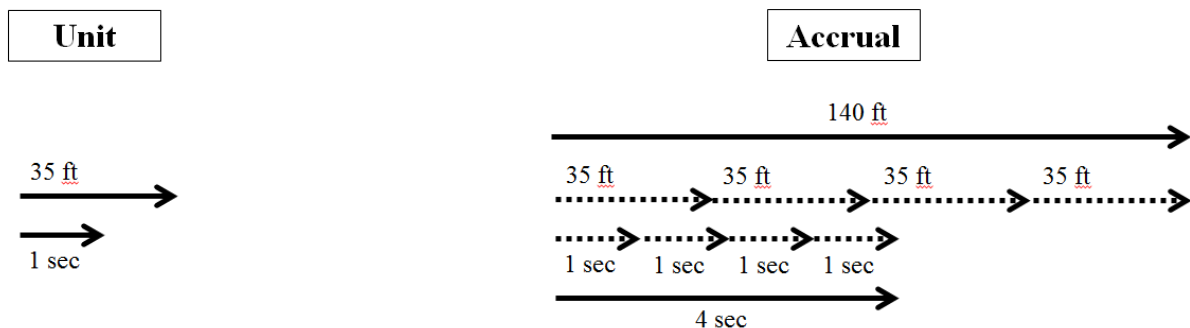
The distinction between the terms ratio and rate is sometimes unclear or ill-defined. Thompson (1994b) offers a distinction between ratio and rate based on the mental operations by which a student conceives a particular situation involving ratios. For Thompson, *ratio* or *internalized ratio* is “the result of comparing two quantities multiplicatively” (p. 190). This

multiplicative comparison is described in terms of associating static, non-varying values of two (usually extensive) quantities. Thompson identifies three ways of comparing multiplicatively two quantities or three ways of producing a ratio: association between particular values of two quantities, association between a particular value of a quantity and a unit of another quantity, and as the quotient between the unknown but fixed values of two quantities. The comparison can involve an association between particular values of two quantities. For instance, one could conceive of concentration of oxygen as an association between 125 liters of oxygen and 500 liters of air, or a simplified version thereof, like 1 liter of oxygen associated with 4 liters of air. The comparison can also involve measuring one quantity in relation to another quantity, which often results in what is commonly known as a *unit rate*. For instance, one could conceive of the value 235 J/m^2 , a measure of energy density, as an association between 235 Joules and 1 square meter. Finally, the comparison may involve the unknown but fixed values of two quantities, comparison that results in a ratio of known and fixed value. Thompson includes this latter comparison as a ratio because the quantities' value, albeit unknown, are present in thought and conceived as fixed. For instance, a student can conceive of 0.23 concentration of oxygen as the result of a ratio between a particular volume of oxygen and a particular volume of air. Although the volumes of oxygen and air may be unknown, the student understands 0.23 as the exclusive result of dividing those two particular values; any change in those values and the ratio would have a different value. This case does not involve the realization that a ratio is a constant multiplicative relationship between values that can vary.

For Thompson, *rate* or *interiorized ratio* represents a more sophisticated conception of ratio. He defines rate as a *reflectively abstracted constant ratio* because it symbolizes the structure of a ratio – dividend, divisor, and quotient – as a whole, but emphasizes the constant

multiplicative relationship between those quantities forming the ratio. In other words, rate is a ratio that can be conceived independently of particular values of the two quantities forming the ratio. Thus, rate involves conceiving of ratio as a quantity that defines a constant multiplicative relationship between two quantities that can vary. Thompson (1994a) indicates that a mature conception of rate involves the coordination of relationships between accumulation of the two quantities forming the rate and the accruals by which these accumulations are built up.

Thompson illustrates this coordination by posing a situation in which a car travels at a constant speed of 35 feet per second. Here, the total distance traveled by the car during 4 seconds can be conceived as the simultaneous accumulation of four 35-foot accruals of distance and four 1-second accruals of time (Figure 2.2). Thus, the car's speed, when conceived as a rate, defines a relationship between distance and time so that, at any particular moment during the trip, the accrued distance and the corresponding accrued time will remain in a constant ratio that equals that ratio between total distance traveled and total time elapsed (i.e., 35 feet per second). Here, there is an implicit understanding of speed as a homogeneous attribute of either the entire situation or any proportional part thereof. In other words, 35 feet per second as a ratio between distance and time is an attribute of the entire trip or any proportional part thereof.



Distance and time accrue simultaneously and in proportional correspondence. One speed-distance or part thereof is made while moving for 1 time-unit or corresponding part thereof. Moving for one time-unit or part thereof implies moving one speed-distance or corresponding part thereof.

Distance and time accrued simultaneously and continuously. Each speed-distance is a fractional part of the total accrued distance. Each time-unit is a fractional part of the total accrued time.

Figure 2.2. Speed as rate, involving the coordination of relationships between accumulation of distance and time by accruals of distance and time. Adopted from “Images of rate and operational understanding of the Fundamental Theorem of Calculus,” by Thompson, 1994, *Educational Studies in Mathematics*, 26(2-3), p. 5.

Conceptual Framework: Conceptions of Intensive Quantities

I created a conceptual framework based on a review of the literature and themes emerged during the analysis of my data. I used this framework to assess the PSTs’ conceptions of the intensive quantities concentration, energy density, and energy flux density (Table 2.2). The framework is built on Simon’s and Placa’s (2012) distinction between *Ratio as Per-One Conception* and *Ratio as Measure Conception* of intensive quantities. When holding a Ratio as Per-One Conception, they explain that

Students tend to interpret per-one quantities as the extensive quantity associated with 1 unit of another extensive quantity, thus, continuing to think exclusively about extensive quantities. Further, we do not assume that the per-one notion of intensive quantities

brings with it other important ideas, such as the invariant multiplicative relationship between co-varying quantities (p. 39).

In their explanation, Simon and Placa emphasized two important aspect of the Ratio as Per-One conception of intensive quantities. First, an intensive quantity is conceived as an association of two extensive quantities. For example, a student holding a Ratio as Per-One conception may not conceptualize energy density as an intensive quantity. Energy density for this student is a magnitude of energy (extensive quantity) associated with one unit of surface area. Thus, 1,050 Joules (J) of energy distributed over a surface with 3 square meters (m^2) of area is conceptualized as 350 J associated with the first m^2 , 350 J associated with the second m^2 , and 350 J associated with the third m^2 . Notice that value 350 is a magnitude of energy and adding the three magnitudes of energy results in the total energy absorbed by the surface. Ratio as Per-One conception is consistent with applying division to define (and represent) a many-to-one correspondence between two quantities (division as multiplicative comparison). Dividing 1,050 J by 3 m^2 defines an association between 350 J and 1 m^2 . Second, Simon and Placa emphasized that Ratio as Per-One conception may not involve understanding an intensive quantity as expressing a constant multiplicative relationship between quantities that can vary. This is consistent with Thompson's (1994b) construct of *ratio* or *internalized ratio* in which the intensive quantity is multiplicative comparison of static, non-varying values of two (usually extensive) quantities. For example, the student holding a Ratio as Per-One conception may think of 350 J/m² as the result of the particular multiplicative comparison of 1,050 J and 3 m^2 , or even without these values, the student may think of 350 J/m² as the results of a particular multiplicative comparison of two unknown, but fixed values. In both cases, the student may not

conceptualize energy and surface area as varying quantities that maintain the same relationship: a ratio's value of 350 J/m^2 .

Table 2.2

Conceptual Framework for Assessing PSTs' Conceptions of Intensive Quantities

<u>Conception</u>	<u>Example</u>	<u>Level of Meaning</u>	<u>Use of Division</u>
Ratio as Per-One The ratio is conceptualized exclusively in terms of a magnitude of quantity A associated with one unit of quantity B. The magnitude of constituent quantity A is conceptualized as the measure of an attribute.	Energy density of 300 J/m^2 represents 300 Joules of energy associated with 1 square meter. The magnitude 300 Joules is considered the measure of temperature.	Internalized Ratio or Ratio (Association of Extensive Quantities)	Part-Whole (Association) Multiplicative Comparison
Ratio as Measure The ratio is conceptualized in terms of a single intensive quantity defining a constant multiplicative relationship between two quantities that can vary. Additionally, the ratio is conceptualized as the measure of an attribute.	Energy density of 300 J/m^2 is a quantity in itself expressing a magnitude of energy averaged over a magnitude of surface area. The relative magnitude 300 J/m^2 is a measure of temperature.	Interiorized Ratio or Rate (Single Intensive Quantity)	Part-Whole (Relationship) Rate

Ratio as Measure conception involves a more sophisticated understanding of intensive quantities than Ratio as Per-One conception. Simon and Placa (2012) stated that

Conceptualizing ratio-as-measure involves knowing that the ratio m/n implies an invariant multiplicative relationship between m and n and that m/n is a measure of the strength of that invariant relationship, that is, the measure of a particular attribute ... It is only in the context of ratio-as-measure, that an intensive quantity makes sense. An intensive quantity is the unit by which the strength of an attribute of this type (e.g., speed, density) is measured (p. 40).

In their explanation, Simon and Placa emphasized two important aspect of the Ratio as Measure conception of intensive quantities. First, a student holding a Ratio as Measure conception is able to conceive of an intensive quantity as a constant multiplicative relationship between quantities

that can vary. For instance, the student can think of 0.21 concentration of oxygen in a diving tank as a constant multiplicative relationship between the volume of oxygen and the volume of air.

The student can envision these volumes varying and still maintaining the same relationship (i.e., a constant 0.21 ratio). Ratio as Measure conception of intensive quantities is consistent with Thompson's (1994b) construct of *rate* or *interiorized ratio* in which the intensive quantity symbolizes the structure of a ratio – dividend, divisor, and quotient – as a whole, but emphasizes the constant multiplicative relationship between those quantities forming the ratio. Second, the student no longer conceives an intensive quantity as an association between two extensive quantities. Rather, the ratio is seen as a quantity in its own right, a quantity measuring an attribute of a situation. For example, the student can conceive of 0.21 concentration of oxygen as a measure of the strength of a part-whole relationship; that is, a measure of a volume of oxygen relative to the volume of air in which it is contained. This is consistent with applying division to define (and represent) the relative magnitude of an intensive quantity (division as rate).

Interpreting the relative magnitude of an intensive quantity as a measure of an attribute distinguishes Ratio as Measure conception from Ratio as Per-One. For instance, a student holding Ratio as Measure conception can conceive energy density as a magnitude of energy *averaged* over an object's surface area. The student can thus conceive of this relative magnitude (energy averaged over a surface) as a measure of the surface temperature of that object. In contrast, a student holding Ratio as Per-One conception may think of energy density as a magnitude of energy associated with one are unit, which represent an absolute magnitude (amount of energy). The student may then conceive of this absolute magnitude as a measure of the surface temperature of that object, which is incorrect.

Concerning Co-variation and Covariational Reasoning

In this section, I reviewed and summarized research literature concerning topics related to the second part of my research: PSTs' ability to reason about co-variation while learning about introductory mathematical models for global warming. I have divided this literature review into four sections: image and co-variation, covariational reasoning, chunky and smooth images of co-variation, and attending to the variation in the intensity of change. I conclude this section with the discussion of Carlson's et al. (2002) Co-variation Framework, which I used to examine PSTs' abilities to reason about co-variation while learning about three important functions: the Forcing by CO₂ function, the Planetary Energy Imbalance function, and the Average Surface Temperature function.

Images and Co-variation

Thompson (1994a) used the term *image* to refer to a set of fragments of experiences involving sensory-motor actions (e.g. vision, smell, movements, touch, taste, etc.), affective states (e.g. fear, joy, struggle, etc.), and cognitive process (e.g. imagining, inferring, deciding, etc.) that one collects and coordinates when reasoning in particular ways about particular situations. Images are less well-defined than schemes and tend to be idiosyncratic. In the context of covariation, the concept of image allows us to elaborate upon that "something" in the student's mind when the student talks about something changing or something accumulating.

Co-variation refers to a relationship in which two quantities are changing simultaneously and interdependently. Thompson (2011) provided a definition of co-variation in terms of a person's images of variation. Thompson explained that variation results from the person anticipating a quantity's measure as having different values at different moments in (conceptual) time. Variation always occurs over an interval of length ϵ so that the domain over which

conceptual time t ranges is covered by intervals $(t, t + \varepsilon)$. This variation in the values of a quantity x can be represented with $x_\varepsilon = x(t_\varepsilon)$, where t_ε represents the interval $[t, t + \varepsilon)$ and t varies through conceptual time. This characterization of variation implies that the student can imagine the quantity varying in intervals of conceptual time with the understanding that the quantity also varies within any interval of completed variation. According to Thompson, covariation can be defined as an extension of variation, extension involving a person imagining two quantities co-varying. This can be represented with $(x_\varepsilon, y_\varepsilon) = (x(t_\varepsilon), y(t_\varepsilon))$, where the pair $(x_\varepsilon, y_\varepsilon)$ represents a person's image of uniting in mind two quantities, and then varying them simultaneously over intervals of conceptual time.

Covariational Reasoning

Covariation represents a more intuitive approach to develop the concept of function (Thompson, 1994b). In addition, research in mathematics education has documented that reasoning about quantities that vary together is central for understanding functions, variation, rate of change, and concepts in calculus (Carlson et al., 2002; Johnson, 2012; Oehrtman, Carlson, Thompson, 2008; Zandieh, 2000). The mental operations involved in this reasoning form what is known as *covariational reasoning*. Saldanha and Thompson (1998) have defined covariational reasoning as

Someone holding in mind a sustained image of two quantities' values (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one's understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity's value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (pp. 1-2).

For Saldanha and Thompson, someone's images of covariation undergo several developmental stages. First, at a pre-operational level, the student is able to reason about particular values of two quantities independently from each other. In other words, the student can think of a quantity's value, then another quantity's value, then another value of the first quantity, then another value of the second quantity, and so on. Second, the student's later images of covariation would involve understanding time as a continuous quantity so that the student conceives of variation in each quantity independently. The student can imagine each quantity separately as varying from one value to another by taking all intermediate values over a period of time.

Later, at the operational level, a student's images of covariation entail tracking of both quantities' values changing over a period of time so that a correspondence between the quantities' values emerges as a property of the images of covariation. Finally, a student's images of continuous covariation entail imagining quantities as simultaneously and correspondingly having different values at different times with the understanding that these quantities assume all intermediate values as they vary together. Saldanha and Thompson asserted that interpreting and constructing graphs as records of continuous covariation between two quantities are non-trivial tasks for students. In their teaching experiment with an 8th grader, Saldanha and Thompson examined the mental operations involved in envisioning continuous covariation. They found that the student was able to develop operative images of covariation in which he conceived of independent variation in two quantities. The student, however, struggled to develop images of covariation in which changes in both quantities were not tightly coupled yet.

Chunky and Smooth Images of Co-variation

Castillo-Garsow, Johnson, & Moore (2013) theorized that students may develop images of co-variation which can be classified in one of two types: *chunky images* and *smooth images*.

Each type of image can lead to a way of reasoning about quantities that vary together, which Castillo-Garsow and colleagues referred to as chunky thinking and smooth thinking. *Chunky thinking* is characterized by two main features: (a) a unit or chunk building up the variation through iteration and (b) no variation occurs within this unit or chunk. The chunky thinker conceives of ongoing change as “a sequence of equal-sized chunks, and this makes measuring change essentially about counting how many chunks have occurred” (p. 33). The chunky thinker reasons in discrete or atomic units (chunks) and only attends to what is happening at the end points of these chunks (i.e., the end points of an interval) so that no variation occurs within the chunks. The chunky thinker thinks *in* intervals, but not *about* intervals. Castillo-Garsow and colleagues explained that chunky thinking is asynchronous. In other words, there is a mismatch between the temporal ordering according to which a quantity’s values vary and the ordering in which the chunky thinker perceives the quantity varying from one value to another. The chunky thinker may need to determine a quantity’s values at the end of a chunk before being able to determine the quantity’s values within that chunk. For instance, Castillo-Garsow and colleagues discussed the case of a high schooler, Tiffany (pseudonym), who was tasked with determining the value of a saving account at one tenth of a second. Tiffany initially determined the value of the account at one second before she was able to determine the value at one tenth of a second. Castillo-Garsow and colleagues argued that “this mismatch makes it seemingly impossible for a student using chunky thinking to imagine a situation dynamically while simultaneously imagining the mathematics of it” (p. 34).

In contrast, smooth thinking can support conceiving of change as occurring progressively and continuously. From the perspective of smooth thinking, Castillo-Garsow and colleagues explained that

Ongoing change is generated by conceptualizing a variable as always taking on values in the continuous, experiential flow of time. A smooth variable is always in flux. The change has a beginning point, but no end point. As soon as an endpoint is reached, the change is no longer in progress. (p. 34)

The smooth thinker conceives of change in a quantity as a continuous co-variation with respect to another quantity in the flow of (conceptual) time. Although the smooth thinker can also conceive of change in chunks, the smooth thinker can imagine how change occurs within the chunk before determining a quantity's values at the end of a chunk, avoiding the asynchronous nature of chunky thinking. Castillo-Garsow and colleagues emphasized that smooth thinking is not a refinement of chunky thinking since the chunky thinker will not develop smooth thinking by solely considering smaller and smaller chunks in the independent variable. In other words, the chunky thinker will not conceive of variation within a chunk regardless of how the size of the chunks. However, Castillo-Garsow and colleagues proposed that smooth thinking could serve as a powerful cognitive root for chunky thinking. They discussed the case of a high schooler, Hannah (pseudonym), who was presented with a situation in which a bottle is being filled by a liquid at a constant rate. Then, Hannah was presented with a graph showing the relationship between the volume (dependent variable) and the height of liquid (independent variable) in the bottle. Hannah was tasked with drawing the bottle that is being filled. Looking at the graph, Hannah attended to the variation in the rate of change of the volume as the height was continuously increasing on the horizontal axis. Castillo-Garsow and colleagues interpreted this as Hannah imagining the bottle being filled as change in progress from beginning to end (smooth thinking). Then, she identified three sections in the graph showing different *rates* of increase in the volume and associated each section with particular shape in the bottle. Hannah's sections

represented chunks, but these chunks were the result of Hannah's smooth thinking so that variation occurred within each chunk. Hannah was able to construct the correct shape of the bottle in the situation. This example illustrates how smooth thinking not only represents a more advance form of reasoning about co-variation, but also a powerful root for a sophisticated version of chunky thinking.

Attending to the Variation in the Intensity of Change

Johnson (2012) distinguished between two perspectives in co-variation: the *static perspective* and the *dynamic perspective*. A *static perspective* involves associating amounts in one quantity with amounts in another quantity. For instance, the forcing by CO₂ function $F(C)$ expresses a covariational relationship between the magnitude of a forcing F and the atmospheric concentration of CO₂, denoted by C . From a static perspective, a person envisions $F(C)$ through associating *movements* between successive values in C and F . For instance, a student may associate C moving from 278 ppmv to 556 ppmv with F moving from 0 J/m²/s to 4 J/m²/s. In contrast, a *dynamic perspective* of covariation involves associating changes in one quantity with change in another quantity. The dynamic perspective can be discrete or continuous. In a discrete dynamic perspective, the student coordinates amounts of change in one quantity with amounts of change in another quantity. For instance, a student can envision $F(C)$ by associating amounts of change in F with changes in C , specifically associating decreasing increments of F with equal increments in C . In contrast, a continuous dynamic perspective involves the coordination between continuous change in one quantity with continuous change in another quantity. Here, the student attends to the *variation in the intensity of change* of one quantity with respect to another quantity. For example, the student can envision $F(C)$ as a relationship in which F increases at a decreasing rate as C increases continuously.

Johnson (2012) also examined and discussed how a secondary student named Hannah (pseudonym), prior to a formal introduction to calculus, reasoned about co-varying quantities involved in the rate of change. Johnson conducted five individual, task-based interviews with Hannah, who worked on seven tasks involving multiple representations of co-varying quantities and the analysis of the *variation in the intensity of change* of one quantity with respect to another quantity. Johnson found that Hannah's reasoning about co-varying quantities involved in the rate of change was consistent with a dynamic perspective on co-variation. This dynamic perspective allowed Hannah to reason numerically and non-numerically about co-varying quantities involved in rate of change. Johnson characterized Hannah's reasoning as the process of "systematically varying one quantity and simultaneously attending to variation in the intensity of change in a quantity indicating a relationship between co-varying quantities" (p. 327). In other words, Hannah considered the quantity being systematically varied as the independent variable, while considering the quantity indicating variation in the intensity of change between the co-varying quantities as the dependent quantity. Johnson concluded that Hannah's *dynamic perspective on co-variation* and *attention to the variation in the intensity of change* were powerful mathematical tools that can provide three advantages when reasoning about co-variation. First, these abilities can help students reason about co-variation beyond solely recognizing the direction in which two co-varying quantities change in a dynamic situation. Second, a dynamic perspective can allow students to mentally run over the variation in the intensity of change of quantity y with respect to quantity x (dependent variable) while envisioning quantity x (independent variable) changing continuously. This ability appeared central to formally reason about co-variation in terms of the rate of change. Third, a dynamic perspective and attention to the variation in the intensity of change can inform how students coordinate *smooth thinking* with *chunky thinking* when

reasoning about co-variation before formally introducing the rate of change in calculus. The student can use *smooth chunks* to reason about the variation in the intensity of change so that the co-varying quantities are envisioned as simultaneously and continuously changing.

The Co-variation Framework

Carlson, Jacobs, Coe, Larsen, & Hsu (2002) defined covariational reasoning as “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). Based on this definition, Carlson and colleagues developed the Covariation Framework in order to define theoretical basis for examining and assessing someone’s covariational reasoning. Their framework describes five mental actions that a student can engage in when reasoning about quantities that vary together. The framework also describes illustrative behaviors as indicators for each mental action (Table 2.3). The mental actions described in the framework are used to classify a student’s behaviors and verbalizations as the student engage in tasks involving situations in which two quantities co-vary. Then, the collection of mental actions inferred from the student’s behaviors is examined as a whole to determine an “overall image that appears to support the various mental actions that he or she exhibited in the context of a problem or task” (Carlson et al., 2002, p. 357). Finally, the student’s overall image of covariation is classified as being at a particular level of development. There are five levels of covariational reasoning, each being more sophisticated than and built upon the previous one: *dependency of change* (L1: y changes when x changes), *direction of change* (L2: y increases as x increases), *amounts of change* (L3: a change Δy in y correspond to a change of Δx in x), *average rate of change* (L4: y increases more rapidly for successive equal changes Δx in x), and *instantaneous rate of change* (L5: y increases more rapidly as x continuously increases). If a student’s overall image of covariation is classified at a particular

level, then it is implied that the student's images of covariation support the mental action associated with that particular level *and* the mental actions associated to all previous levels. For instance, a student's overall image of covariation is said to be at L5 if the student is able to reason using MA5 and to unpack that mental action to reason about co-variation in terms of MA4 and MA3 (note that MA3 includes MA1 and MA2).

Carlson and colleagues emphasized the importance of examining the complete collection of the student's mental actions in order to assign a particular level of covariational reasoning. They explained that a student can show behaviors and verbalizations that appear as evidence of certain mental action, but upon further questioning it becomes clear that the student does not have an understanding to support that mental action. They referred to this type of behavior as *pseudo-analytical behavior* and describe the mental action associated with it as *pseudo-analytical mental action*. For example, a student can exhibit behaviors associated with MA5, but upon further questioning the student may not be able to unpack MA5 to reason in terms of MA4 and MA3, which means the student's images of covariation are not at L5 level of development.

Carlson and colleagues used the Covariation Framework to examine and assess the covariational reasoning abilities of 20 college students, who had recently completed a second semester calculus course. The students worked on several mathematical tasks in which they were asked to interpret and construct graphs representing quantities that varied together. They found that most students were able to recognize what quantities were changing together (MA1) as well as the direction of this change (MA2). Additionally, students were often able to coordinate images of amounts of change in one variable with amounts of change in the other variable (MA3). These students were able apply L3 consistently since they consistently coordinated

quantities changing together, direction of change, and amounts of change between quantities (MA1 through MA3).

Table 2.3
Mental Actions of the Covariation Framework

Mental Action	Description of Mental Action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other.	<ul style="list-style-type: none"> Labeling the axes with verbal indications of coordinating the two variables (e.g. y changes with changes in x).
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable.	<ul style="list-style-type: none"> Constructing an increasing straight line. Verbalizing an awareness of the direction of change of output while considering changes in the input.
Mental Action 3 (MA3)	Coordinating the amounts of change of one variable with changes in the other.	<ul style="list-style-type: none"> Plotting points/constructing secant lines. Verbalizing an awareness of the amount of change of the output while considering changes in the input.
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	<ul style="list-style-type: none"> Constructing contiguous secant lines for the domain. Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input.
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function.	<ul style="list-style-type: none"> Constructing smooth curve with clear indications of concavity changes. Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflexion points are correct).

Note. Adopted from “Applying covariational reasoning while modeling dynamic events: a framework and a study” by M. Carlson, S. Jacobs, E. Coe, S. Larsen, and E. Hsu, 2002, *Journal for Research in Mathematics Education*, 33(5), pp. 357.

However, students showed signs of struggle when interpreting and representing the rate of change in dynamic situations. In particular, students were unable to interpret information

about the average rate of change (MA4) in a consistent manner. In other words, students were unable to apply L4 consistently since they struggled to coordinate the average rate of change for fixed changes in the independent variable. Finally, students have difficulties in constructing images of a rate of change changing continuously in dynamic situations (i.e. instantaneous rate of change) as well as interpreting the meaning of inflection points (MA5). The students were unable to apply L5 since they seldom coordinated images of a continuously changing rate of change for a dynamic functional situation and struggled to interpret the meaning of smoothness and inflection points.

CHAPTER 3

METHODOLOGY

In this chapter, I discuss the methods and approaches I used in my dissertation study. The chapter is organized into four parts: research design, participants, data collection, and data analysis. This discussion will help the reader understand the rationale behind the methodological decisions made for the study.

Research Design

I designed my study in two parts. The first part involved gathering information about PSTs' conceptions of intensive quantities commonly used to model global warming. The second part involved examining PSTs' ability to reason about co-variation when learning about introductory models for global warming. Because little is known about PSTs' understanding of intensive quantities and mathematical models related to global warming, I decided to conduct a detailed case-study to gain insights into the previously mentioned inquiries. Yin (2013) explains that a case study is a qualitative research methodology that involves the investigation of a case in order to unveil "operational links needing to be traced over time, rather than mere frequencies or incidence" (p. 10). In particular, Yin offers a two-part definition of case study: the first part refers to the scope of case studies and the second part refers to the main features of case studies.

Regarding the scope, Yin states that "a case study is an empirical inquiry that investigates a contemporary phenomenon (the "case") in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (p. 16). Case studies are preferred to investigate contemporary events, especially when the researcher cannot

(or may not want to) manipulate relevant behaviors. To this end, case studies can make use of “direct observation of events being studied and interviews of the persons involved in the events” (p. 12). In my study, I conducted an in-depth examination of a contemporary phenomenon within a real-world context: PSTs’ understanding of intensive quantities and mathematical models related to global warming. The phenomenon and the context are not clearly distinguished because PSTs’ conceptions and reasoning is closely related to the context of global warming. It is in relation to that context (global warming) that I attempt to examine the phenomenon (PSTs’ conceptions and reasoning) by observing and examining PSTs’ responses and actions as they work on several mathematical tasks.

The second part of Yin’s definition describes the main features of case studies. For Yin, these features come as a result of the close relationship between the phenomenon being studied and its context. In particular, Yin states that the case study research design:

- Copes with the technically distinctive situation in which there will be many more variables of interest than data points, and as one result
- Relies on multiple sources of evidence, with data needing to converge in a triangulation fashion, and as another result
- Benefits from the prior development of theoretical propositions to guide data collection and analysis. (p. 17)

My study examined the responses of a small sample of PSTs to several mathematical tasks. Their responses provided evidence for several variables of interest including interpretations, conceptions, misconceptions, and meanings regarding quantities, ratios, co-variation of quantities, rate of change, and graphs. These variables were more than the number of PSTs (data points) participating in the study. As a result, my study relies on multiple source of evidence

including video recordings of responses and actions, transcripts of interviews, implementation of several mathematical tasks, and PSTs' work on worksheets. Additionally, the data collection and analysis were guided by previously defined theoretical propositions on literature concerning intensive quantities and co-variational reasoning. Therefore, the main features of case study appear advantageous for my study given the nature of the research goals.

I carefully selected three secondary PSTs for my study (see Participants section), each being a particular case under investigation. Because my study involved investigating more than one case, I used a multiple-case study research design. An important feature of a multiple-case study is to consider the multiple cases as multiple experiments by following a replication logic (Yin, 2013). When conducting multiple experiments, a researcher attempts to replicate important findings from a single experiment in a second, third, or more subsequent experiments. For a multiple-case study, Yin explained that "each individual case study consist of a "whole" study, in which convergent evidence is sought regarding the facts and conclusions for the case; each case's conclusions are then considered to be information needing replication by other individual cases." I carefully selected the cases for my study in order to investigate theoretical replication or the replication of "contrasting results but for anticipatable reasons" (Yin, 2013, p. 57). I selected three PSTs enrolled in the same mathematics education program and with similar mathematics course work background (see Participants section). I expected different results regarding PSTs' understanding of introductory models for global warming, different results that could be explained by differentials in PSTs' conceptions of intensive quantities and ability to reason about co-variation when learning about these introductory models.

My multiple-case study follows an exploratory and embedded design. Since there is no substantial research connecting mathematics education and global warming, little is known about

PSTs' understanding of intensive quantities and introductory mathematical models related to global warming. My study represents an initial approach to investigate these inquiries, and thus it is an exploration of the phenomenon. Additionally, my multiple-case study involves an embedded design because, "within a single case, attention is also given to a subunit or subunits" (Yin, 2013, 53). I am interested in unveiling PSTs' conceptions and reasoning regarding particular mathematical and scientific concepts related to global warming. Therefore, each PST (case) is investigated in relation to their responses to six particular mathematical tasks (see Mathematical Tasks subsection) designed for this study. These tasks represent subunits units of analysis within each case.

Participants

I used *purposeful sampling* or *criterion-based selection* in order to select participants for my study. Maxwell (2005) provides the following definition of purposeful sampling

A strategy in which particular settings, persons, or activities are selected deliberately in order to provide information that can't be gotten as well from other choices ... Selecting those times, settings, and individuals that can provide you with information that you need in order to answer your research questions is the most important consideration in qualitative selection decisions (p. 88).

Qualitative studies often rely on a small sample since the goal is to study a particular group of informants or events under particular conditions. Traditional quantitative sampling strategies such as random selection may not be appropriate to draw small samples because of "the high likelihood of substantial chance variation" (Maxwell, 2005, p. 89). Purposeful sampling allows the research to deal with this problem and to determine a small sample suitable for the study. For my study, I was interested in exploring the conceptions and ways of reasoning of a particular

group of individuals: preservice secondary mathematics teachers (PSTs). I was interested in exploring their conceptions and ways of reasoning under particular conditions: while learning about introductory mathematics models for global warming. Thus, I deliberately recruited PSTs enrolled in the secondary Grades (6-12) program at a large public university in the Southeast.

To be admitted into the program, PSTs must have completed three mathematics content courses: Calculus I, Calculus II, and Introduction to Higher Mathematics. The two calculus courses covered topics such as limits, derivatives, curve sketching, optimization, integrals, separable differential equations, and science or engineering applications. The introduction to higher mathematics course is a preparation in mathematical reasoning and proof-writing necessary for upper division mathematics courses. The content covered in this course included logic, sets and relations, and functions (injectivity and surjectivity). I created a sequence of mathematical tasks (see details in the next section) that required PSTs to have experience with topics such as limits, rate of change and derivatives, curve sketching, and applications. Additionally, the tasks required PSTs to draw logical conclusions involving relations between quantities as functions. Therefore, participants with experience in these topics were preferred. The mathematics education program offers two mathematics content courses for secondary teachers with a focus on ratio and proportional reasoning, measurement, functions (algebraic and transcendental), connections to rate of change, and modeling. Since the tasks demanded experience working with and reasoning about ratios, measurement, functions, and rate of change, PSTs who attended or were attending one of the courses were selected.

Nineteen students were enrolled in the mathematics education program at the time the study took place. Twelve students were completing their first semester in the program (beginning of junior year), while seven students were completing their third semester in the program

(beginning of senior year). I made class visits to recruit volunteers from these two cohorts of PSTs. During these visits, I provided a brief overview of my study and collected contact information from those PSTs who expressed interest in receiving more information about participating in the study. I collected email addresses of ten students who were in their first semester and three students who were in their third semester in the program. Then, I emailed all these PSTs with more information about the study and to formally invite them to participate. Of the 13 PSTs I initially emailed, six replied expressing their decision of taking part in the study. The six volunteers included four first semester students (two females and two males) and two third semester students (one female and one male). The four first-semester students were completing a mathematics content course for secondary grades, while the two third-semester students had already completed one mathematics content course for secondary teachers. All six PSTs had no previous experience learning about scientific or mathematical concepts related to global warming. Nonetheless, they believed that global warming was real and that human activity was at least partially responsible for it.

All six initial volunteers participated during the first stage of data collection; that is, each one participated in two individual, task-based interviews during which they completed four mathematical tasks. However, the three male participants were unable or decided not to continue participating during the second stage of data collection. The first stage of data collection focuses on gathering information concerning PSTs' conceptions of quantities commonly used to model global warming. This information is used in the analysis of PSTs' responses to the tasks created for the second stage of data collection, in which PSTs learn about introductory models for global warming. Since the three male participants did not continue into the second stage, their responses were excluded from the study. Therefore, three (female) PSTs participated during both

stages of data collection completing all the tasks created for the study. The results and conclusions I report in this dissertation study correspond to the analysis of the responses of these three PSTs, whom I hereafter referred as Pam, Kris, and Jodi (pseudonyms).

Data Collection

Data were collected from individual task-based interviews. Pam, Kris, and Jodi participated in five task-based interviews in which each one of them completed a total of six mathematical tasks. First, I discuss the six mathematical tasks followed by the process for conducting the task-based interviewing.

Mathematical Tasks

There is no substantial research connecting mathematics education and global warming. As a result, tasks describing situation involving quantities such as concentration, energy density, and energy flux density are rare. Similarly, tasks designed to explore co-variational reasoning in the context of modeling global warming are also rare. Therefore, I created an original sequence of tasks for my study, along with interview protocols that could probe into PSTs' reasoning concerning quantities and introductory models related to global warming. I initially created a total of 13 mathematical tasks aligned with the research goals. Six tasks were created keeping in mind the first part of my study: assessing PSTs' conceptions of intensive quantities commonly used to model global warming. These six tasks described situations in which intensive concentration, energy density, and energy flux density were needed to solve a problem. The remaining seven tasks were created keeping in mind the second research goal: examining PSTs' reasoning about co-variation while learning about introductory models for global warming. These seven tasks described situations in which the solver needed to attend to how two quantities varied together in order to sketch a graph of the relation.

I piloted the 13 mathematical tasks in order to assess their effectiveness at unveiling and prompting into PSTs' conceptions and reasoning. I designed an interview protocol for each task containing questions that could help me gather information concerning PSTs' conceptions and reasoning. The interview protocols provide standard prompts ensuring that all participants would receive similar probing questions at the beginning of each interview. Thus, it was important to test my interview protocols during the pilot in order to eliminate, create, and refine questions for the interviews. I also wanted to evaluate the degree of difficulty of each task. Since global warming represents a complex scientific phenomenon, I need to examine whether the tasks were applicable for participants with the mathematical background of a future secondary teacher. Also, by piloting the tasks, I identified necessary background information that participants needed to know before working on each task. Finally, I needed to have a sense of how much time participants may need in order to complete a particular task.

Seven individuals volunteered for that pilot study and were students in the program mathematics education program at the same large public university in the Southeast. Three individuals were students in the secondary Grades (6-12) program during their third semester in the program (beginning of senior year). They had recently completed a content course with a focus on mathematical modeling at the time of the pilot. Because the tasks were to be implemented with PSTs, I piloted the tasks with similar PSTs in order to assess their adequacy for the targeted group. The remaining four individuals taking part in the pilot were doctoral students in mathematics education. Two doctorate students were in their third year, while the other two were in their fourth year in the program. Doctoral students in mathematics education are expected to take upper-level mathematics courses equivalent of a master's degree. I decided to pilot the tasks with doctoral students because I was interested in their feedback. In particular, I

was interested in their suggestions concerning the tasks' effectiveness and level of difficulty as well as suggestion concerning the questions in my interview protocol. Based on the pilot results, I selected six mathematical tasks for the final study. The tasks were divided into two sets according to the two overarching research goals. Set 1 included four tasks describing situations in which PSTs were required to work with and reason about concentration, energy density, or energy flux density. Set 2 included three tasks describing situations in which PSTs were required to reason about co-variation while learning about introductory models for global warming. Next, I present each mathematical task along with its research objective and the expected solution.

Set 1 task 1 (S1T1): concentration. Concentration is an intensive quantity defined by measuring the average abundance of a substance relative to the volume of a mixture containing the substance. Mathematically, concentration is the ratio between the amount of a substance, expressed in units of mass (e.g., pounds or kilograms) or volume (e.g., ounces or cubic centimeters), and the volume of the mixture. For instance, air is a mixture of different gases such as Nitrogen, Oxygen, Argon, and other trace gases (including Carbon Dioxide). Concentration emerges as a way of measuring the average abundance of a gas (substance) relative to a volume of air (mixture). Particularly, concentration emerges in its fractional form when defined as the ratio between the volume of a gas and the volume of air (dimensionless). S1T1 described a situation in which concentration emerges in its fractional form (Figure 3.1).

The air you breathe is a mixture of gases such as nitrogen (N), oxygen (O₂), and other trace gases. Diving tanks also contain air and come in different sizes. Consider two diving tanks of different size and with different air composition (see Figure below).

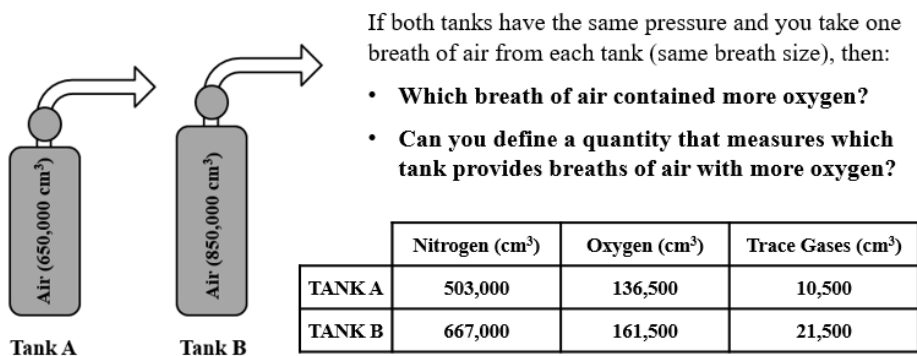


Figure 3.1. Set 1 task 1 (S1T1): concentration in fractional form.

When modeling global warming, concentration is used to measure abundance of greenhouse gases in the atmosphere. I created S1T1 to be a simplification of the problem of finding the concentration of a gas mixed in the atmospheric air. The research objective of this task was to assess PSTs' conception of the intensive quantity concentration in its fractional form. In particular, concentration in its fractional form emerges as a way to anticipate higher average abundance of oxygen in a breath of air. PSTs were expected to determine the concentration of oxygen for each tank (i.e., the ratio between the volume of oxygen and volume of air) based on the tanks' air composition:

$$Q_A[O_2] = \frac{\text{volume of oxygen}}{\text{volume of air}} = \frac{136,500 \text{ cm}^3}{650,000 \text{ cm}^3} = 0.21$$

$$Q_B[O_2] = \frac{\text{volume of oxygen}}{\text{volume of air}} = \frac{161,500 \text{ cm}^3}{850,000 \text{ cm}^3} = 0.19$$

Here, $Q_A[O_2]$ and $Q_B[O_2]$ denoted the oxygen concentration for tank A and tank B, respectively.

Tank A had an oxygen concentration of 21%, which was higher than the oxygen concentration of 19% for tank B. Therefore, a breath of air from tank A would contain more oxygen than a breath of air from tank B.

Set 1 task 2 (S1T2): concentration in ppmv. When concentrations are relatively small, they are often measured in parts per million by volume or ppmv. This unit is defined as the average volume of a substance per one million volume units of the mixture. For instance, today's atmospheric concentration of carbon dioxide (CO₂) is approximately 406 ppmv, which means that there is an average of 406 cm³ of carbon dioxide per 1,000,000 cm³ of air in the atmosphere. When modeling global warming, ppmv is the preferred unit to define concentration of greenhouse gases in the atmosphere. Therefore, I created S1T2 to be a simplification of the problem of finding the atmospheric concentration of CO₂ measured in ppmv (Figure 3.2).

Diving tanks also contain a small volume of carbon dioxide (CO₂) in their air. The table below shows two diving tanks' total air volume and the CO₂ volume contained in their air.

Tank	Air Volume (cm ³)	CO ₂ Volume (cm ³)
A	4,000,000	1,448
B	800,000	316

When concentrations are small, they are often measured in ppmv units (parts per million by volume units). This unit is defined as the average volume of a gas (in cm³) per 1,000,000 cm³ of air. Diving tanks must have a CO₂ concentration between 350 ppmv and 400 ppmv in order to have safe air for a diver.

Calculate each tank's CO₂ concentration in ppmv and check whether these tanks have safe air for a diver.

Figure 3.2. Set 1 task 2 (S1T2): concentration in parts per million by volume (ppmv).

The research objective of this task was to assess PSTs' conception of the intensive quantity concentration measured in ppmv. In particular, concentration in ppmv emerges as a way to anticipate the average abundance of CO₂ in the air of the tanks. PSTs were expected to determine the concentration of CO₂ in ppmv and to decide whether the air of a particular tank was safe for a diver (i.e., a concentration of CO₂ between 350 ppmv and 400 ppmv):

$$P_A[CO_2] = \frac{\text{volume of } CO_2 \text{ (in } cm^3\text{)}}{\text{volume of air (in } 10^6 cm^3\text{)}} = \frac{1,448 \text{ } cm^3}{4 \times 10^6 cm^3} = 362 \frac{cm^3}{10^6 cm^3} = 362 \text{ ppmv}$$

$$P_B[CO_2] = \frac{\text{volume of } CO_2 \text{ (in } cm^3\text{)}}{\text{volume of air (in } 10^6 cm^3\text{)}} = \frac{316 \text{ } cm^3}{0.8 \times 10^6 cm^3} = 395 \frac{cm^3}{10^6 cm^3} = 395 \text{ ppmv}$$

Here, $P_A[O_2]$ and $P_B[O_2]$ denoted the CO_2 concentration of tank A and tank B, respectively. Both tanks contained safe air for a diver since their CO_2 concentration (362 ppmv for tank A and 395 ppmv for tank B) were within the safe range.

Set 1 task 3 (S1T3): energy density. I first discussed with PSTs the concept of radiation and its connection with heat and temperature. Radiation is the emission of energy from one object (source) to another object (receiver). Broadly speaking, heat represents a type of radiation and is closely related to the concept of temperature. However, these two concepts are different: temperature is a measure of the average internal energy of an object, while heat is a measure of the magnitude of energy flowing from one object to another due to their temperature difference. For instance, the sun (source) radiates energy, in the form of heat, toward the Earth's surface (receiver). Following the previous definitions, the magnitude of solar energy incident to the planet's surface averaged over the planet's surface area is a measure of the average surface temperature of the planet. In this context, energy density can be defined as the magnitude of energy incident to an object's surface averaged over the object's surface area. Mathematically, energy density is a ratio between a magnitude of energy, measured in Joules (J), and a magnitude surface area, measured in square meters (m^2). S1T3 described a situation in which energy density emerges as a way to anticipate higher surface temperature (Figure 3.3).

Heat can be transmitted from one object (source) to another (receiver). This can rise the receiver's temperature since it increases its internal energy

In an experiment, two (very thin) rectangular sheets (same material) were positioned at the same distance from two devices radiating heat (energy) toward them (see Figure). If both sheets were initially at room temperature and both devices started and stopped radiating heat at the same time, then:

- What sheet had a higher temperature by the end of the experiment?
- Can you define a quantity that indicates the sheet with higher temperature by the end of the experiment?

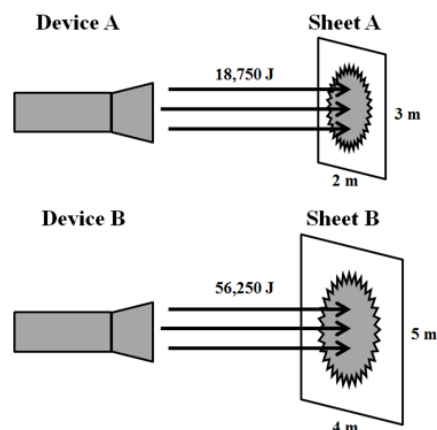


Figure 3.3. Set 1 task 3 (S1T3): energy density.

The situation described in S1T3 used energy distributed over a flat sheet as a simplification of the solar energy distributed over a sphere, such as the planet Earth. The research objective of this task was to assess PSTs' conception of the intensive quantity energy density in relation to two meanings: energy density as a ratio between energy and surface area and energy density as a measure of surface temperature. PSTs were expected to determine the energy density of each sheet and used it to conclude which sheet had higher temperature:

$$S_A = \frac{\text{energy added}}{\text{surface area}} = \frac{18,750 \text{ J}}{6 \text{ m}^2} = 3,125 \text{ J/m}^2$$

$$S_B = \frac{\text{energy added}}{\text{surface area}} = \frac{56,250 \text{ J}}{20 \text{ m}^2} = 2,812.5 \text{ J/m}^2$$

Here, S_A and S_B denoted the energy density for sheet A and sheet B, respectively. Since surface temperature is the magnitude of energy averaged over the sheet's surface area, energy density is a measure of surface temperature by its definition. Therefore, sheet A had a higher temperature than sheet B by the end of the experiment since sheet's A energy density was larger than sheet's B energy density.

Set 1 task 4 (S1T4): energy flux density. Energy flux density is an intensive quantity defined by measuring the radiation rate of energy incident to an object's surface averaged over

the object's surface area. Mathematically, energy flux density is the ratio between a magnitude of radiation rate, measured in Joules per second (J/s), and a magnitude of surface area, measured in square meters (m^2). When modeling global warming, the transfer of energy, in the form of heat, between the sun, the Earth's surface, and the atmosphere is often measured in terms of energy flux density. For instance, the sun radiates energy toward the Earth's surface at a certain rate, let's called it solar radiation rate for convenience. The magnitude of the solar radiation rate incident to the planet's surface averaged over the planet's surface area is a measure of the rate of change of the average surface temperature of the planet. Therefore, I created S1T4 to be a simplification of the solar radiation rate distributed over the Earth's surface (Figure 3.4).

Energy density is directly proportional to temperature. For instance, when the energy density of a metallic sheet increases by $2,500 \text{ J/m}^2$, the sheet's temperature rises by $4 \text{ }^\circ\text{C}$.

In an experiment, two (very thin) rectangular sheets (same material) were positioned at the same distance from two devices radiating heat (energy) toward them (see Figure). Device A transfers heat to Sheet A at a rate of 750 J/s , while Device B transfers heat to Sheet B at a rate of $1,200 \text{ J/s}$. If both sheets were initially at room temperature and both devices started radiating heat at the same time, then:

- **What sheet's temperature will first reach $25 \text{ }^\circ\text{C}$?**
- **Can you define a quantity that indicates the sheet's temperature that will first reach $25 \text{ }^\circ\text{C}$?**

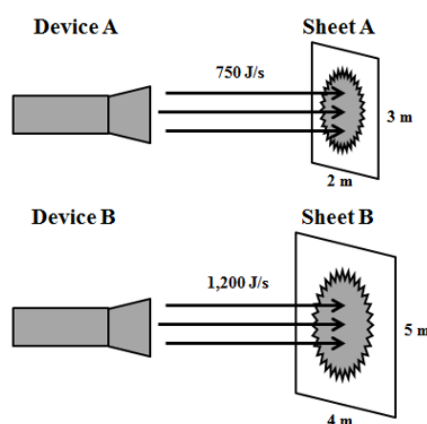


Figure 3.4. Set 1 task 4 (S1T4): energy flux density

The research objective of this task was to assess PSTs' conception of the intensive quantity energy flux density in relation to two meanings: energy flux density as a ratio between radiation rate and surface area and energy flux density as a measure of the rate of change of surface temperature. PSTs were expected to determine the energy flux density of each sheet and used it to conclude which sheet's surface temperature was increasing most rapidly:

$$R_A = \frac{\text{rate of energy radiation}}{\text{surface area}} = \frac{750 \text{ J/s}}{6 \text{ m}^2} = 125 \frac{\text{J/s}}{\text{m}^2}$$

$$R_B = \frac{\text{rate of energy radiation}}{\text{surface area}} = \frac{1,200 \text{ J/s}}{20 \text{ m}^2} = 60 \frac{\text{J/s}}{\text{m}^2}$$

Here, R_A and R_B denoted the energy flux density for sheet A and sheet B, respectively. Based on these values, PSTs were expected to conclude that the surface temperature of sheet A was increasing more rapidly than the surface temperature of sheet B since sheet's A energy flux density was larger than sheet's B energy flux density.

Set 2 task 1 (S2T1): the forcing by CO₂ function. Before defining the forcing by CO₂ function, I will first discuss the concepts of Earth's energy budget, greenhouse effect, and planetary energy imbalance. Earth's climate system is powered by the sun and there is a continuous flow of energy between the sun, the planet's surface, and the atmosphere. This continuous flow of energy is known as the Earth's energy budget (Figure 3.5). The sun warms the planet's surface (S). As the surface warms up, it radiates (infrared) energy to the atmosphere (R), the majority of which is absorbed by greenhouse gases (GHG) such as water vapor (H₂O), carbon dioxide (CO₂), and methane (CH₄) (B). The atmosphere re-radiates the absorbed energy in both directions toward space and toward the surface (A). This continuous energy exchange between the surface and the atmosphere is known as the greenhouse effect and influences the planet's average surface temperature. The energy flows S, R, B, L, and A are all measured in terms of energy flux density (J/m²/s), while the abundance of atmospheric GHG is measured in terms of concentration (ppmv). The parameter $0 < g < 1$ is related to the greenhouse effect. Quantifying changes in the energy fluxes due to changes in the abundance of GHG is central to accurately model global warming. My study focuses on the particular relationship between a quantity known as planetary energy imbalance and the atmospheric concentration of CO₂.

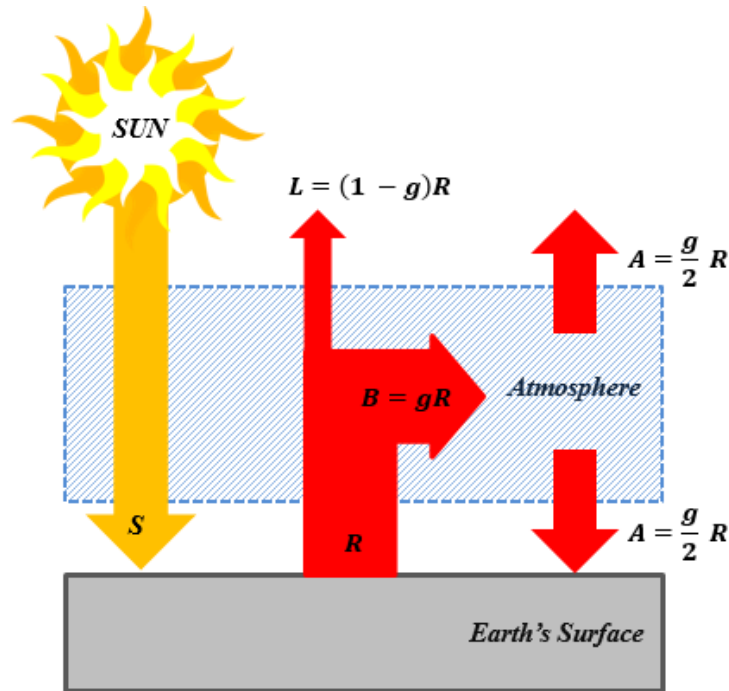


Figure 3.5. The Earth's energy budget assuming a one-layered atmosphere

The *planetary energy imbalance* (N) is a measure of the energy imbalance in the Earth's energy budget. In particular, N can be defined as a difference between the combine magnitude of all downward energy flows and the combine magnitude of all upward energy flows at the planet's surface (Figure 3.5), or mathematically $N = (S + A) - R$. The Earth's energy budget is said to be in radiative equilibrium when $N = 0$, which implies that the average surface temperature of the planet remains constant. In contrast, if $N \neq 0$, then the budget experiences an energy imbalance. This imbalance results in a variation of the average surface temperature. $N > 0$ is of particular interest for my study. This results in a warming effect over the planet's surface (increase in temperature), which causes the surface's upward energy flow (R) to increase. This flow will continue to increase until the magnitude of the upward energy flows equals that of the downward energy flows ($N = 0$) so that a new radiative equilibrium is reached.

There are several factors, commonly known as *forcing agents* that can cause an energy imbalance. A change in the planetary energy imbalance (N) cause by a particular forcing agent is

known as *radiative forcing* or simply *forcing*. The forcing by CO₂ (F) is of particular interest for my study because F is the change in N caused by a change in the atmospheric concentration of CO₂ (C). When C changes, the composition of the atmosphere changes, resulting in changes in the energy flows B and A. For instance, an increase in C results in an increase in the atmosphere's capacity to absorb infrared energy, or an increase in B. This causes the energy flow A to increase since $A = \frac{1}{2}B$ (Figure 3.5), which in turn causes $N = (S + A) - R$ to increase. This value of N is the magnitude of the forcing by CO₂. I created S2T1 to examine PSTs' reasoning about co-variation while making sense of the variation of F with respect to C (i.e., the forcing by CO₂ function), which I denoted as F(C) (Figure 3.6).

The atmosphere's capacity to absorb infrared energy depends on the concentration of greenhouse gases in the atmosphere. Thus, a change in the concentration of CO₂ can result in an imbalance in the Earth's energy budget. An imbalance created by a change in the concentration of CO₂ is known as forcing by CO₂. Let F and C be the forcing by CO₂ in J/m²/s and the concentration of CO₂ in ppmv, respectively. Based on the concepts of Earth's energy budget and greenhouse effect, determine how F varies with respect to C and sketch a graph for the relationship F(C).

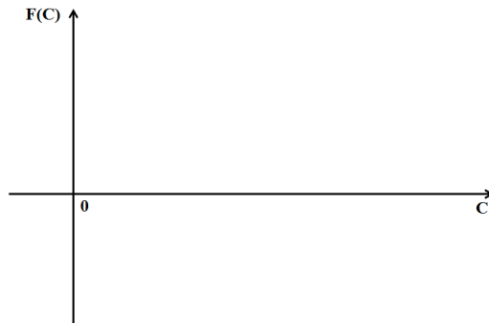


Figure 3.6. Set 2 task 1 (S2T1): the forcing by CO₂ function.

PSTs were given a diagram of the Earth's energy budget (see Figure 3.5 above) showing the initial values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, $B = 300 \text{ J/m}^2/\text{s}$, and $A = 150 \text{ J/m}^2/\text{s}$. As a simplification, PSTs were told to assume that CO₂ is the only GHG responsible for the atmosphere's capacity to absorb infrared energy. With this information, PSTs were expected to determine that F increases as C increases. Next, I directed PSTs' attention to two extreme theoretical values of F(C). The minimum theoretical value, which was $-150 \text{ J/m}^2/\text{s}$, occurs for a

totally transparent atmosphere. This case implies that $C = 0$ ppmv and that the atmosphere does not absorb any of the infrared energy emitted by the planet's surface (R). The maximum theoretical value, which was $45 \text{ J/m}^2/\text{s}$, occurs for a *totally opaque atmosphere*. This case implies that C has such a high value that the atmosphere absorbs practically all the infrared energy emitted by the planet's surface (i.e., if $C \rightarrow \infty$, then $B \rightarrow R$). The latter case implies that $F(C)$ must be increasing at a decreasing rate involving values that asymptotically approach to the maximum theoretical value. Thus, the expected response involved drawing an increasing concave-down (logarithmic-like) curve representing $F(C)$ (Figure 3.7).

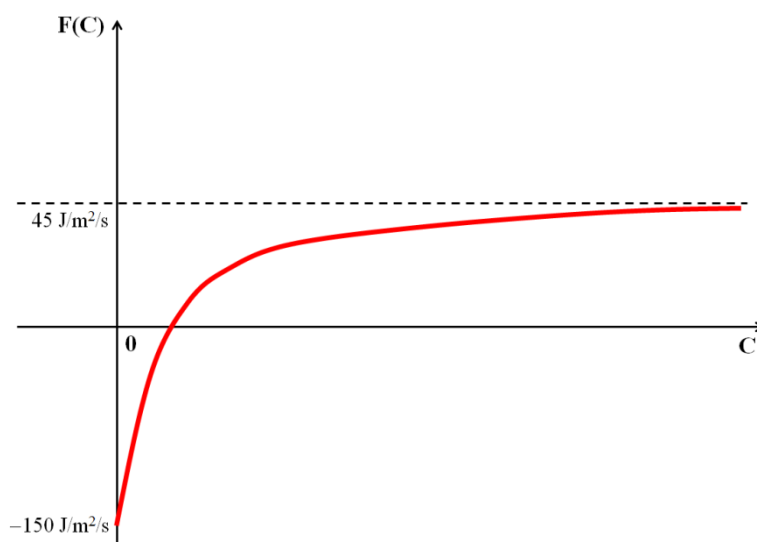


Figure 3.7. The expected response for S2T1: the graph of $F(C)$.

Set 2 task 2 (S2T2): planetary energy imbalance and average surface temperature. I created S2T2 to examine PSTs' reasoning about co-variation regarding two functional relations: the planetary energy imbalance with respect to time and the planet's average surface temperature with respect to time, after the concentration of CO_2 is instantaneously increased at time zero (positive forcing). To work on S2T2, PSTs were given a brief overview of the concept of planetary energy imbalance and a diagram of the Earth's energy budget (see Figure 3.5 above).

Then, S2T2 required PSTs to think about and graph the two functional relations previously described. As a simplification, the task assumed that the concentration then remained constant and no other forcing agent affects the energy budget (Figure 3.8).

An increase in the atmospheric concentration of CO_2 results in an energy imbalance in the Earth's energy budget. This initial imbalance is known as forcing by CO_2 . We want to examine how the planetary energy imbalance N and the planet's average surface temperature T vary over time, after the forcing has occurred. Use what you learned about the Earth's energy budget and the greenhouse effect, and the definition of $N = (S + A) - R$ to determine:

- How N vary over time t (in years) and sketch the graph of $N(t)$
- How T vary over time t (in years) and sketch the graph of $T(t)$

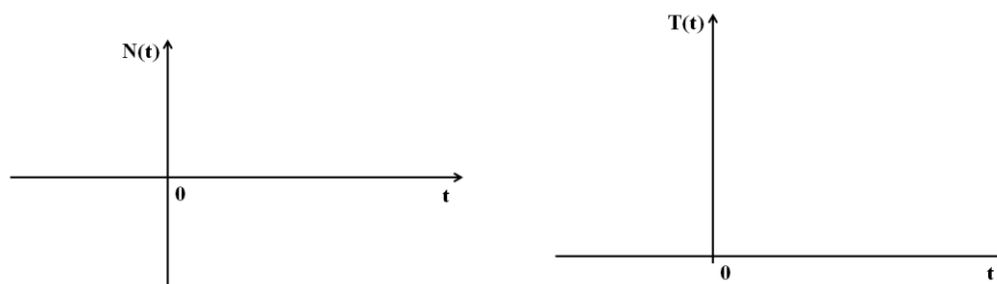


Figure 3.8. Set 2 task 2 (S2T2): planetary energy imbalance and average surface temperature.

The planetary energy imbalance as a function of time can be defined by $N(t) = (S + A(t)) - R(t)$. The energy flow S is approximately constant and is known as the solar constant. When the concentration of CO_2 is instantaneously increased at $t = 0$, a positive forcing of magnitude $F > 0$ is created, which in turn implies that $N(0) = F$. This means that the downward energy flow ($S + A$) exceed in magnitude the upward energy flow R , which produces a warming effect over the planet's surface, which in turn causes the surface's radiation R to increase. The surface's radiation R will continue to increase until it equals the downward energy flow ($S + A$) and a new radiative equilibrium is reached (i.e., $N = 0$). Mathematically, we have that $A = \frac{g}{2}R$ for $0 \leq g \leq 1$ (see Figure 3.5 above), which implies that

$$N(t) = (S + A(t)) - R(t) = S - \left(1 - \frac{g}{2}\right)R(t)$$

Therefore, as R increases, the magnitude of the energy imbalance decreases (i.e., if $t \rightarrow \infty$, then $R(t) \rightarrow (S + A(t))$ and $N(t) \rightarrow 0$). This realization leads to the conclusion that $N(t)$ must be decreasing at an increasing rate¹ involving values that asymptotically approach to zero. The latter implies that the Earth's energy budget is reaching a new radiative equilibrium. Therefore, the expected response for the graph of $N(t)$ involved drawing a decreasing concave-up curve similar to exponential decay (Figure 3.9).

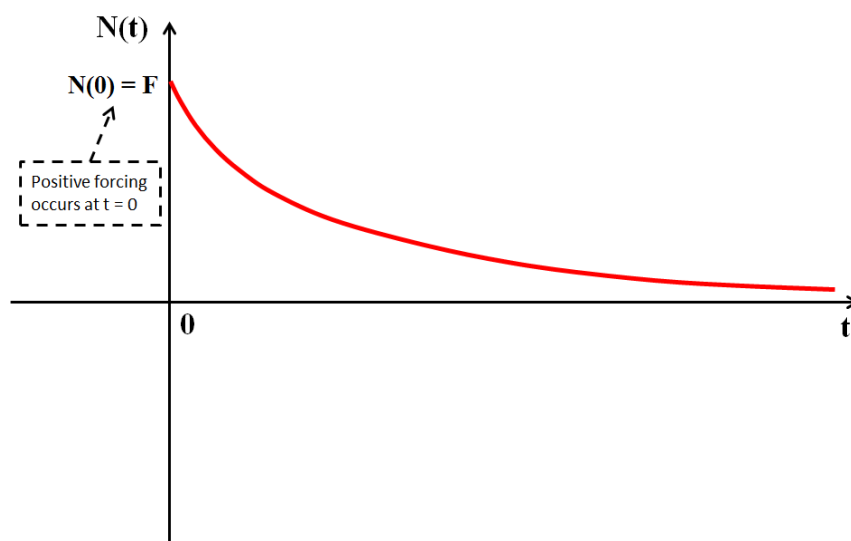


Figure 3.9. The expected response for S2T2: the graph of $N(t)$.

For the second part of S2T2, PSTs used their graph of $N(t)$ and what they learned about the energy budget in order to graph the variation of the planet's average surface temperature over time $T(t)$. As a simplification, PSTs are told to assume that $T(t)$ depends exclusively on the (simplified) relationship $N(t) = (S + A(t)) - R(t)$. A crucial realization for this task involved attending to the meaning of $N(t) > 0$. This implies that the downward energy flow ($S + A$) is

¹ Mathematically speaking, the rate of change of $N(t)$ is negative for $t > 0$ (assuming a positive forcing occurs at $t = 0$) since $N(t)$ is decreasing in magnitude over time. Therefore, the rate of change of $N(t)$ is increasing from negative values towards zero. PSTs referred to the rate of change of $N(t)$ as a decreasing rate, which can be seen as mathematically incorrect. However, what they meant (and they explicitly expressed so) is that $N(t)$ is decreasing by smaller and smaller amounts, which is correct.

larger in magnitude than the upward energy flow $R(t)$ for $t > 0$. This implies that a warming effect is being produced over the planet's surface. Therefore, $N(t) > 0$ implies that $T(t)$ must be increasing in magnitude. Additionally, the decreasing magnitude of $N(t)$ implies that $T(t)$ must be increasing at a decreasing rate since a new radiative equilibrium is being reached. This in turn implies that $T(t)$ grows asymptotically towards a new equilibrium temperature. Therefore, the expected response involved drawing an increasing, concave-down curve representing the function $T(t)$ (Figure 3.10). An additional outcome of this task may include the realization that $N(t)$ is a measure of the rate of change of $T(t)$ since $N(t) > 0$ defines an increasing $T(t)$ and the rate of change of $N(t)$ defines the concavity of $T(t)$. This leads to the reasonable conclusion that $\frac{\partial T}{\partial t} = \alpha N(t)$ for a constant $\alpha > 0$.

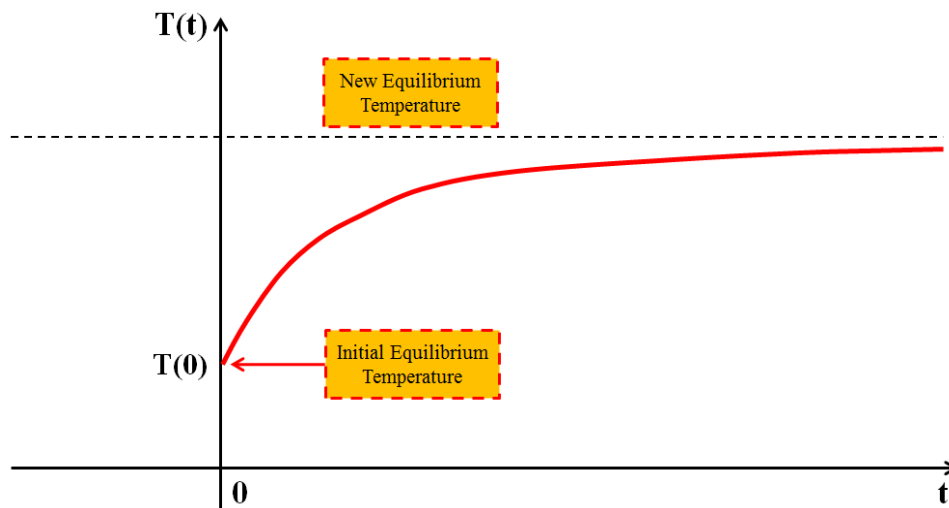


Figure 3.10. The expected response for S2T2: the graph of $T(t)$.

The Interview Process

I collected data through semi-structured task-based interviews (Goldin, 2000). I created interview protocols containing pre-defined questions to guide the interview process and to ensure all participants received similar probes during the interviews (structured part). Additionally, a semi-structured interview approach allowed me to explore emergent aspects regarding each

participant's particular responses and reasoning. This provided me with flexibility to spontaneously react to participants' responses and reasoning by asking additional follow-up questions not included on the protocols (unstructured part). Nonetheless, throughout the interview process, I adopted a facilitator role, avoiding intervening in their reasoning or directing their thinking in any particular way. I carefully listened to their responses and asked for clarifications, further explanation, or arguments for their claims.

All interviews involved participants working on mathematical tasks. Goldin (2000) used the term *task-based interviews* to refer to these research instruments in mathematics education. Goldin provided the following definition:

Task-based interviews for the study of mathematical behavior involve minimally a subject (the problem solver) and an interviewer (the clinician), interacting in relation to one or more tasks (questions, problems, or activities) introduced to the subject by the clinician in a preplanned way. The latter component justifies the term *task-based*, so that the subject's interactions are not merely with the interviewers, but with the task environment (p. 519).

I was interested in examining PSTs' conceptions of intensive quantities and ability to reason about co-variation related to introductory models for global warming. As participants worked on mathematical tasks regarding these quantities and mathematical models, my focus was to examine their cognitive processes rather than the correctness of their responses. The task-based interviews appeared advantageous for the goals of my research because "task-based interviews make it possible to focus research attention more directly on the subjects' process of addressing mathematical tasks, rather than just on the patterns of correct and incorrect answers in the results they produce" (Goldin, 2000, p. 520). It is important to note that all interview situations affect, to

a certain extent, the responses obtained and it is difficult to know to what extent the interview situation and questions may have influenced the results. However, Goldin explained that this does not represent a limitation of the methodology because “what is observed is always the subject’s behavior in the presence of the structured interventions. What may be inferred are aspects of the subject’s internal cognitions and/or affect in the presence of the interventions” (p. 521).

The interviews were conducted individually with each participant in a quiet room where the participant and I sat close to each other at a table. In order to capture participants’ responses and work, I utilized a video camera and an audio recording device. I placed the video camera across the table from where I sat and aimed it to capture the participant’s written work. The audio recording device was placed on the table close to the participant in order to better capture their verbal responses and improve audio quality. I provided each participant with markers, a ruler, white paper, and a calculator during each interview. The typical layout for the interviews is shown in Figure 3.11. Each interview lasted between 60 and 80 minutes, with interviews including Set 1 tasks being shorter than interviews including Set 2 tasks.

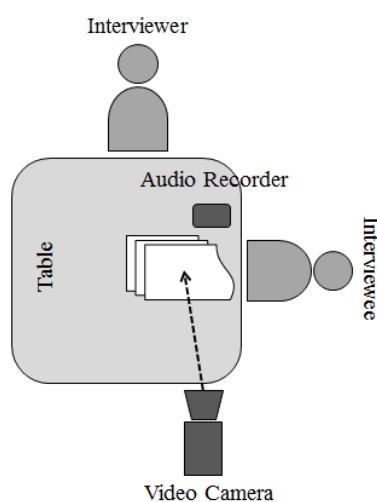


Figure 3.11. Interview Layout

I conducted a total of four individual, semi-structured task-based interviews with each participant. The participants completed tasks S1T1 and S1T2 during interview 1 and tasks S1T3 and S1T4 during interview 2 (Table 3.1). I gave PSTs time to read the statement of a task and develop a solution. Since the goal of these tasks was to examine PSTs' conceptions of the intensive quantities concentration, energy density, and energy flux density, I initially avoided providing PSTs with any formal or informal definition regarding these quantities so I did not influence their responses. I introduced the formal name for each quantity after a participant had determined and worked with the corresponding ratio or when a participant had difficulties working on the task and a direct prompt was needed.

Table 3.1

Correspondence between Study Parts, Interviews, and Tasks

Part I: Intensive Quantities				Part II: Introductory Mathematical Models	
Interview 1		Interview 2		Interview 3	Interview 4
S1T1 (Concentration)	S1T2 (Concentration)	S1T3 (Energy Density)	S1T4 (Energy Flux Density)	S2T1 (Forcing by CO ₂)	S2T2 (Energy Imbalance & Temperature)

The participants completed tasks S2T1 and S2T2 during interviews 3 and 4, respectively (Table 3.1). At the beginning of interview 3, I provided participants with important background information concerning the concept of Earth's energy budget, radiative equilibrium, and greenhouse effect. The participants needed this information in order to make sense of what S2T1 and S2T2 would ask them to do. First, I showed each participant a 7-minute video introducing the concepts of Earth's energy budget, radiative equilibrium, and greenhouse effect. This video is part of a series of educational videos available on the NASA Earth Observatory YouTube channel. After they watched the video, I answered any questions they may have had concerning the concepts discussed in that video. Second, I showed each participant a diagram of the Earth's energy budget (see Figure 3.5 above). I used this diagram to summarize the concept of Earth's

energy budget discussed in the video and to illustrate the planetary energy imbalance (N). In particular, I told participants that N measures the energy imbalance in the Earth's energy budget and is defined as a difference between the downward energy flows and the upward energy flows, or mathematically $N = (S + A) - R$. I explained to them that the Earth's energy budget was in radiative equilibrium when $N = 0$ and was in state of energy imbalance when $N \neq 0$. I spent time answering all questions PSTs may have had about this quantity in order to ensure they learned its definition and meaning. Finally, I gave PSTs time to read the statement of a task and subsequently described the concept of forcing by CO_2 for them.

For interview 4, I revisited the concepts of Earth's energy budget and planetary energy imbalance N . I answered any questions the participants may have had about these concepts. Then, I gave them time to read the statement of the first part of S2T2 and envision the graph of $N(t)$. Once participants had drawn a graph of $N(t)$, I gave them time to read the statement of the second part of S2T2, the part concerning the graph of the planet's average surface temperature $T(t)$. After they read the statement, I re-explained to them the meaning of $N(t) > 0$; that is, $N(t) > 0$ implies that the downward energy flow ($S + A$) is larger than the upward energy flow $R(t)$. Next, I gave participants time to envision the graph of $T(t)$.

Data Analysis

The video recordings of interviews and PSTs' work during the interviews constituted the data sources for my study. I used the *Framework Analysis* approach to analyze my interview data. This qualitative method of data analysis "involves several inter-related but distinct stages that allow for theme-based or case-based analysis, or a combination of the two, through the development of charts that may be read across (cases) or downwards (themes)" (Ward, Furber, Tierney, & Swallow, 2013, p. 2425). The inter-related stages in framework analysis provide a

transparent and systematic process for data analysis, in which there is a clear transition from interviews and transcript to themes and interpretations of results (Ward et al, 2013). Through these stages, the researcher creates and refines framework analysis' distinctive feature: the *matrix output*. Gale, Heath, Cameron, Rashid, and Redwood (2013) explained that the matrix output involves “rows (cases), columns (codes), and ‘cells’ of summarized data, providing a structure into which the researcher can systematically reduce the data, in order to analyze it by case and by code” (p. 2). The matrix output allows the researcher to “compare with ease data *across* cases as well as *within* individual cases” (Gale et al., 2013, p. 2). Since my multiple-case study involves three PSTs working on six different tasks, the framework analysis approach appears advantageous to analyze responses across cases (between PSTs) or within a case (between tasks).

Framework analysis is commonly used in the analysis of semi-structured and unstructured interview transcripts since it allows the researcher to treat each individual interviewee as a single case under study (Gale et al., 2013). Additionally, framework analysis allows the researcher to define themes in accordance with the type of research being conducted; that is, framework analysis can be used in *inductive* or *deductive* research. Gale and colleagues drew the distinction based on how themes are selected

In the deductive approach, themes and codes are pre-selected based on previous literature, previous theories or the specifics of the research question; whereas in the inductive approach, themes are generated from the data through open (unrestricted) coding, followed by refinement of themes. In many cases, a combination approach is appropriate when the project has some specific issues to explore, but also aims to leave

space to discover other unexpected aspects of the participants' experience or the way they assign meaning to phenomena (p. 3).

Framework analysis was pertinent for my study since I used interviews to gather information regarding PSTs' conceptions and reasoning. Additionally, framework analysis provided me with the flexibility to utilize a deductive-inductive combination approach for the analysis of PSTs' responses during the interviews. To assess PSTs' conceptions of intensive quantities commonly used to model global warming, I pre-selected themes from the literature in mathematics education concerning quantitative reasoning (Thompson 1994a, 1994c; & Thompson, 2011) and intensive quantities and conceptions of ratio (Howe et al., 2010; Nunes et al., 2003; & Simon & Placa, 2012). Similarly, to examine PSTs' ability to reason about co-variation while learning about introductory models for global warming, I pre-selected themes from the literature in mathematics education concerning covariational reasoning (Carlson et al., 2002; Johnson, 2012, 2015; & Saldanha & Thompson, 1998). However, I also left space for coding emergent themes regarding PSTs' responses during the data analysis.

In general, the framework analysis method consists of five distinctive stages: *familiarization with data, developing an analytic framework, indexing and pilot charting, summarizing data in analytic framework, and synthesizing data by mapping and interpreting* (Ward et al., 2013). What follows is a description of each stage along with what I did during each stage in order to analyze the data for my study.

Stage 1: Familiarization with Data

The researcher becomes familiar with the data in order to develop a sense of the content of each interview. This can be accomplished by conducting the interviews, listening to the video/audio recordings, transcribing interviews, and reading interview transcripts. I conducted all

task-based interviews for my study. Then, I watched the videos all interview videos to develop as sense of the content of my interviews. As I watched the videos, I took notes about topic shifts in the conversation; these notes were used to divide the interview videos into smaller more manageable episodes of 10 to 15 minutes each. The episodes guided the transcription process, which took place during the next round of video watching.

Next, I conducted a second round of video watching in order to transcribe the episodes identified during the first round of video watching. I transcribed conversations between the participant and I, which included questions asked and the participant's responses to such questions. I also included description of the participant's actions and gestures along with my interpretation in order to provide context for the participant's responses and actions during the interview. The following excerpt (Table 3.2) is a transcription sample and illustrates how interviews were transcribed. In the excerpt, I was trying to help Kris reason about the variation in the planetary energy imbalance by revising the concept of forcing by CO₂. The excerpt also shows the conversation that Kris and I had about the way she was thinking of the situation:

I developed what I called an *interview map* for each interview so that there were four interview maps, each corresponding to interview 1, 2, 3, and 4. An interview map is a table-like arrangement formed by three columns and several rows. The column headings were Location, Behaviors or Events, and Notes, while the row headings represented a 10 to 15-minute episode from the interview videos. An interview map starts with general information concerning the participant's pseudonym, the task the participant worked on, and the duration of the video recording. The *Location column* provides information concerning the duration of an episode (initial time stamp) and the number assigned to that episode. A *cell* formed by the intersection between the *Behaviors or Events column* and a particular row contained transcript of PSTs'

responses and actions during a particular episode (Figure 3.12). The transcripts also included additional information to provide context in which the participant's responses and actions are taken place.

Table 3.2

Transcription Sample: Excerpt from Kris, Interview 4, S2T2.

In order to help Kris make sense of how N varies, I showed her a diagram of the budget. Then, I explained to Kris that an instantaneous increase in the concentration of CO₂ would result in a change in the value of B and, consequently, A. To show this, I wrote $B = 320 \text{ J/s/m}^2$ and $A = 160 \text{ J/s/m}^2$ on the diagram as an example of what a positive forcing would create in terms of imbalance. I showed Kris that the value of N right at the moment of instantaneous increase in C (value at $t = 0$) would be $N = 10 \text{ J/s/m}^2$. After that, Kris and I had the following exchange:

- Me: All that I just explained to you is what happens at $t = 0$. How do you think the flows of energy and N will change as time passes?
- Kris: That means 400 is going to leave (she refers to the new value of R). Whatever percentage (she refers to the fraction of R being absorbed by the atmosphere) 320 of 390 is going to stay [she points at B]. That (the new value of B) is going to be whatever percentage that is multiply by 400, which is the new {value of R}, it is going to give you a higher value for B
- Me: So, we will get a new value for N
- Kris: Uh-huh, and that is going to be more positive, I feel like /pause/ Maybe not
- Me: Do you what to calculate the new value of N?
- Kris: Sure [she grabs the calculator]. Ok, so we are going to do 320 divided by 390 is about 82.1 percent. So, we are saying 82.1 percent of R is absorbed by the atmosphere. So, if we multiply that number by 400, which is our new R value, it'll give us what is absorbed by the atmosphere, which is about 328. So, we can write that [she crosses 320 out and writes 328 for B]. So, the new one (value of B) is 328, so then we do 328 divided by 2, which is our new A value that is being absorbed by the Earth, so, 164 [she writes 164 next to A]
- Me: So, up until here, I see you were right because R is increasing and A is also increasing
- Kris: Yes! I trying to see how the difference between these values [she draws a square bracket between 320 and 328 and another between 160 and 164] like, OK! Since we did the initial increase from 300 to 320, I just want to see how these values change [she points at the values $A = 160$ and $A = 164$] because if /pause/ OK!, I am just going to do it so you see what I see

This exchange shows Kris's attempt to visualize how the flows of energy in the budget vary over time after the positive forcing has occurred. She became to coordinate values of the flows at different moments in time, which seemed to assist her in visualizing the variation in the budget. Kris decided to repeat the process described in the excerpt above two more times. In other words, Kris determined values for R, B, and A for two more moments in time, obtaining the values $R = 404 \text{ J/s/m}^2$, $B = 331.5 \text{ J/s/m}^2$, and $A = 165.75 \text{ J/s/m}^2$ for the first moment and the values $R = 405.75 \text{ J/s/m}^2$, $B = 332.917 \text{ J/s/m}^2$, and $A = 166.46 \text{ J/s/m}^2$. After these calculations, Kris concluded "what I am seeing is that we are reaching a new equilibrium because, after this initial increase of absorption of 320 (she is referring to the value of B for the moment after the forcing), the next cycle (she is referring to the next value of B) is going to absorb 328, which is an increase of 8. The next one (the next value of B) is going to increase by an amount of 3.5, which is less than 8, and then the next cycle (the next value of B) causes the absorption to increase by 1.417. Well, this difference right here between 320 and 328 is greater than the difference between these two values the 328 and 331.5, and the difference between these two [she points at 332.917 and 331.5] is less than those [she points at 328 and 331.5], which is less than that [she points at 320 and 328]. That tells me that there is eventually going to be a limit. Yeah! It (the Earth's energy budget) is going to reach a new equilibrium point somewhere."

Initial Time Stamp & Episode Number	General Information	Pseudonym	PAM	Task	S1T3	Video duration	00:53:47	Notes, Interpretations, & Codes
00:00:00 [1]	<p>I ask Pam to explain the relationship between temperature and energy density based on the given definition for temperature on the task. Pam struggles to understand what we meant by internal energy of an object. I remind Pam we said both sheets had same temperature at the beginning. Then, I ask her to tell me the meaning of that in terms of energy density. Pam says "they had different internal energies," but she really doesn't know how to interpret this in terms of energy density. I remind Pam she concluded which sheet will be hotter by looking at energy densities. So, I restate the question about the relationship between energy density and temperature. Pam struggles to see that energy density takes into account both energy and size. She realizes that the highest energy density implies the highest temperature, but she keeps focusing only on energy because, according to her, "size doesn't matter, but the amount of Joules going towards it [1 m²] matters."</p> <p>I ask Pam what the energy densities were for sheet A and B at the beginning of experiment (assuming same initial temperature). Pam says</p> <p>Pam: I think this [sheet B] had a bigger one...</p> <p>Me: energy density?</p> <p>Pam: No, smaller. But, if they [the sheets] had the same room temperature, this one [sheet B] had a higher internal energy because it had to keep that room temperature the same for a bigger sheet, and this one [sheet A] was a smaller sheet, so it didn't have to have as much internal energy</p> <p>I asked Pam again what relationship there was between the sheets' energy density at the beginning of the experiment if they had the same temperature. Pam drew a table comparing information between sheet A and sheet B. This information includes internal energy at the beginning (smaller or higher), the sheet's dimensions, the radiation incident to the sheet, and the energy density at the end (smallest or highest). Pam says "I find it interesting that although this has smaller internal energy and it has less Joules radiated at it, it ended up with the highest energy density. But I guess that makes sense because it is smaller ... so here, it had a higher energy at the beginning, and it had more Joules radiated, and they had more [where] to spread out in a 5 by 4 sheet because there was more space to go, so less density." Then, Pam concludes that "the higher the energy density, the higher the temperature," which means that, at the beginning of the experiment, the sheets' energy densities "are the same because the temperature was the same"</p>							<p>INABILITY TO SEE ENERGY DENSITY AS A MEASURE OF TEMPERATURE</p> <p>CONCEIVING OF ENERGY DENSITY IN TERMS OF ENERGY ONLY</p> <p>RATIO AS AN ASSOCIATION BETWEEN AMOUNT OF ENERGY AND ONE AREA UNIT</p> <p>ENERGY DENSITY IS CONCEIVED OF AS A SINGLE EXTENSIVE QUANTITY/ (ENERGY), DESCRIBING AN ASSOCIATION</p> <p>BEGAN TO CONCEIVE OF ENERGY DENSITY AS A CONSISTANT MULT. RELATIONSHIP BETWEEN ENERGY & SURFACE AREA</p> <p>BEGAN TO RECOGNIZE THE ROLE OF SURFACE AREA IN DEFINING ENERGY DENSITY</p> <p>CONCEIVING OF ENERGY DENSITY AS A MEASURE OF TEMPERATURE</p>

Figure 3.12. Interview Map sample

During this first stage of data analysis, the researcher can identify some initial recurring categories or codes to be used later in developing an analytic framework. After I completed all interview maps, I read them in order to identify initial codes or interpretations concerning PSTs' conceptions of quantities and ways of reasoning about co-variation. I first read all three interview maps concerning interview 1 (one interview map per participant), which concerned the mathematical tasks S1T1 and S1T2 both related to the intensive quantity concentration. As I read an interview map, I used the *Notes column* to record initial codes or interpretations of the participants' responses (see Figure 3.12 above). I looked and coded responses and actions that provided evidence of how each participant was:

- Thinking about the situation (e.g., "concentration is a property of both a breath of air and the whole tank (assuming all gases are well mixed)"),

- Conceiving of concentration (e.g., “interpreting concentration as the fraction or percentage of air that O₂ represents” or “interpreting concentration of 0.21 as 0.21 cm³ in 1 cm³ of air”),
- Using division or ratio (e.g., “recognizing that amplifying or simplifying volume concentration ratios creates equivalent ratios, which all have the same value”),
- Reasoning about the relationship between volumes in the ratio (e.g., “recognizing ppmv concentration is an invariant ratio’s value between volume of CO₂ and the number of 1,000,000 cm³ of air in a tank” or “recognizing that volume concentration and ppm concentration measure the same attribute because 362 ppmv means “362 cm³ of CO₂ in a one-million-cm³ tank,” which has a value of 0.000362 which equals volume concentration”), and
- Measuring by using a ratio (e.g., “understanding concentration measures volume of O₂ in relation to volume of air, and this allows a comparison between tanks” or “recognizing concentration as measuring abundance of a gas relative to air (‘out of 850,000 cm³ of air in tank B, 19% is oxygen’”).

When I finished coding the interview maps of interview 1, I read all three interview maps of interview 2, which concerned the mathematical tasks S1T3 and S1T4 related to the intensive quantities energy density and energy flux density, respectively. I recorded codes or interpretations under the *Notes column* of each interview map. As before, I looked for responses and actions that provided evidence of how each participant was: thinking about the situation, conceiving of energy density or energy flux density, using division or ratio, reasoning about the relationship between volumes in the ratio, and measuring by using a ratio. Finally, I organized my codes and interpretations in each interview map according to their frequency of occurrence

and their meaningfulness in providing evidence for the participants' conceptions of intensive quantities commonly used to model global warming. I discarded all codes that appeared rarely and were not meaningful in answering the aforementioned research inquiry.

For interviews 3 and 4, the process was slightly different since codes did not emerge from the analysis of transcripts. These interviews provided data for the second part of my study: examining PSTs' ability to reason about co-variation while learning about introductory mathematical models for global warming. I identified four pre-defined themes from the literature on covariational reasoning. In particular, I utilized four mental actions from the Covariation Framework (Carlson et al., 2002):

- Mental action 1 (MA1) involves recognizing what quantities are varying together in a situation, or the dependency of change (e.g., y varies with changes in x);
- Mental action 2 (MA2) involves identifying the direction in which quantities vary together, or the direction of change (e.g., y increases as x increases);
- Mental action 3 (MA3) involves coordinating amounts of change in a quantity with changes in another quantity (e.g., a change Δy in y correspond with a change Δx in x); and
- Mental action 4 (MA4) involves coordinating rates of change of a quantity with changes in another quantity (e.g., y increases faster during a change Δx in x).

Although the Covariation Framework describes five mental actions, I did not find evidence of mental action 5 (MA5) during the analysis of PSTs' responses. Therefore, I did not include MA5 into my analytic framework. Additionally, I included the constructs of *chunky thinking* and *smooth thinking* (Castillo-Garsow et al., 2013) as two additional themes for my analytic framework. These two constructs describe two different ways of reasoning about co-variation. Chunky thinking refers to reasoning about co-variation through iteration of discrete units or

chunks. The *chunks* are intervals of values (for either the dependent or independent variable) that are conceived as single discrete units so that no variation occurs within them. In contrast, smooth thinking refers to reasoning about co-variation as occurring progressively and continuously. In other words, a quantity varies continuously with respect to another quantity during the flow of conceptual time. I read all six interview maps of interviews 3 and 4, which concerned the forcing by CO₂ function $F(C)$, the planetary energy imbalance function $N(t)$, and the average surface temperature function $T(t)$. As I was reading each interview map, I looked for evidence of the mental actions and type of thinking described above. When I interpreted a particular transcript text as showing evidence of a mental action or type of thinking, I used the *Notes column* to record the appropriate theme or themes for that transcript text.

Stage 2: Developing an Analytic Framework

In this stage, the researcher organizes the codes or categories identified during the Familiarization Stage (Stage 1) into emerging themes. The researcher may also incorporate pre-defined themes from literature or theory. Then, the themes are used to create an analytic framework that is applied back to the interview transcripts in order to analyze participants' responses. For my study, I used a combination of emerging themes identified during Stage 1 and pre-defined themes from literature and theory in order to create two analytic frameworks: one for analyzing data from interviews 1 and 2 and another for analyzing data from interviews 3 and 4. The two analytic frameworks correspond with each of my research goals: (a) assessing PSTs' conceptions of intensive quantities commonly used to model global warming; (b) and examining PSTs' ability to reason about co-variation while learning about introductory mathematical models for global warming. I created the first analytic framework by identifying relations among the codes and interpretations defined during Stage 1. Related codes and interpretations were

grouped under overarching themes showing evidence of the participants' conceptions of the intensive quantities concentration, energy density, and energy flux density. Particularly, I defined three overarching themes under which I grouped all codes and interpretations defined in Stage 1 for interviews 1 and 2: *Use of Division*, *Constant Multiplicative Relationship*, and *Measurable Attribute*. Under theme 1, I grouped all codes and interpretations providing information about the meaning a participant demonstrated for the value of the ratios concentration, energy density, and energy flux density. Under theme 2, I included all codes and interpretations providing information about whether a participant understood that a ratio represents a constant multiplicative relationship between two quantities that can vary. Under theme 3, I grouped all codes and interpretations providing information about what a participant recognized as the attribute measured by a particular ratio. Table 3.3 shows examples of codes and interpretations defined in Stage 1 that were grouped under a particular theme during this stage (Stage 2). Finally, I related these themes with the theoretical constructs of *Ratio as Per-One* conception and *Ratio as Measure* conception (Simon & Placa, 2012) in order to create the analytic framework for assessing PSTs' conceptions of intensive quantities commonly used to model global warming (Table 3.4).

The second analytic framework was developed to examine PSTs' ability to reason about co-variation while learning about introductory mathematical models for global warming. The themes for this analytic framework came from the literature on covariational reasoning and included those used during Stage 1. For this stage (Stage 2), I related the mental actions 1 through 4 with the two constructs of chunky thinking and smooth thinking in order to create the analytic framework for the second part of my study. For instance, when working on S2T2, PSTs needed to visualize how the energy flows were varying over time in the Earth's energy budget.

Table 3.5 shows two excerpts: one showing Pam’s reasoning as she visualize how these energy flows were varying, and another showing Kris’s reasoning as she visualize how these energy flows were varying. Both excerpts were coded as showing evidence of mental actions 1 and 2 (MA1 and MA2). Pam and Kris identified what energy flows were varying as time increased (MA1) and visualized the direction in which the energy flows were varying as time increased (MA2).

Table 3.3

The Three Overarching Themes (First Part of Study)

<u>Theme</u>	<u>Initial Codes</u>
Use of Division	<ul style="list-style-type: none"> • Interpreting concentration as the fraction or percentage of air that O₂ represents. • Interpreting concentration of 0.21 as 0.21 cm³ in 1 cm³ of air. • Interpreting energy density as average amount of energy per area unit (“[is] about 2,812.5 Joules per m²”). • Interpreting energy density exclusively as an amount of energy within an area unit (“how much [energy] is in like 1 meter by 1 meter”). • Seeing energy density as enough information to draw conclusions about temperature because it allows comparing amount of energy relative to 1 m² across all three sheets (sheet A, B, and C). • Interpreting energy flux density as “every second, how many Joules are to 1 meter.”
Constant Multiplicative Relationship	<ul style="list-style-type: none"> • Concentration is a property of both a breath of air and the whole tank (assuming all gases are well mixed). • Recognizing ppmv concentration is an invariant ratio’s value between volume of CO₂ and number of 1,000,000 cm³ of air in a tank. • Reflecting on the relationship between energy and area and how this relationship relates to energy density. • Realizing same energy density means same temperature because “it doesn’t matter how big or small the sheet is. As long as [relation energy-area] is proportional they’re going to have the same energy density.” • Deciding which sheet is hotter if they had same size by comparing amounts of energy. However, struggle to decide which sheet would be hotter if they had different size but same radiation. • Thinking energy flux density is insufficient information to determine which sheet’s temperature will first reach 25 °C. Needing to know either size or radiation rate.
Measurable Attribute	<ul style="list-style-type: none"> • Recognizing concentration as measuring abundance of a gas relative to air (“out of 850,000 cm³ of air in tank B, 19% is oxygen”). • Recognizing that having same temperature implies having the same energy density because “energy density and temperature are kind of the same.” • Recognizing energy flux density measures rate of change of temperature (“as long as we know the increase in energy density per second, then we can tell immediately which [sheet] is going to reach [25 °C] faster.” • Energy flux density does not measure anything in the situation.

Table 3.4

The Analytic Framework for the First Part of Study (Conceptions of Intensive Quantities)

	<u>Ratio as Per-One</u>	<u>Ratio as Measure</u>
<i>Use of Division</i>	The ratio is conceptualized exclusively in terms of a magnitude of constituent quantity A associated with one unit of constituent quantity B.	The ratio is conceptualized in terms of the relative magnitude of constituent quantity A with respect to constituent quantity B
<i>Constant Multiplicative Relationship</i>	Conceiving of a ratio as a multiplicative relationship between (possibly unknown) fixed values of quantities.	Conceiving of a ratio as a constant multiplicative relationship between quantities that can vary.
<i>Measurable Attribute</i>	The magnitude of constituent quantity A is conceptualized as the measure of an attribute.	The ratio's relative magnitude is conceptualized as the measure of an attribute.

Notice, however, that Pam's excerpt showed evidence of chunky thinking since she associated particular values of energy flows to fixed and discrete values of time. This suggests that Pam visualized the variation in the energy flows as occurring in discrete moments. In contrast, Kris's excerpt showed evidence of smooth thinking since she did not need particular values to reason about the direction of change of the energy flows with respect to time. This suggests she visualized the energy flows as taking different values over time without having to determine (and see) particular values corresponding to particular (and discrete) values of time. By comparing Pam's and Kris's excerpts, I identified and defined how mental actions 1 and 2 look like across chunky and smooth thinking. Particularly, I notice that there is no distinction between chunky thinking and smooth thinking when referring to MA1. However, MA2 looked different under Pam's chunky thinking than it did under Kris's smooth thinking. I followed this procedure of comparing mental actions across chunky and smooth thinking in order to create the second analytic framework, which was used to examine PSTs' ability to reason about co-variation while learning about introductory mathematical models for global warming (Table 3.6).

Table 3.5

Example of Relating Mental Actions 1 and 2 with Chunky/Smooth Thinking

<u>Participant</u>	<u>Excerpt</u>	<u>Theme(s)</u>
Pam	<p>We have 390 here, originally [she writes 390 for R on her diagram]. Then, we increased {the concentration of CO₂} in the atmosphere. So, this increased to 310 [she writes 310 for B]. Then, this spiked to be 155 [she writes 155 for A]. So then, 240 plus 155 is 395, so this is our new {value of R} [she writes 395 for R]. Then, we have 79.5 percent {of R} is getting absorbed as B into the atmosphere [she writes 79.5% on her diagram]. So, 79.5 percent of 395 is that [she writes 313.97 for B] ...</p> <p>Then, we have to split {the value B = 313.97} in half to get that [she writes 156.98 for A], which would increase {the value of R} again ... [She adds 240 and 156.98 with the calculator] so then, {the value of R} goes up to 396.98 [she writes 396.985 for R]. [She calculates 79.5% of 396.98 with the calculator] so then, {the value of B} increases {to} 315.6. So then, divide {the value B = 315.6} by two, and we have 157.8 [she writes 157.8 for A]. And then, that increases {the value of R} again.</p>	<p>Mental action 1 Mental action 2 Chunky thinking</p>
Kris	<p>Right before {the concentration of CO₂} increases, it is going to be like the same R as before it increased. And then, a bigger proportion or percentage of the R value that is released into the atmosphere is going to get absorb by the atmosphere, and so then more of it is going to go back into the surface ... Yeah! So, the B portion of {R} is going to be larger also. So, that means A, which is one half of B, it's also going to be increased ... So then the input of S and A together, which is right here [she points at the formula $N = (S + A) - R$], is going to be increased. And then, R is also going to increase because we absorbed more energy or heat into the surface.</p>	<p>Mental action 1 Mental action 2 Smooth thinking</p>

Table 3.6

The Analytic Framework for the Second Part of Study (Reasoning about Co-variation)

	<u>Chunky Thinking</u>	<u>Smooth Thinking</u>
<i>MA1</i>	Dependency of change is established by noticing that the dependent variable y changes when the independent variable x changes.	
<i>MA2</i>	Direction of change is established by associating particular values of the dependent variable y with discrete values of the independent variable x .	Direction of change is established by visualizing variation in the dependent variable y as the independent variable x varies continuously.
<i>MA3</i>	Amounts of change Δy in the dependent variable y are associated with entire intervals I (chunks) of the independent variable x . The change Δy occurs instantaneously in interval I , rather than occurring through interval I (no variation within a chunk).	Amounts of change Δy in the dependent variable y are associated with amounts of change Δx in the independent variable x . The change Δy builds itself as the independent variable changes from x to $x + \Delta x$.
<i>MA4</i>	Average rates of change $\frac{\Delta y}{\Delta x}$ of the dependent variable y are associated with entire intervals I (chunks) of the independent variable x . The value $\frac{\Delta y}{\Delta x}$ is the <i>actual</i> rate of change of y at the particular interval I of x , rather than rate of change of y <i>averaged over</i> that interval.	Average rates of change $\frac{\Delta y}{\Delta x}$ of the dependent variable y are associated with amounts of change Δx in the independent variable x . Therefore, the value $\frac{\Delta y}{\Delta x}$ represents the rate of change of y <i>averaged over</i> the interval $(x, x + \Delta x)$.

Stage 3: Indexing and Pilot Charting

The researcher applies the analytic framework back to the interview transcripts. Notes can be made to identify what categories or themes are reflected in transcripts excerpts. As the analytic framework is applied back to the transcripts, the researcher can refine categories and themes and verify that they reflect appropriate interpretations of transcript excerpts. During this stage, I applied each analytic framework back to their corresponding interview maps. This represented a second round of transcript coding. I began by applying the analytic framework shown in Table 3.4 back to interview maps from interview 1 and 2. I read all coded transcript text in each interview map in order to verify appropriate interpretation and record the

corresponding themes under the interview map's *Notes column*. In this way, I assigned themes to all initially coded transcript text. For instance, consider the following excerpt from interview map 2 (interview 2):

I ask Jodi to interpret this result ($125 \text{ J/m}^2/\text{s}$). Jodi draws a 2×3 rectangle formed by 6 little squares representing sheet A. Then, she says “in one square unit of area, we need to increase the energy density by 125 Joules per second in order to increase the overall sheet's temperature by $10 \text{ }^\circ\text{C}$.” Jodi then repeats the ratio for sheet B and gets $60 \text{ J/m}^2/\text{s}$. She interprets this ratio in the following way: “as the energy density of 1 m^2 increase by 60 J/s , after 104 seconds, the energy density of 1 square unit would've increased 6,250 Joules.”

This excerpt had three initial codes: *Jodi interprets the ratio's value ($125 \text{ J/m}^2/\text{s}$) as association* (e.g., “in one square unit of area, we need to increase the energy density by 125 J/s ”); *Jodi thinks of $\text{J/m}^2/\text{s}$ units as “weird”*; and *Interpreting energy flux density of $60 \text{ J/m}^2/\text{s}$ as association* (e.g., “the energy density of 1 m^2 increase by 60 J/s ”). Since all these three codes were related to the theme Meaning of Ratio's Value (Theme 1, Analytic Framework in Table 3.4), I recorded this theme next to the initial codes between square brackets, as shown below:

- *Jodi interprets the ratio's value ($125 \text{ J/m}^2/\text{s}$) as association* (e.g., “in one square unit of area, we need to increase the energy density by 125 J/s ”) [**Meaning of Ratio's Value**];
- *Jodi thinks of $\text{J/m}^2/\text{s}$ units as “weird”* [**Meaning of Ratio's Value**]; and
- *Interpreting energy flux density of $60 \text{ J/m}^2/\text{s}$ as association* (e.g., “the energy density of 1 m^2 increase by 60 J/s ”) [**Meaning of Ratio's Value**].

When I was not sure what theme better represented a particular transcript text, I went back and watched the interview video from which the transcript text came. This allowed me to form a

better idea of what theme was being reflected in that particular transcript text. I only indexed transcript text that reflected themes described in the analytic framework. Transcript texts that did not reflect any theme from this analytic framework were not indexed since they did not provide meaningful information to answer the research questions. After I finished indexing the interview maps from interviews 1 and 2, I applied the analytic framework shown in Table 3.6 back to interview maps from interview 3 and 4. I followed a similar indexing procedure as the one described for the first two interviews. As before, I read all coded transcript text in each interview map in order to verify appropriate interpretation and assignment of themes under the interview map's *Notes column*. Through reading each interview map and indexing transcript texts, I became further immersed in the data and was able to refine my existing themes as well as ensuring appropriate interpretations of transcript texts.

Stage 4: Summarizing Data in Analytic Framework

During this stage, the researcher reduces data into “understandable but brief summaries of what was said by the participants” (Ward et al., 2013, p. 2427). The summarized pieces of data should provide enough information about participants’ responses so that the researcher can make sense of them without frequently having to refer back to the original data. It is also important to develop references to the original transcript to summarize pieces of data in order to track back where to locate a particular piece of data. I started with the interview maps from interviews 1 and 2. I identified and read all transcript texts categorized under a particular theme. As I read through these transcript texts, I identified and summarized those showing more representative evidence regarding the theme. For example, Then, I repeated this process for the next transcript texts categorized under the next theme. Again, I identified and summarized transcript texts showing more representative evidence regarding the theme. Once I completed

this process for all themes for all interview maps concerning interviews 1 and 2, I organized the selected summarized pieces of data in the corresponding analytic framework in order to create a matrix output per quantity: one for concentration, another for energy density, and finally one for energy flux density. Each matrix contained four columns (Meaning of Ratio's Value, Constant Multiplicative Relationship, Measurable Attribute, Conception) and three rows (Pam, Kris, and Jodi). I charted the selected, summarized pieces of data into the corresponding cell formed by an intersection between one of the first three columns and a row. Once a matrix had all its corresponding selected, summarized pieces of data, I read all pieces of data for a particular participant in order to infer what conception they had about a particular quantity. I annotated my inference into the cell formed by the *conception column* and the corresponding row for that participant. An example of the matrix output for energy density is shown in Figure 3.13. I added the episode number in square brackets next to each selected, summarized piece of data. This number in square brackets corresponds to the episode number specified in each interview map. This allowed me to track each selected, summarized piece of data back to the original transcript in the interview maps.

	Meaning of Ratio's Value	Constant Multiplicative Relationship	Measurable Attribute	Conception
Pam	How many Joules are per square meter [2] Fixing area to compare energy [3]	Beginning to consider both energy and surface area to define energy density [7]	More energy to heat up something bigger [4] Energy as a measure of temperature [6]	Ratio as Per-One
Kris	Energy density as energy averaged over surface [1]	Ratio between energy and surface defines energy density [2] Energy and surface remain proportional [5] Energy density as a constant ratio between energy and surface [5]	Energy density as a measure of temperature [2] Energy density is like temperature [6]	Ratio as Measure
Jodi	Energy IN one meter square [3] Fixing area to compare energy [4]	Size probably matter when defining energy density [2] Beginning to consider both energy and surface area to define energy density [8]	Only amount of heat defines temperature [1]	Ratio as Per-One

Figure 3.13. The matrix output for energy density.

I followed a similar process for the interview maps concerning interviews 3 and 4. After I completed the process for all themes for all interview maps concerning interviews 3 and 4, I organized the selected summarized pieces of data in the corresponding analytic framework in order to create three matrices output: one for the forcing by CO₂ or F(C), another for the planetary energy imbalance or N(t), and finally one for the planet's average surface temperature or T(t). Each matrix contained five columns, one for each mental action 1 through 4 and one for level of covariational reasoning. As before, each matrix had three rows, one for each participant (Pam, Kris, and Jodi). I charted the selected, summarized pieces of data into the corresponding cell formed by an intersection between a mental action column and a row. Once a matrix had all its corresponding selected, summarized pieces of data, I read all pieces of data for a particular participant in order to infer their level of covariational reasoning relative to a particular function (i.e., relative to either F(C), N(t), or T(t)). I annotated my inference into the cell formed by the *level of reasoning column* and the corresponding row for that participant. An example of the matrix output for the planetary energy imbalance N(t) is shown in Figure 3.14. I added the episode number in square brackets next to each selected, summarized piece of data. This number in square brackets corresponds to the episode number specified in each interview map. This allowed me to track each selected, summarized piece of data back to the original transcript in the interview maps.

	MA1	MA2	MA3	MA4	Level of Reasoning
Pam	Recognizing energy flows change over time [1]	Associating particular values of energy flows to discrete moments in time (chunky thinking) [1] Direction of change of $N(t)$ from video [2] Determining particular values of $N(t)$ for discrete moments in time (chunky thinking)[2]	Associating change in $N(t)$ with chunks in time (chunky thinking) [2]	Not Observed	Amounts of Change Level (L3) (Chunky Thinking)
Kris	Recognizing energy flows change over time [1] Identifying which energy flows vary and which ones remain constant in $N = (S+A)-R$ [3]	Visualizing direction of change of energy flows over continuous time (smooth thinking) [1] Visualizing value of $N(0)$ without the aid of particular values [1] Associating particular values of $N(t)$ with discrete values of time (chunky thinking) [3]	Associating amounts of change with changes in time and attending to variation in amounts of change [3] Justifying concavity by using amounts of change level reasoning (smooth thinking) [3]	Awareness of rate of change [3]	Amounts of Change Level (L3) (Smooth Thinking & Chunky Thinking)
Jodi	$N(t)$ does not vary with respect to tie, but rather with respect to C [1]	Associating particular values of energy flows to discrete moments in time (chunky thinking) [2] Determining particular values of $N(t)$ for discrete moments in time (chunky thinking)[2]	Interpreting graph as decreasing at decreasing rate (awareness of rate). But, explanation was in terms of amounts of change (chunky thinking) [5]	Awareness of rate of change [5]	Amounts of Change Level (L3) (Chunky Thinking)

Figure 3.14. The matrix output for the planetary energy imbalance function.

Stage 5: Synthesizing Data by Mapping and Interpreting

In the final stage, the researcher compares themes and sub-themes and checks them against original transcript and video/audio recordings to ensure appropriate interpretations. The researcher identifies differences and commonalities between themes across the data set in order to construct concepts and explore relationships or causality (Gale et al., 2013). During this stage, I checked the selected, summarized pieces of data described in the previous stage against original transcripts in order to ensure rigor of interpretations. No further changes were made to the themes involved in both analytic frameworks, and thus these frameworks were confirmed as expressing appropriate interpretations of the participants' responses. For instance, in my analysis of interviews 1 and 2, Kris held a Ratio as Measure conception for all three intensive quantities (concentration, energy density, and energy flux density). In contrast, Pam held a Ratio as Measure conception only for concentration and Ratio as Per-One conceptions for energy density and energy flux density.

Similarly, for the second part of my study, I checked all selected, summarized pieces of data against original transcript to ensure appropriate examination of the participants' reasoning

about co-variation when learning about introductory mathematical models for global warming. For example, Kris consistently reasoned about co-variation at the amounts of change level (L3) regarding the function forcing by CO₂ ($F(C)$), planetary energy imbalance ($N(t)$), and average surface temperature ($T(t)$). In contrast, Jodi reasoned about co-variation at the direction of change level (L2) regarding $F(C)$, at the amounts of change level (L3) regarding $N(t)$, and at the direction of change level (L2) regarding $T(t)$. In addition to confirming interpretations, I explored relationships between the participants' responses across tasks. In particular, I examined two relationships across participants' responses: (a) the relationship between conceptions of energy flux density and conceptualization of the planetary energy imbalance and (b) the relationship between level of covariational reasoning and understanding of forcing by CO₂ and connection between energy imbalance and temperature. Exploring these relationships allowed me to find common aspects across the participants' conceptions and level of covariational reasoning, and how these common aspects relate to their understanding of scientific concepts related to global warming such as forcing by CO₂, planetary energy imbalance, and average surface temperature.

CHAPTER 4

RESULTS: PSTS' COCEPTIONS OF INTENSIVE QUANTITIES COMMONLY USED TO MODEL GLOBAL WARMING

In this chapter, I characterized PSTs' conceptions of intensive quantities commonly used to model global warming. I have organized the chapter into three cases: the case of Pam, the case of Jodi, and the case of Kris. Each case is divided into four subsections. The first subsection is a synthesis of the PST's conceptions of the three intensive quantities involved in this study. The synthesis subsection is followed by three subsections: concentration, energy density, and energy flux density. In each of these subsections, I present evidence supporting the conjectures regarding the PST's conception of one particular intensive quantity.

The Case of Pam

Synthesis

Pam's responses to S1T1 and S1T2 suggest that she held a *Ratio as Measure Conception* of concentration. Pam identified the part-whole relationship between a gas (part) and air (whole) as the *measurable attribute of concentration*. In other words, she constructed concentration by quantifying *how much* of the whole is formed by a particular part. Pam used measurement division as the *measurement process of concentration*. She stated that division allowed her to "take the part of the whole," thus defining a magnitude of concentration. Pam in fact interpreted values of concentration in terms of *percentage* or *proportion* of air formed by a gas. She also interpreted values of concentration in terms of multiplicative comparison; that is, concentration expresses a volume of a gas per volume unit of air (e.g., 0.21 as 0.21 cm³ of oxygen *per* 1 cm³ of

air or 362 ppmv as 362 cm^3 of CO_2 *per* $1,000,000 \text{ cm}^3$ of air). Finally, Pam demonstrated conceptualizing concentration as a constant multiplicative relationship between co-varying quantities. These are the defining characteristics of the Ratio as Measure conception of an intensive quantity.

Pam's responses to S1T3 suggest that she held a *Ratio as Per-One Conception* of energy density. She identified a correspondence between energy and the number of 1×1 squares covering a sheet's surface as the *measurable attribute of energy density*. To quantify energy density, Pam equally distributed energy among several 1×1 squares. This suggests that she used partitive division as the *measurement process of energy density*. As a result, she conceived of energy density as a magnitude of energy *per* 1×1 square. This conception seemed to downplay the role of surface area in defining a magnitude of energy density, which resulted in sometimes conceiving energy density exclusively in terms of energy. This represented a cognitive obstacle to understand: (a) energy density as expressing a multiplicative relationship *between quantities* and (b) the relationship between energy density and temperature. Pam overcame such obstacles by reasoning about energy density as a multiplicative relationship between energy and surface area. This encouraged Pam to consider energy relative to surface area in order to define a magnitude of energy density. She, however, did not demonstrate imagining energy density as a constant multiplicative relationship between quantities that could vary. This, I presume, prevented her from developing a *Ratio as Measure* conception of energy density. Nonetheless, she conceptualized energy density as a measure of temperature. This involved no longer thinking of temperature as a magnitude of energy, but rather as a magnitude of the multiplicative relationship between energy and surface area. Pam's new conception of energy density and understanding of the relationship between such quantity and temperature suggest that she

developed a *Ratio as Per-One* conception of energy density. She, however, may have remained reasoning about this intensive quantity as a magnitude of energy per 1×1 square.

Pam's responses to S1T4 suggest that she held a *Ratio as Per-Composite-One Conception* of energy flux density. The term Per-Composite-One reflects the structure of a conception involving a quantity *per* one composite unit formed by two quantities. Pam did not use division to anticipate the answer; instead, she mentally compared radiation rates to respective surface areas. This suggests that she reasoned in terms of such comparison to determine how rapidly the temperature was increasing. She, however, did not demonstrate seeing such comparison in terms of division. When Pam was told to divide radiation rate and surface area, she demonstrated difficulties interpreting the unit $\text{J}/\text{m}^2/\text{s}$. This suggests that Pam did not acknowledge intensive quantities that could relate more than two quantities through division. After using division to define energy flux density, she interpreted values of this intensive quantity as magnitudes of energy *per* time-area composite object. These composite objects represented an abstract object formed by combining 1s and 1m^2 . Since Pam demonstrated conceptualizing 1m^2 as an actual square, the composite object can be thought of as an abstract box with a 1×1 square base and a height of 1s. Pam used partitive division to define energy flux density by equally distributing a magnitude of energy among several time-area composite objects. Partitive division, however, was not used as a measurement process of energy flux density because she did not identify a *measurable attribute energy flux density*. It seemed that her conception of energy flux density prevented her from conceptualizing this intensive quantity as measures of the *rapidity of increase of energy density*. This may explain why Pam did not demonstrate relating energy flux density and temperature. She conceptualized energy density as a measure of temperature, but she saw no connection between energy density and energy flux

density. Thus, she did not conceptualize energy flux density as a measure of the *rapidity of increase of temperature*. The way Pam conceived energy flux density suggests that her conception of this intensive quantity was an extension of the *Ratio as Per-One* conception. Such extension involved a magnitude of an extensive quantity (energy) per an object formed by combining two quantities (time-area composite objects). Thus, Pam's conception was termed *Ratio as Composite-Per-One* conception of energy flux density.

Concentration

After reading S1T1, Pam explained that, although tank B had more oxygen, it also had “more air in it, so oxygen is a lesser percent of it.” Next, she calculated a ratio between the volume of oxygen and the volume of air for each tank. Pam obtained the values 0.21 for tank A and 0.19 for tank B. She concluded that a breath of air taken from tank A had more oxygen than one taken from tank B since “21% of tank A is oxygen and 19% of tank B is oxygen.” Pam's approach to complete S1T1 suggests that she identified the *relationship* between the volume of oxygen and the volume of air as the measurable attribute of concentration (e.g., “more air in it, so oxygen is a lesser percent of it”). Her use of division to define such *relationship* suggests that she conceptualized division as the measurement process of concentration. To gather more information about Pam's understanding of division, I asked Pam to elaborate on why she used division to find an answer to S1T1. Pam explained that she wanted to find out “which [volume of oxygen] made up more of the tank. So, it is kind of part out of a whole. Like part of a tank is oxygen ... So, you just take the part of the whole.” Pam's response suggests that she used measurement division to define a magnitude of concentration. She used division to “take the part of the whole,” which involved interpreting concentration as the volume of oxygen relative to the volume of air containing it. This interpretation is consistent with Pam interpreting the values 0.21

and 0.19 as the percentage of a tank being oxygen. Also, Pam defined the measurable attribute, or relationship between oxygen and air, as a part-whole relationship between quantities. For Pam, concentration involved the quantification of the part-whole relationship between the volume of oxygen and the volume of air. To further explore Pam's conception of concentration, I asked her to provide an alternative interpretation of the value 0.21, an interpretation not involving percentage. Pam interpreted 0.21 as "out of that 1 cm^3 of air, point twenty-one cm^3 (0.21 cm^3) are oxygen." She then drew a cube representing 1 cm^3 of air and colored a blue section at the bottom in order to illustrate her interpretation (Figure 4.1). Pam interpreted 0.21 in terms of a multiplicative comparison between oxygen and air. When Pam said "out of 1 cm^3 of air, 0.21 cm^3 are oxygen," she was creating a many-to-one correspondence in which 0.21 cm^3 of oxygen corresponded to 1 cm^3 of air.



Figure 4.1. Pam's illustration of the meaning of 0.21 as 0.21 cm^3 of oxygen per 1 cm^3 of air

Since Pam's answer to S1T1 involved considering concentration as remaining the same for any breath of air, I decided to further examine how Pam was thinking of concentration as a constant multiplicative relationship between co-varying quantities. I asked Pam whether any breath of air taken from tank A would have 0.21 concentration of oxygen. She stated that any

breath of air would be formed by 21% of oxygen since “[concentration] is a proportion ... the proportion is not going to change.” Pam illustrated this by writing the fraction $\frac{21}{100}$ to represent a breath of air with volume of 100 cm^3 . Her response shows that Pam saw concentration as a homogeneous property in the tank. In other words, concentration as a multiplicative relationship between oxygen and air remained constant for the entire tank and any part thereof (e.g., any breath of air). Notice that I did not mention any particular size for that breath of air, rather it was of arbitrary size. Although Pam made use of particular values, it seemed that Pam did this to illustrate her reasoning rather than to understand that the ratio 0.21 remained constant. Her illustration showed that the volume of oxygen and the volume of air can vary, but they do so in constant ratio.

For S1T2, Pam first defined a fraction formed by the volume of carbon dioxide (CO_2) over the volume of air, obtaining $\frac{1,448}{4,000,000}$ for tank A and $\frac{316}{800,000}$ for tank B. Next, she determined a new pair of fractions, equivalent to the previous fractions, whose denominators were 1,000,000; she used these new fractions to determine the concentration of CO_2 measured in ppmv. Pam multiplied the numerator and denominator of $\frac{1,448}{4,000,000}$ by $\frac{1}{4}$, obtaining $\frac{362}{1,000,000}$; similarly, she multiplied the numerator and denominator of $\frac{316}{800,000}$ by $\frac{5}{4}$, obtaining $\frac{395}{1,000,000}$. Based on these new fractions, Pam concluded that the concentration of CO_2 in tank A was 362 ppmv, while that in tank B was 395 ppmv, and thus both tanks had safe air for a diver. Pam used equivalent fractions to define a multiplicative comparison between CO_2 and air in which a volume of CO_2 corresponded to $1,000,000 \text{ cm}^3$ of air. Pam used equivalent fractions to establish that 362 cm^3 of CO_2 corresponded to $1,000,000 \text{ cm}^3$ of air for tank A and that 395 cm^3 of CO_2 corresponded to $1,000,000 \text{ cm}^3$ of air for tank B.

Although Pam created multiplicative comparisons to define concentration measured in ppmv, it was unclear whether Pam saw 362 ppmv as a constant multiplicative relationship between co-varying quantities. For instance, she might have seen 362 ppmv as a volume of CO₂ for the particular case of a tank holding 1,000,000 cm³ of air. I decided to further explore Pam's conception of concentration measured in ppmv. When I asked Pam to interpret the value 362 ppmv, she provided the following interpretation

It is kind of in that [*points at ppmv*]; like this is parts per million by volume ... So, every million you have, it is 362 cm³ of carbon dioxide. So, if you have 2,000,000, it is 724 cm³ of carbon dioxide

Pam's interpretation suggests that she conceived of the value 362 ppmv as expressing a constant multiplicative relationship between co-varying quantities. She conceived 362 ppmv as expressing a multiplicative relationship between CO₂ and air; that is, 362 ppmv is not just the volume of CO₂ for the particular case of a tank holding 1,000,000 cm³ of air. Notice how Pam stated that the volume of CO₂ must increase proportionally as the volume of air increases to maintain the same relationship: 362 cm³ of CO₂ for every 1,000,000 cm³ of air. This is similar to thinking of the value 362 ppmv as a rate in which *one unit* increase in volume of air (from 1,000,000 cm³ to 2,000,000 cm³) corresponds to a constant increase of 362 cm³ of CO₂ (from 362 cm³ to 724 cm³). This further support the conclusion that Pam conceptualized concentration measured in ppmv as a multiplicative comparison in which a volume of CO₂ corresponded to 1,000,000 cm³ of air. Since Pam did not used the decimal values 0.000362 and 0.000395, which express concentration of CO₂ in decimal form, I decided to examine how Pam thought about the relationship between the values 0.000362 and 362 ppmv. I asked Pam to elaborate on that

sheet, obtaining $\frac{18,750 J}{6 m^2} = 3,125 J/m^2$ for sheet A and $\frac{56,250 J}{20 m^2} = 2,812.5 J/m^2$ for sheet B.

She, however, did not know how to proceed after obtaining these values. It seemed that Pam had not yet seen a connection between energy density and temperature. Also, it is unclear how Pam was using division to define energy density. Thus, I decided to ask her to interpret these values in the context of the problem. Pam stated that these values represented “how many Joules are per square meter.” Pam then concluded that sheet A should have a higher temperature by the end of the experiment because.

This is the same amount of space [*simultaneously points at 3,125 and 2,812.5*], they’re both one square meter. So, this one [*points at sheet A*] has way more Joules in that one square meter than this one does [*points at sheet B*]. So, sheet A has got more Joules going to the same amount of space than [*sheet B*]

Pam’s approach to complete S1T3 included doing a ratio between energy and surface area. This suggests that she identified the relationship between energy and a sheet’s surface as the measurable attribute of energy density. Also, since Pam used division to measure such attribute, she conceptualized division as the measurement process of energy. Pam’s interpretation of the values $3,125 J/m^2$ and $2,812.5 J/m^2$ suggests that she used partitive division to define energy density. She used division to equally distribute a magnitude of energy, such as 18,750 J or 56,250 J, over the sheet’s surface area. For instance, she distributed 18,750 J (total number of elements) over a surface area of $6 m^2$ (number of groups), obtaining $3,125 J$ *per every* $1 m^2$ (number of elements per group). Thus, she used partitive division to define energy density as a rate, or a magnitude of energy per area unit. Pam compared these rates to conclude that sheet A had the highest temperature (e.g., “This [*points at sheet A*] has way more Joules in that one

square meter than this one does [*points at sheet B*]). This suggests that Pam began to construct a connection between energy density and temperature.

Since Pam concluded that sheet A had higher temperature, I decided to further examine how Pam was thinking about the relationship between energy density and temperature. I told Pam to imagine a third sheet (sheet C) of arbitrary dimensions, which was also involved in the experiment along with sheet A and sheet B. At the end of the experiment, sheet C had an energy density of $3,000 \text{ J/m}^2$ (it was assumed sheet C had the same initial temperature as sheet A and sheet B). Then, I asked Pam to order the sheets from the one with highest temperature to the one with lowest temperature. To make sense of the situation, Pam drew two rectangles: one formed by six little squares representing sheet A and another formed by 20 little squares representing sheet B. Next, she wrote the values $3,125 \text{ J/m}^2$, $2,812.4 \text{ J/m}^2$, and $3,000 \text{ J/m}^2$ representing the energy density of sheet A, sheet B, and sheet C, respectively. Pam then drew three little 1×1 squares: one next to sheet A, one next to sheet B, and one next to sheet C (Figure 4.3). Finally, Pam provided the following response for the problem.

If sheet A ended up being just like a 1-by-1, and sheet B was a 1-by-1, and sheet C was a 1-by-1, then this [*points at sheet A*] would heat up the fastest because it's got 3,125 Joules towards it, and then C, and then B ... Here [*points at the little 1×1 squares*], the areas are all the same. So, it doesn't matter what size [sheet C] is because it all gets down to one meter square.

Pam's response suggests an interesting aspect of her conception of energy density. Pam interpreted the values of energy density in term of rate; that is, a magnitude of energy per area unit. Her drawing, however, suggests that she was thinking of energy density as a magnitude of energy corresponding to (or literally contained in) a 1×1 square (a geometric object). Notice that

Pam drew little squares right next to the values $3,125 \text{ J/m}^2$, $2,812.5 \text{ J/m}^2$, and $3,000 \text{ J/m}^2$ (Figure 4.3). This suggests that she used division to *equally distribute* energy over several little 1×1 squares so that she could focus on comparing magnitudes of energy across squares. This suggests that Pam saw partitive division as a way to homogenize one quantity: surface area. In other words, she *took* a 1×1 square (a geometric object) from each sheet and compare the magnitudes of energy within them. This suggests that Pam may have thought of surface area as the number of 1×1 squares covering a sheet.

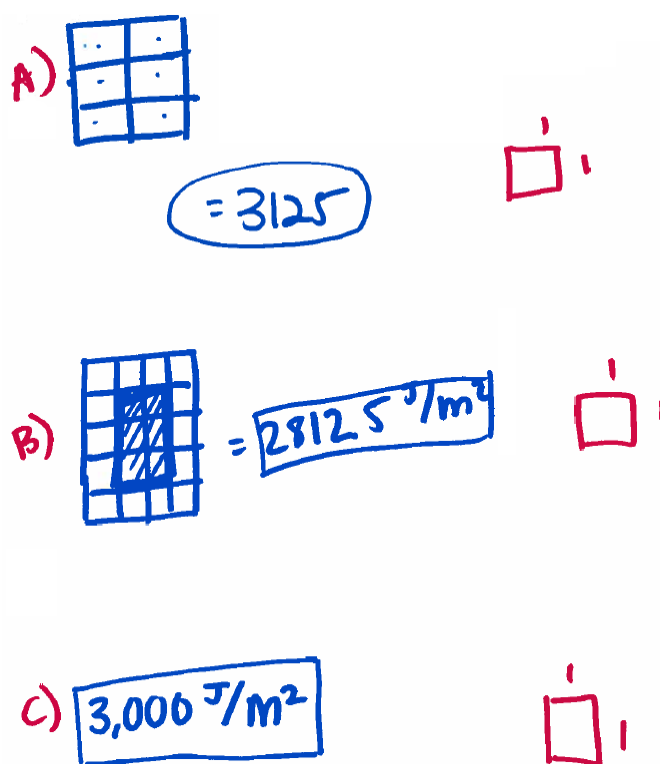


Figure 4.3. Pam drew three little 1×1 squares to order sheet A, sheet B, and sheet C according to temperature

Since Pam conceptualize energy density as a magnitude of energy corresponding to a 1×1 square, I decided to explore that hypothesis further and test whether Pam could conceive of energy density as a constant multiplicative relationship between co-varying quantities. I asked

Pam to explain how the energy density of sheet A and sheet B compared to each other at the beginning of the experiment (i.e., when both sheets had the same temperature). Pam demonstrated difficulties trying to answer this question. The following exchange illustrates her difficulties.

Me: How did the energy density of sheet A and sheet B relate to each other at the beginning of the experiment, when both sheet had the same temperature?

Pam: These (sheet A and sheet B) had the same temperature, so the size must've affected it [*long pause*]. These are different sizes, but I knew that this one [*points at sheet A*] had a higher temperature at the end of the experiment, and this one [*points at sheet B*] had the lowest temperature. And this one [*points at sheet B*] was bigger than this [*points at sheet A*], but it didn't matter, size didn't matter. [*Pauses*] but the amount of Joules going toward it mattered

Me: Do you think that is possible that both quantities matter?

Pam: I think I am overthinking this

Me: So, if sheet A and sheet B had the same temperature at the beginning, then what were their energy density like at the beginning?

Pam: I think this had a bigger one [*points at sheet B and pauses*]. No, smaller! [*Pauses*] if they had the same room temperature, this one [*points at sheet B*] had a higher internal energy because it had to keep that room temperature the same for a bigger sheet. And this one [*points at sheet A*] was a smaller sheet so it didn't have to have as much internal energy

Me: Ok, so what about their energy density?

Pam: Ok, let me write this down

It seemed that the source of Pam's difficulties was her inability to reason about energy density in terms of a multiplicative relationship between two quantities. Pam's conception of energy density seemed to involve ignoring surface area and emphasizing energy (e.g., "size didn't matter. [Pauses] but the amount of Joules going toward it mattered"). Close to the end of the exchange, Pam began to move her focus toward the relationship between energy and surface area. She began to consider the role of surface area in defining a magnitude of energy density. This suggests that her conception of energy density (and its connection to temperature) were changing.

Continuing her reasoning, Pam wrote a summary of quantities involved in the situation: each sheet's internal energy at the beginning of the experiment, each sheet's dimensions, energy added to each sheet during the experiment, and each sheet's energy density at the end of the experiment (Figure 4.4). Then, Pam said the following.

I find it interesting that although this [points at sheet A] has smaller internal energy and it has less Joules radiated at it, it ended up with the highest energy density. But I guess that makes sense because it is smaller ... So here, [sheet B] had a higher energy at the beginning, and it had more Joules radiated, and they had more [surface area] to spread out in a 5×4 sheet because there was more space to go, so less density

At the beginning, she was surprised that a sheet with lower internal energy and receiving lower radiation ended up with a higher energy density. Notice that Pam then attended to surface area to make sense of the situation; she began to consider energy relative to surface area in order to define a magnitude of energy density (e.g., "it had more Joules radiated, and they had more [surface area] to spread out in a 5×4 sheet because there was more space to go, so less density"). Her response, however, did not explicitly include arguing that energy density was a constant

multiplicative relationship between energy and surface area, which suggests that Pam's conception of energy density remained as *Ratio as Per-One*. Nonetheless, Pam concluded that sheet A and sheet B had the same energy density at the beginning "because the temperature was the same." Pam's conclusion suggests that she began to think of energy density, a magnitude of the multiplicative relationship between energy and surface area, as a measure of temperature. This contrasted with her initial conception of temperature as a magnitude of energy.

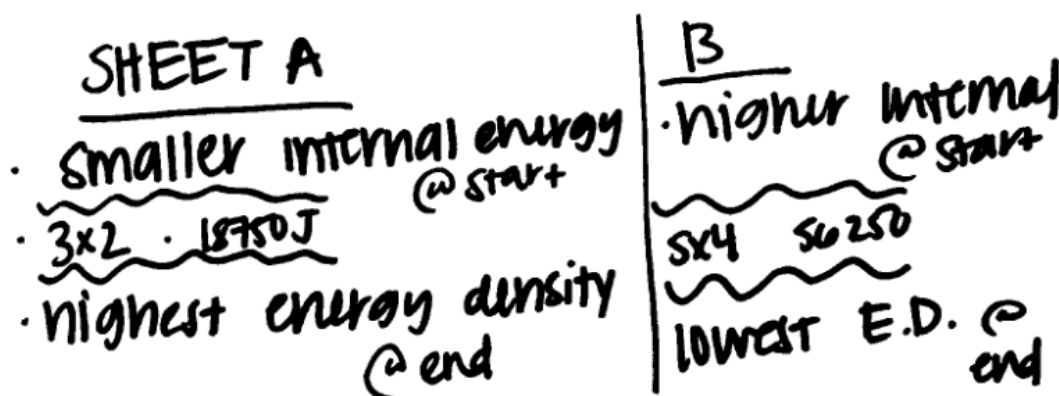


Figure 4.4. Pam compared each sheet's: level of energy at the beginning of the experiment, dimensions, energy added, and level of energy density at the end of the experiment.

Energy Flux Density

After reading S1T4, Pam stated that the temperature of both sheets needed to increase by 10 °C (from the initial temperature of 15 °C to the final temperature of 25 °C). Based on the information provided in the statement of S1T4 (a 2,500 J/m² increase in energy density results in a 4 °C increase in temperature), Pam calculated that the energy density of both sheets must increase by $2.5 \times 2,500 \text{ J/m}^2 = 6,250 \text{ J/m}^2$ since a 10 °C increase is 2.5 times larger than a 4 °C increase. Next, Pam determined the total amount of energy needed to be added to each sheet in order to increase their energy density by 6,250 J/m². She calculated that $6,250 \text{ J/m}^2 \times 6 \text{ m}^2 = 37,500 \text{ Joules}$ needed to be added to sheet A and $6,250 \text{ J/m}^2 \times 20 \text{ m}^2 = 125,000 \text{ Joules}$ needed to

be added to sheet A. Pam's strategy further supports the hypothesis that she saw energy density as a rate defining a magnitude of energy per area unit (this unit was an actual square). Then, Pam calculated the ratio between total energy added and radiation rate to determine the time it took for each sheet's temperature to increase up to 25 °C. That is, she calculated that it took $\frac{37,500 J}{750 J/s} = 50 \text{ sec}$ for sheet's A temperature to reach 25 °C and $\frac{125,000 J}{1,200 J/s} \approx 104.17 \text{ sec}$ for sheet's B temperature to reach 25 °C. Based on these results, Pam concluded that sheet's A temperature would first reach 25 °C. Her last calculation suggests that Pam used measurement division to find the answer. She divided the total number of elements (total energy added) by the number of elements per group (radiation rates) in order to obtain the number of groups (number of seconds). This suggests that Pam conceptualized the values 750 J/s and 1,200 J/s as rates defining magnitudes of energy per time unit. It seemed that Pam was able to think of ratios in terms of rates as long as they multiplicatively related *two* quantities (i.e., Joules *per* square meter or Joules *per* second).

Pam's initial approach did not involve determining or using energy flux density. For this reason, I asked Pam if she could think of another way to anticipate which sheet's temperature would first reach 25 °C, a way that did not involve determining the time it took for their temperatures to reach that value. Pam reflected on my question for a moment, and then said.

If I had to guess, I'd probably say sheet A anyway because it's got a smaller surface area ... 1,200 [J/s] is not too far over 750 [J/s], but there is a big difference between 6 m² and 20 m² ... So, if I was just looking at them, I'd be like sheet A because there is less surface area ... Smaller things heat up faster, but if they are heated up at a lower rate, then it would kind of depend

In her response, Pam was attending to the relationship between radiation rate and surface area in order to anticipate which sheet's temperature would first reach 25 °C. Pam compared 1,200 J/s to 750 J/s and 20 m² to 6 m² to establish that the relationship between radiation rate and surface area for sheet A was *stronger* or *more intense* than the relationship for sheet B. It seemed that Pam visualized the radiation rate being distributed over the surface area, but her reasoning did not involve division. Thus, it was unclear whether Pam's reasoning involved conceptualizing such relationship between radiation rate and surface area as a ratio. I decided to prompt Pam to calculate the ratio between radiation rate and surface area for each sheet. Pam determined the values $\frac{750 \text{ J/s}}{6 \text{ m}^2} = 125 \frac{\text{J/s}}{\text{m}^2}$ for sheet A and $\frac{1,200 \text{ J/s}}{20 \text{ m}^2} = 60 \frac{\text{J/s}}{\text{m}^2}$ for sheet B. Here, I introduced the term *energy flux density* to refer to these ratios. Next, I asked Pam to interpret those values in the context of the task. Pam demonstrated difficulties making sense of the units for energy flux density.

Pam: It is Joules per second per square meter. Is that a thing? Can you do that?

Me: Can you tell me a little bit more about that unit? How would you interpret Joule per second per square meter in the context of the task?

Pam: It is like how many Joules [*pauses*]. Like every second, how many Joules are to one meter. It would be like 125 Joules after one second in each square meter

Pam drew a rectangle formed by six little squares representing sheet A. Then, she wrote '125' in each little square and '*after 1 sec. how many Joules are in each m²*' next to the rectangle (Figure 4.5). Pam's comment regarding the measure unit of energy flux density suggests that she was not familiar with (or had not had opportunities to work with) intensive quantities that could relate more than two quantities. Pam's response and drawing suggest that she used a partitive division to define energy flux density. Although Pam did a ratio between radiation rate and surface area,

her interpretation suggests that she equally distributed a magnitude of energy among several *time-area composite objects* (e.g., “125 Joules after one second in each square meter”). These composite objects were formed by abstractly combining 1s and 1m². For instance, in the case of sheet A, Pam used partitive division to equally distribute 750 Joules among six 1s × 1m² composite objects, thus defining energy flux density as a magnitude of energy *per* composite object.

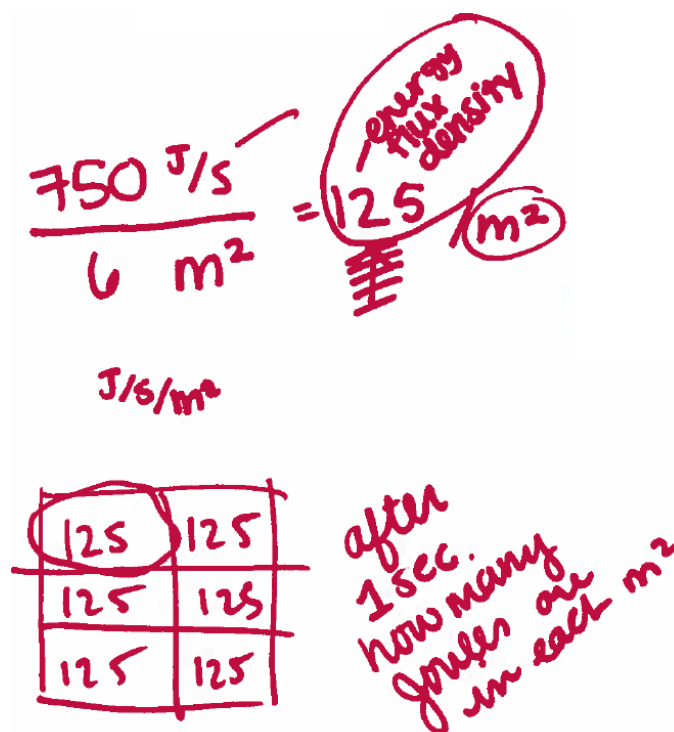


Figure 4.5. Pam drew a rectangle representing sheet A to interpret the value 125 J/m²/s

Before using division, Pam compared radiation rate to surface area to anticipate which sheet’s temperature was increasing most rapidly. When she divided radiation rate by surface area, however, she did not interpret the resulting value as a multiplicative relationship between radiation rate and surface area. This suggests that she may have not been thinking of energy flux density as a measure of how rapidly temperature was increasing. Thus, I decided to further

examine how Pam was thinking about the relationship between energy flux density and temperature. I told Pam to assume that she only knew the energy flux densities for each sheet and that both sheets had the same initial temperature. That is, she neither knew the dimensions of the sheets nor the radiation rates. Then, I asked her whether it was possible to anticipate which sheet's temperature first reached 25 °C. Pam stated that the information was insufficient and provided the following argument

This could be 125 [J/m²/s], but the area could be 20 and the [radiation] rate could be 2,500 [J/s] ... So, you don't know anything. So, I think this (energy flux density) doesn't help you unless you either know how big [the sheet] is or you know the [radiation] rate

Pam's response suggests that she did not conceptualize energy flux density as a measure of how rapidly temperature was increasing. Although Pam conceptualized energy density as a measure of temperature while working on S1T3, she did not demonstrate conceptualizing energy flux density as a measure of how rapidly energy density increases while working on S1T4. This might explain why she did not see a relationship between energy flux density and temperature.

Moreover, Pam's response suggests that she did not conceptualize energy flux density as a quantity multiplicatively comparing radiation rate to surface area. Notice that Pam seemed to prefer to know (or at least have sufficient information to determine) the radiation rates and the surface areas corresponding to each sheet (e.g., "this (energy flux density) doesn't help you unless you either know how big [the sheet] is or you know the [radiation] rate"). Pam possibly needed this information so that she can compare radiation rate to surface area in a non-ratio-based way (as she previously did). This further supports the hypothesis that Pam used partitive division to equally distribute a magnitude of energy among several time-area composite objects, in opposition to equally distributing radiation rate over surface area. Interestingly, Pam seemed

to conceptualize energy flux density as a constant multiplicative relationship between co-varying quantities. Pam stated that if sheet A's surface area was 20 m^2 instead of 6 m^2 , then the radiation rate must be $2,500 \text{ J/s}$ instead of 750 J/s in order to maintain the same relationship ($125 \text{ J/m}^2/\text{s}$). Pam's conception of energy flux density seemed to be an extension of her *Ratio as Per-One Conception* of energy density. Pam's conception of energy flux density involved multiplicatively relating a magnitude of energy with one *unit* of another quantity. The unit was a time-area composite object, which was why I termed Pam's conception of energy flux density as *Ratio as Per-Composite-One* conception of energy flux density.

The Case of Jodi

Synthesis

Jodi's responses to S1T1 and S1T2 suggest that she held *Ratio as Measure Conception* of concentration. Jodi identified the proportion of a gas to air as the *measurable attribute of concentration*. In other words, she constructed concentration by quantifying how the volume of a gas (*multiplicatively*) compared to the volume of air. Jodi used measurement division as the *measurement process of concentration*. She stated that division gives "the proportion of [a gas] within the tank," thus defining a magnitude of concentration. Jodi in fact interpreted values of concentration in terms of *percentage* or *proportion* of air formed by a gas. She also interpreted values of concentration in terms of multiplicative comparison; that is, concentration expresses a volume of a gas per volume unit of air (e.g., 0.21 as 0.21 cm^3 of oxygen *per* 1 cm^3 of air or 362 ppmv as 362 cm^3 of CO_2 *per* $1,000,000 \text{ cm}^3$ of air). Finally, Jodi demonstrated conceptualizing concentration as a constant multiplicative relationship between co-varying quantities. These are the defining characteristics of the *Ratio as Measure* conception of an intensive quantity.

Jodi's responses to S1T3 suggest that she developed a *Ratio as Measure Conception* of energy density. She identified a correspondence between energy and the number of 1×1 squares covering a sheet's surface as the *measurable attribute of energy density*. To quantify energy density, Jodi equally distributed energy among several 1×1 squares. This suggests that she used partitive division as the *measurement process of energy density*. As a result, she conceived of energy density as a magnitude of energy *per* 1×1 square. This conception seemed to downplay the role of surface area in defining a magnitude of energy density, which resulted in sometimes conceiving energy density exclusively in terms of energy. This represented a cognitive obstacle to understand: (a) energy density as expressing a multiplicative relationship *between quantities* and (b) the relationship between energy density and temperature. Jodi overcame such obstacles by reasoning about energy density as a constant multiplicative relationship between co-varying quantities. This encouraged Jodi to consider energy relative to surface area in order to define a magnitude of energy density. Also, she imagined energy and surface area changing proportionally so that energy density remained constant. These two realizations helped Jodi conceptualize energy density as a measure of temperature. This involved no longer thinking of temperature as a magnitude of energy, but rather as a magnitude of the multiplicative relationship between energy and surface area. Jodi's new conception of energy density and understanding of the relationship between such quantity and temperature suggest that she developed a *Ratio as Measure* conception of energy density. She, however, may have remained reasoning about this intensive quantity as a magnitude of energy per 1×1 square.

Jodi's responses to S1T4 suggest that she held a *Ratio as Rate-Per-One Conception* of energy flux density. The term Rate-Per-One reflects the structure of a conception involving a rate quantity *per* one unit of another quantity. Jodi did not use the ratio between radiation rate and

surface area to complete the task. She was told to divide these two quantities to define a third quantity known as energy flux density. This suggests that Jodi did not acknowledge intensive quantities such as energy flux density, which can relate more than two quantities, by reasoning about the situation described in S1T4. After being told to divide radiation rates by surface areas, Jodi began reasoning about such intensive quantities. Jodi interpreted values of energy flux density as magnitudes of energy density corresponding to one 1×1 square *per* second. This suggests that she identified the rapidity of increase of energy density in one 1×1 square as the *measurable attribute of energy flux density*. To quantify such attribute, Jodi equally distributed a magnitude of radiation rate among several 1×1 squares. This suggests that she used partitive division as the *measurement process of energy flux density*. Jodi's conception of energy flux density shaped her understanding of the relationship between this intensive quantity and temperature. Jodi conceptualized energy flux density as a measure of how rapidly the temperature of one 1×1 square was increasing. The way Jodi conceived energy flux density and its relationship with temperature suggests that her conception of this intensive quantity was an extension of the *Ratio as Per-One* conception. Such extension involved a magnitude of a rate quantity (energy density for one 1×1 square) per unit of another quantity (time). Thus, Jodi's conception was termed *Ratio as Rate-Per-One* conception of energy flux density.

Concentration

After reading S1T1, Jodi calculated a ratio between the volume of oxygen and the volume of air for each tank. Jodi obtained the values 0.21 for tank A and 0.19 for tank B. Jodi then said that “this [*points at 0.21*] tells us that tank A has more oxygen per breath since the proportion of oxygen to air is bigger.” When prompted to interpret the value 0.21, Jodi said “I would say that point twenty-one (0.21) or 21% of the total volume of gases is oxygen.” Jodi's

approach to complete S1T1 suggests that she identified the *proportion* of the volume of oxygen to the volume of air as the measurable attribute of concentration. Jodi's use of division to define such *proportion* suggests that she conceptualized division as the measurement process of concentration. Also, it seemed that Jodi used measurement division to define a magnitude of concentration since she interpreted the values 0.21 and 0.19 in terms of the volume of oxygen relative to the volume of air containing it (e.g., "point twenty-one (0.21) or 21% of the total volume of gases is oxygen"). To gather more information about Jodi's understanding of division, I asked her to elaborate on why she used division to find an answer to S1T1. Jodi drew a rectangle representing the composition of air in tank A (Figure 4.6). Then, Jodi provided the following justification for using division.

We know that the amount of oxygen divided by this amount [*points at 650,000 cm³*] gives us a proportion of oxygen within the tank. Then, let's say that you breathe in the entire tank in one breath. Then, the amount of oxygen that you'd breathe in is point twenty-one (0.21) of the total volume of air ... but we know that this proportion is the same for each breath because the air is completely mixed

Jodi's response further supports the hypothesis that she saw the proportion of oxygen to air as the measurable attribute of concentration and division as the measurement process of concentration. Moreover, her response further supports the hypothesis that she was applying measurement division to define concentration. Notice how Jodi interpreted the value 0.21 in terms of the volume of oxygen relative to the volume of air (i.e., the fraction of the total volume of air that oxygen represents). Her response also shows that Jodi saw concentration as a homogeneous property in the tank. In other words, concentration as a multiplicative relationship between oxygen and air remained constant for the entire tank and any part thereof (e.g., any breath of air).

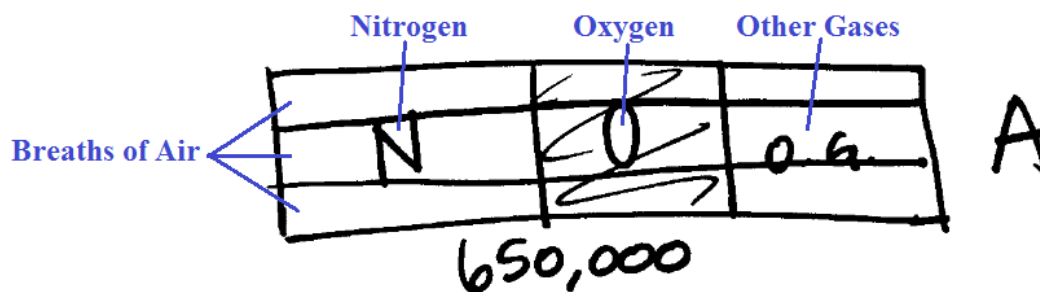


Figure 4.6. Jodi's rectangle represents tank A with air composition (vertical lines) and three breaths of air (horizontal lines).

Since Jodi stated that concentration remained the same for any breath of air, I decided to further examine how Jodi was thinking of concentration as a constant multiplicative relationship between co-varying quantities. I asked Jodi whether any breath of air taken from tank A would have 0.21 concentration of oxygen. She stated that "I'd say it is still point twenty-one (0.21) because if we divide the tank into three breaths, then we can do like this [*draws two horizontal lines* (see Figure 4.6)] ... Then, this proportion of oxygen is still the same." The lines Jodi drew represented three equal breaths of air from tank A. Notice how Jodi drew the lines so that all three breaths of air had the same air composition as the entire tank (see Figure 4.6). This suggests that Jodi understand that the volumes of oxygen and air can vary, but they do so in a way that maintains the same relationship: a constant ratio of 0.21. To further explore Jodi's reasoning about this property of concentration, I asked Jodi whether two tanks with the same concentration of oxygen would have the same volume of oxygen. Jodi said that the two tanks may not have the same volume of oxygen because

You would have to know how much air is in the tanks total to be able to tell if the amount of oxygen in the tanks is the same. Because, this point twenty-one (0.21) could be created by two different numbers as a ratio of each other. So, you could have double the amount

air total ... and then double the amount of oxygen, and then you'll still get the point twenty-one (0.21) proportion of oxygen to air

Jodi's response further supports the hypothesis that she conceptualized concentration as a constant multiplicative relationship between co-varying quantities. Jodi stated that a particular concentration such as 0.21 can be the result of a ratio between varying volume of oxygen and air. These volumes vary accordingly in order to maintain the 0.21 ratio (e.g., "you could have double the amount air total ... and then double the amount of oxygen").

After reading S1T2, Jodi defined a fraction formed by the volume of CO₂ over the volume of air for each tank, obtaining $\frac{1,448}{4,000,000}$ for tank A and $\frac{316}{800,000}$ for tank B. Jodi explained that she was thinking of changing "the proportion of carbon dioxide and air, and make the volume of air 1,000,000 cm³, and then change the carbon dioxide volume in relation to how I change the volume of air." Next, she determined a new pair of fractions, equivalent to the previous fractions, whose denominators were 1,000,000; she used these new fractions to determine the concentration of CO₂ measured in ppmv. Jodi multiplied the numerator and denominator of $\frac{1,448}{4,000,000}$ by 1/4, obtaining $\frac{362}{1,000,000}$; similarly, she multiplied the numerator and denominator of $\frac{316}{800,000}$ by 5/4, obtaining $\frac{395}{1,000,000}$. Jodi also calculated the decimal value of the new pair of fractions, $\frac{362}{1,000,000} = 0.000362$ for tank A and $\frac{395}{1,000,000} = 0.000395$ for tank B. Based on these new fractions, Jodi concluded that the concentration of CO₂ in tank A was 362 ppmv, while that in tank B was 395 ppmv, and thus both tanks had safe air for a diver. Jodi used equivalent fractions to define a multiplicative comparison between CO₂ and air in which a volume of CO₂ corresponded to 1,000,000 cm³ of air. Jodi used equivalent fractions to establish that 362 cm³ of CO₂ corresponded to 1,000,000 cm³ of air for tank A and that 395 cm³ of CO₂

corresponded to $1,000,000 \text{ cm}^3$ of air for tank B. It was also interesting that Jodi calculated the decimal values 0.000362 and 0.000395 for her equivalent fractions. Since it was unclear why she did that calculation, I asked her to elaborate on the relationship between the values 0.000362 and 362 ppmv. Jodi first wrote three equivalent fractions (Figure 4.7), and then provided the following interpretation for them.

This point zero, zero, zero, three, six, two (0.000362) is the same ratio as 362 because we are talking about this [*points at 0.000362*] in terms of 1 cm^3 , but then 362 in terms of $1,000,000 \text{ cm}^3$... we are definitely talking about the same quantity of carbon dioxide as a proportion of air

Jodi referred to the values 0.000362 and 362 ppmv as ratios, which suggests that she saw both values as expressing the same multiplicative comparison between CO_2 and air, but the correspondences were defined for different volumes of air (e.g., “this [*points at 0.000362*] in terms of 1 cm^3 , but then 362 in terms of $1,000,000 \text{ cm}^3$ ”). Moreover, Jodi’s response suggests that she conceptualized the values 0.000362 and 362 ppmv as two measures of the same attribute: the proportion of CO_2 to air (e.g., “we are definitely talking about the same quantity of carbon dioxide as a proportion of air”). The equality between the three fractions in Figure 4.7 further supports the hypothesis that Jodi thought of concentration as being defined by measuring the strength of a constant multiplicative relationship between co-varying quantities.

$$\frac{1,440 \text{ cm}^3}{4,000,000 \text{ cm}^3} = \frac{.000362 \text{ cm}^3}{1 \text{ cm}^3} = \frac{362 \text{ cm}^3}{1,000,000 \text{ cm}^3}$$

Figure 4.7. Jodi wrote three equivalent fractions to explain the relationship between 362 ppmv and 0.000362.

Jodi also showed evidence of conceiving concentration measured in ppmv as a homogeneous property of the tank. I asked Jodi whether a breath of air with volume 500 cm^3 taken from tank A would still have 362 ppmv concentration of CO_2 . Jodi stated that the concentration would remain equal to 362 ppmv since “you can still change this proportion [*points at* $\frac{362}{1,000,000}$] to be over 500, and then ... that would still be the same ratio of parts per million.” Jodi conceptualized 362 ppmv as a homogeneous property of the entire tank and any part thereof (i.e., a breath of air). Notice how Jodi conceived 362 ppmv as expressing a multiplicative relationship between CO_2 and air; that is, 362 ppmv is not just the volume of CO_2 for the particular case of a tank holding $1,000,000 \text{ cm}^3$ of air. Jodi stated that if breath of air had volume of 500 cm^3 , then the volume of CO_2 would change accordingly to maintain the 362 ppmv ratio (e.g., “that would still be the same ratio of parts per million”).

Energy Density

After reading S1T3, Jodi was not sure how to approach the task. Jodi sat silent and reflective for a relatively long period of time. When it became clear that she could not come up with a way to approach the task, I decided to help her by simplifying the problem. First, I asked Jodi to anticipate which sheet would have higher temperature by the end of the experiment if both sheets were to have the same dimensions (assuming the levels of radiation are maintained).

Jodi stated that sheet B would have higher temperature because it was “receiving more heat” than sheet A. Next, I reversed the problem. I asked Jodi to anticipate which sheet would have higher temperature if both sheets were receiving the same radiation level (assuming the sheets had their original dimensions). Interestingly, Jodi needed more time to think about this problem. After being silent for a moment, Jodi stated that it was not possible to anticipate which sheet would have higher temperature. She said that “even though sheet A is smaller, it is receiving the same amount of heat. So, I wouldn’t think that it would need to be hotter than sheet B.” Jodi’s response suggests that she did not consider the size of the sheet relevant when thinking about temperature; for her, only the magnitude of heat (or energy) was relevant to determine temperature. This may explain why Jodi did not think of division to complete S1T3; Jodi did not think of the relationship between energy and surface area as relevant since surface area was not relevant. To help Jodi see the need for a ratio, I discussed with her a situation involving a swimming pool and a cup of water. I asked Jodi whether the pool and the cup require the same magnitude of heat to reach the same temperature at the same time. Jodi provided the following response.

The heat from the sun is affecting both of them. If you take the temperature of them at maybe midday, then the cup of water would be hotter than the swimming pool. So, size probably does have a role in how hot an object is

Jodi then concluded that if both the pool and the cup were to have the same temperature at the same time, then we either “make the cup of water have the same volume as the swimming pool or we could put more heat into the swimming pool.” Her response suggests she began to identify the relationship between energy and size as relevant attribute of the situation. It was unclear, however, whether Jodi thought of quantifying such relationship through a division.

To explore whether Jodi would use division to complete S1T3, I redirected her attention toward the original problem described in the task. This time Jodi calculated the ratio between energy and surface area for each sheet, obtaining $\frac{18,750 J}{6 m^2} = 3,125 J/m^2$ for sheet A and $\frac{56,250 J}{20 m^2} = 2,812.5 J/m^2$ for sheet B. Then, Jodi concluded that sheet A should have a higher temperature by the end of the experiment because “there is more energy in the same amount of space.” It seemed that Jodi’s use of division was consistent with partitive division. To explore this hypothesis, I asked her to interpret the value $3,125 J/m^2$ in the context of the problem. Jodi provided the following interpretation.

We could talk about how much [energy] is in like one meter by one meter. So, if we could divide 18,750 by 6, since there are 6 units of one-by-one meters, then we would get in one-by-one meter cube [sic.] there is 3,125 Joules

Jodi’s use of division suggests that she identify the relationship between energy and a sheet’s surface as a measurable attribute of energy density. Jodi’s interpretation suggests that she used partitive division to define energy density. Her response, however, also suggests that Jodi used partitive division to equally distribute the magnitude of energy among six 1×1 squares (six geometric objects). Thus, Jodi may have conceptualized surface area as the number of squares *covering* a sheet and energy density as a magnitude of energy corresponding to one 1×1 square. To further explore this hypothesis, I asked Jodi to elaborate on why she used division to complete the task. Jodi drew two rectangles: one formed by six little squares representing sheet A and one formed by 20 little squares representing sheet B. She also wrote the corresponding magnitudes of energy associated with ‘ $1 \times 1 m$ ’ (Figure 4.8). Then, she said

I knew it would be easier to talk about how much energy is in one unit of sheet A [*colored one little square*] and one unit of sheet B ... If the heat is evenly distributed

throughout sheet A and sheet B, then each square in sheet A should have this amount of energy [points at 3,125 J]. Same for sheet B, each square should have this amount of energy [points at 2,812.5 J]

Jodi's response further supports a couple of hypothesis. First, she was using partitive division to define energy density. In other words, Jodi used division to evenly distribute the magnitude of energy among several little squares. Second, Jodi's way of speaking of the unit square meter suggests that she consider 1 m^2 as a literal square (a geometric object), rather than one unit of a quantity (e.g., "how much energy is in one unit of sheet A" or "each square in sheet A should have this amount of energy"). This further supports the hypothesis that Jodi conceptualized energy density as a magnitude of energy corresponding to a 1×1 square.

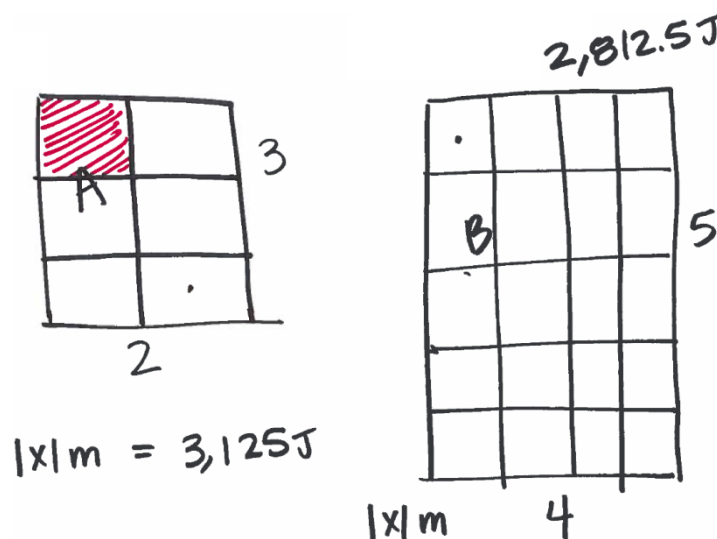


Figure 4.8. Jodi drew two rectangles representing sheet A and sheet B, respectively.

Since Jodi used energy density to conclude which sheet had higher temperature, I decided to further explore the way Jodi was thinking about the relationship between energy density and temperature. I told Jodi to imagine a third sheet (sheet C) of arbitrary dimensions, which was also involved in the experiment along with sheet A and sheet B. At the end of the experiment, sheet C had an energy density of $3,000 \text{ J/m}^2$ (it was assumed sheet C had the same initial

temperature as sheet A and sheet B). Then, I asked Jodi to order the sheets from the one with highest temperature to the one with lowest temperature. To make sense of the situation, Jodi drew two rectangles representing two possible dimensions for sheet C: one representing a $1\text{m} \times 1\text{m}$ sheet C and another representing a $20\text{m} \times 10\text{m}$ sheet C (Figure 4.9). Then, she reasoned about the multiplicative relationship between energy and surface area, and how such relationship defined energy density.

The overall energy in this sheet [*points at 1×1 sheet C*] is 3,000 Joules, but the overall energy in this sheet [*points at 20×10 sheet C*] is 3,000 times 200 since there are 200 units (200 m^2) in this one ... So, the overall energy of this sheet is 600,000 Joules

The way that Jodi determined each sheet's *overall* energy further supports the hypothesis that she used partitive division to define energy density. The strategy followed a rate-problem structure in which a partitive division is *reversed*: energy density (number of elements per group) \times surface area (number of groups) = *overall* energy (total number of elements). Additionally, it is important to point out that Jodi spoke as if she was making sense of the situation (she was not explaining something to me, but to herself). This suggests that Jodi began to reason about energy density as expressing a constant multiplicative relationship between two quantities that could vary. In her response, she stated that the magnitude of energy and the magnitude of surface area vary proportionally in order to maintain the same relationship: $3,000\text{ J/m}^2$ of energy density. Jodi finally answered my question in the following way.

Sheet A is hotter than sheet C, which is hotter than sheet B. If we only know the energy density, that's enough to know which sheet is hotter. Just because 600,000 Joules of energy were added to sheet C, if sheet C were 20 by 10, that is a lot of energy over a

larger amount of area. So, a larger energy doesn't necessarily mean a sheet is hotter because the energy has to be dispersed throughout the sheet

Jodi's response suggests that her conception of temperature (and its relationship with energy density) changed. Initially, Jodi seemed to think of temperature as a magnitude of energy. This downplayed the role of surface area in defining a magnitude of temperature. In the excerpt above, Jodi stated that energy density is defined by considering energy relative to surface area (e.g., "if sheet C were 20 by 10, that is a lot of energy over a larger amount of area"). Then, she stated that the magnitude of energy was not sufficient to define a magnitude of temperature because surface area also needed to be considered (e.g., "a larger energy doesn't necessarily mean a sheet is hotter because the energy has to be dispersed throughout the sheet"). This suggests that she began to conceptualize temperature as involving a comparison between energy and surface area as well. This conception contrasted with her initial conception of temperature as a magnitude of energy.

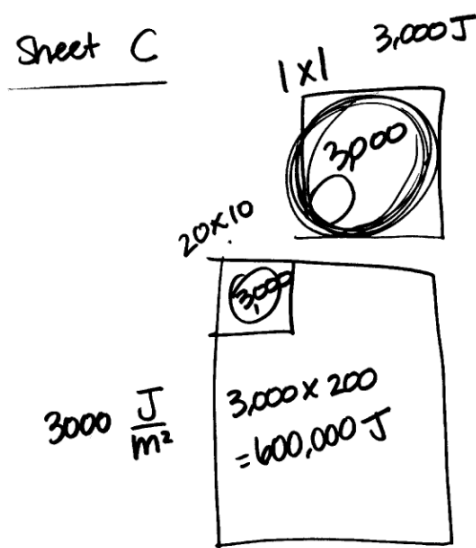


Figure 4.9. Jodi drew two possible sheets C, a 1×1 sheet and a 20×10 sheet.

Energy Flux Density

After reading S1T4, Jodi stated that the temperature of both sheets needed to increase by 10 °C (from the initial temperature of 15 °C to the final temperature of 25 °C). Based on the information provided in the statement of S1T4 (a 2,500 J/m² increase in energy density results in a 4 °C increase in temperature), Jodi determined how much the sheets' energy density needed to increase to produce a 10 °C increase in their temperatures. She wrote the equation $\frac{2,500 \text{ J/m}^2}{4^\circ\text{C}} = \frac{x}{10^\circ\text{C}}$ and solve it for x , obtaining the solution $x = 6,250 \text{ J/m}^2$. Next, Jodi determined the total amount of energy needed to be added to each sheet in order to increase their energy density by 6,250 J/m². She calculated that $6,250 \text{ J/m}^2 \times 6 \text{ m}^2 = 37,500 \text{ Joules}$ needed to be added to sheet A and $6,250 \text{ J/m}^2 \times 20 \text{ m}^2 = 125,000 \text{ Joules}$ needed to be added to sheet B. Jodi's strategy further supports the hypothesis that she saw energy density as a rate defining a magnitude of energy per area unit (this unit was an actual square). Then, Jodi calculated the ratio between total energy added and radiation rate to determine the time it took for each sheet's temperature to increase up to 25 °C. That is, she calculated that it took $\frac{37,500 \text{ J}}{750 \text{ J/s}} = 50 \text{ sec}$ for sheet's A temperature to reach 25 °C and $\frac{125,000 \text{ J}}{1,200 \text{ J/s}} \approx 104.17 \text{ sec}$ for sheet's B temperature to reach 25 °C. Based on these results, Jodi concluded that "it would take less time for device A to add 10 °C of heat to sheet A. So, sheet A would reach 25 °C faster." Her last calculation suggests that Jodi used measurement division to find the answer. She divided the total number of elements (total energy added) by the number of elements per group (radiation rates) in order to obtain the number of groups (number of seconds). This suggests that Jodi conceptualized the values 750 J/s and 1,200 J/s as rates defining magnitudes of energy per time unit. It seemed that Jodi was able to think of ratios in

terms of rates as long as they multiplicatively related *two* quantities (i.e., Joules *per* square meter or Joules *per* second).

Jodi's initial approach did not involve determining or using energy flux density. For this reason, I asked Jodi if she could think of another way to anticipate which sheet's temperature would first reach 25 °C, a way that did not involve determining the time it took for their temperatures to reach that value. Jodi did not seem to understand my question; she sat silent and reflective for a relatively long period of time. When it became clear that she did not know how to approach my question, I decided to rephrase it. I reminded Jodi that she previously stated that each sheet's energy density needed to increase by 6,250 J/m² in order to produce a 10 °C increase in their temperatures. Then, I asked Jodi whether she could anticipate the answer by knowing how rapidly each sheet's energy density was increasing. Based on her facial expression and her prolonged silence, I concluded that Jodi did not understand or did not know how to approach my question once more. This suggests that Jodi did not demonstrate conceiving of a quantity defined by measuring how rapidly energy density was increasing; that is, she did not demonstrate conceiving of an intensive quantity indicating a magnitude of energy density per second. Moreover, Jodi did not think of dividing radiation rates by surface areas, which suggests that she may not have seen such ratio as indicating a measure of how rapidly energy density was increasing.

I decided to be more direct and prompted her to calculate the ratio between the radiation rate and the surface area for each sheet. Jodi determined the values $\frac{750 \text{ J/s}}{6 \text{ m}^2} = 125 \frac{\text{J/s}}{\text{m}^2}$ for sheet A and $\frac{1,200 \text{ J/s}}{20 \text{ m}^2} = 60 \frac{\text{J/s}}{\text{m}^2}$ for sheet B. Here, I introduced the term *energy flux density* to refer to these ratios. Next, I asked Jodi to interpret those values in the context of the task. Jodi drew two rectangles: one formed by six little squares representing sheet A and another formed by 20 little

squares representing sheet B. Then, she wrote ' $125 \text{ J/m}^2/\text{s}$ ' next to the rectangle representing sheet A and ' $60 \text{ J/m}^2/\text{s}$ ' next to the rectangle representing sheet B (Figure 4.10). Next, Jodi provided the following interpretation for the value $125 \text{ J/m}^2/\text{s}$: "in one square unit of area [*colors one little square*], we need to increase the energy density by 125 J/s in order to increase the overall sheet's temperature by $10 \text{ }^\circ\text{C}$." She also provided the following interpretation for the value $60 \text{ J/m}^2/\text{s}$: "the energy density of one square unit increases by 60 J/s . After 104 seconds, the energy density of one square unit would've increased 6,250 Joules." Jodi's interpretations of the values $125 \text{ J/m}^2/\text{s}$ and $60 \text{ J/m}^2/\text{s}$ suggest that she used partitive division to define energy flux density. Notice that Jodi used partitive division to equally distribute a *magnitude of radiation rate* (total number of elements) among several 1×1 squares (number of groups). She, however, did not interpret this ratio as a magnitude of radiation rate per 1×1 square. Instead, she stated that $60 \text{ J/m}^2/\text{s}$ represented the rate at which the energy density corresponding to one 1×1 square was increasing (e.g., "the energy density of one square unit increases by 60 J/s . After 104 seconds, the energy density of one square unit would've increased 6,250 Joules"). Jodi's interpretation was consistent with her conception of energy density as a magnitude of energy corresponding to a square (a geometric object). Notice that she did not include area units in her response, all units involve Joules or Joules per second. Jodi's interpretation also suggests that she identify *rapidity of increase of energy density for one square* as the measurable attribute of energy flux density. In other words, energy flux density is a rate defining a magnitude of energy density corresponding to one 1×1 square *per second*.

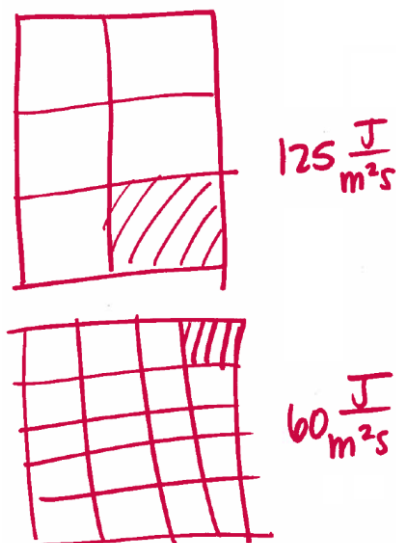


Figure 4.10. Jodi drew two rectangles representing sheet A (top) and sheet B (bottom).

Jodi conceptualized energy density as a measure of temperature while working on S1T3. Moreover, she conceptualized energy flux density as a magnitude of energy density corresponding to one square *per* second, while working on S1T4. I was interested in exploring how these two conceptions would shape Jodi's understanding of the relationship between energy flux density and temperature. I told Jodi to assume that she only knew the energy flux densities of sheet A and sheet B, and that both sheets had the same initial temperature. That is, she neither knew the dimensions of the sheets nor the radiation rates. Then, I asked her whether it was possible to anticipate which sheet's temperature would first reach 25 °C solely based on their energy flux densities. Jodi stated that the information was sufficient and provided the following argument.

Yes! Because, for sheet A, [in] each square unit ... the temperature is increasing faster. So, for one second, the temperature increases 125 Joules. But, for one second of sheet B, the temperature per unit is increasing 60 Joules. So, sheet A is going to reach 25 °C faster than sheet B

Jodi's response suggests that she conceptualized energy flux density as indicating how rapidly the temperature was increasing. The way she conceptualized it resembled the way she conceptualized the relationship between energy flux density and energy density. For Jodi, energy flux density was a measure of how rapidly the temperature of one 1×1 square was increasing (e.g., "for one second of sheet B, the temperature per unit is increasing 60 Joules"). Notice that this is consistent with her conception of energy flux density as a magnitude of energy density corresponding to one square per second. Jodi's conception of energy flux density appeared as an extension of a *Ratio as Per-One* conception. Through partitive division, Jodi constructed energy flux density by equally distributing a magnitude of radiation rate among several 1×1 squares. Then, she interpreted such ratio as a magnitude of energy density corresponding to one 1×1 square *per* second. This suggests that Jodi conceptualized energy flux density as a rate (energy density corresponding to one 1×1 square) per one unit of another quantity (time). Thus, I decided to term Jodi's conception as *Ratio as Rate-Per-One Conception* of energy flux density.

The Case of Kris

Synthesis

Kris's responses to S1T1 and S1T2 suggest that she held *Ratio as Measure Conception* of concentration. Kris identified both the proportion of a gas to air and the part-whole relationship between a gas and air as *measurable attributes of concentration*. In other words, she constructed concentration by quantifying: how the volume of a gas (*multiplicatively*) compared to the volume of air or *how much* of the whole (air) is formed by a particular part (gas). Kris used measurement division as the *measurement process of concentration*. She used division to define the proportion of a gas to air or the percentage of air formed by a gas, thus defining a magnitude of concentration. Kris in fact interpreted values of concentration in terms of *percentage* of air

formed by a gas. She also interpreted values of concentration in terms of multiplicative comparison; that is, concentration expresses a volume of a gas per volume unit of air (e.g., 0.21 as 0.21 cm³ of oxygen *per* 1 cm³ of air or 362 ppmv as 362 cm³ of CO₂ *per* 1,000,000 cm³ of air). Finally, Kris demonstrated conceptualizing concentration as a constant multiplicative relationship between co-varying quantities. These are the defining characteristics of the Ratio as Measure conception of an intensive quantity.

Kris's responses to S1T3 suggest that she developed a *Ratio as Measure Conception* of energy density. She identified a correspondence between energy and the number of 1×1 squares covering a sheet's surface as the *measurable attribute of energy density*. To quantify energy density, Kris equally distributed energy among several 1×1 squares. This suggests that she used partitive division as the *measurement process of energy density*. As a result, she conceived of energy density as a magnitude of energy *per* 1×1 square. This conception seemed to downplay the role of surface area in defining a magnitude of energy density, which resulted in sometimes conceiving energy density exclusively in terms of energy. This represented a cognitive obstacle to understand: (a) energy density as expressing a multiplicative relationship *between quantities* and (b) the relationship between energy density and temperature. Kris overcame such obstacles by reasoning about energy density as a constant multiplicative relationship between co-varying quantities. This encouraged Kris to consider energy relative to surface area in order to define a magnitude of energy density. Also, she imagined energy and surface area changing proportionally so that energy density remained constant. These two realizations helped Kris conceptualize energy density as a measure of temperature. This involved no longer thinking of temperature as a magnitude of energy, but rather as a magnitude of the multiplicative relationship between energy and surface area. Kris's new conception of energy density and

understanding of the relationship between such quantity and temperature suggest that she developed a *Ratio as Measure* conception of energy density. She, however, may have remained reasoning about this intensive quantity as a magnitude of energy per 1×1 square.

Kris's responses to S1T4 suggest that she held a *Ratio as Rate-Measure Conception* of energy flux density. The term Rate-Measure reflects the structure of a conception involving a ratio (energy flux density) measuring the rate of change of a ratio (energy density). Kris was the only PST who determined and used energy flux density to complete S1T4. She used energy flux density to determine how long it took for each sheet's energy density to increase by certain amount. This suggests that she identified the rapidity of increase of energy density as the *measurable attribute of energy flux density*. To quantify such attribute, Kris first *decoupled* radiation rate in terms of a magnitude of energy *and* one second. Then, she applied partitive division to equally distribute that magnitude of energy among several 1×1 squares, thus defining a magnitude of energy density. Finally, Kris *re-coupled* that energy density with one second, creating a rate defined by a magnitude of energy density *per* second. This suggests that she used partitive division as the *measurement process of energy flux density*. Kris's conception of energy flux density shaped her understanding of the relationship between this intensive quantity and temperature. She conceptualized energy density as a measure of temperature while working on S1T3. Here, she conceptualized energy flux density as a measure of how rapidly energy density was increasing. As a result, Kris concluded that energy flux density informed her about the rapidity of increase of the temperature as well. Moreover, she imagined energy flux density as the slope of a line representing the magnitude of a sheet's energy density in function of time. This suggests that she conceptualized energy flux density as a constant multiplicative relationship between co-varying quantities. The way Kris conceived energy flux density and its

relationship with temperature suggests that her conception of this intensive quantity was an extension of the *Ratio as Measure* conception. Such extension involved a magnitude of a rate quantity (energy density) per unit of another quantity (time). Also, such intensive quantity measures the rate of change of a ratio (rapidity of increase of energy density). Thus, Kris's conception was termed *Ratio as Rate-Measure* conception of energy flux density.

Concentration

After reading S1T1, Kris stated that “assuming we take the same amount of air from each tank, then the one [breath of air] that has more oxygen should be the one that has the higher proportion of oxygen in the tank.” Then, she calculated ratios between the volume of oxygen and the volume of air for each tank, obtaining 0.21 for tank A and 0.19 for tank B. Kris's approach to complete S1T1 suggests that she identified the *proportion* of the volume of oxygen to the volume of air as the measurable attribute of concentration. Kris's use of division to define such *proportion* suggests that she conceptualized division as the measurement process of concentration. To gather more information about Kris's understanding of division, I asked her to interpret the values 0.21 and 0.19. Kris stated that the value 0.21 represented “21% of the 650,000 cm³ of the air volume in tank A is oxygen,” while 0.19 meant that “out of 850,000 cm³ of the air volume in tank B, 19% is oxygen.” Kris concluded tank A provided more oxygen per breath because “the proportion of oxygen in tank A is greater than the proportion of oxygen in tank B.” Kris's interpretation suggests that she used measurement division to define a magnitude of concentration. Notice that she interpreted the values 0.21 and 0.19 in terms of the percentage of air formed by oxygen (e.g., “21% of the 650,000 cm³ of the air volume in tank A is oxygen”). This is consistent with applying division to define a measure of an intensive quantity, such as

concentration, by multiplicatively relating the measures of two other quantities, such as volume of oxygen and volume of air.

Since Kris's answer to S1T1 involved considering concentration as remaining the same for any breath of air, I decided to further examine how she was thinking of concentration as a constant multiplicative relationship between co-varying quantities. I asked Kris whether any breath of air taken from tank A would have the same concentration of oxygen of 0.21. She stated the following

I guess we are assuming all the gases are sort of uniformly distributed throughout the tank. Because, if not, I guess you could argue that you might get more of one [gas] than the others. But, if it is uniformly distributed throughout the tank, then yes, you would be getting 21% of oxygen

Her response shows that Kris saw concentration as a homogeneous property in the tank. In other words, concentration as a multiplicative relationship between oxygen and air remained constant for the entire tank and any part thereof (e.g., any breath of air). Notice that I did not mention any particular size for that breath of air, rather it was of arbitrary size. This suggests that Kris conceptualized concentration a constant ratio independent of particular volumes of oxygen and air. She emphasized the importance of having well-mixed gases in the tank for breaths of air to have the same concentration. That is, Kris saw oxygen as forming 21% of any volume of air (e.i., any part of the tank). Thus, Kris conceptualized the value 0.21 as a constant multiplicative relationship between co-varying quantities such as the volume of oxygen and the volume of air. To gather more information about how Kris was thinking of concentration as a multiplicative relationship between quantities, I asked her to elaborate on the difference between volume of oxygen and concentration of oxygen. Kris stated that volume of oxygen is “the amount [of

oxygen] that is there [in the tank],” whereas the concentration of oxygen involves “comparing the volume of a certain gas ... to the total amount of gases in [the tank].” Notice that Kris’s *definition* of concentration was consistent with applying measurement division to define such quantity; that is, division is used to define a measure of an intensive quantity, such as concentration, by multiplicatively comparing the volume of oxygen to the volume of air. Adding to her explanation, Kris provided an example involving two new tanks, which she named tank C and tank D (Figure 4.11). Both tanks had the same volume of oxygen, which was $136,500 \text{ cm}^3$. Then, she assigned different volumes of air to each tank: tank C had $200,000 \text{ cm}^3$ and tank D had $500,000 \text{ cm}^3$. Finally, Kris finalized her example by stating that tank D had a lower concentration of oxygen because the volume of oxygen “is taking a smaller fraction of the tank’s volume.” Kris’s response further supports the hypothesis that she identified the *proportion* or *fraction* of the volume of air formed by oxygen as the measurable attribute of concentration. She also used division to define a measure for such attribute so that concentration was defined by multiplicatively comparing the volume of oxygen to the volume of air.

Concentration

136,500 cm³
of O₂ in Tank C

Tank C
200,000 cm³
total of
Volume

136,500 cm³
of O₂ in Tank D

Tank D
had
500,000 cm³
total of
Volume

Figure 4.11. Kris used an example involving two new tanks (tank C and tank D) to illustrate the difference between volume of oxygen and concentration of oxygen.

For S1T2, Kris calculated a ratio between the volume of CO₂ and volume of air for each tank. She obtained the values 0.000362 for tank A and 0.000395 for tank B. Next, Kris multiplied each value by 1,000,000 in order to find the concentration of CO₂ in ppmv. She obtained the values 362 ppmv for tank A and 395 ppmv for tank B, and then concluded that both tanks have a safe concentration of CO₂ for a diver. To understand why Kris decided to use the product $0.000362 \times 1,000,000$ to determine concentration measured in ppmv, I asked her to state her reasons for her strategy. Kris provided the following justification.

Me: Why did you multiply 0.000362 by 1,000,000 to get the concentration in ppmv?

Kris: [0.000362 is] the proportion of carbon dioxide out of the whole tank ... if we take 1,000,000 cm³ of air from tank A and kind of transfer it to a new tank ... then the proportion would be the same in the second tank. So, we can take this proportion [points at 0.000362] and multiply it by the volume of the second tank to get the volume of carbon dioxide in the second tank, which is that [points at 362]

Me: Why did you transfer 1,000,000 cm³ of air from tank A to a new tank?

Kris: In order to measure the ppmv level or units, whatever, we have to see the amount of CO₂ contained in 1,000,000 cm³ of air. So, given that it was uniformly distributed kind of like the first problem, putting 1,000,000 of this [*points at tank A*] into a new tank would maintain the concentration, but we would be able to more easily see the ppmv

Kris's response further supports the hypothesis that she conceptualized concentration as a multiplicative comparison between quantities, the volume of CO₂ and the volume of air in this particular case. Her response suggests that she used multiplication because she saw the value 0.000362 as indicating that the volume of CO₂ was 0.000362 times as large as the volume of air. By doing the product $0.000362 \times 1,000,000$, she determined the volume of CO₂ that maintained the 0.000362 ratio when the volume of air was 1,000,000 cm³, which for her represented the concentration of CO₂ in ppmv (e.g., "we can take this proportion [*points at 0.000362*] and multiply it by the volume of the second tank to get the volume of carbon dioxide in the second tank"). Kris also stated that concentration, or the proportion of CO₂ to air, remained constant after transferring 1,000,000 cm³ of air from tank A to another tank. This suggests that Kris conceptualized the value 362 ppmv as expressing a constant multiplicative relationship between the volume of CO₂ and the volume of air. Although this conception involved seeing 362 ppmv as a ratio, it was unclear whether Kris conceptualized such value as an alternative measure of the *fraction* of the volume of air formed by CO₂. To gather information about that, I asked Kris to elaborate on the relationship between the values 0.000362 and 362 ppmv. Kris stated that

The ppmv of tank A, that's 362, represents the volume of carbon dioxide contained in 1,000,000 cm³ of air ... then, we could set up a new proportion (Figure 4.12) ... you

know that you're looking at a one-million-cm³ tank here [*points at 362 ppmv*], and so 362 out of that 1,000,000 is going to give you this proportion [*points at 0.000362*]

Kris's response further supports the hypothesis that she conceptualized concentration measured in ppmv as multiplicative comparison relating a volume of CO₂ with 1,000,000 cm³ of air. Her analogy involving a tank containing 1,000,000 cm³ of air suggests that she thought of that particular volume of air as the *unit of measure*; that is, a value of concentration, such as 362 ppmv, involved quantifying concentration as a volume of CO₂ for every *unit* of 1,000,000 cm³ of air. Moreover, her response suggests that she conceptualized the values 0.000362 and 362 ppmv as two measures of the same attribute: the proportion of CO₂ to air (e.g., "so 362 out of that 1,000,000 is going to give you this proportion [*points at 0.000362*]").

$$\frac{362 \text{ cm}^3 \text{ O}_2}{1,000,000 \text{ cm}^3 \text{ total air}} = 0.000362$$

$$\frac{362 \text{ ppmv}}{1,000,000}$$

Figure 4.12. Kris set up a proportion to illustrate why the product $0.000362 \times 1,000,000 = 362$ represents the concentration of CO₂ in ppmv.

Kris's analogy of transferring air from tank A to a smaller tank suggested she conceptualized concentration measured in ppmv as a constant multiplicative relationship between co-varying quantities. Particularly, Kris stated that the transfer of air from tank A to another tank would not change the concentration. I asked Kris to explain why she stated that the concentration remained the same after the transfer. Kris provided the following explanation

This proportion [*points at 0.000362*] is going to stay the same no matter how big or how small the tank is. But, this number [*points at the numerator 362*] is only looking similar to this [*points at 0.000362*] because we're specifically looking at a tank of 1,000,000 cm³. Because, if it was 2,000,000 cm³, no worrying about ppmv units, this [*points at the numerator 362*] would be 724. But, the proportion of carbon dioxide in the two-million tank would be 0.000362


Kris stated that the volume of CO₂ increases by 362 cm³ every time the volume of air increases by 1,000,000 cm³. This is consistent with her conception of concentration measure in ppmv as a many-to-one correspondence (multiplicative comparison) between CO₂ and air. This relationship remained constant “no matter how big or how small the tank is.” This suggests that Kris conceptualized concentration measured in ppmv as expressing a constant multiplicative relationship between co-varying quantities. She stated that the volume of CO₂ increased proportionally as the volume of air increased in order to maintain the proportion of 0.000362. Kris's response also suggests that she conceptualized the values 0.000362 and 362 ppmv as two measures of the same attribute: the proportion of CO₂ to air.

Energy Density

After reading S1T3, Kris was not sure how to approach the task. After reflecting on the problem for a moment, Kris said “I guess the quantity that would show this (what sheet is hotter) would be the ratio of energy added divided by total area, and you can compare those two [ratios] between sheet A and sheet B.” Next, she calculated the ratio between energy added and surface area for each sheet, obtaining $\frac{18,750 J}{6 m^2} = 3,125 J/m^2$ for sheet A and $\frac{56,250 J}{20 m^2} = 2,812.5 J/m^2$ for sheet B. Then, Kris stated the following conclusion.

For any square meter that we pick in sheet A, the energy in that square meter is going to be about 3,125 Joules, and that the one in sheet B is going to be about 2,812.5 Joules per square meter. So, I would say that sheet A had a higher temperature by the end

Kris's approach to complete S1T3 included doing a ratio between energy and surface area. This suggests that she identified a relationship between energy and the sheet's surface as the measurable attribute of energy density. Since Kris used division to measure such relationship, this suggests that she conceptualized division as the measurement process of energy density (e.g., "I guess the quantity that would show this (what sheet is hotter) would be the ratio of energy added divided by total area"). The way that Kris used division to define energy density appears consistent with applying partitive division. Kris's interpretation of the values $3,125 \text{ J/m}^2$ and $2,812.5 \text{ J/m}^2$ suggests that she used division to equally distribute a magnitude of energy, such as 18,750 J or 56,250 J, over the sheet's surface area. For instance, she distributed 18,750 J (total number of elements) over a surface area of 6 m^2 (number of groups), obtaining $3,125 \text{ J per every } 1 \text{ m}^2$ (number of elements per group). Thus, she used partitive division to define energy density as a rate, or a magnitude of energy per area unit. To test this hypothesis, I asked Kris to elaborate on the meaning of $3,125 \text{ J/m}^2$. Beneath the ratio $\frac{18,750 \text{ J}}{6 \text{ m}^2} = 3,125 \text{ J/m}^2$, she drew a rectangle formed by six little squares representing sheet A. Then, Kris stated "in each of these little squares ... each of them would have 3,125 Joules of energy added to it," as she drew arrows pointing to each little square and wrote ' $3,125 \text{ J of energy}$ ' (Figure 4.13). Kris's drawing and interpretation further support the hypothesis that she used partitive division to define energy density. Her drawing also suggests that Kris used division to equally distribute energy over six 1×1 squares (six geometric objects).

$$\frac{18750 \text{ J}}{6 \text{ m}^2} = \frac{3125 \text{ J}}{\text{m}^2}$$


1 m² 3125 J of energy

Figure 4.13. Kris drew a rectangle formed by six little squares representing sheet A to illustrate her interpretation of the value $3,125 \text{ J/m}^2$

Although Kris provided the correct answer to S1T3, it was unclear how she was thinking of the relationship between temperature and energy density. Kris stated that it was the ratio between energy and surface area that showed which sheet had the highest temperature by the end of the experiment. This could suggest that she conceptualized energy density, which is defined by such ratio, as a measure of temperature. However, more information was needed; that is why I asked Kris to explain what would need to happen for both sheet to have the same temperature by the end of the experiment. Kris stated that “if we want to get sheet B up to the same temperature as sheet A, we would set up the proportion that says 18,750 Joules per 6 m^2 is equal to some amount of temperature over 20 m^2 .” Next, Kris wrote the equation $\frac{18,750 \text{ J}}{6 \text{ m}^2} = \frac{x}{20 \text{ m}^2}$ and solve it for x , obtaining $x = 62,500 \text{ Joules}$ (Figure 4.14). Kris then stated “in order for sheet B to come up to the same temperature as sheet A, instead of 56,250 Joules of energy, we would have to add 62,500 Joules.” Kris’s response to my question was intriguing. On one hand, Kris’s equation suggests that she thought that both sheets would have equal temperature as long as they have the

same ratio between energy and surface area. This may suggest that Kris conceptualized *the ratio* defining energy density as a measure of temperature. In other words, energy density, as a *multiplicative relationship* simultaneously involving a measure of energy and a measure of surface area, is a measure of temperature. On the other hand, Kris did not use the values $3,125 \text{ J/m}^2$ or $2,812.5 \text{ J/m}^2$ to answer my question (e.g., sheet A would have the same temperature as sheet B if both have an energy density of $3,125 \text{ J/m}^2$). Her response was framed exclusively in terms of magnitudes of energy (e.g., “in order for sheet B to come up to the same temperature as sheet A, instead of 56,250 Joules of energy, we would have to add 62,500 Joules”).

Additionally, Kris used the word *temperature* inconsistently throughout her response. When she said “we would set up the proportion that says 18,750 Joules per 6 m^2 is equal to some amount of *temperature* [emphasis added] over 20 m^2 ,” she used the word *temperature* to refer to the energy incident to the sheet’s surface. When she said “for sheet B to come up to the same *temperature* [emphasis added] as sheet A, instead of 56,250 Joules of energy, we would have to add 62,500 Joules,” she used the word *temperature* to refer to temperature. This suggests that Kris also thought of temperature as being a magnitude of energy rather than a magnitude of the relationship between energy and surface area. This seemed to contradict the hypothesis that Kris saw the *ratio defining energy density* as a measure of temperature. Her conception of energy density as a magnitude of energy corresponding to a 1×1 square may explain Kris’s inconsistent use of the term temperature. In this conception, energy density was not unlike energy since $3,125 \text{ J/m}^2$ were 3,125 Joules of energy *within* a square; that is, energy density was in fact energy in a smaller piece of sheet A. Thus, using the word temperature to refer to both energy and the multiplicative relationship between energy and surface area appeared consistent with conceiving energy density as a magnitude of energy *within* a 1×1 square.

$$\frac{18750 \text{ J}}{6 \text{ m}^2} = \frac{X?}{20 \text{ m}^2}$$

$$\frac{20 \text{ m}^2}{6} (18750 \text{ J}) = \frac{6 \text{ m}^2}{6} (X) \text{ J}$$

$$X = 62500 \text{ J to sheet B}$$

Figure 4.14. Kris set up an equation to determine how sheet A and sheet B could have the same temperature.

I decided to further explore Kris's conception of energy density (and its connection to temperature) by reversing the problem. I told Kris to imagine two new sheets, sheet C and sheet D, without specific dimensions and having the same energy density. Then, I asked her to explain how the temperature of sheet C and sheet D related to each other. This prompted Kris to ask the question "temperature as in energy added or just the temperature?" Kris's question further supports the hypothesis that her conception of temperature was unstable, which appeared to be an obstacle to reason about the relationship between energy density and temperature. As a reply to her question, I asked her to elaborate on how she was thinking about the relationship between energy density and temperature. Kris sat silent and reflective for a relatively long time. Then, she said "if the area is larger, the energy added has to be larger, and then if the area is smaller, the energy added has to be smaller to keep energy density the same." Kris's response suggests that she began to reason about energy density in terms of constant multiplicative relationship between *quantities* that could vary. Notice how Kris stated that energy and surface area increased or

decreased in a way that can maintain energy density constant. She next expanded this idea further by stating the following.

We can have the energy density of object 1 and object 2 equaling each other, and then we can have it equal 10 or whatever. But 10 can be 100 divided by 10 or it can be 50 divided by 5. So, the energy added could be different

Once more, Kris stated that energy changes and surface area change proportionally so that the ratio between them remained constant. It seemed that reasoning about energy density as constant multiplicative relationship helped Kris realized that this intensive quantity is defined by considering energy relative to surface area. Also, Kris stated that two objects having the same energy density not necessarily would have the same energy. This suggests that Kris realized that both quantities, energy and surface area, play a role in defining a magnitude of energy density.

Kris decided to go back and review her initial answer for S1T3 (sheet A had the highest temperature at the end of the experiment since it had the highest energy density). It seemed that she was making sense of the relationship between energy density and temperature. Kris then provided the following conclusion regarding the problem involving sheet C and sheet D.

I guess if they have the same energy density, they're going to have the same temperature because it doesn't matter how big or small the sheet is. As long as *it is proportional* [emphasis added], they're going to have the same energy density ... So, if they have the same energy density, I would assume that the temperatures are the same

Kris's response suggests that her conception of temperature (and its relationship with energy density) changed. Initially, Kris seemed to think of temperature as a magnitude of energy. This downplayed the role of surface area in defining a magnitude of temperature. In the excerpt above, Kris stated that energy density expressed a constant multiplicative relationship between

energy and surface area. Notice how she used the expression “it is proportional” to emphasize this. Then, she associated that conception of energy density with temperature. This suggests that she began to conceptualize temperature as involving a comparison between energy and surface area as well. This was further supported when Kris continued her reasoning by adding that energy density and temperature “are like the same. The energy density is like comparing the object’s internal energy and the size just like the temperature is.”

Energy Flux Density

After reading S1T4, Kris stated that the temperature of both sheets needed to increase by 10 °C (from the initial temperature of 15 °C to the final temperature of 25 °C). Based on the information provided in the statement of S1T4 (a 2,500 J/m² increase in energy density results in a 4 °C increase in temperature), Kris determined that a 625 J/m² increase in energy density must correspond to a 1 °C increase in temperature. Since the temperature of both sheets needed to increase by 10 °C, Kris determined that each sheet’s energy density needed to increase by 625 J/m² × 10 = 6,250 J/m². Next, Kris calculated the ratio between the radiation rate and the surface area for each sheet, obtaining the values $\frac{750 \text{ J/s}}{6 \text{ m}^2} = 125 \frac{\text{J/s}}{\text{m}^2}$ for sheet A and $\frac{1,200 \text{ J/s}}{20 \text{ m}^2} = 60 \frac{\text{J/s}}{\text{m}^2}$ for sheet B. Since Kris was the only PSTs who (unprompted) calculated the energy flux density, I asked her to interpret these values before continuing with her work. She provided the following interpretation.

We know that, for every second device A is running, 750 Joules of radiation get put into sheet A ... So, then we divide 750 by 6, which is the area of sheet A, so then we’ve got 125 J/m² increase in energy density per second that device A is running

Kris used partitive division to define energy flux density. The interesting aspect of Kris’s use of partitive division laid on the quantity that was equally distributed. Kris attended to a radiation

rate, such as 750 J/s, and *decoupled* that rate into a magnitude of energy (750 Joules) *and* one second. She then used partitive division to equally distribute the *magnitude of energy* over the sheet's surface area (number of 1×1 squares) in order to define a magnitude of energy density (energy per 1×1 square). Next, Kris *re-coupled* 125 J/m² with one second to form a rate, or a magnitude of energy density *per* second. This suggests that Kris conceptualized energy flux density as a measure of how rapidly energy density was increasing (e.g., “we’ve got 125 J/m² increase in energy density per second that device A is running”).

Continuing with her work, Kris used the values 125 J/m²/s and 60 J/m²/s to determine how long it took for each sheet's energy density to increase by 6,250 J/m². She calculated that it took $\frac{6,250 \text{ J/m}^2}{125 \frac{\text{J/s}}{\text{m}^2}} = 50 \text{ sec}$ for sheet's A temperature to reach 25 °C and $\frac{6,250 \text{ J/m}^2}{60 \frac{\text{J/s}}{\text{m}^2}} \approx 104.17 \text{ sec}$ for sheet's B temperature to reach 25 °C. Based on these results, Kris concluded that sheet's A temperature would first reach 25 °C. Kris used measurement division to find an answer: she divided the increase in energy density (total number of elements) by energy flux density (number of elements per group) in order to determine how long it would take for each sheet's temperature to reach 25 °C (number of groups). This further supports the hypothesis that Kris conceptualized energy flux density as a measure of the *rapidity of increase of energy density*. Kris was the only PST who determined and used energy flux density without being prompted to do so. Yet, it was intriguing that Kris did not anticipate the answer just by comparing the values 125 J/m²/s and 60 J/m²/s. This suggests that, albeit Kris conceptualized energy flux density as a measure of how rapidly energy density was increasing, she might not have yet conceptualized a relationship between energy flux density and temperature.

Next, I decided to examine how Kris was thinking about the relationship between energy flux density and temperature. I asked her if she could think of another way to anticipate which

sheet's temperature would first reach 25 °C, a way that did not involve determining the time it took for their temperatures to reach that value. When I finished asking my question, Kris said “now that you ask, I guess I could ...” and directed her attention to the values 125 J/m²/s and 60 J/m²/s. Kris drew a coordinated plane and labeled the vertical axis with ‘*ED*’ (energy density) and the horizontal axis with ‘*t, sec*’ (time). Next, she drew two straight lines: one with slope 125 representing sheet's A energy density and another with slope 60 representing sheet's B energy density. She then wrote ‘*125 J/m²/s > 60 J/m²/s*’ next to the coordinate plane (Figure 4.15). Finally, Kris provided the following response to my question

It [*points at 125 J/m²/s*] is the rate at which the energy is being added (radiation rate) divided by the area, so it is Joules per meter square per second. So, I know that more Joules are being added per second per area unit to sheet A than sheet B, and because I know we are trying to increase the same amount of energy density, it is going to take longer for sheet B to do the same increase in energy density that it is for A. So, I really didn't have to calculate the time. I could've just said the energy density is going to increase faster for sheet A than it is for sheet B

Kris's drawing and response revealed that she developed a sophisticated conception of energy flux density. First, her drawing suggests that she conceptualized energy flux density as expressing a constant multiplicative relationship between co-varying quantities (energy density and time). The lines she drew suggest that she envisioned both energy density and time continuously increasing. Associating the slopes with the values 125 J/m²/s and 60 J/m²/s suggests that she conceptualized energy flux density as remaining constant as both energy density and time increase. Second, Kris's response further supports the hypothesis that Kris conceptualized energy flux density as a measure of how rapidly energy density was increasing. By comparing

the values $125 \text{ J/m}^2/\text{s}$ and $60 \text{ J/m}^2/\text{s}$, Kris determined that the energy density of sheet A was increasing the fastest (e.g., “I really didn’t have to calculate the time. I could’ve just said the energy density is going to increase faster for sheet A than it is for sheet B”). Although Kris did not mention temperature, she demonstrated conceptualizing energy density as a measure of temperature while working on S1T3. Since she saw energy flux density as a measure of how rapidly energy density was increasing, it seemed a natural extension for Kris to conceptualize energy flux density as a measure of how rapidly the temperature was increasing. Since Kris’s conception of energy flux density involved conceiving a ratio (energy flux density) as a measure of the rate of change of another ratio (energy density), I termed her conception as *Ratio as Rate-Measure Conception*.

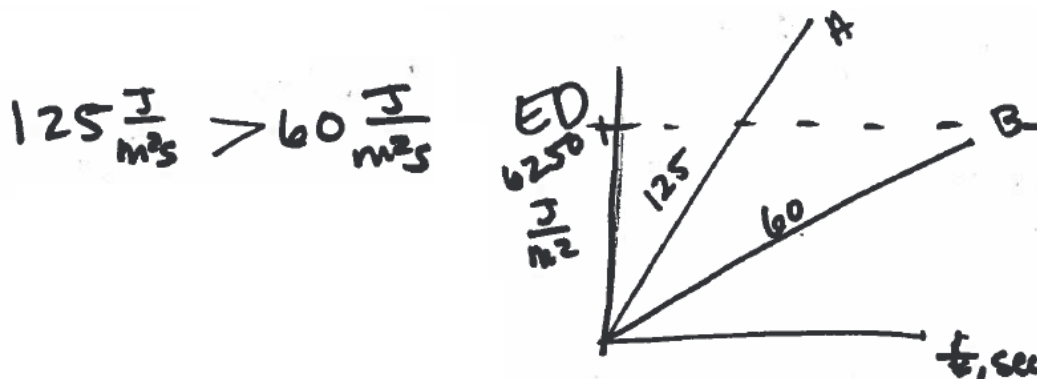


Figure 4.15. Kris drew two straight lines representing each sheet’s energy density as a function of time. The slope of each line represented the energy flux density for each sheet

CHAPTER 5

RESULTS: PSTS' REASONING REGARDING FUNCTIONS INVOLVED IN INTRODUCTORY MATHEMATICAL MODELS FOR GLOBAL WARMING

In this chapter, I characterized PSTs' reasoning regarding the functions commonly used to model global warming as presented in tasks S2T1 and S2T2. In these tasks, PSTs are asked to construct the graph of three functions: the forcing by CO₂ function $F(C)$, the planetary energy imbalance function $N(t)$, and the mean surface temperature function $T(t)$. I have organized the chapter into three cases: the case of Pam, the case of Jodi, and the case of Kris. Each case begins with a synthesis of the PST's reasoning regarding the three functions involved in this study. The synthesis is followed by three subsections, each describing the PST's reasoning regarding a particular function.

The Case of Pam

Synthesis

Pam's responses to the tasks suggest that her reasoning varied from *the direction of change level (L2)* and *the amounts of change level (L3)*. In other words, she reasoned about co-variation from identifying the direction in which the change occurs (e.g., y increases as x increases) and coordinating amounts of change in the dependent variable with changes in the independent variable. For S2T1, Pam demonstrated reasoning about the co-variation represented by the forcing by CO₂ function $F(C)$ at the *direction of change level (L2)*. She needed particular values to determine what energy flows were changing with changes in the atmospheric CO₂ concentration C (dependency of change level or L1). She also made use of particular values to

imagine the direction in which the forcing F was changing as C was increasing (direction of change level or L2). This suggests that Pam's L2 covariational reasoning was dependent on particular values. To construct the graph of $F(C)$, Pam tried to relate amounts of change in F with change in C in order to determine the *slope of a line* as the graph of $F(C)$, rather than to confirm whether the graph was a line. This suggests that Pam assumed that the amounts of change in F remained constant for equal increments in C . Such assumption was not supported by any quantitative argument. When working with the completely transparent atmosphere and the completely opaque atmosphere, Pam did not demonstrate difficulties determining the minimum (transparent) and maximum (opaque) values of forcing. She, however, could not reason about how *rapidly* F was increasing between these two values, and she ended up drawing a piece-wise graph for $F(C)$ formed by two lines: a line with positive slope for $0 \leq C \leq C_{\text{opa}}$ and a horizontal line for $C > C_{\text{opa}}$, where C_{opa} represented the concentration of CO_2 for a completely opaque atmosphere. This suggests that she imagined C as continued to increase after taking the value $C = C_{\text{opa}}$. She, however, was not able to reason about co-variation at the amounts of change level (L3), which may explain why she did not conceive *asymptotic variation* when imagining $F(C)$. Conceiving asymptotic variation requires imaging $F(C)$ increasing at a decreasing rate as C increases.

Pam's responses to the first part of S2T2 suggest that she reasoned about the co-variation represented by the planetary energy imbalance function $N(t)$ at the *amounts of change level* (L3). She needed particular values to determine what energy flows were changing over time t (dependency of change level or L1). She did not need particular values to imagine the direction in which the energy imbalance N was changing as t was increasing (direction of change level or L2). Pam imagined the direction of change based on the understanding that the energy budget

tends to radiative equilibrium after a forcing occurs, which meant that $N(t)$ decreased over time. She constructed a concave-up, decreasing curve to represent $N(t)$. The curve showed $N(t)$ decreasing from an initial value $N(0) = F > 0$ down to zero. To construct her graph, Pam determined particular values of $N(t)$ and attended to the amounts by which N was changing for equal changes in time t . This way of thinking is consistent with reasoning about co-variation at *the amounts of change level* (L3). Pam interpreted the decreasing magnitude of $N(t)$ as indicating that the energy budget was returning to the *original radiative equilibrium*, rather than reaching a *new radiative equilibrium*. It seemed that Pam conceptualized $N(t)$ as the magnitude of energy in the planet's surface, rather than a magnitude of energy flux density indicating how *rapidly* the surface energy was changing. Since $N(t)$ decreases after the positive forcing, she thought that the planet's surface began to lose energy as time increased. For her, this meant that the energy budget returned to the original equilibrium after the departure from it brought about by the forcing. In other words, after a positive forcing created an energy imbalance, the planet's surface began to lose energy so that planet's surface radiation could be equal to the solar radiation, returning to the original equilibrium. Finally, Pam was the only PST who demonstrated conceiving *asymptotic variation* when imagining $N(t)$. When attending to the amounts of change in $N(t)$, she noticed that the values were decreasing by smaller and smaller amounts. She interpreted this as $N(t)$ asymptotically approaching to zero. There were three realization that supported Pam's conception of asymptotic variation: (a) she conceptualized $N = 0$ as the minimum possible energy imbalance (after a positive forcing); (b) she conceived time as continue to increase without a limit; and (c) she reasoned about co-variation at L3 which allowed her to imagine $N(t)$ approaching to a limit value while t continue to increase.

Pam's responses to the second part of S2T2 suggest that she reasoned about the co-variation represented by the mean surface temperature function $T(t)$ at the *direction of change level* (L2). She attended to the variation of the energy flows R and A over time in order to establish the direction in which the surface temperature T was changing as the time t increases (the direction of change level or L2). Pam conceptualized R and A as magnitudes of energy being absorbed and released by the planet's surface. She noticed that $R > A$ for all t and thought of it as an indication that the planet's surface was losing more energy than the energy it was absorbing. For Pam, this meant that the temperature of the planet's surface was decreasing over time after a positive forcing (direction of change). Thus, she constructed a decreasing curve with concavity shift for $T(t)$ (the concavity switched from *concave-down* to *concave-up* at some $t > 0$). The curve showed $T(t)$ decreasing from an initial value $T(0) = T_0 > 0$ down to the *original equilibrium temperature* (i.e., the temperature before the positive forcing occurred). This was consistent with her idea that the budget returns to the original radiative equilibrium after a forcing. Pam interpreted the increasing magnitude of the energy flow R as an indication that the planet's surface was trying to "cooling itself off" after the forcing. For her, this meant that $T(t)$ must decrease after the positive forcing, which in turn meant that the temperature of the planet's surface was returning to the original equilibrium temperature. Pam constructed a curve with concavity shift, but she did not provide any quantitative argument regarding the meaning of each concavity and the concavity shift. This suggests that Pam did not reason about co-variation beyond stating the direction in which the change was occurring (i.e., $T(t)$ was decreasing over time). What follows is a description of Pam's responses to the tasks and my interpretation of such responses.

The Forcing by CO₂ Function F(C)

After Pam read S2T1, I asked her to describe the way in which the energy flows B and A change when the concentration of CO₂, denoted by C, increases. Pam did not think that B and A could change because C could not increase since the atmosphere was already full of CO₂. Pam and I had the following exchange.

Pam: The atmosphere can handle so much, right? It's already got a lot of CO₂, so I don't think it could handle, it could bring more up, which means more CO₂ would be stuck [*pauses*]

Me: Actually, the atmosphere has more than enough room to absorb more CO₂

Pam: I picture a bunch of marshmallows in there so nothing else could fit. OK, so yes, if CO₂ increases, more would come down [*pauses*]

Notice that Pam did not finish her thought. This suggests that she did not have a clear idea of the direction in which B and A change as C increases. I decided to re-explain the concept of forcing to Pam in order to clarify her understanding of it. I explained to her that more concentration of CO₂ results in an atmosphere with more capacity to absorb radiation, which translates in an increase of the energy flow B. To complete the explanation, I presented Pam with an example involving particular values for the energy flows. I showed Pam a diagram of the Earth's energy budget showing the values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, and $B = 300 \text{ J/m}^2/\text{s}$. Notice that these values implied that $A = 150 \text{ J/m}^2/\text{s}$, $L = 90 \text{ J/m}^2/\text{s}$, and no forcing exists since $F = (240 + 150) - 390 = 0$. Next, I told Pam to imagine that a certain increase in C caused B to increase from $300 \text{ J/m}^2/\text{s}$ to $340 \text{ J/m}^2/\text{s}$ since the atmosphere's capacity to absorb energy increased. Then, I asked Pam to determine the value of flow A and the forcing F corresponding to that increase in C. Pam and I had the following exchange.

Me: Let's imagine we increase the concentration of CO₂ by a certain amount. This results in B growing from 300 J/m²/s to 340 J/m²/s. So, this flows changed [*point at B*], this flow change [*point at L*], and these two change [*point at A*]

Pam: They've just got bigger

Me: Exactly, could you now calculate the value of N corresponding to the new values of energy flows?

Pam: But, I don't know what S is now

Me: It is still 240 because we are just making changes in the atmosphere, and S does not depend on the atmosphere's composition. So, S is 240 and R remains at 390 as well ...

Pam: Except B. [*Writes the formula $N^2 = (S + A) - R$*] ... So, B is 340; that means A is now 170. So, we have 240 plus 170 minus 390 ($F = (240 + 170) - 390 = 20$)

Me: This value [*point at 20*] is a change in N, or the planetary energy imbalance, caused by a change in the concentration of CO₂. That is a forcing by CO₂; that is the value of F

Pam: Ah! So, when the CO₂ increases, F increases

The excerpt above suggests that the example helped Pam understand what is meant by forcing by CO₂ and how C related to such forcing. Pam's first realization involved identifying what energy flows change when C changes. In this case, when C changes, B, L, and A change while S and R remained constant. A key realization here involved identifying what energy flows depended on the atmospheric composition. After I explained to Pam that B, L, and A depended on the

² The expression $N = (S + A) - R$ defines the value of the planetary energy imbalance N according to the values of the energy flows S, A, and R. The forcing F is a change in N caused solely by a change in C; that is, F does not depend on time. In this context, the formula $N = (S + A) - R$ can be used to determine the value of F because we are ignoring variation over time.

atmospheric composition (unlike S and R), she appeared to construct a relationship (dependency of change) between such flows and C. Based on the example, Pam determined values for L and A, which allowed her to use the expression $N = (S + A) - R$ to determine a value of N corresponding to the increase in C. When I told Pam that such value represented F, she appeared to reason about the forcing by CO₂ function, denoted by F(C), in terms of direction of change. Since F increased from 0 to 20 after C increased, Pam conceptualized that F must increase when C increases.

Since Pam developed a sense of F(C) as a function, I wanted to examine if she could construct a graph of such relationship. Thus, I re-directed her attention to the original task: sketching the graph of the forcing by CO₂ function F(C). Pam decided to draw a line³ with positive slope to represent such function (Figure 5.1). The following exchange illustrates how Pam constructed her graph

Pam: It increases obviously, but I don't know how steep the increase is [*moves the marker indicating a line*]

Me: May I ask you something? So, if C increases, what is happening to F?

Pam: It increases

Me: OK, so if C increases, F increases

Pam: Yeah, I just didn't know how [*pauses and moves marker indicating two lines with different slopes*] steep the increase was, but I am just going to do like a rough [*draws a line with positive slope starting at the origin*]. I know it is positive. I don't know if it is linear. I just know that it is positive

Me: OK

³ I used the term *line* to refer to a graph showing a *straight line*, and I used the term *curve* to refer to a graph showing a *curved line*.

Pam: So, when we added 40 to that [*points at B*], how much parts per million the concentration went up?

Me: Well, I didn't really give you a value for the concentration. We just said it increased by some amount

Pam: OK, so then I am just going to say yes

Me: What does this point mean [*point at the origin*]? Is that (0, 0)?

Pam: Oh gosh. No because the concentration [*pauses*] There is always a concentration

Me: Yeah, there is always CO₂ in the atmosphere

Pam: So, here there was a concentration, but my N was still zero [*points at $N = (240 + 150) - 390 = 0$*]. So, there was a concentration [*pauses*] but it was zero [*pauses*]. Something like that maybe [*draws a line with positive slope and positive C-intercept (Figure 5.1)*]

Me: So, here your line is negative [*point at the line's negative section*]. So, if I decrease the concentration of CO₂, it means we are going to have

Pam: A negative N

Me: Which is an imbalance as well, right?

Pam: There is like a happy medium where there is concentration, but N is still zero [*points at the line's C-intercept*]

Pam started by stating that the graph should show F increasing in magnitude as C increases. She, however, did not immediately attempt to draw a graph because she did not know "how steep the increase was." Pam seemed to be using the word *steep* to describe the slope of a line. Notice how Pam moved the marker indicating two possible lines at the same time that she said "how steep the increase was." It is also possible that Pam was using the word *steep* to describe how rapidly F

increase with respect to C because she later said “when we added 40 to that [*points at B*], how much parts per million the concentration went up?” This question suggests that Pam wanted to construct an object to coordinate amounts of change in F with changes in C . Although such object appeared similar to the concept of rate of change, it is unclear whether Pam was in fact thinking of rate of change when referring to the *steepness* of a line. Thus, it seemed that Pam drew a line to communicate that all she knew about $F(C)$ was that F increased when C increased; that is, she solely knew the direction of change of F with respect to C . In addition to drawing a line for $F(C)$, Pam was able to interpret her graph in relation to the equilibrium concentration of CO_2 . Initially, Pam drew a line passing through $(0, 0)$, which she discarded because she stated that the concentration of CO_2 is never zero. This is in fact true since CO_2 is a naturally occurring gas in the atmosphere, and thus its concentration is not zero. Based on that realization, she drew a new line showing a C -intercept for some $C = C_0 > 0$. Pam interpreted such C -intercept as a “happy medium” for which there is no forcing ($F = 0$). Pam’s response (and new graph) suggests that she developed a sense of the notion of equilibrium concentration of CO_2 (concentration for which there is zero forcing).

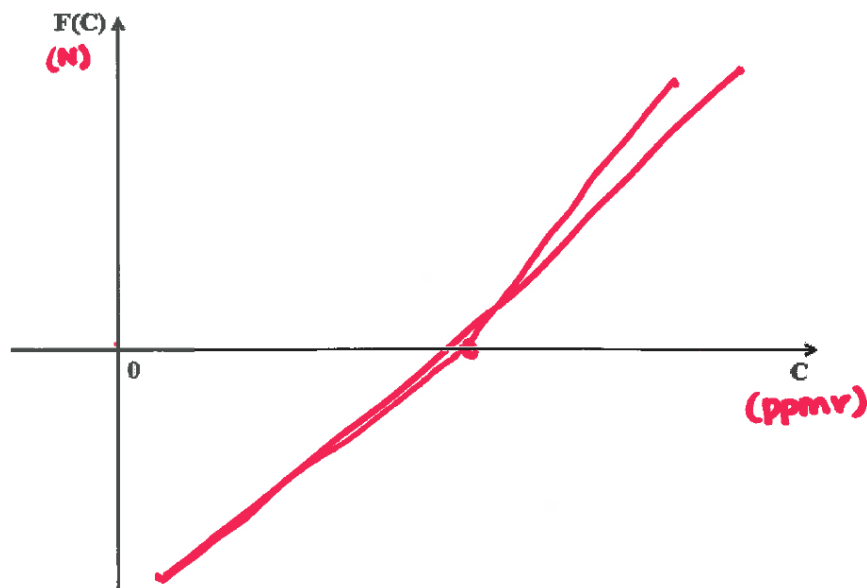


Figure 5.1. Pam's second attempt to sketch the graph for $F(C)$

Conceptualizing the direction of change of F with respect to C represents an important and foundational step in modeling $F(C)$. However, it was also important to examine whether Pam could hold that direction of change in mind while attending to how F was changing in relation to C . I decided to direct Pam's attention to the theoretical minimum and maximum values of $F(C)$, which correspond to the cases of a completely transparent atmosphere and a completely opaque atmosphere, respectively (see Chapter 3, pp. 17-18). I decided to discuss such cases with Pam in order to encourage her to think about the rate of change of $F(C)$. I specifically wanted Pam to notice that the maximum value for $F(C)$, a completely opaque atmosphere, implies that $F(C)$ increases at decreasing rate since F approaches to an upper limit as C continues increasing. Thus, I told Pam to reconsider the diagram of the Earth's energy budget that I showed her earlier during the interview. The original values on the diagram were $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, and $B = 300 \text{ J/m}^2/\text{s}$. Notice that these values implied that $A = 150 \text{ J/m}^2/\text{s}$, $L = 90 \text{ J/m}^2/\text{s}$, and no forcing exists since $F = (240 + 150) - 390 = 0$. I directed Pam's attention to the first case: a completely transparent atmosphere. I told Pam to imagine all CO_2 from the atmosphere was

removed so that the concentration is zero. Also, I told her to assume that CO₂ is the only greenhouse gas in the atmosphere. This meant that the atmosphere would not absorb any of the radiation emitted by the surface. Next, I asked Pam to determine the value of F corresponding to this situation. She demonstrated some difficulties conceptualizing dependency of change between energy flows and C. Particularly, Pam thought that the value of R changes after C changes.

Pam: So, this is going to go straight in [*points at S*], and the R is going to go straight out [*points at R*]. Because there is nothing in the atmosphere to absorb, so it is like getting rid of all that [*covers B and A with her hand*] ... So, N would be S minus R ($N = S - R$) because [*the radiation*] would be just coming and leaving

Me: What would the value of F be then?

Pam: Zero? [*Pauses*] I guess that is wrong. The value of the forcing would be [*pauses*] So, there is nothing here to absorb anybody [*circles her hand over the atmosphere*]. So, it goes in 240 [*points at S*] and it leaves 240 [*points at R*]

The excerpt above suggests that Pam was making sense of the Earth's energy budget in terms of what energy flows vary when C changes. She quickly realized that $B = A = 0 \text{ J/m}^2/\text{s}$ when $C = 0$ ppmv because "there is nothing in the atmosphere to absorb." Notice, however, that Pam thought that $R = S = 240 \text{ J/m}^2/\text{s}$ because "[*the radiation*] would be just coming and leaving." Pam appeared to imagine the situation as follow: since the atmosphere absorbs nothing, $B = A = 0$, then the surface receives solar radiation alone, which was $240 \text{ J/m}^2/\text{s}$, and thus the surface can only emit such magnitude of radiation toward the atmosphere. From a mathematical perspective, Pam's visualization of the situation involved having R as ultimately dependent on C; that is, R changes when C changes. This suggests that Pam was in the process of coordinating what

quantities were changing together (and what quantities were not). I told Pam that that R remained at $390 \text{ J/m}^2/\text{s}$ since changes in C only affect the atmospheric flows B, L, and A. It was unclear whether she interpreted this comment as stating that R is constant with respect to C. She, nonetheless, stated that “so, R is still 390, and S is going in at 240, but then there would be no B because nothing is being absorbed [*covers B with her hand*]. So, A would zero.” Then, she determined that $F = (240 + 0) - 390 = -150 \text{ J/m}^2/\text{s}$ for the case of a completely transparent atmosphere.

Next, I directed Pam’s attention to the second case: a completely opaque atmosphere. Pam showed less difficulties thinking about this situation. Particularly, she was able to identify what energy flows change when C changes. To make sense of the situation, she drew a diagram representing the energy budget for a completely opaque atmosphere (Figure 5.2).

This still is getting half and half [*points at A*], but that doesn’t exist [*points at L*]. So, this N would be [*pauses*]. Ok, so S goes in at 240 [*draws a downward arrow and writes 240*]. R comes out at 390 [*draws an upward arrow and writes 390*]. But B is the same [*draws a right arrow and writes 390*] and it gets split out into 195 and 195 [*draws an upward arrow and a downward arrow and writes 195*]

After drawing the diagram, Pam stated that $F = 45 \text{ J/m}^2/\text{s}$ at the top of the atmosphere and at the surface level (“So the space would be 45 and the surface would be 45 as well”). It seemed that Pam used the diagram to help her reason about the dependency of change between energy flows and C. Notice how Pam went through her diagram establishing the value of every flow while keeping in mind that the atmosphere was absorbing all surface radiation. It was unclear, however, how Pam was thinking about the variation in C.

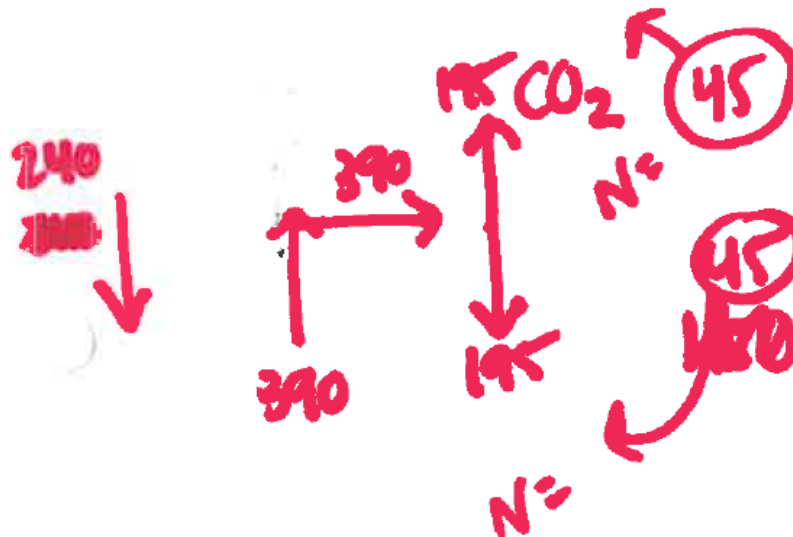


Figure 5.2. Pam's diagram of the Earth's energy budget for a completely opaque atmosphere.

To examine how Pam was thinking about the variation in C , I asked her to draw a new version of the graph of $F(C)$ based on what she learned about the minimum ($F = -150 \text{ J/m}^2/\text{s}$) and maximum ($F = 45 \text{ J/m}^2/\text{s}$) values of $F(C)$. Starting at the *lowest section* of the vertical axis, Pam drew a line with positive slope and with positive C -intercept. Next, she made a mark for the F -intercept and wrote '-150' next to it. She then made a mark at the *highest section* of the vertical axis and wrote '45' next to it. From that mark, Pam drew a horizontal, dashed segment incident to her line. From that intersection, she drew a vertical, dashed segment incident to the horizontal axis. She made a mark at that intersection and said "this would be the concentration that we just shoved in here (completely opaque atmosphere)" (Figure 5.3). The intersection between dashed segments and the line defined a point representing the maximum value $F = 45 \text{ J/m}^2/\text{s}$ for some arbitrary value of C ; for convenience, I refer to such value as $C = C_{\text{opa}}$. Notice how Pam's line *started* at $F(0) = -150 \text{ J/m}^2/\text{s}$, passed through $F(C_{\text{opa}}) = 45 \text{ J/m}^2/\text{s}$, and continued to include points (C, F) such that $C > C_{\text{opa}}$ and $F > 45 \text{ J/m}^2/\text{s}$. The way Pam constructed this line unveiled interesting aspects regarding how she was thinking about the situation. First, Pam

plotted a point representing $F(C_{opa}) = 45$, which suggests that she thought that the forcing reaches $45 \text{ J/m}^2/\text{s}$ for a finite value of $C = C_{opa}$. She interpreted this value as the concentration of CO_2 that creates a completely opaque atmosphere (e.g., “this would be the concentration that we just shoved in here (completely opaque atmosphere)”). Second, Pam’s line continued for $C > C_{opa}$, which might suggest that she visualized C as increasing passed C_{opa} and $F(C)$ increasing passed $45 \text{ J/m}^2/\text{s}$. This, however, was unclear since she *first* drew the line *and then* mark the point $F(C_{opa}) = 45 \text{ J/m}^2/\text{s}$. To examine how Pam was thinking about this, Pam and I discussed some of the properties of her graph of $F(C)$.

Me: What does this point represent [*point at (C_{opa} , 45)*]?

Pam: That point represents the amount of imbalance we have when there is a big concentration of CO_2 . So, there is a lot of forcing, it is a high forcing by CO_2

Me: OK, so what does this point represent [*plot a point (C , F) on the line such that C > C_{opa} and F > 45*]?

Pam: That would be [*pauses*] This is the amount of force[ing] when there is more concentration [*pauses*] I don’t know, I don’t think [*pauses*] it would maybe just stop there [*points at the point (C_{opa} , 45)*]

Me: Why?

Pam: I think it just stops there because we are at the point where nothing is leaving. So, it is all being absorbed [*pauses*] I just think it just stops [*scratches the section of the line for C > C_{opa} (Figure 5.3)*]

Me: Do you think that $F(C)$ can’t get bigger than $45 \text{ J/m}^2/\text{s}$?

Pam: So, S is still coming in at 240 [*draws a downward arrow and writes 240*] ... So, R is still 390 [*draws an upward arrow and writes 390*] ... So, B can increase

[*pauses*]. If B increases, then L decreases. I guess 390, but if this [*points at B*] takes more [*radiation*] out, less leaks because I thought what if this [*points at B*] took 380 [*draws a right-pointing arrow and pauses*]

Me: If the atmosphere absorbs $380 \text{ J/m}^2/\text{s}$, then only $10 \text{ J/m}^2/\text{s}$ would leak through it

Pam: OK! So, if I take 380 [*writes 380*], 10 leaks [*draws an upward arrow and writes 10*] [*draws a downward arrow and an upward arrow and writes 190 for each one*] [*uses the calculator*]. That was only 40 [*pauses*]! It can't because if this was 390, this doesn't change [*points at R*], so we already had the highest amount 390 coming back. So, 45 is the highest [*value of F(C)*]

Me: So, should your line continue?

Pam: No, it stops

Me: So, the graph of $F(C)$ just stops when F reaches 45. Is that right?

Pam: I kind of want to change it now. I was thinking like, maybe it will go, go like this [*moves the marker indicating a horizontal segment at $F = 45$ and for $C > C_{opa}$*]. Because, we just discussed 45 was the highest [*value of F*]. I am thinking it maybe goes like a plateau here [*draws a horizontal segment (Figure 5.3)*]

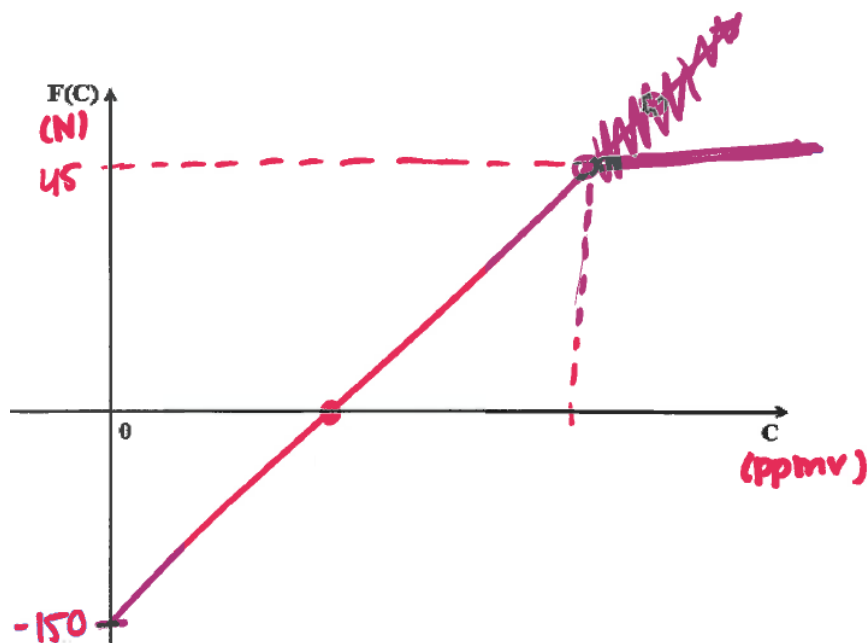


Figure 5.3. Pam's final sketch of the graph of $F(C)$

Pam's final graph of $F(C)$ consisted of two distinct sections: a line with positive slope for $0 \leq C \leq C_{\text{opa}}$ and a horizontal line for $C > C_{\text{opa}}$, where C_{opa} represents Pam's concentration of CO_2 for which the atmosphere is completely opaque. Her graph suggests that Pam continued reasoning about co-variation in terms of the direction of change of F with respect to C . For instance, Pam drew a line with positive slope to represent the direction in which C and F were changing before the atmosphere was completely opaque (i.e., F increases as C increases). Then, Pam drew a horizontal line to represent that F remains constant as C increases. Both sections were drawn to show direction of change of F with respect to C . This may explain why Pam *first* drew the line *and then* plotted the points for $F(0) = -150$ and $F(C_{\text{opa}}) = 45$. She drew the line to indicate that F increased when C increased and plotted the points to indicate that such increase went from $-150 \text{ J/m}^2/\text{s}$ to $45 \text{ J/m}^2/\text{s}$. Pam, however, did not demonstrate thinking about how F was varying in relation to C when increasing from $-150 \text{ J/m}^2/\text{s}$ to $45 \text{ J/m}^2/\text{s}$. For that, Pam needed to reflect on the meaning of the value $45 \text{ J/m}^2/\text{s}$ as the maximum value and horizontal

asymptote of $F(C)$. Although she had some initial difficulties, Pam did demonstrate conceptualizing the value $45 \text{ J/m}^2/\text{s}$ as the maximum value of $F(C)$. She, however, initially stated that $F(C)$ should *stop* at $F(C_{\text{opa}}) = 45$. This suggests that she might have thought of both quantities, F and C , as taking their maximum values ($C = C_{\text{opa}}$ and $F = 45$) so that neither of them can take a larger value. She might have thought of $C = C_{\text{opa}}$ as the maximum concentration of CO_2 the atmosphere could possibly have. Pam then corrected this by drawing a horizontal segment at $F = 45$ and for $C > C_{\text{opa}}$. This suggests that she may have conceived of C as keep increasing passed $C = C_{\text{opa}}$. Nonetheless, Pam seemed to think that there was a finite value $C = C_{\text{opa}}$ for which $F(C_{\text{opa}}) = 45 \text{ J/m}^2/\text{s}$. This may explain why she did not demonstrate conceptualizing $F = 45$ as a horizontal asymptote of $F(C)$. Also, Pam did not demonstrate thinking beyond the direction of change. To conceive of the asymptote of $F(C)$, Pam needed to reasoning in terms of how F varies in relation to C , or in terms of the rate of change of $F(C)$.

The Planetary Energy Imbalance Function $N(t)$

After Pam read S2T2, I showed her a diagram of the Earth's energy budget having the initial values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, $B = 300 \text{ J/m}^2/\text{s}$, and $A = 150 \text{ J/m}^2/\text{s}$. These values defined an energy budget in equilibrium since $N = (S + A) - R = (240 + 150) - 390 = 0$. While working on the previous task, Pam did not demonstrate imagining the flows changing without the aid of particular values. Thus, I decided to let Pam use particular values from the beginning while working on S2T2. I next told Pam that C is instantaneously increased at $t = 0$, creating a positive forcing. I changed the values of the flows B , L , and A to the new values $B = 310 \text{ J/m}^2/\text{s}$, $L = 80 \text{ J/m}^2/\text{s}$, and $A = 155 \text{ J/m}^2/\text{s}$ for $t = 0$. These values defined a positive forcing by CO_2 of magnitude $5 \text{ J/m}^2/\text{s}$ since $F = (S + A) - R = (240 + 155) - 390 = 5$. Then, I asked Pam if she could determine the value of $N(0)$.

Me: What is the value of N at $t = 0$?

Pam: Well, if we increase here [*points at $B = 310$*], immediately at $t = 0$, then N would be greater than zero, it would be positive. So, it would just be five because we just increased that by five [*points at $A = 155$*]. So, we have 240 plus 155 minus 390, it is just five ($N = (240 + 155) - 390 = 5$). So, in the surface [N] would be five at t zero because N increases suddenly at $t = 0$

Me: So, the value of N is 5 at $t = 0$, is that correct?

Pam: Um-hum

Pam determined that $N(0) > 0$ by reasoning about the direction of change of the energy flow A with respect to the concentration of CO_2 . Although she utilized particular values, her response suggests that she develop an understanding of the forcing as the initial value of $N(t)$. Notice that she coordinated N being positive with $t = 0$ (e.g., “in the surface [N] would be five at t zero because N increases suddenly at $t = 0$ ”). Next, I helped Pam use these values of the flows at $t = 0$ to determine new values for R , B , and A corresponding to the arbitrary times $t = 0 + h = h$ and $t = 2h$ (for some positive h). I told Pam to base the calculations on the assumption that $R([i + 1]h) = S + A(ih)$ for $i = 0, 1, 2, \dots$. To make sense of the situation, Pam drew her own diagram of the energy budget (Figure 5.4).

We have 390 here, originally [*writes 390 for R on her diagram*]. Then, we increased [the concentration of CO_2] in the atmosphere. So, this increased to 310 [*writes 310 for B*]. Then, this spiked to be 155 [*writes 155 for A*]. So then, 240 plus 155 is 395, so this is our new R [*writes 395 for R*]. Then, we have 79.5 percent [of R] is getting absorbed as B into the atmosphere. So, 79.5 percent of 395 is that [*writes 313.97 for B*] ... Then, we have to split [$B = 313.97$] in half to get that [*writes 156.98 for A*], which would increase R again

... [Adds 240 and 156.98] so then, [R] goes up to 396.98. [Calculates 79.5% of 396.98] so then, [B] increases [to] 315.6. So then, divide it by two, and we have 157.8 [writes 157.8 for A]. And then, that increases [R] again

In total, Pam had three successive values for each energy flow R, B, and A on her diagram (Figure 5.4), values corresponding to the time values $t = 0$, $t = h$, and $t = 2h$. Pam coordinated values of the flows with time values to make sense of the direction of change of the flows over time (as time passes, the flows increase in magnitude). This suggests that she may have developed a sense of these flows increasing over time; that is, the flows were in function of time. The next step involved examining whether Pam could develop a sense of $N(t)$ as a function of time.

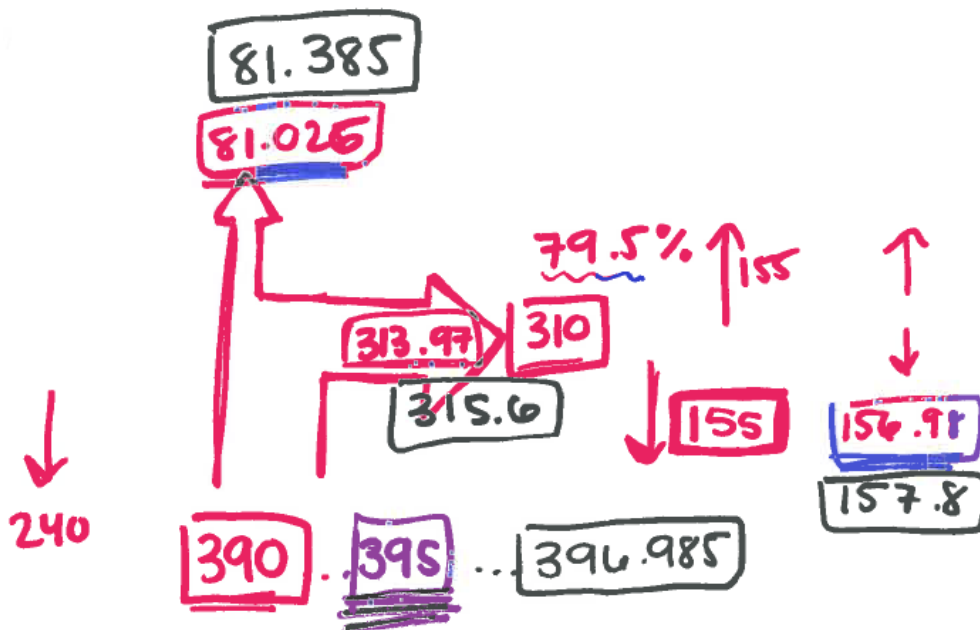


Figure 5.4. Pam's version of the diagram of the Earth's energy budget. Pam used this diagram to explain how the flows of energy B, A, and R vary over time.

Since Pam coordinated values of flows with time values, I decided to examine whether Pam could imagine how N was varying over time based on knowing how the flows were varying over time. I re-directed Pam's attention to the original task: drawing the graph of $N(t)$.

Me: So, how does N vary after the positive forcing?

Pam: It will balance itself out

Me: OK, so you said N was five at $t = 0$. Would N increase or decrease after that?

Pam: It was five, and then it would decrease to be zero again [*pauses*]. Let me think about this one more time. So, this was the original [*points at R = 390*], it was 390 like 240 plus 150. That was when [N] was zero, and then we increased the concentration, just here [*points at B*] to affect everything else in the atmosphere. Then, N was five because [B] increased to 310 ... So, at $t = 0$, N was five because at that time is when we increased [the concentration of CO_2], and that is when N turned into five, right? So, $t = 0$, N was five [*writes $t_0 : N = 5$*]. But then, 395 got absorbed, wait [*pauses*] ... OK! This is what I am thinking. We increased time zero (the concentration of CO_2 was increased at $t = 0$), but I am pretty sure [N] would balance itself back out

Me: OK! Did you learn that by watching the video⁴?

Pam: That is what I heard. That is what I am thinking I got it from

Me: So, we know the energy imbalance should decrease over time

Pam: Yeah, from five back down to zero. So, I think [the graph of $N(t)$] could go from like here, like this was five at $t = 0$ [*points at the top section of the vertical axis*]. I

⁴ I showed each PST a 7-minute long, introductory video from the NASA Earth Observatory channel on YouTube. This video introduces the concepts of Earth's energy budget, radiative equilibrium, and greenhouse effect. The goal was for PSTs to gain elementary knowledge concerning the aforementioned concepts so that they could work on the tasks of Set 2.

think it is going to go [*moves marker indicating a concave-up, decreasing curve*].
 But I think it is going to hit zero pretty quickly [*moves marker indicating a line with negative slope*]. But I think it is going to hit this zero, this axis [*points at horizontal axis*] because we know N can be zero

In the exchange, it appears that Pam imagined the graph of $N(t)$ as a decreasing curve. Pam did not demonstrate imagining a decreasing $N(t)$ based on her analysis of how the flows were changing over time. She arrived to such conclusion based on what she learned about radiative equilibrium after watching a video. In other words, she imagined $N(t)$ decreasing because, as it was said in the video, the energy imbalance must approach to zero as the budget approaches to an equilibrium. Also, notice that Pam did not attempt to (or was unable to) determine and coordinate values of $N(t)$ for the time values $t = 0$, $t = h$, and $t = 2h$. This suggests that Pam might not have yet coupled the variation N with the variation in time, which in turn suggests that Pam might not have yet imagine N and t as two quantities varying together (i.e., $N(t)$ as a function of time).

To help Pam develop a sense of $N(t)$ as a function of time, I directed her attention toward how the flows were varying after a positive forcing. Particularly, I asked Pam to determine the value of N corresponding to each time value $t = 0$, $t = h$, and $t = 2h$ by utilizing the relationship $N(ih) = (S + A(ih)) - R(ih)$. Pam determined the values $N(0) = 5 \text{ J/m}^2/\text{s}$, $N(h) = 1.98 \text{ J/m}^2/\text{s}$, and $N(2h) = 0.82 \text{ J/m}^2/\text{s}$. After Pam determined these values, I asked her to describe how $N(t)$ was changing over time.

Me: Could you tell me how N changes over time?

Pam: So, N decreased by about three (from $N(0) = 5$ to $N(h) = 1.98$), and then it decreased by about one-ish, a little more than one (from $N(0) = 1.98$ to $N(2h) =$

0.82). It decreased pretty quickly, like three units of Joules per second per meter square. Then, it decreased by about one Joules per second per meter square ... I am assuming [N] is going to decrease by a little bit, and a little bit, and a little bit, until it reaches zero again

Me: OK, so what would the graph look like?

Pam: I think it'd go like that [*moves marker indicating a concave-up, decreasing curve*]

After this exchange, I asked Pam to draw a graph of $N(t)$; she first made a mark on the top section of the vertical axis and wrote '5' next to it. From that mark, Pam drew a concave-up, decreasing curve asymptotically (at least in appearance) approaching to zero (Figure 5.5). I asked her to elaborate on whether her graph of $N(t)$ had an asymptote at $N = 0$ or whether it intersected the horizontal axis. Pam provided the following response.

I can't decide if [the curve] crosses that axis or not [*points at the horizontal axis*] ... It would have to balance back out, but I don't [*pauses*]. Looking at this [*points at the values of N*], it is like continually decreasing, but I don't know if it would ever reach zero ... I think it would be really, really close to zero

Pam coordinated values of $N(t)$ with values of time. Then, she attended to the variation in the amounts of change in $N(t)$ between time values to construct the graph of $N(t)$ (e.g., "Then, it decreased by about one Joule per second per meter square"). Although Pam made use of intensity adjective to describe change, which could be considered rate of change, Pam's object of reasoning was the amounts by N was changing over time (e.g., "It decreased *pretty quickly* [*emphasis added*], like *three units* [*emphasis added*] of Joules per second per meter square"). More specifically, she compared the amounts of change in N across successive time values in order to imagine how $N(t)$ was changing with respect to time. Additionally, Pam demonstrated

some difficulties deciding whether the graph of $N(t)$ should intersect the horizontal axis. It seemed that Pam's idea of radiative equilibrium conflicted with how she thought $N(t)$ was decreasing by observing the value $N(0) = 5$, $N(h) = 1.98$, and $N(2h) = 0.82$. On one hand, Pam thought $N(t)$ took zero as a value because the energy budget must "balance itself out." At some point in time, the energy budget reaches a radiative equilibrium, which meant that $N(t)$ must be zero for some (finite) t . On the other hand, Pam observed a pattern in the succession of values $N(0) = 5$, $N(h) = 1.98$, and $N(2h) = 0.82$ suggesting that $N(t)$ asymptotically decreased to zero. By attending to the variation in the amounts of change in $N(t)$, she ultimately decided that $N(t)$ would ever reach zero, but "it would be really, really close to zero."

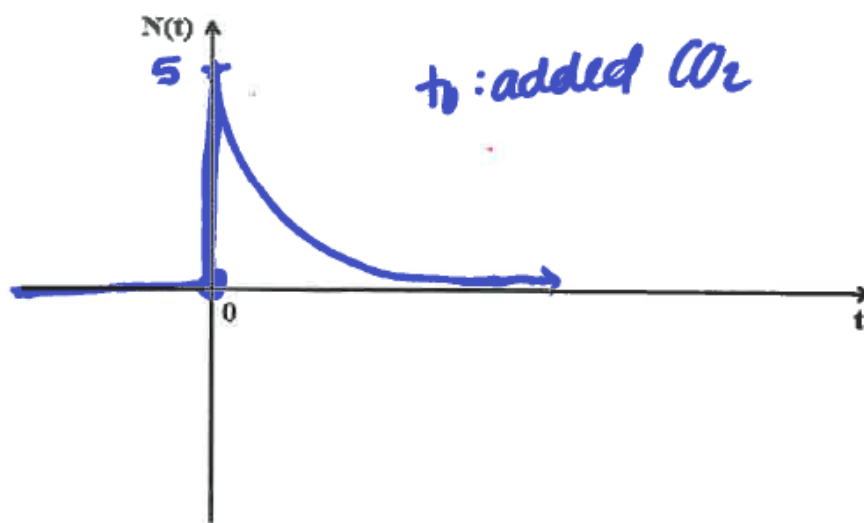


Figure 5.5. Pam's final version of her graph of $N(t)$

I decided to investigate if Pam could interpret her graph of $N(t)$ in relation to the Earth's energy budget. Pam demonstrated difficulties interpreting the meaning of $N(t)$ in this context. Pam's difficulties seemed to arise from her conception of the quantities R , A , and $N(t)$. Pam's responses suggest she conceptualized all three quantities as actual magnitudes of energy rather

than magnitudes of energy flux density. The following excerpt from the discussion we had about her graph of $N(t)$ illustrates Pam's struggle.

Me: According to your graph, is $N(t)$ positive or negative after the forcing?

Pam: N is positive

Me: What does that mean in terms of energy in the planet's surface? Is the planet's surface gaining or losing energy?

Pam: I think it is losing [*pauses*]. No, it is gaining! If we look, [the energy flows] keep increasing, my numbers are getting bigger [*circles her hand over R, B, and A*]. So, I think the energy increases

Me: OK, let's look at these two values of N that you determined before [*draw a circle around $N(0) = 5$ and $N(2h) = 0.82$*]. For which value of N is the surface gaining energy?

Pam: Well, from here to here, it is losing energy (from $N(0) = 5$ to $N(2h) = 0.82$). But, I thought [*pauses*]. I am thinking of energy as in the amounts ... These numbers are increasing [*points at R and A on her diagram (see Figure 5.4)*]. But, the N is the planetary energy imbalance. Oh! So, it is decreasing. I don't know why I said increasing

Me: OK, but you said N was positive after the forcing. If N is positive, then the downward radiation is larger than the upward radiation. Wouldn't that mean that the planet's surface is gaining energy?

Pam: [*Long pause*] yeah

Me: Here, we have two positive values of N [*point at $N(0) = 5$ and $N(2h) = 0.82$*]. So, for which value is the surface gaining energy then?

Pam: We are gaining more energy here [*points at $N(0) = 5$ and pauses*] No, we are gaining more energy here [*points at $N(2h) = 0.82$*]. Oh shut! I am looking at these numbers, and this number is bigger [*points at A on her diagram*]. So, more [radiation] is going in, but also more [radiation] is coming out [*points at R on her diagram*]. So, I want to say that this [*points at $N(0) = 5$*] has higher energy than this [*points at $N(2h) = 0.82$ and pauses*]. I don't know, I just don't know

The source of Pam's difficulties seemed to be a conflict between two notions regarding $N(t)$. On the one hand, Pam seemed to conceptualize $N(t)$ as the difference between downward radiation and upward radiation. She stated that a positive $N(t)$ indicated the planet's surface was gaining energy. She supported her statement by indicating that R and A were increasing in magnitude, which meant that the surface's energy was increasing. She then corrected herself and stated that the surface's energy was actually decreasing because $N(t)$ was decreasing. This suggests that Pam also conceptualized $N(t)$ as a magnitude of energy rather than a magnitude of energy flux density. For her, $N(t)$ represented the magnitude of energy (or heat) contained in the planet's surface. In such context, a decreasing $N(t)$ showed that the planet's surface must be losing energy. Notice how Pam solved this conflict by comparing the magnitudes of A and R . Since $A < R$ for all $t > 0$, Pam argued that the surface was absorbing less energy than the energy it was radiating (e.g., "this number is bigger [*points at A on her diagram*]. So, more [radiation] is going in, but also more [radiation] is coming out [*points at R on her diagram*]"). For Pam, A and R are actual magnitudes of energy *coming into* and *leaving* the surface, respectively. Thus, she conceptualized the difference $N(t) = (S + A(t)) - R(t)$ as the energy *in* in the surface, which meant that a decreasing $N(t)$ indicated a loss of energy.

The Surface Temperature Function $T(t)$

I was interested in exploring how Pam's interpretation of a decreasing $N(t)$ (i.e. as indication of both energy loss and returning to the original equilibrium) would influence her understanding of the average surface temperature function, denoted by $T(t)$. I directed Pam's attention to the second part of S2T2: drawing the graph of $T(t)$. I started by asking Pam to interpret her graph of $N(t)$ in terms of the surface temperature. Pam seemed to conceptualize $N(t)$ as a measure of $T(t)$ rather than as a measure of the rate of change of $T(t)$. For Pam, the decreasing magnitude of $N(t)$ meant that $T(t)$ was decreasing in magnitude.

Me: When is the surface temperature at its highest? Is it at the beginning, at the moment the concentration of CO_2 was increased, or later, after the concentration was increased?

Pam: The beginning, yes

Me: So, if the surface temperature is at its highest at the beginning, wouldn't that mean the temperature decreases over time?

Pam: A lot [of radiation] is going in, but more is coming out ... So, as our A increases, R increases, but our S is staying the same, but our A is always less than R . So, more [radiation] is coming out [*pauses*]. So, the Earth is trying to cool itself off, so the temperature is decreasing from here to here (from $N(0) = 5$ to $N(2h) = 0.82$), yes

Me: So, for which value of N is the surface temperature at its highest?

Pam: Here [*points at $N(0) = 5$*]

Me: For which value of N is the surface temperature changing most rapidly? This is a different question

Pam: Here [*points at $N(0) = 5$*]? We are getting more hotter here [*points at $N(0) = 5$*] because more [radiation] is going into the Earth and less is coming out. But here [*points at $N(2h) = 0.82$*], more is going in, but more is coming out than here [*points at $N(0) = 5$*]. So, [the surface] is cooler, it is getting cooler because if N is high, then the temperature is high

Pam concluded that the temperature must be decreasing over time because the magnitude of energy *leaving* the surface was larger than that *coming into* the surface. Notice how Pam conceptualized R and A as magnitudes of energy rather than magnitudes of energy flux density; that is, Pam's response suggests that she did not see R and A as rates at which the energy is being transferred. This appeared consistent with Pam's conception of $N(t)$ as the magnitude of energy *in* the surface. For Pam, S and A represented the energy absorbed by the surface, while R represented the energy *released* by the surface. Thus, the difference between energy absorbed and energy released, $N = (S + A) - R$, represented the energy *left in* the surface. Since $N(t)$ was decreasing, $T(t)$ must also be decreasing. Pam's interpretation regarding an increasing R also led her to conclude that $T(t)$ was decreasing over time. Notice that Pam interpreted an increasing R as indicating that "the Earth [was] trying to cool itself off." This suggests that Pam saw the planet as having agency in the sense that it started emitting more radiation to decrease its temperature. Finally, notice that Pam did not address my question regarding when $T(t)$ was changing most rapidly. She did not demonstrate understanding $N(t)$ as a measure of how rapidly $T(t)$ was changing. In fact, she did not demonstrate using $N(t)$ to construct the graph of $T(t)$.

Pam concluded that $T(t)$ was decreasing over time, but she did not refer to how rapidly $T(t)$ might have been decreasing. I was also interested in examining how Pam's notion that "the Earth [was] trying to cool itself back off" would influence her understanding of $T(t)$. Thus, I

asked her to sketch the graph of $T(t)$ based on her graph of $N(t)$. The following exchange illustrates how Pam constructed a graph of $T(t)$ (Figure 5.6).

Pam: We have a temperature of zero before this time starts (before $t = 0$) [*pauses*]. No, we don't have a temperature of zero, we have a temperature of balance. Then, as t hits zero, [the temperature] increases. So, it is about here [*points at the top section of the vertical axis*]. Then, it cools off until it is back in its balancing point [*moves her finger indicating a decreasing, concave-down curve*]

Me: Could you draw the graph of $T(t)$ for me?

Pam: Yes [*draws a decreasing, concave-down curve intersecting the horizontal axis (Figure 5.6)*]

Me: According to your graph, the temperature will decrease back to zero. Is that correct?

Pam: No, it is just going to be [*scratches and replaces a section of her curve by a decreasing, concave-up curve (Figure 5.6)*]. This is its balancing point, like it is normal equilibrium [*draws a horizontal line representing equilibrium temperature*]

Her construction of $T(t)$ was exclusively done in terms of direction of change. Although Pam drew a curve with concavity shift, she did not demonstrate an awareness of its meaning in terms of the variation in $T(t)$. Pam did not provide an argument for the concave-down section of her graph and drew a concave-up section only to avoid having $T(t)$ intersecting the horizontal axis. Pam also seemed to conceptualize radiative equilibrium as returning to the *original* state of equilibrium rather than reaching a *new* one. She stated that the planet cooled itself down over time so that it could return to “its balancing point” or “normal equilibrium.” If T_e represents the

original equilibrium temperature, then Pam's graph showed that $T(t)$ instantaneously increased from T_e at $t = 0$ just to return to it as t increased. This suggests that Pam conceptualized reaching a radiative equilibrium as returning to the original equilibrium.

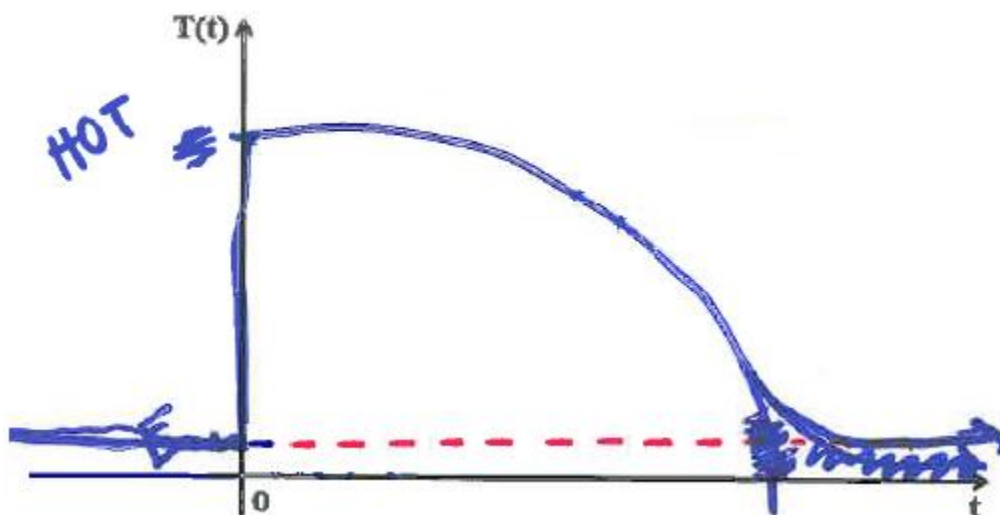


Figure 5.6. Pam's final version of her graph of $T(t)$.

The Case of Jodi

Synthesis

Jodi's responses to the tasks suggest that she consistently reasoned about co-variation at *the amounts of change level (L2)*. This level involves identifying the direction in which the dependent variable is changing (i.e., increasing, decreasing, or not changing) as the independent variable increases. For S2T1, Jodi did not need particular values to determine what energy flows were changing with changes in the atmospheric CO_2 concentration C (dependency of change level or L1). She also did not need particular values to imagine the direction in which the forcing F was changing as C was increasing (direction of change level or L2). This suggests that Jodi had no difficulties reasoning about co-variation at the first two levels. To construct the graph of $F(C)$, Jodi tried to relate amounts of change in F with change in C in order to determine the *slope of a line* as the graph of $F(C)$, rather than to confirm whether the graph was a line. This suggests that

Jodi assumed that the amounts of change in F remained constant for equal increments in C . Such assumption was not supported by any quantitative argument. When working with the completely transparent atmosphere and the completely opaque atmosphere, Jodi had some difficulties determining the minimum (transparent) and maximum (opaque) values of forcing. These difficulties were rooted in her conception of F as an unknown quantity that must be subtracted from $S + A$ or R so that downward radiation = upward radiation. For Jodi, F was not defined by the difference between these two radiations. After determining the minimum and maximum values of F , Jodi could not reason about how *rapidly* F was increasing between these two values, and she ended up drawing a line segment with positive slope for $0 \leq C \leq C_{\text{opa}}$, where C_{opa} represented the concentration of CO_2 for a completely opaque atmosphere. This suggests that she did not conceptualize C as continued to increase after taking the value $C = C_{\text{opa}}$. As a result, she did not demonstrate conceiving *asymptotic variation* when imagining $F(C)$. Conceiving asymptotic variation for $F(C)$ requires imagining C to continue to increase without limit as F approaches to an upper limit value. It is possible that Jodi conceptualized $C = C_{\text{opa}}$ as representing an atmosphere completely formed by CO_2 , which would explain why C would not continue to increase.

Jodi's responses to the first part of S2T2 suggest that she reasoned about the co-variation represented by the planetary energy imbalance function $N(t)$ at the *direction of change level* (L2). She needed particular values to determine what energy flows were changing over time t (dependency of change level or L1). She also needed particular values to imagine the direction in which the energy imbalance N was changing as t was increasing (direction of change level or L2). Jodi was surprised to notice that $N(t)$ was decreasing over time. To make sense of that, she used the understanding that the energy budget tends to radiative equilibrium after a forcing

occurs. In other words, Jodi justified $N(t)$ decreasing over time because the budget tends to equilibrium over time. To construct her graph, Jodi associated values of N with discrete time intervals she called *revolutions*. By plotting these pairs of values as points on a coordinate system, Jodi drew a concave-up, decreasing curve to represent $N(t)$. The curve showed $N(t)$ decreasing from an initial value $N(0) = F > 0$ down to zero. Although her graph showed concavity, she did not deduce such concavity from reasoning about the situation (i.e., how the difference between downward radiation and upward radiation varies after a positive forcing). Instead, the graph's concavity was the result of joining several points by a curve. This suggests that Jodi did not reason beyond L2 when imagining $N(t)$. Jodi interpreted the decreasing magnitude of $N(t)$ as indicating that the energy budget was returning to the *original radiative equilibrium*, rather than reaching a *new radiative equilibrium*. It seemed that Jodi conceptualized $N(t)$ as the magnitude of energy in the planet's surface, rather than a magnitude of energy flux density indicating how *rapidly* the surface energy was changing. Since $N(t)$ decreases after the positive forcing, she thought that the planet's surface began to lose energy as time increased. For her, this meant that the energy budget returned to the original equilibrium after the departure from it brought about by the forcing. In other words, after a positive forcing created an energy imbalance, the planet's surface began to lose energy so that planet's surface radiation could be equal to the solar radiation, returning to the original equilibrium. Finally, Jodi did not demonstrate conceiving *asymptotic variation* when imagining $N(t)$. She constructed a graph incident to the horizontal axis, which suggests that she did not conceptualize t as continued to increase without limit as N approaches to zero. It is possible that Jodi thought of the energy budget as reaching equilibrium for a finite time, which would explain why t would not continue to increase.

Jodi's responses to the second part of S2T2 suggest that she reasoned about the co-variation represented by the mean surface temperature function $T(t)$ at the *direction of change level* (L2). She attended to the variation of the energy flows R and A over time in order to establish the direction in which the surface temperature T was changing as the time t increases (the direction of change level or L2). Pam conceptualized R and A as magnitudes of energy being absorbed and released by the planet's surface. She noticed that $\Delta R = \Delta A$ between revolutions; that is, an amount of increase in energy absorbed by the surface at revolution i was followed the same amount of increase in energy released by the surface at revolution $i + 1$. She interpreted this as an indication that the magnitude of energy in the surface was increasing and decreasing by the same amount between revolutions. For her, this meant that the surface temperature $T(t)$ must have also been increasing and decreasing between revolutions. As a result, Jodi drew two different graphs for $T(t)$, both representing a periodic function. Then, Jodi attended to the amounts by which the energy flow B increased for equal changes in time. She noticed that B was increasing by smaller and smaller amounts, which she interpreted as $T(t)$ must decrease after the positive forcing, which in turn meant that the temperature of the planet's surface was returning to the original equilibrium temperature. Since she initially thought of a periodic function, Jodi drew a quasiperiodic curve approaching to the *original equilibrium temperature* (i.e., the temperature before the positive forcing occurred) as the graph of $T(t)$. This was consistent with her idea that the budget returns to the original radiative equilibrium after a forcing. She, however, did not provide any quantitative argument for the concavity of her curve. This suggests that Jodi did not reason about co-variation beyond stating the direction in which the change was occurring (i.e., $T(t)$ was oscillating as time increases). What follows is a description of Kris' responses to the tasks and my interpretation of such responses.

The Forcing by CO₂ Function F(C)

After Jodi read S2T1, I asked her to describe the way that flows B and A change when the concentration of CO₂, denoted by C, increases. Jodi and I had the following discussion regarding changes in energy flows in the Earth's energy budget.

Me: How do you think B would change if C increases?

Jodi: B would increase

Me: Why?

Jodi: Because if there is more CO₂ in the atmosphere, then there is more absorption ...

After the surface absorbs the energy from the sun [*points at S*], and then it is repealed by R [*points at R*], then CO₂ absorbs more energy in B. So B would increase, which means A and L would increase

Me: Would L increase?

Jodi: Since B is increasing, you have a number [*pauses*]. L might decrease

Me: Remember that B and L are both part of R. B is the part that is absorbed and L the part that leaks. So, if R remains constant and B increases, then how should L change?

Jodi: L has to decrease

Jodi showed no difficulties visualizing B increasing when C increased; she stated that a higher atmospheric concentration of CO₂ results in "more absorption." She did show some difficulties conceptualizing R as remaining constant and the relationship between B and L. Nonetheless, Jodi realized that an increase in C resulted in B and A increasing, while L decreased since the atmosphere absorbed more radiation. Notice that Jodi did not necessitate particular values to reason about the direction of change of the energy flows with respect to C. This suggests that she

imagine the magnitudes of energy flows increasing when C increased. I decided to examine whether Jodi could reason about the direction of change of the forcing, denoted by F, with respect to C. I asked Jodi to describe how F varied when C increased. Jodi reasoned about what flows were changing and what flows were constant when C changed while attending to the relationship $N = (S + A) - R$. Based on that dependency of change of flows with respect to C, Jodi reasoned about the direction of change of F with respect to C.

[F] would be positive because if A is increasing, but S and R are staying the same, then you have more of an input [*simultaneously points at S and A*] than an output [*points at R and pauses*] ... So, it would need to be positive. So N would positive

Once more, Jodi identified the direction of change of F with respect to C without the aid of particular values. Jodi imagined the magnitude of F increasing when C increased. This suggests that Jodi might have developed a sense of forcing by CO₂ as a function of C, denoted by F(C).

Since she developed a sense of F(C), I wanted to examine if she could construct a graph of such relationship. Thus, I re-directed her attention to the original task: sketching the graph of the forcing by CO₂ function F(C). Jodi first made a mark around the middle of the horizontal axis (C axis) and wrote '50' underneath it. Jodi chose C = 50 ppmv as an arbitrary atmospheric concentration of CO₂ for which there was no forcing (F = 0). She then made another mark at the bottom section of the vertical axis and wrote '-50' next to it. Finally, Jodi drew a straight line passing through these two marks (Figure 5.7). Jodi and I had the following discussion concerning her graph of F(C).

Me: What does this point represent [*point at (50, 0)*]?

Jodi: The amount of concentration in the atmosphere [*pauses*]. When the amount of concentration is what we would [*pauses*]. So, if we know the amount of

concentration in atmosphere is 50, then we would expect, if it stays 50, for F to be zero. But if it were to increase, F would also increase, and if it were to decrease, F would decrease

Me: How did you figure that the graph of $F(C)$ was a line?

Jodi: I guess I don't [pauses]. Wait, I might, I might ... So, is B the amount of concentration of CO_2 in the atmosphere?

Me: Not really. B represents the radiation coming from the surface and absorbed by the atmosphere

Jodi: OK [pauses]. I think that if there is more CO_2 in the atmosphere, then the temperature increases, and I am making the assumption that the temperature of the atmosphere [points at B] increases as a linear relationship with how much CO_2 is in the atmosphere. So, I made that assumption, and then that would mean that the amount of heat or temperature that is released back to the surface [points at A] is also linear with the amount of CO_2 in the atmosphere. So, since I assumed this relationship was linear [points at B] and this relationship was linear [points at A], the amount of energy that we would need to subtract from S plus A [to] equal R would be linear also (F such that $(S + A) - F = R$)

Jodi demonstrated reasoning about the direction of change of F with respect to C. She interpreted the C-intercept of the line as the value of C for which $F = 0$. Then, she stated that F decreases when C decreases from 50 ppmv and F increases when C increases from 50 ppmv. This suggests that Jodi coordinated variation in F with variation in C to develop a sense of $F(C)$ as an increasing function. Her choice of plotting $(0, -50)$ was intriguing. It seemed that Jodi arbitrarily chose -50 for the F-intercept in the same way she arbitrarily chose 50 for the C-intercept. This

suggests that Jodi was not aware that plotting $(0, -50)$ implied several assumptions regarding $F(C)$ (e.g., $F(0) = -50 \text{ J/m}^2/\text{s}$, rate of change is 1, or $F(C) = C - 50$). She did recognize assuming that $F(C)$ was a linear relationship. She first tried to reason about how F was changing in relation to C in order to justify (or discard) that $F(C)$ was linear. But, she could not coordinate changes in F with changes in C since she could not *see* C in the diagram of the energy budget (she, in fact, thought of B as being the concentration of CO_2). In other words, Jodi recognized she assumed $F(C)$ was linear, but she did not demonstrate having an argument for why such relationship should be linear. In an important note, Jodi also demonstrated conceptualizing the energy flows in terms of temperature rather than in terms of energy flux density. This suggests that Jodi had not yet conceptualized the relationship between these two quantities (energy flux density measures how rapidly the temperature is changing).

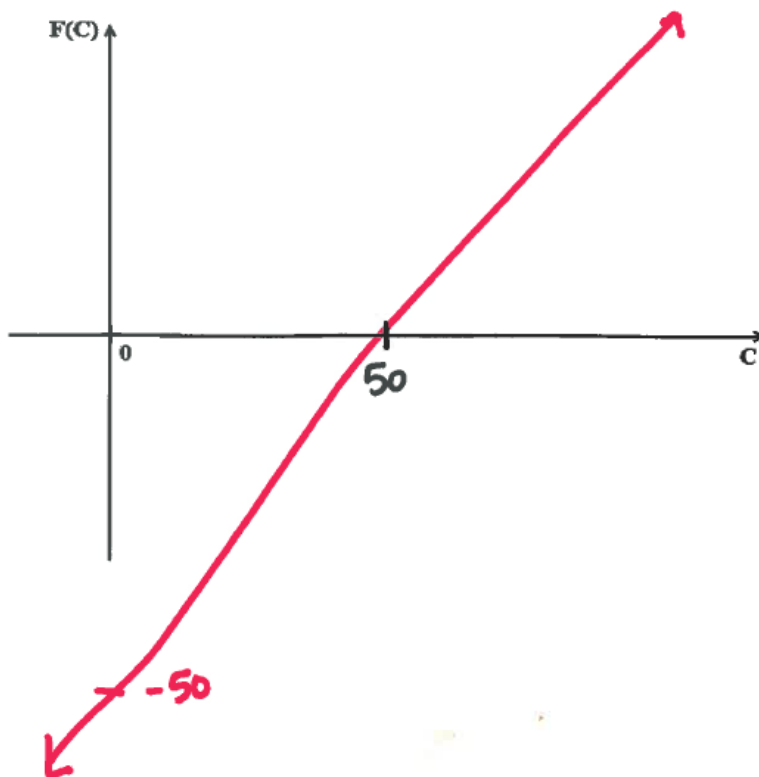


Figure 5.7. Jodi's first attempt to sketch the graph of $F(C)$

Conceptualizing the direction of change of F with respect to C represents an important and foundational step in modeling $F(C)$. However, it was also important to examine whether Jodi could hold that direction of change in mind while attending to how F was changing in relation to C . I decided to direct Jodi's attention to the theoretical minimum and maximum values of $F(C)$, which correspond to the cases of a completely transparent atmosphere and a completely opaque atmosphere, respectively (see Chapter 3, pp. 17-18). I decided to discuss such cases with Jodi in order to encourage her to think about the rate of change of $F(C)$. I specifically wanted Jodi to notice that the maximum value for $F(C)$, a completely opaque atmosphere, implies that $F(C)$ increases at decreasing rate since F approaches to an upper limit as C continues increasing. Thus, I presented Jodi with a diagram of the Earth's energy budget showing the values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, and $B = 300 \text{ J/m}^2/\text{s}$. Notice that these values implied that $A = 150 \text{ J/m}^2/\text{s}$, $L = 90 \text{ J/m}^2/\text{s}$, and no forcing exists since $F = (240 + 150) - 390 = 0$. I directed Jodi's attention to the first case: a completely transparent atmosphere. I told Jodi to imagine all CO_2 from the atmosphere was removed so that the concentration was zero. Also, I told her to assume that CO_2 is the only greenhouse gas in the atmosphere. This meant that the atmosphere would not absorb any of the radiation emitted by the surface. Next, I asked Jodi to determine the value of F corresponding to this situation. Jodi and I had the following discussion regarding this case.

Me: So, if we had zero concentration of CO_2 , can you tell me the value of F ?

Jodi: [*Thinking*]. It would need to be R minus S ($F = R - S$) because [*pauses*]. Yeah, I mean it would need to equal A , or what A would be if heat was being absorbed by the atmosphere [*points at B*]. But, so if nothing is being absorbed by the atmosphere, then we would not have A .

Me: OK, so A would be zero

Jodi: We would need N to be some amount that when we add it to S equals R to keep the equilibrium

Me: Could you calculate the value of F for me?

Jodi wrote $390 = 240 + N$ and solved for N, obtaining $N = 150$. This suggests Jodi did not see N as a difference between downward radiation and upward radiation. Jodi seemed to see N, and for now F, as an unknown value that is added to *either side* of the *equation* (downward radiation) = (upward radiation) so that the equality is maintain in case that either (downward radiation) < (upward radiation) or (downward radiation) > (upward radiation). This conception of N has the conceptual drawback that all values of N would be positive, regardless if they represent a positive or a negative forcing. I asked her to interpret $F = 150 \text{ J/m}^2/\text{s}$ in the context of the energy budget.

Me: So, you obtained a positive value there, right? Could you tell me what that value represent in terms of the budget gaining or losing energy?

Jodi: Gaining energy

Me: OK, so we are in the case of a completely transparent atmosphere, right? Do you think that the budget would be gaining or losing energy in that case?

Jodi: Yeah, we would be losing energy in that case ... Yeah, I didn't think I was doing my equation right when I was writing it

Jodi seemed to realize that her interpretations were contradictory. She, however, did not seem to identify why her result contradicted her understanding of the budget. This suggests that Jodi had not yet conceptualized the forcing as a difference of downward radiation and upward radiation, difference that distinguished between an excess of downward radiation (positive forcing) and an excess of upward radiation (negative forcing). I encouraged Jodi to use the relationship $F = (S +$

A) – R to think about the value of F. Although she had no difficulties determining that $F = (240 + 0) - 390 = -150$, it was unclear whether she conceptualized F as the aforementioned difference.

Next, I directed Jodi's attention to the second case: a completely opaque atmosphere. Although Jodi did not show difficulties identifying what energy flows changed when C increased, Jodi did not use the relationship $N = (S + A) - R$ to determine the value of the forcing. She, once more, treated F as an unknown value that is added to *either side* of the *equation* (downward radiation) = (upward radiation) so that the equality is maintain.

Me: Let's discuss the opposite case. Let's imagine that we keep filling the atmosphere with enough CO₂ until it can absorb practically all surface radiation. What is the value of the forcing?

Jodi: [*Thinking*]. A would be 390 over 2, which is going to be [*uses calculator and writes 195*]. So, A is 195, and we would need, we would want S to equal A [*writes S = A*]. But, since 240 is greater than 195, we would need to add N [*writes 240 = 195 + N*]. And, that would make $N = 45$ [*writes N = 45*]. In the case we add more CO₂ to the atmosphere and L no longer is emitted

Notice how Jodi wrote the equation $240 = 195 + N$ and solved for N, instead of using the formula $N = (S + A) - R$. Although she obtained the correct value $N = 45$, she did not demonstrate conceiving F as a difference measuring the excess of downward radiation or the excess of upward radiation. Jodi's conception of F may not involve imagining this quantity as varying as C varies since she treated it as an unknown, but fixed value. Jodi conceptualized that F was larger if C was larger, but she may not imagine the change as continuously (and *co-variationally*) occurring.

To further examine how Jodi was thinking about the co-variation of F and C , I asked her to draw a new version of the graph of $F(C)$ based on what she learned about the minimum ($F = -150 \text{ J/m}^2/\text{s}$) and maximum ($F = 45 \text{ J/m}^2/\text{s}$) values of $F(C)$. Jodi first made a mark on the negative section of the vertical axis and wrote ‘ -150 ’ next to it. She then made a mark on the positive section of the vertical axis and wrote ‘ 45 ’ next to it. Jodi was not sure what to do next since she did not know the value of C such that $F(C) = 45$.

Jodi: We added more CO_2 , so the concentration is positive [*moves her index finger to the right over the horizontal axis*]. Should I just make a point somewhere?

Me: What do you think? Do you have an idea of how the graph should look like?

Jodi: Well, it should be positive like this [*moves marker indicating a line with positive slope*]. I just don’t know where the x -intercept is, or where how much CO_2 we would need to add

Me: We don’t really know either of those values. Would you be able to draw a graph?

Jodi decided to draw a line passing through $(0, -150)$; her line contained points (C, F) such that $C < 0$ and $F < -150$ and such that $F > 45$ (Figure 5.8). Interestingly, Jodi scratched the section of the line containing points such that $C < 0$ and $F < -150$. This suggests that she did not think that such values for C and F had meaning in the context of the situation. She, however, did not do the same for the section of the line containing points such that $F > 45 \text{ J/m}^2/\text{s}$. This suggests that Jodi might not have yet conceptualized $45 \text{ J/m}^2/\text{s}$ as the maximum value for the forcing. I decided to examine whether Jodi could conceptualize $45 \text{ J/m}^2/\text{s}$ as the upper limit of $F(C)$. I was interested in encouraging Jodi to think about how F was varying in relation to C (F must increase at a decreasing rate as C increases so that $F(C)$ would have an upper limit at 45).

Me: Can F be larger than 45 ?

Jodi: No! Because the reason why we made $F = 45$ was because we didn't have L. So, we had to figure out what number we could add to A, to equal S, to create an equilibrium ($S = A + F$) ... We already added enough concentration so that A is not going to increase and S is not going to increase ... So, we won't have to add anymore to A to create an equilibrium

Me: Can I draw something on your graph [*draw a dashed horizontal line at $F = 45$ (Figure 5.8)*]? Do you think that your graph crosses that line?

Jodi: No [*scratches the line's section containing points such that $F > 45$*]

Me: So, how does the graph of $F(C)$ look like?

Jodi: We have an end point here [*marks the point $(0, -150)$*] and [*re-marks the line*] an end point here [*marks the intersection between her line and the dashed line*]

Me: So, do you think the graph is a line that just stops there [*point at the intersection between her line and the dashed line*]?

Jodi: Uh-huh

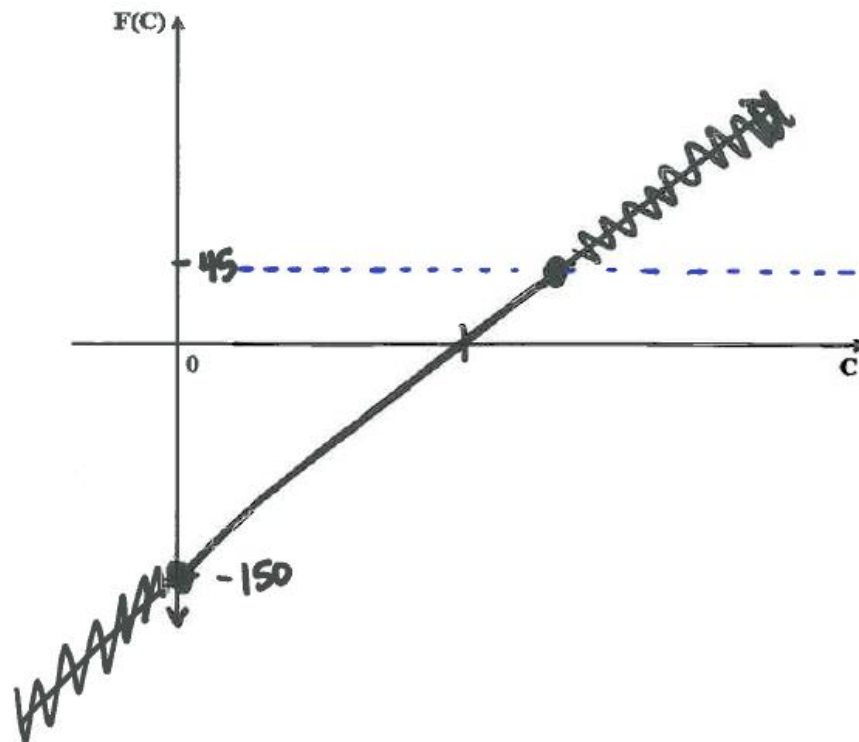


Figure 5.8. Jodi's second attempt to sketch the graph of $F(C)$

Jodi's response suggests that she conceptualized F as a quantity that allows for an equilibrium between downward radiation and upward radiation. This conception appears different from conceptualizing F as a measure of energy imbalance caused by changes in concentration of CO_2 . Jodi's conception of F may explain why she thought her line just stopped at a point $(C_{\text{opa}}, 45)$, where $C = C_{\text{opa}}$ represents a concentration of CO_2 that creates a completely opaque atmosphere. For Jodi, F can range from -150 to 45 in the equation $S = A + F$ for $0 \leq A \leq 195$. This conception may not involve thinking of F varying in relation to C . Unfortunately, I did not ask Jodi whether she thought of C as stop varying at $C = C_{\text{opa}}$. Her response shed some light about it though; Jodi seemed to believe that there is a finite value of C for which the atmosphere has turned completely opaque to surface radiation (e.g., "We already added enough concentration so that A is not going to increase and S is not going to increase"). From this point of view, her graph suggests that Jodi might have seen $C = C_{\text{opa}}$ as the maximum amount of CO_2

that the atmosphere could hold. Once this limit was reached, C could not continue varying since no more CO_2 could be put into the atmosphere, thus the line should stop at $(C_{\text{opa}}, 45)$. Jodi's line also suggests that she remained reasoning about co-variation in terms of the direction of change level; that is, Jodi's line indicated what she knew: F increases as C increases. She did not demonstrate thinking about how F was varying in relation to C in order to construct her final graph of $F(C)$.

The Planetary Energy Imbalance Function $N(t)$

After Jodi read S2T2, I asked her to describe how the flows of energy B , A , and R were varying after the positive forcing occurred. Jodi demonstrated great difficulties finding a way to approach my question. I decided to direct her attention toward finding the value of $N(0)$. Jodi continued to struggle greatly and did not seem to know how to approach this question either. The following exchange illustrates Jodi's struggle

Me: What is the value of N at the moment the instantaneous, positive forcing occurs?

Jodi: Positive?

Me: OK, so how will N vary with respect to time? How will N vary after the positive forcing has occurred? For instance, do you think N will increase or decrease?

Jodi: I guess it would increase? [*Pauses*] but, I don't see an argument for why it wouldn't stay the same

Me: OK. Remember that N measures the magnitude of the energy imbalance in the budget. Do you think that the imbalance would keep getting larger over time?

Jodi: [*Long pause*] I think that the value of N will stay the same after we have [*pauses*]. If we increase [the concentration of CO_2] by just some number, then we would

increase N by some number. And, it wouldn't increase or decrease if CO_2 is kept stable

The exchange above suggests that Jodi was still developing an understanding of the planetary energy imbalance as a function of time, denoted by $N(t)$. Jodi stated that N would not change if C remained constant (e.g., “[N] wouldn't increase or decrease if CO_2 is kept stable”). This suggests that Jodi may have not yet distinguished N from the forcing by CO_2 F . These two quantities are strongly related, but they are different. F measures a magnitude of energy imbalance caused by changes in C , while N measures the magnitude of the total energy imbalance at any given time. While working on S2T1 (previous task), she saw the value of N being equal to the value of F . This might explain why she did not think N varied over time if C remained constant. To help Jodi develop a sense of N as a function of time, I presented her with a diagram of the Earth's energy budget having the values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, $B = 300 \text{ J/m}^2/\text{s}$, and $A = 150 \text{ J/m}^2/\text{s}$ for $t = 0$. These values defined an energy budget in equilibrium since $N = (S + A) - R = (240 + 150) - 390 = 0$. I next told Jodi that C is instantaneously increased at $t = 0$, creating a positive forcing. I changed the values of the flows B , L , and A to the new values $B = 310 \text{ J/m}^2/\text{s}$, $L = 80 \text{ J/m}^2/\text{s}$, and $A = 155 \text{ J/m}^2/\text{s}$ for $t = 0$ (Figure 5.9). These values defined a positive forcing by CO_2 of magnitude $5 \text{ J/m}^2/\text{s}$ since $F = (S + A) - R = (240 + 155) - 390 = 5$. Then, I help Jodi use these values to determine two new successive values for R , B , and A , which corresponded to two additional, equally-spaced values of time (for convenience, let's denoted them as $t = 0 + h = h$ and $t = 2h$ for some positive h). I told Jodi to base the calculations on the assumption that $R([i + 1]h) = S + A(ih)$ for $i = 0, 1, 2, \dots$. Following my directions, Jodi obtained the values $R(h) = 395 \text{ J/m}^2/\text{s}$; $B(h) = 313 \text{ J/m}^2/\text{s}$, and $A(h) = 157 \text{ J/m}^2/\text{s}$, and $R(2h) = 397 \text{ J/m}^2/\text{s}$; $B(2h) =$

$315 \text{ J/m}^2/\text{s}$, and $A(2h) = 158 \text{ J/m}^2/\text{s}$. In summary, Jodi had three successive values for the flows R, B, and A (Figure 5.9), values corresponding to the time values $t = 0$, $t = h$, and $t = 2h$.

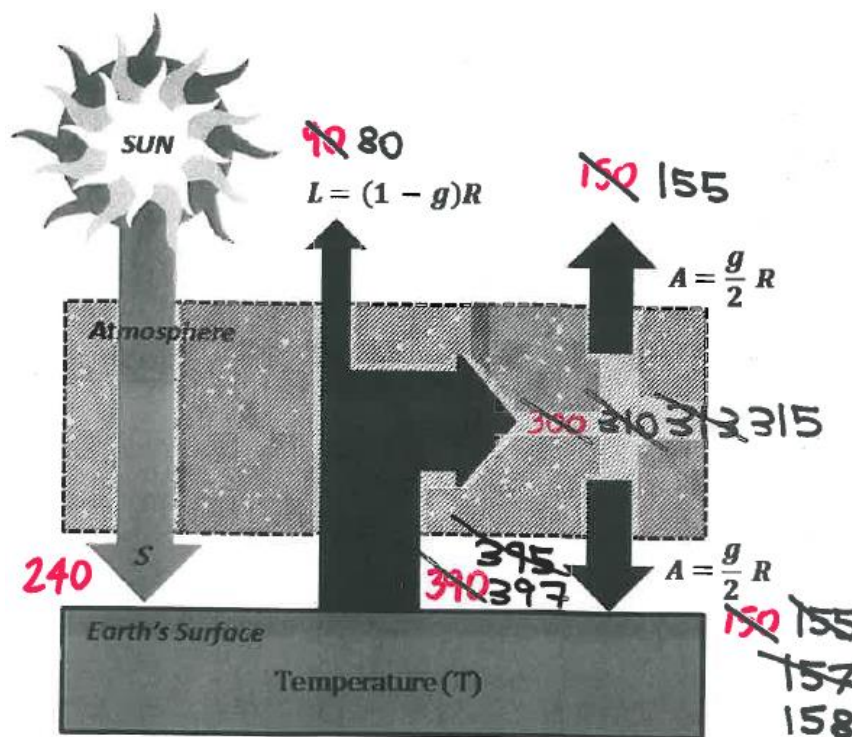


Figure 5.9. Jodi had three successive values of R, B, and A for times $t = 0$, $t = h$, and $t = 2h$.

After determining new values for the energy flows, I asked Jodi whether the value of N would increase or decrease over time. Jodi maintained that “N is increasing as time goes on.” Since Jodi had three values for the flows R, B, and A, I asked her to determine the corresponding values of $N(t)$ for $t = 0$, $t = h$, and $t = 2h$ by utilizing $N(ih) = (S + A(ih)) - R(ih)$. Jodi determined the values $N(0) = 5 \text{ J/m}^2/\text{s}$, $N(h) = 2 \text{ J/m}^2/\text{s}$, and $N(2h) = 1 \text{ J/m}^2/\text{s}$. Jodi did not seem to expect N to be decreasing over time. Thus, I decided to direct Jodi’s attention toward interpreting the results in the context of the energy budget. The following exchange illustrates how Jodi interpreted it

Me: Do your values make sense?

Jodi: No, I thought that N would be larger

Me: Why do you think N decreased in magnitude? What could that mean in terms of the energy budget?

Jodi: That we are going back to an equilibrium, or we are not as far from equilibrium as we were

Me: Do you remember they said something similar in the video you watched?

Jodi: Yeah, once the temperature of the Earth heats, then I guess like we release more [*points at R*]. And then, we go back to an equilibrium

The exchange above suggests that Jodi thought of a decreasing $N(t)$ as indicating the energy budget was *going back* to radiative equilibrium. Jodi demonstrated developing an understanding of $N(t)$ as a measure of the magnitude of energy imbalance in the budget. This understanding seemed to emerge from reasoning about the situation and helped her make sense of why $N(t)$ was decreasing over time. Her response, however, also suggests that Jodi conceptualized a decreasing $N(t)$ as indicating that the budget was actually returning to the *original radiative equilibrium*, rather than reaching a *new radiative equilibrium*. Jodi's last response (i.e., "once the temperature of the Earth heats, then I guess like we release more [*points at R*]. And then, we go back to an equilibrium") suggests that she may be thinking of the planet as cooling down after a positive forcing occurred. Thus, for Jodi, the planet's surface temperature was actually returning to the original equilibrium. I decided to examine this issue later in the interview (see The Surface Temperature Function $T(t)$ section) since Jodi had not yet constructed a graph of $N(t)$.

Since Jodi appeared to develop a sense of N as a function of time, I re-directed her attention toward the original task of drawing the graph of $N(t)$. Jodi made three marks on the vertical axis and wrote the numbers 5, 2, and 1 next to them. She then made three equally-spaced marks on the horizontal axis and wrote the numbers 1, 2, and 3 beneath them. Using these marks,

Jodi plotted three points: $(0, 5)$, $(1, 2)$, and $(2, 1)$; as she was plotting the points, Jodi said “once we increase the CO_2 , N spikes to 5. And then, after one revolution, and I am going to call that one unit of time, N went back to 2.” For Jodi, a *revolution* seemed to be a discrete unit of time representing a complete transfer of heat between sun, surface, and atmosphere. After she plotted the points, Jodi drew a decreasing, concave-up curve incident to all three points and said “we are decreasing at a decreasing rate” (Figure 5.10). Jodi did not continue the curve beyond $(2, 1)$. Jodi did not demonstrate reasoning about co-variation beyond the direction of change while constructing the graph of $N(t)$. Notice that Jodi coordinated values of $N(t)$ with discrete values of time (revolutions), which does not require reasoning beyond direction of change. The *curvature* of her graph did not seem to be a deduction based on reasoning about the situation, but rather a result of connecting the points $(0, 5)$, $(1, 2)$, and $(2, 1)$. Also, it was unclear whether Jodi conceived of variation within a revolution. It seemed that she coordinate a value of $N(t)$ with an entire revolution so that $N(t)$ varied *between* revolutions but might not vary *within* a given revolution. When I asked Jodi about why she did not continue her curve beyond $(2, 1)$, she replied

Maybe we can go to three. Maybe at three, N would be back to zero, but we can't go pass N equal zero [*points at the negative section of the vertical axis*] because, once we added CO_2 and the temperature increased, you would go back to equilibrium and nothing would change from there

Next, Jodi continued her curve to include the point $(3, 0)$. Her response suggests that Jodi conceptualized $N = 0$ as indicating the budget reaching radiative equilibrium, where no change occur. This represents a central understanding about the meaning of the planetary energy imbalance. Her response also suggests that Jodi did not conceived of the curve as having an

asymptote at $N = 0$. Jodi seemed to believe that there is a finite number of revolutions r for which $N(r) = 0 \text{ J/m}^2/\text{s}$ (for her, r might be three). This might explain why Jodi did not use her understanding of equilibrium ($N(t)$ cannot take a value less than zero) to draw her curve.

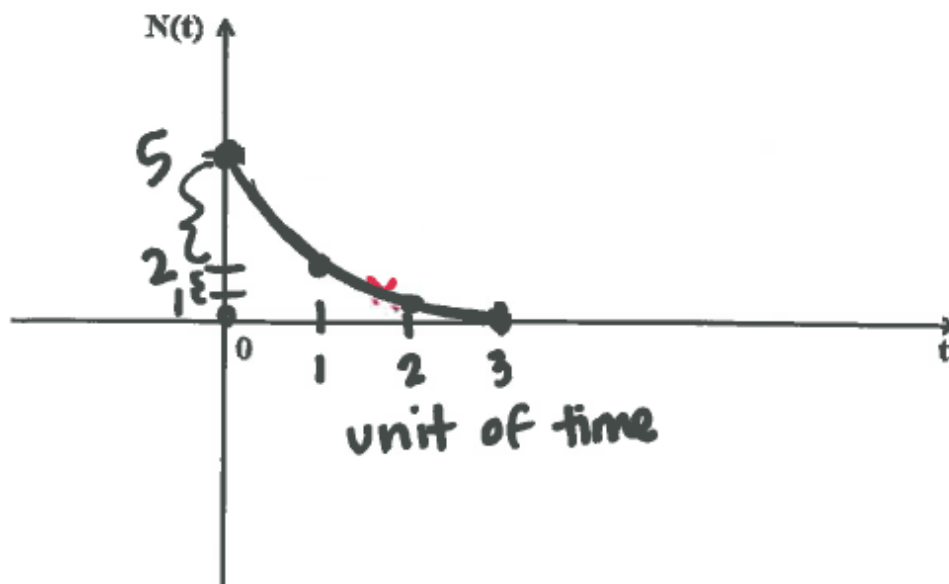


Figure 5.10. Jodi's final version of the graph of $N(t)$

Jodi also mentioned the rate of change of $N(t)$. She stated that $N(t)$ was decreasing at a decreasing rate *after* she completed her graph of $N(t)$. Thus, I decided to further examine how Jodi was thinking about the rate of change of $N(t)$. I asked Jodi to elaborate on how she realized that $N(t)$ was decreasing at a decreasing rate. She provided the following argument

Each time we are increasing t , we are decreasing N by smaller amounts. Like here, we decrease 3 [points at the interval $[2, 5]$ on the vertical axis (see Figure 5.10)], and then we decrease 1 [points at the interval $[1, 2]$ on the vertical axis]. So, it would make [pauses]. I am trying to make sure I know what the graph looks like. OK, when you have a graph and you do like this [draws a concave-up decreasing curve (Figure 5.11)]. This is one [makes a mark on the horizontal axis and writes 1] and this is two [makes a mark on the horizontal axis and writes 2]. You would be decreasing by smaller amounts each

time. The same thing what we are doing here [*draws two curly brackets on the vertical axis: one between 5 and 2 and one between 2 and 1 (see Figure 5.10)*], so I want to say that the graph looks like this [*points at Figure 5.11*]: decreasing at a decreasing rate

The excerpt above suggests that Jodi was reasoning in terms of amounts of change to explain why $N(t)$ was decreasing at a decreasing rate. Rather than the rate of change of $N(t)$, her object of reasoning was the amounts of change in N coordinated with changes in time. It is important to highlight that Jodi's justification regarding the rate of change of $N(t)$ was based on the shape of her curve. She did not demonstrate reasoning about the rate of change of $N(t)$ based on the situation of the energy budget. Jodi even drew a new curve to reason about why $N(t)$ was decreasing at a decreasing rate. This suggests that Jodi reasoned about co-variation in terms of direction of change when thinking of the situation, while she reasoned in terms of amounts of change when thinking of her already finished graph of $N(t)$.

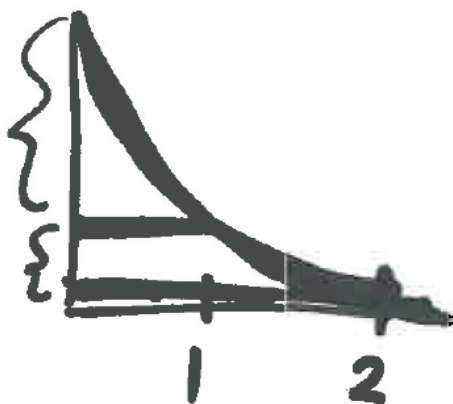


Figure 5.11. Jodi drew a new curve to confirm that $N(t)$ was decreasing at a decreasing rate.

I decided to investigate if Jodi could interpret her graph of $N(t)$ in relation to the Earth's energy budget. Jodi demonstrated difficulties interpreting the graph of $N(t)$ in this context. It seemed that Jodi was holding in mind two conflicting notions regarding the planetary energy imbalance $N(t)$. The following exchange illustrates Jodi's conflict.

- Me: According to your graph, is the planet's surface gaining or losing energy?
- Jodi: Losing?
- Me: How do you figure that out?
- Jodi: Because the line is going in the negative direction [*moves marker indicating a decreasing line*], and we know that as N decreases, the surface is losing energy
- Me: Your graph does show a decreasing $N(t)$. But, your graph also shows a positive $N(t)$. What does a positive $N(t)$ mean in terms of downward and upward radiation?
- Jodi: It says that the energy is greater than equilibrium [*pauses*]. But, that doesn't imply that the surface is losing or gaining energy because like here [*points at her curve for $N(t)$*] N is greater than zero. So, we know that the Earth is absorbing more energy, so it is like [*pauses*]. OK, let me re-think this. For S plus A to equal R [*writes $(S + A) = R$*], this would be an equilibrium ... like in this situation (positive forcing) ... that means that this [*points at $(S + A)$*], this side is going to be greater? [*pauses*] Oh! OK, let's see about input, output. So, when the input is greater than the output [*points at $(S + A)$*], then we need to decrease N [*writes $-N$ next to $(S + A)$*] so we can get back to equilibrium. So, when we are subtracting N , we know N is positive like in this case [*points at her curve for $N(t)$*], but we are also trying to subtract it [*pauses*]. That is not really right, is it?
- Me: OK, so when N is positive, what do you think is happening? I mean, is the budget losing or gaining energy, or the sign of N has nothing to do with that?
- Jodi: [*Long silence*]

The source of Jodi's difficulties seemed to be a conflict between two notions regarding $N(t)$. On the one hand, Jodi seemed to conceptualize $N(t)$ as a magnitude of energy rather than a magnitude of energy flux density. She appeared to think of $N(t)$ as the magnitude of energy (or heat) contained in the planet's surface. For her, a decreasing $N(t)$ showed that the planet's surface must be losing energy (e.g., "we know that as N decreases, the surface is losing energy"). On the other hand, Jodi also conceptualized $N(t)$ as the difference between downward radiation and upward radiation. She stated that a positive $N(t)$ indicated the downward radiation exceeded in magnitude the upward radiation. She then recognized that as indicating that the planet's surface was gaining energy (e.g., "we know that the Earth is absorbing more energy, so it is like [pauses]"). Jodi immediately seemed to realize that such conclusion was contradicting her first claim stating that the surface was losing energy.

Since Jodi did not clearly state whether the surface was gaining or losing energy, I decided to draw her attention toward the meaning of $N(t)$ as a difference between downward radiation and upward radiation. I wanted to examine if Jodi could think about $N(t)$ in the context of the diagram of the energy budget. I showed her the diagram and then we had the following exchange.

Me: So, if the downward radiation exceeds the upward radiation, then the surface is experiencing an increase of energy, it is gaining energy

Jodi: OK! I guess I was thinking about change in N . Since N is decreasing, we are going back to an equilibrium at some point

Me: That is true. The budget is reaching a new equilibrium because N , the measure of the energy imbalance, is decreasing back to zero. However, the question remains, is the surface gaining or losing energy as N decreases toward zero?

Jodi: [Long pause] this makes me want to say gaining [points at the values of R and A in Figure 5.9]

Me: OK. So, why do you think the surface is losing energy?

Jodi: Just intuition that, between each cycle, we would need to be losing energy so that we can go back to equilibrium

This exchange suggests that Jodi's conflict was rooted in her understanding of the concept of radiative equilibrium. To be clear, having $N(t)$ approaching to zero does show that the energy budget is reaching a radiative equilibrium. Jodi, however, thought that the surface was losing energy because that was the only way to go back to the *original state* of equilibrium. In other words, $N(t)$ approaching zero meant that the energy budget was going back to the original state of equilibrium rather than reaching a *new* radiative equilibrium. Jodi's response suggests that she was more inclined to think of $N(t)$ as the magnitude of energy contained in the planet's surface since this idea was consistent with her understanding of radiative equilibrium. Another possibility also involves her graph of $N(t)$. I previously asked Jodi to explain how she concluded that the surface was losing energy. Jodi initial argument involved the decreasing aspect of her curve (e.g., "Because the line is going in the negative direction [*moves marker indicating a decreasing line*]"). This might have led her to think that the surface energy was decreasing since decreasing curves show decreasing quantities. In either case, Jodi did not demonstrate conceptualizing $N(t)$ as a rate, particularly as the rate at which the surface's energy was changing.

The Surface Temperature Function $T(t)$

I was interested in exploring how Jodi's interpretation of a decreasing $N(t)$ (i.e. as indication of both energy loss and returning to the original equilibrium) would influence her

understanding of the average surface temperature function, denoted by $T(t)$. I directed Jodi's attention to the second part of S2T2: drawing the graph of $T(t)$. Jodi initially attended to the variation of the energy flows R and A in order to make sense of $T(t)$. Intriguingly, Jodi concluded that $T(t)$ was fluctuating between two values. The way Jodi arrived to that conclusion is illustrated in the following exchange.

Jodi: Well, I don't expect the temperature of the Earth to be zero. So, I wouldn't think it would start here [*points at the origin*] ... Let's say that the temperature of the Earth is like here [*points around the middle of the vertical axis*]. So, the temperature of the Earth at equilibrium is there [*makes a mark on the vertical axis*]

Me: So, is that the initial surface temperature?

Jodi: Uh-huh! I am trying to look at the differences. So here, the change was five (A changing from $150 \text{ J/m}^2/\text{s}$ to $155 \text{ J/m}^2/\text{s}$). Then, the change was two (A changing from $155 \text{ J/m}^2/\text{s}$ to $157 \text{ J/m}^2/\text{s}$) [*Long pause*] is it not changing?

Me: What isn't changing?

Jodi: OK! So, the flow of energy increased by five (A changing from $150 \text{ J/m}^2/\text{s}$ to $155 \text{ J/m}^2/\text{s}$). Then, it decreased by five (R changing from $390 \text{ J/m}^2/\text{s}$ to $395 \text{ J/m}^2/\text{s}$). Then, it increased by two, and then it decreased by two [*pauses*]. So, it is almost as if there was no change in temperature because I associate energy as kind of having a relationship with temperature. So, if the energy increases, then the temperature increases. But, in this scenario [*circles her hand over the diagram (see Figure 5.9)*], an equal change in energy was an equal change in output [*simultaneously points at A and R*]

Me: Would you be able to draw the graph of the surface temperature now?

Jodi: *[Makes three equally-spaced marks on the horizontal axis and labels them as 1, 2, and 3]* I want to say that it stays the same, but maybe it like *[pauses]* OK, *[a]* cycle started here *[points at B]*, and here the Earth's temperature would've been something ... So, when the cycle started, there was an input, and then it got released *[circles her hand clockwise over the diagram starting at B, moving to the right toward A, continuing to R, and returning to B]*. Then, another cycle starts: input of energy, release of energy *[circles her hand over the diagram]*. So, it would almost be like *[draws periodic curve 1 (Figure 5.12)]*

Me: Why does your graph look like that?

Jodi: The Earth would heat up because it got more energy *[points at A]*, but then it would release it within the same cycle *[points at R]*. I mean, if we started the cycle here *[points at R]*, then it would be like zero cycle. *[Circles her hand clockwise starting at R, moving to B, continuing to A, and returning to R]* at the end of cycle 1, we would have an increased temperature. But then, we would go back and it would almost be something that looks like this *[draws periodic curve 2 (Figure 5.12)]*, where at the beginning of a cycle, we will be back to a normal temperature, equilibrium temperature

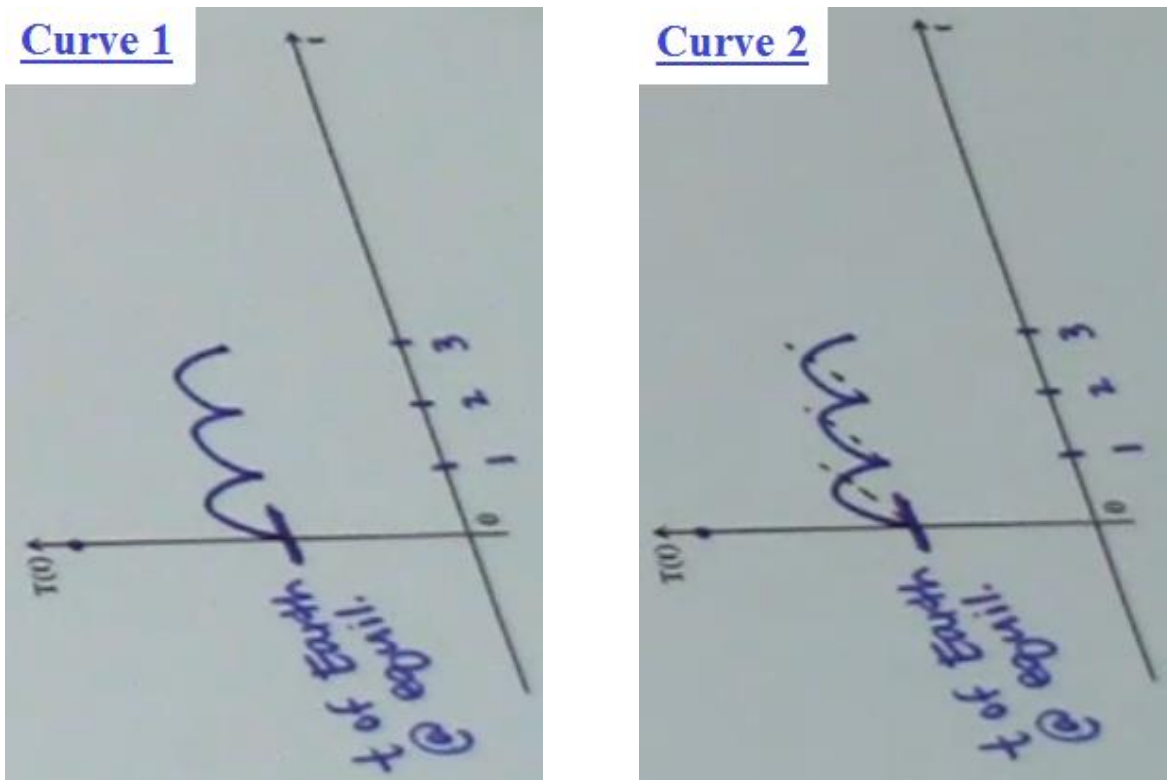


Figure 5.12. From left to right, Jodi's first attempt to draw the graph of $T(t)$ (Curve 1) and Jodi's second attempt to draw the graph of $T(t)$ (Curve 2).

Jodi's construction of the graph of $T(t)$ unveiled two interesting aspects of her reasoning. First, Jodi appeared to reason about time variation in a discrete way. She conceptualized time as increasing in discrete units that she called *cycles* or *revolutions*. Jodi did not conceive of time variation within a cycle and conceptualized R and A as having a value for each entire cycle. For instance, Jodi coordinated the values $R = 390 \text{ J/m}^2/\text{s}$ and $A = 155 \text{ J/m}^2/\text{s}$ with cycle 0 and $R = 395 \text{ J/m}^2/\text{s}$ and $A = 157 \text{ J/m}^2/\text{s}$ with cycle 1. Notice how particular values of R and A are associated with entire intervals of time, which suggests that Jodi did not conceive R and A varying *within* an interval but taking different values *between* intervals. This might explain why Jodi thought of the temperature as remaining constant (e.g., "it is almost as if there was no change in temperature"). Since the downward radiation ($S + A$) had the same magnitude as the

upward radiation R for any particular cycle, the magnitude of energy, which she associated with temperature, was not changing. Second, Jodi did not demonstrate reasoning about co-variation beyond the direction of change. She exclusively attended to the direction in which $T(t)$ was changing for successive values of time (i.e., that is, as time increases, $T(t)$ increases and decreases in magnitude). Although Jodi coordinated amounts of change in A and R with changes in time (e.g., by noticing that A “increased by five ... then, it increased by two”), she did not attend to how these changes were increments of decreasing magnitude. Instead, Jodi compared amounts of change in A (downward radiation) with amounts of change in R (upward radiation) between successive cycles. Since Jodi assumed that $R([i + 1]h) = S + A(ih)$ for $i = 0, 1, 2, \dots$, she noticed that $\Delta A = A(ih) - A([i - 1]h)$ was equal in magnitude to $\Delta R = R([i + 1]h) - R(ih)$. From her perspective, this meant that the surface’s energy was fluctuating between cycles, which in turn meant that $T(t)$ was fluctuating between cycles. Notice that Jodi did not justify her choice of curve type to represent that fluctuation.

After Jodi drew the two periodic curves in Figure 5.12, she decided to attend to the amounts of change in B between cycles. Jodi noticed that the amounts by which B was increasing were decreasing in magnitude. Therefore, she concluded that her graph of temperature needed to be corrected.

The amount of energy was decreasing, like each time. Because in here, $[B]$ increased 10, then 3, and then 2 (B taking the values 300, 310, 313, and 315 (see Figure 5.9)). So, maybe these should be like smaller [*points at periodic curve 1 (see Figure 5.12)*]. You know, like they wouldn’t be the same size. Because, the Earth’s temperature wouldn’t increase that much [*draws a quasiperiodic curve approaching to the initial equilibrium*

temperature (Figure 5.13)] because the increases in temperature are smaller during the cycles.

Jodi coordinated amounts of change in B with cycles in order to reason about how much $T(t)$ changed between cycles. This may suggest that Jodi reasoned about co-variation in terms of amounts of change (coordinating amounts of change in B with different cycles). Jodi reasoned about amounts of change in a discrete manner since the change occurred between cycles. Also, Jodi did not demonstrate conceptualizing the energy flows as magnitude of energy flux density, or rates of radiation. She associated an *amount of change* in B with a *magnitude* of $T(t)$, which suggests that she saw increments in B as a measure of temperature. Additionally, notice how Jodi concluded that $T(t)$ should be *decreasing* for each cycle so $T(t)$ was to return to the original equilibrium temperature. This further supports the hypothesis that Jodi conceptualized radiative equilibrium as unique; that is, the budget goes back to equilibrium, rather than reaching a new one. Thus, when the budget returned to equilibrium, $T(t)$ must also return to the original temperature.

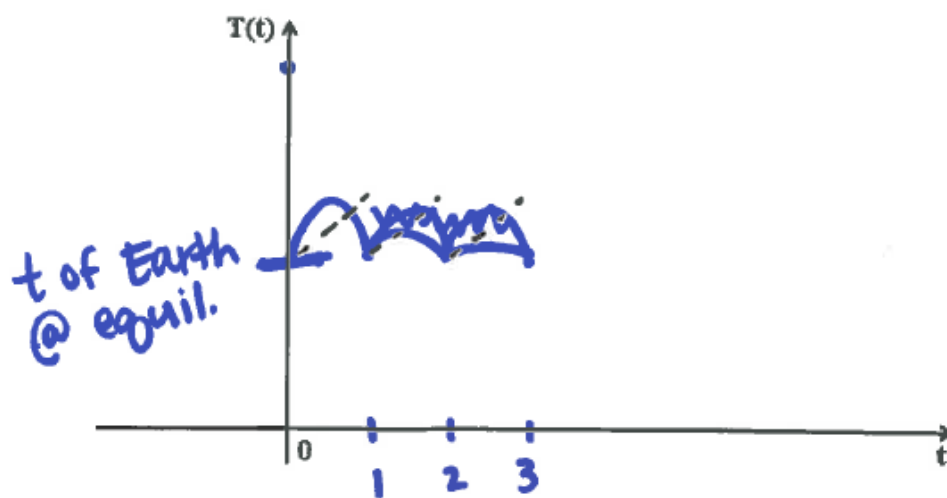


Figure 5.13. Jodi's finale version of the graph of $T(t)$

The Case of Kris

Synthesis

Kris's responses to the tasks suggest that she consistently reasoned about co-variation at *the amounts of change level (L3)*. This level involves coordinating amounts of change in the dependent variable with change in the independent variable. For S2T1, Kris did not need particular values to determine what energy flows were changing with changes in the atmospheric CO₂ concentration C (dependency of change level or L1). She also did not need particular values to imagine the direction in which the forcing F was changing as C was increasing (direction of change level or L2). This suggests that Kris had no difficulties reasoning about co-variation at the first two levels. To construct the graph of F(C), Kris tried to relate amounts of change in F with change in C in order to reason about how *rapidly* F was increasing with respect to C. This way of thinking is consistent with reasoning at L3. She, however, could not do that because she needed particular values of F and C in order to determine amounts of change in F and C, and such information was not available for S2T1. This suggests that Kris's L3 covariational reasoning was dependent on particular values of F and C. When working with the completely transparent atmosphere and the completely opaque atmosphere, Kris did not demonstrate difficulties determining the minimum (transparent) and maximum (opaque) values of forcing. She, however, could not reason about how rapidly F was increasing with respect to C, and she ended up drawing a line segment with positive slope for $0 \leq C \leq C_{\text{opa}}$, where C_{opa} represented the concentration of CO₂ for a completely opaque atmosphere. This suggests that she did not conceptualize C as continued to increase after taking the value $C = C_{\text{opa}}$. As a result, she did not demonstrate conceiving *asymptotic variation* when imagining F(C). Conceiving asymptotic variation for F(C) requires imagining C to continue to increase without limit as F approaches to

an upper limit value. It is possible that Kris conceptualized $C = C_{opa}$ as representing an atmosphere completely formed by CO_2 , which would explain why C would not continue to increase.

Kris's responses to the first part of S2T2 suggest that she reasoned about the co-variation represented by the planetary energy imbalance function $N(t)$ at the *amounts of change level* (L3). She did not need particular values to determine what energy flows were changing over time t (dependency of change level or L1). She, however, did need particular values to imagine the direction in which the energy imbalance N was changing as t was increasing (direction of change level or L2). She drew a concave-up, decreasing curve to represent $N(t)$ decreasing from an initial value $N(0) = F > 0$ down to zero. Kris constructed the graph by attending to the amounts by which the energy flow B was changing for equal changes in time t . This way of thinking is consistent with L3 covariational reasoning. Additionally, Kris interpreted the decreasing magnitude of $N(t)$ as indicating that the energy budget was reaching a *new radiative equilibrium*. In other words, after a positive forcing created an energy imbalance, the energy budget was evolving so that the planet's surface radiation could equal the solar radiation, creating a new equilibrium. Through this realization, Kris demonstrated understanding that the temperature of the planet's surface must increase for there be a new radiative equilibrium. It was unclear, however, whether Kris conceptualized $N(t)$ as a magnitude of energy flux density indicating how rapidly the surface temperature was changing. Finally, Kris did not demonstrate conceiving *asymptotic variation* when imagining $N(t)$. Although her curve appeared to have an asymptote, she stated that the budget must reach radiative equilibrium eventually so that $N(t)$ must be zero eventually. This suggests that she did not conceptualize t as continued to increase without limit

as N approaches to zero. Instead, Kris thought of the energy budget as reaching equilibrium for a finite time, which would explain why t would not continue to increase.

Kris's responses to the second part of S2T2 suggest that she reasoned about the co-variation represented by the mean surface temperature function $T(t)$ at *the amounts of change level* (L3). She attended to the variation of the energy flow A over time in order to establish the direction in which the surface temperature T was changing as the time t increases (the direction of change level or L2). Kris attended to the amounts by which the energy flow A was changing for equal changes in time t in order to construct the graph of $T(t)$. She noticed that A was increasing by smaller and smaller amounts. For Kris, the planet's surface was absorbing an increasing magnitude of energy, magnitude that was increasing by smaller and smaller amounts. She interpreted that as $T(t)$ increasing by smaller and smaller amounts. Thus, she constructed a concave-down, increasing curve to represent $T(t)$ increasing from an initial value $T(0) = T_0 > 0$ and approaching to a new (higher) equilibrium temperature. This also suggests that Kris was reasoning about co-variation at L3 when imagining $T(t)$. Another important realization involved conceiving the energy budget reaching a *new radiative equilibrium* after a positive forcing occurred. Kris demonstrated understanding that the surface temperature must increase for the budget to reach a new radiative equilibrium. She represented that in her graph of $T(t)$ by drawing a horizontal asymptote at $T = T_{eq}$, where $T_{eq} > T(0)$ represented the new equilibrium temperature. What follows is a description of Kris' responses to the tasks and my interpretation of such responses.

The Forcing by CO₂ Function F(C)

After Kris read S2T1, I asked her to describe how the energy flows B and A change when the concentration of CO₂, denoted by C, increases. Kris and I had the following discussion regarding changes in energy flows in the Earth's energy budget.

Me: How do you think B would change if C increases?

Kris: If you increase the concentration of CO₂, B is going to increase because there are more CO₂ molecules to absorb the heat, so less it is going to be leaked [*points at L*]

Me: So, if C increases, then B increases

Kris: Yeah

Me: OK, so how does F change when C increases?

Kris: [*Writes $N = (S + A) - R$*] ok, so right now R stays the same [*pauses*]

Me: Remember, S also stays the same because it is the solar constant. These flows will change though [*sequentially point at B, L, and A*]

Kris: OK, so S stays the same [*pauses*]. Wait, hold on, [*Writes $A = B/2$*] if B increases, then A is going to increase, and S and R stay the same [*pauses*]. So, N is going to be positive

Kris showed no difficulties visualizing B increasing when C increased; she stated that a higher atmospheric concentration of CO₂ results in an atmosphere with “more CO₂ molecules to absorb the heat.” Kris identified R as remaining constant when C increased. She was also told that S remained constant while B, L, and A varied when C increased. Kris thought about this information (dependency of change of energy flows with respect to C) while attending to the relationship $N = (S + A) - R$ in order to establish the direction of change of F with respect to C.

Notice that Kris did not necessitate particular values to reason about such direction of change. This suggests that she imagined the magnitude of energy flows and F increasing when C increased. This in turn suggests that Kris might have developed a sense of forcing by CO_2 as function of C , denoted by $F(C)$. Thus, I decided to examine if she could construct a graph of such relationship. Thus, I asked her to work on the original task: sketching the graph of the forcing by CO_2 function $F(C)$. Kris did not attempt to draw any particular graph for $F(C)$. She stated that she did not have sufficient information to do it, especially she did not know how F varied in relation to C . The following exchange illustrates Kris's difficulties.

Kris: What are we assuming is our like equilibrium amount of CO_2 ?

Me: That is a great question. Do you think that would help you?

Kris: I am just trying to think about [*pauses*]. I don't know, I am thinking about how much change in C causes how much change in N

Me: Would you prefer to work with values?

Kris: I rather work with quantities ... The only think I know is like we already said that S stays the same and R stays the same (S and R in $N = (S + A) - R$). So A is the only thing that's changing. So, as long as I know how much change in C [*points at the horizontal axis*] causes how much change in A [*points at A in $N = (S + A) - R$*], then I should be able to do it

Kris's responses suggest that she was trying to conceptualize how F was changing in relation to C . This in turn suggests that she was reasoning beyond direction of change. She demonstrated her reasoning about co-variation in terms of coordinating amounts of change in F with changes in C (e.g., "how much change in C causes how much change in N ").

Since Kris tried to conceptualize how F changed in relation to C , I decided to direct Kris's attention to the theoretical minimum and maximum values of $F(C)$, which correspond to the cases of a completely transparent atmosphere and a completely opaque atmosphere, respectively (see Chapter 3, pp. 17-18). I decided to probe Kris' thinking to see if she would notice that the maximum value for $F(C)$, a completely opaque atmosphere, implies that $F(C)$ increases at decreasing rate since F approaches to an upper limit as C continues increasing. I presented Kris with a diagram of the Earth's energy budget showing the values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, and $B = 300 \text{ J/m}^2/\text{s}$. Notice that these values implied that $A = 150 \text{ J/m}^2/\text{s}$, $L = 90 \text{ J/m}^2/\text{s}$, and no forcing exists since $F = (240 + 150) - 390 = 0$. I began with the first case: a completely transparent atmosphere. I told Kris to imagine all CO_2 from the atmosphere was removed so that the concentration was zero. Also, I told her to assume that CO_2 is the only greenhouse gas in the atmosphere. This meant that the atmosphere would not absorb any of the radiation emitted by the surface. Next, I asked Kris to determine the value of F corresponding to this situation. Kris and I had the following discussion regarding this case.

Me: So, if we had zero concentration of CO_2 , can you tell me the values of the flows?

Kris: If there is no CO_2 , then [*pauses*]

Me: What would happen to the radiation coming from the surface?

Kris: It is just going to leak. Everything is going to go outside of the atmosphere because there is nothing to absorb [the radiation]

Me: OK, let's play with those values. So, S would be 240 and R would be 390. What is the value of B ?

Kris: Zero

Me: And the value of L ?

Kris: 390!

Me: And these [*point at A*]?

Kris: Zero

Me: So, could you determine the value of the forcing then?

Kris: Let's find out! [*Writes $N = 240 + 0 - 390 = -150$*] negative one fifty

Kris did not demonstrate difficulties conceptualizing dependency of change between energy flows and C. In other words, she seemed to be aware of what flows change with changes in C and what flows remain constant with changes in C. Next, I asked Kris to consider the second case: a completely opaque atmosphere. Kris showed some difficulties thinking about this situation. Particularly, she was troubled by not having a particular value of C for which all surface radiation was absorbed. The following exchange illustrates how Kris thought about the case of completely opaque atmosphere.

Me: Let's discuss the opposite case. Let's imagine that we keep filling the atmosphere with enough CO₂ until it can absorb practically all surface radiation. What is the value of the forcing?

Kris: But [*moves the marker over the horizontal axis and pauses*]. Not talking about CO₂ right now [*pauses*]. Let's just say if whatever value that C is that's going to cause R to be completely absorbed, where L is zero, we can calculate it by doing N equals 240 plus A minus R, which is 390 [*writes $N = 240 + A - 390$ and pauses*] ... Because if we absorb absolutely all of this [*points at R*], then exactly half of it would be coming down [*points at A*] and half of it would be going up [*points at A*]

Me: Well, if the atmosphere absorbs all of R, what is the value of A then?

Kris: So, it would be 390 over 2 [*scratches A and writes 390/2*]. So, it is going to be negative 150 plus 195 [*writes $N = -150 + 195 = 45$*] ... So that would be positive 45!

Kris initially expected a particular value of C , similar to the previous case in which $C = 0$ ppmv (completely transparent atmosphere). Since she did not identify a particular value of C , Kris decided to assigned an arbitrary value of C for which all surface radiation was absorbed; for convenience, I refer to this value as $C = C_{\text{opa}}$. This suggests that Kris imagined a finite concentration of CO_2 that created a completely opaque atmosphere. This conception may be an obstacle to think about $F(C)$ as having a horizontal asymptote at $F = 45 \text{ J/m}^2/\text{s}$.

To examine whether Kris was able to conceptualize the asymptote, I asked Kris to draw a graph of $F(C)$ based on what she learned about the minimum ($F = -150 \text{ J/m}^2/\text{s}$) and maximum ($F = 45 \text{ J/m}^2/\text{s}$) values of $F(C)$. First, Kris made three equally spaced marks on the negative section of the vertical axis. She then wrote the values -50 , -100 , and -150 next to them. Next, Kris made a mark on positive section of the vertical axis and wrote 50 next to it. She also made a mark on the horizontal axis and wrote '*where all energy is absorbed*' (this mark represented the value I refer to as $C = C_{\text{opa}}$). Finally, Kris plotted two points: one representing $F(0) = -150$ and another representing $F(C_{\text{opa}}) = 45$. After she plotted the points, Kris stopped and did not draw any graph for $F(C)$. It seemed that Kris did not want to draw a graph of $F(C)$ because she was not sure about how F was increasing in relation to C .

Me: So, how would the graph of $F(C)$ look like?

Kris: I am tempted to say that it looks like that [*draws a concave-down, increasing curve (Figure 5.14)*], kind of like a square root graph. But, because none of these

terms (terms in $N = (S + A) - R$) have like any square powers or square roots, I want to say it is constant. But I also don't think it is constant

Me: Why don't you think it is constant?

Kris: I don't know. Whenever, I don't know. I know the numbers don't have any like powers or square roots to make it look like this form mathematically [*points at Figure 5.14*]. But, you know how like [*pauses*] ... We know there is an equilibrium temperature for the Earth. Maybe?

Me: Yes, when the budget reaches a new balance, there is a new equilibrium temperature

Kris: Ok, so between here it is going to be the balance [*points at the middle of the horizontal axis*]. I don't know

Me: Let me ask you something. Can F exceed $45 \text{ J/m}^2/\text{s}$?

Kris: Assuming we keep with this number [*points at R*], I don't think so

Me: OK, so how can you graphically represent a function that will never exceed 45 [*draw a horizontal dashed line at $F = 45$*]? For instance, when F is increasing, can the graph of $F(C)$ cross this line?

Kris: No

Me: How can you graphically represent an increasing $F(C)$ that doesn't cross that line?

Kris: It is an asymptote! Wait! It gets to it but never crosses it, or it doesn't even get to it. Well, it has to get to it because [F] is 45. [*Pauses*] I mean, just a line? [*Moves the marker indicating a line passing through $(0, -150)$ and $(C_{opa}, 45)$*]. See? But that would imply that it is like a constant rate of change with C and N

Me: If the graph is linear, then would the line just stop here [*point at $(C_{opa}, 45)$*]?

Kris: Um-hum, I think so. So, tentatively, this is my line [draws a (dashed) line passing through $(0, -150)$ and $(C_{opa}, 45)$ (Figure 5.15)]

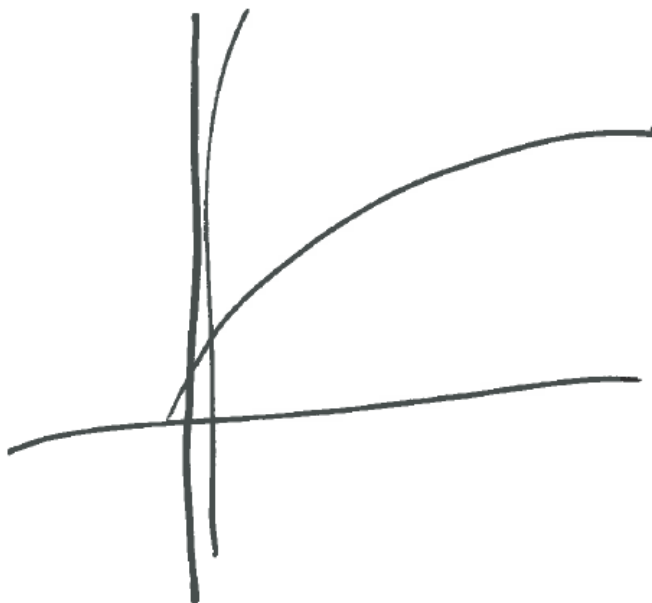


Figure 5.14. Kris's first attempt to sketch the graph of $F(C)$

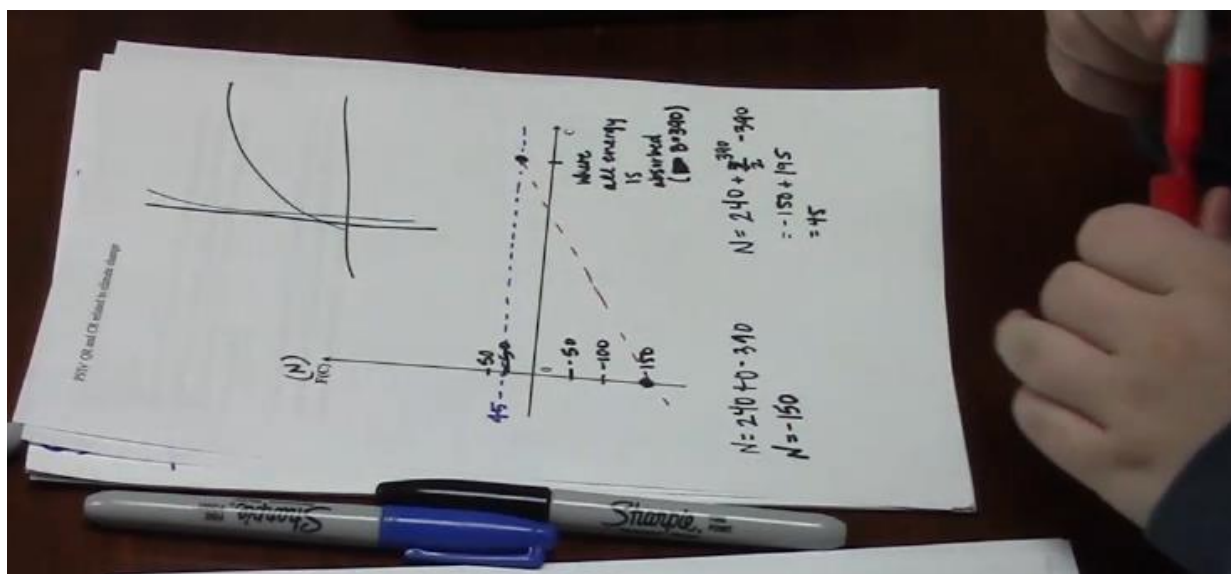


Figure 5.15. Kris's final version of the graph of $F(C)$

Kris initially believed that the graph of $F(C)$ should be a concave-down, increasing curve (see Figure 5.14). She, however, was also considering a line as the graph of $F(C)$. It seemed that Kris's indecision was rooted in her inability to identify how F was varying in relation to C . To make sense of the situation, Kris first attended to the relationship $N = (S + A) - R$; she attempted to discard a concave-down, increasing curve for $F(C)$ because the equation $N = (S + A) - R$ was linear (in terms of S , A , and R). This suggests that Kris had not yet *tightly coupled* the variation in F with the variation in C . Although F linearly depends on S , A , and R , she needed to focus on whether F was linearly dependent on C . By tightly coupling F and C , she may have noticed that the equation $N = (S + A) - R$ was not expressing a relationship in terms of C , and thus its *linearity* must not be applied to a graph expressing $F(C)$. Since Kris knew that F could not exceed $45 \text{ J/m}^2/\text{s}$, I prompted her to think about how such variation could be graphically represented. Notice how she initially proposed a horizontal asymptote, but then she discarded it. It seemed that Kris discarded the asymptote based on her understanding that F could take the maximum value of $45 \text{ J/m}^2/\text{s}$. For Kris, this meant that the graph of $F(C)$ intersects the line $F = 45$. It is also important to notice that Kris previously conceptualized a finite value of $C = C_{\text{opa}}$ so that $F(C_{\text{opa}}) = 45$ when thinking about a completely opaque atmosphere. This may explain why she discarded a curve with an asymptote and drew a line that *just stopped* at $(C_{\text{opa}}, 45)$.

Nonetheless, Kris was not convinced that the graph of $F(C)$ was a line since “that would imply that it is like a constant rate of change with C and N .” This suggests that Kris was reasoning about co-variation beyond the direction of change level; she attempted to reason about how F was varying in relation to C , which may involve reasoning in terms of amounts of change. She also demonstrated being aware that drawing a line involves stating that the rate of change is constant.

The Planetary Energy Imbalance Function $N(t)$

After Kris read S2T2, she began working on the task by making sense of how the flows of energy in the Earth's energy budget vary after a positive forcing occurred. Kris described the variation of the flows of energy in the following way.

Right before [the concentration of CO_2] increases, it is going to be like the same R as before it increased. And then, a bigger proportion or percentage of the R value that is released into the atmosphere is going to get absorb by the atmosphere, and so then more of it is going to go back into the surface ... Yeah! So, the B portion of $[R]$ is going to be larger also. So, that means A , which is one half of B , it's also going to be increased ... So then the input of S and A together, which is right here [*points at $N = (S + A) - R$*], is going to be increased. And then, R is also going to increase because we absorbed more energy or heat into the surface [*simultaneously points at S and A*]. So, that means in order for the Earth to go back to its original state, R is going to have to increase in order to take the temperature down

Kris's response suggests that she imagined the energy flows increasing over time, which suggests reasoning about co-variation in terms of direction of change. Kris did not demonstrate difficulties stating what and how energy flows were changing, which suggests that she was developing a sense of how the budget evolves over time. It is important to notice that Kris did not necessitate particular values to reason about the direction of change. This suggests that she may have imagined the energy flows increasing continuously over time. Also, Kris stated that as R increased, the temperature of the planet's surface decreased as if it was cooling down. This suggests that she may not yet conceptualize that an increase in upward radiation is the result of an increase in temperature. Another possibility involves Kris thinking of returning to equilibrium

as actually going back to the original radiative equilibrium. I decided to focus on these issues later in the interview (see The Surface Temperature Function $T(t)$ section). At this moment, I was interested in examining how Kris would construct the graph of $N(t)$.

Since Kris described how the flows change over time, I asked her to describe how the planetary energy imbalance N changed over time after a positive forcing. She started by making sense of the value $N(0)$ based on what she knew about the energy budget and the forcing by CO_2 .

So, the S is always going to be constant [*points at S in $N = (S + A) - R$*], so that doesn't really matter right now. So, we need to see the relationship between A and R and N [*points at those letters in $N = (S + A) - R$*]. So, initially the normal amount of heat is radiated into the atmosphere (initial value of R), but because we have a positive forcing, then B is going to be larger than normal, which means this A is going to be larger than normal [*points at A in the diagram*], which means this A is going to increase [*points at A in $N = (S + A) - R$*], which means N is going to be positive because this stays the same [*points at S in $N = (S + A) - R$*]. And if everything were balanced, then N would be zero, so S plus A would equal R . But since we are increasing the term S plus A and keeping R the same right as the CO_2 concentration is increased, then N is going to be positive ... So, we are going to start out with N being somewhere up here [*makes a mark on the top section of the vertical axis*]

Kris determined that $N(0) > 0$ by reasoning about the direction of change of the energy flows R , B , and A with respect to the concentration of CO_2 . She did not utilize particular values for the flows to think about the value of N at the moment a positive forcing occurred. She also demonstrated conceptualizing N as a difference between downward radiation and upward radiation. Her response also suggests that Kris develop an understanding of the forcing as the

initial value of $N(t)$. Notice that she coordinated N being positive with $t = 0$ (e.g., “So, we are going to *start out* [emphasis added] with N being somewhere up here”). Kris, however, did not answer my question about how $N(t)$ varied over time. When I repeated the question, Kris and I had the following exchange.

Kris: I think what is going to happen is, well eventually to compensate for the imbalance, the Earth is going to radiate more because that is what it says in the video. Like, whenever the Earth gets too hot, it releases more longwave radiation [pauses]. I don't know

Me: That's correct. The Earth will start radiating more energy because the surface is getting hotter. Since A increased at the moment of the forcing, the Earth will absorb more radiation, which means R will increase. Then, the atmosphere will have more radiation to absorb [point at B], which means A will get larger again. Then, R will increase again, and so on

Kris: So, that means that R is just going to [pauses]. That means R increases [points at R in $N = (S + A) - R$], which means A and R are increasing?

Me: That's correct. They (A and R) are related: A is half of a part of R

Kris: So, as R increases, A increases [points at those letters in $N = (S + A) - R$] ... [R] is not going to stay constant because the new R is influenced by S plus A . So, when A increases, R is going to be [pauses]. It can't just keep increasing!

Me: What quantity can't keep increasing?

Kris: N of t , the imbalance

Me: What do you think? Will the imbalance continue to increase over time?

Kris: That doesn't make [*pauses*]. But then, the whole like staying at some equilibrium wouldn't happen if the imbalance kept increasing. So, somewhere [N] has to come back to zero

Kris demonstrated reasoning in terms of the direction of change of the flows R and A with respect to time. Notice that she stated that R and A were increasing over time after a positive forcing. She came up with such conclusion by thinking about the energy budget and how the flow of radiation evolves after a positive forcing. Kris's difficulties appeared when she tried to reason about the direction of change of $N(t)$. Kris anticipated that $N(t)$ should decrease over time because the budget must reach a radiative equilibrium, which meant that "[N] has to come back to zero." She, however, did not see this reflected on the mathematical relationship $N = (S + A) - R$. She seemed to think that if R and A were increasing over time, then $N = (S + A) - R$ must also be increasing over time. This suggests that Kris needed particular values to develop a sense that $N(t)$, as a difference between downward radiation and upward radiation, might still decrease even if R and A were increasing.

To help Kris see how $N(t)$ was changing with respect to time, I showed her a diagram of the Earth's energy budget having the values $S = 240 \text{ J/m}^2/\text{s}$, $R = 390 \text{ J/m}^2/\text{s}$, $B = 300 \text{ J/m}^2/\text{s}$, and $A = 150 \text{ J/m}^2/\text{s}$. These values defined an energy budget in equilibrium since $N = (S + A) - R = (240 + 150) - 390 = 0$. I next told Kris that C is instantaneously increased at $t = 0$, creating a positive forcing. I changed the values of the flows B, L, and A to the new values $B = 320 \text{ J/m}^2/\text{s}$, $L = 70 \text{ J/m}^2/\text{s}$, and $A = 160 \text{ J/m}^2/\text{s}$ for $t = 0$ (Figure 5.16). These values defined a positive forcing by CO_2 of magnitude $10 \text{ J/m}^2/\text{s}$ since $F = (S + A) - R = (240 + 160) - 390 = 10$. Then, I helped Kris use these values to determine a new value for R, B, and A corresponding to the time $t = 0 +$

$h = h$ (for some positive h). I told Kris to base the calculations on the assumption that

$R([i + 1]h) = S + A(ih)$ for $i = 0, 1, 2, \dots$ Kris reasoned as follow

That means 400 is going to leave ($R(h) = 240 + 160 = 400$) ... [The new value of B] is going to be whatever percentage 320 out of 390 is, multiplied by 400 ... So, we are going to do 320 divided by 390. It is about 82.1 percent ... So, if we multiply that number by 400 ... it'll give us what is absorbed by the atmosphere, which is about 328 ($B(h) = 328$). So, we can write that [*writes 328 for B (Figure 5.16)*] ... So then, we do 328 divided by 2, which is our new A value that is being absorbed by the Earth [*writes 164 for A*]

Kris would repeat the above process to determine two additional values of R, B, and A corresponding to two additional, equally-spaced times (for convenience, I refer to them as $t = 2h$ and $t = 3h$). Kris wrote the values $B(2h) = 331.5 \text{ J/m}^2/\text{s}$ and $B(3h) = 332.917 \text{ J/m}^2/\text{s}$ on the diagram of the Earth's energy budget (Figure 5.16). In total, Kris had four successive values for R, B, and A corresponding to the values of time $t = 0, t = h, t = 2h$, and $t = 3h$ (she only wrote the values of B on the diagram though, but she needed to determine values of R and A in order to determine such values of B). Kris had previously demonstrated reasoning in terms of the direction of change of the flows with respect to time. Determining particular values for the flows, however, may have helped her notice how rapidly the flows were increasing over time. Kris coordinated amounts of change in B with changes in time in order to examine whether the energy budget was actually reaching a radiative equilibrium.

Well, this difference right here between 320 and 328 is greater than the difference between these two values the 328 and 331.5. And, the difference between these two [*points at 331.5 and 332.917*] is less than those [*points at 328 and 331.5*], which is less

than that [points at 320 and 328]. That tells me that there is eventually going to be a limit.

Yeah, [the energy budget] is going to reach a new equilibrium point somewhere

This excerpt suggests that Kris arrived to two important realizations. First, Kris realized that “there is eventually going to be a limit” for the growth of B. Notice how Kris attended to the variation in amounts of change of B in order to arrive to this realization. Particularly, Kris coordinated amounts of change in B with changes in time and noticed that B was increasing by smaller and smaller amounts. For her, this was a confirmation that the energy budget was approaching to a radiative equilibrium over time. Second, Kris thought of the budget as approaching to new radiative equilibrium, rather than thinking of the energy budget as going back to the *original* radiative equilibrium. This is a crucial realization that would help her interpret the graph of $N(t)$ in terms of the planet’s average surface temperature; that is, the surface temperature is increasing when $N(t)$ is positive (even if it is decreasing).

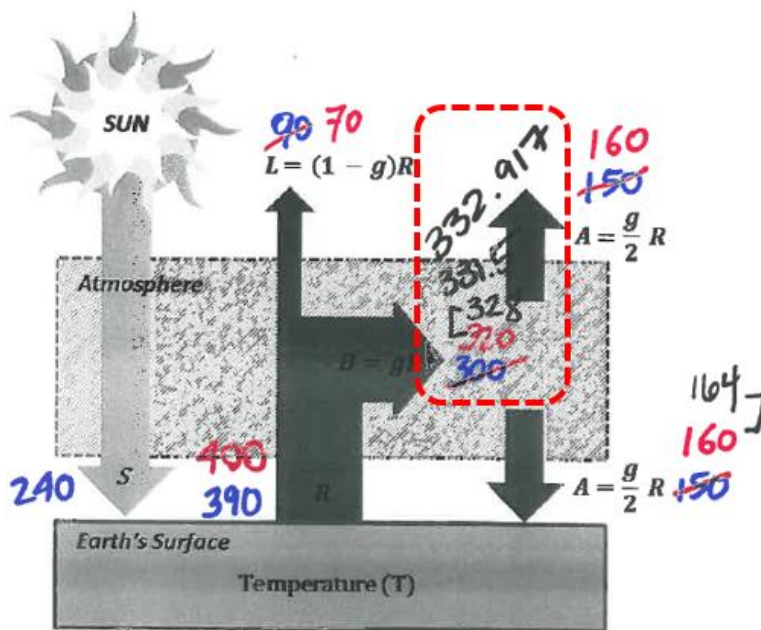


Figure 5.16. Kris had four successive values of B (dashed-line rectangle).

Kris confirmed that the budget was reaching a new radiative equilibrium. She, however, had not yet described how $N(t)$ was changing over time. Thus, I decided to re-direct her attention to the original task: drawing the graph of $N(t)$.

Me: Initially, we determined that $N = 10 \text{ J/m}^2/\text{s}$ for $t = 0$. This was the instant in which the forcing occurred. What do you think the value of N should be right after $t = 0$?
Would this new value of N be larger or smaller than $10 \text{ J/m}^2/\text{s}$?

Kris: Smaller [*pauses*] because that would mean [the budget] is reaching a new equilibrium

Me: Could you determine the value of N for the next time value?

Kris: That is going to be 4 [*calculates* $N = (240 + 164) - 400 = 4$]

Me: Does that make sense?

Kris: Yes, because it is decreasing. So, that goes with what I was thinking, which means that the imbalance is going to look like [*draws a decreasing, concave-up curve (Figure 5.17)*] ... Theoretically, I think it gets equal to or equals zero eventually, but I am not sure how long it would take to actually get there [*pauses*].
I am just going to assume it gets really, really, really close to zero

Kris's response suggests that she imagined $N(t)$ as a quantity decreasing over time. Notice that she anticipated $N(t)$ as a decreasing quantity because such variation was consistent with the idea of reaching a radiative equilibrium. Earlier, Kris had difficulties seeing $N(t)$ as a decreasing quantity in the mathematical relationship $N = (S + A) - R$. She did not see why $N(t)$ would be decreasing if the energy flows R and A were increasing. It seemed that realizing that there was a limit for the growth of B helped her reconcile her idea of having a decreasing $N(t)$ when the energy flows R and A were increasing. By attending to the variation in amounts of change of B

for change in time, Kris may have conceptualized the difference between downward radiation and upward radiation as having a limit value. The idea that such value was zero may have been based on her understanding of $N(t)$ as a measure of the energy imbalance in the budget. After looking at her graph, Kris stated that “this makes sense because as t increases, $N(t)$ decreases at a decreasing rate ... So, for equal changes in time, the amount by which $N(t)$ decreases is getting smaller.” This suggests that she related the variation in B with the variation in N . For her, having $N(t)$ decreasing by smaller and smaller amounts reflected how B was approaching to a limit value. It is important to notice that, although Kris talked about rate of change, her object of reasoning exclusively involved comparing amounts of change in $N(t)$ with changes in time (e.g., “for equal changes in time, the amount by which $N(t)$ decreases is getting smaller”).

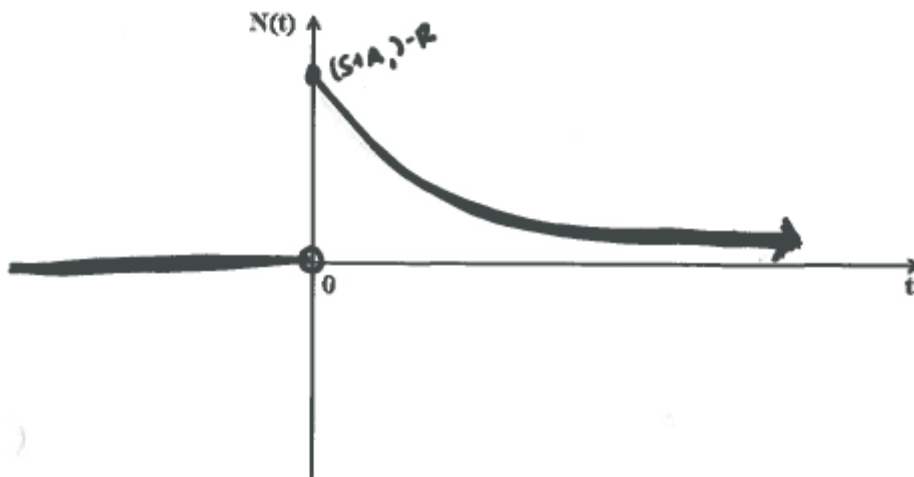


Figure 5.17. Kris’s final version of the graph of $N(t)$

The Surface Temperature Function $T(t)$

I was interested in exploring how Kris’s interpretation of a decreasing $N(t)$ (i.e. as indication of reaching a new radiative equilibrium) would influence her understanding of the average surface temperature function, denoted by $T(t)$. I directed Kris’s attention to the second

part of S2T2: drawing the graph of $T(t)$. I asked Kris to describe whether the surface temperature was increasing or decreasing. She used the diagram of the energy budget to answer my question.

Well, [the planet's surface] keeps intaking. I think it is warming up because once we added more CO_2 , that is less of the emitted energy that is getting just like shut out passed the atmosphere, leaks from it. Then, more [radiation] is going to be absorbed by the atmosphere ... Whatever is absorbed by the atmosphere [*points at B*], half of that plus the sun's energy is going to be absorbed by the Earth ($\frac{1}{2}B + S = A + S$), which is going to keep increasing as we saw with like the 400 [*points at R = 400*]. Then, from the A value, just with the A, [the planet's surface] absorbs 160 ... Then, it absorbs 164 ... So, I think it is going to keep increasing [*draws an increasing, concave down curve (Figure 5.18)*]. So, it is going to like [*pauses*]. Because the difference of the temperature, or whatever, was the four initially, well we changed it to ten, then it went to four [*points at the values of A = 150, 160, and 164*]. So, it is increasing at a decreasing rate

Kris attended to the variation of the flows in the energy budget to make sense of how $T(t)$ was changing over time. She noticed that the magnitude of radiation being absorbed by the surface kept increasing over time. In particular, she described how the continuous exchange of radiation between the surface and the atmosphere involved increasing magnitudes of radiation. This represents an elemental (and central) understanding of the concept of the greenhouse effect, which refers to such exchange of radiation between surface and atmosphere. Next, Kris attended to the variation in the amounts of change in A to justify the concavity of her curve. She stated that $T(t)$ was increasing at a decreasing rate because A was increasing by smaller and smaller amounts. Although she mentioned the rate of change, Kris was reasoning about co-variation in terms of amounts of change. It is also important to notice that Kris did not demonstrate using the

graph of $N(t)$ to draw conclusions about the graph of $T(t)$. This suggests that she may have not yet developed a connection between these two quantities.

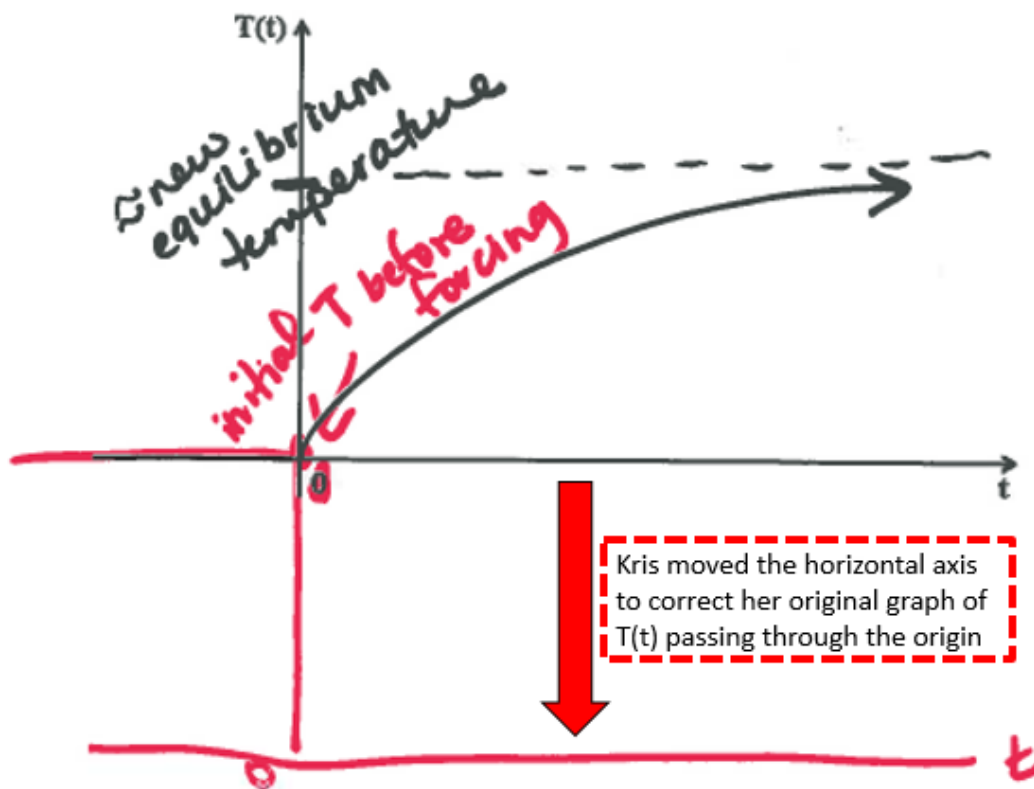


Figure 5.18. Kris's final version of her graph of $T(t)$

Kris's initial graph of $T(t)$ included the origin $(0, 0)$. She immediately recognized her mistake and corrected it by *moving* the horizontal axis down. Kris explained that the graph of $T(t)$ did not pass through the origin "because the temperature of the Earth is not zero." Then, Kris drew a horizontal, dashed line representing a horizontal asymptote limiting the growth of her curve. Kris wrote 'new equilibrium temperature' next to that horizontal asymptote. Finally, Kris wrote 'initial temperature before forcing' to indicate the meaning of the intersection between her graph and the vertical axis (see Figure 5.18). The way Kris constructed her graph unveiled interesting aspects of her reasoning regarding the energy budget and the surface temperature. First, Kris constructed an increasing curve for $T(t)$ because she noticed that the

magnitude of A was increasing over time. She interpreted this as the planet's surface absorbing an increasing magnitude of heat or energy, which in turn meant that the surface was warming up. This represents a central understanding concerning the greenhouse effect, which refers to the exchange of radiation between the planet's surface and the atmosphere, exchange that regulates the surface temperature. Second, Kris justified the concavity of her curve by attending to the amounts by which A was changing for equal changes in time t . She noticed that A was increasing by smaller and smaller amounts. For Kris, the magnitude of heat or energy being absorbed by the planet's surface was increasing by smaller and smaller amounts. She interpreted that as $T(t)$ increasing by smaller and smaller amounts, which justified the concavity of her curve. This also suggests that Kris was reasoning about co-variation at L3 when imagining $T(t)$. Finally, Kris explicitly distinguished between the energy budget's *original* radiative equilibrium and *new* radiative equilibrium. Notice how Kris translated this distinction to the context of the surface temperature. She labeled the T -intercept of her graph as initial temperature before forcing and drew an asymptote to represent the new equilibrium temperature (see Figure 5.18). The later represents a crucial realization regarding the impact of carbon dioxide pollution on global warming. After polluting the atmosphere, the temperature does not go back to its original value, but rather it reaches a new higher value.

CHAPTER 6

DISCUSSION AND CONCLUSIONS

In this chapter, I present the results of this study across cases so that common aspects are established and discussed. I also situate the findings in the research literature in mathematics education and discuss the implications of this study for the field of mathematics education and future research. Thus, I have organized the chapter into three sections: (1) discussion of results, (2) conclusions and implications, and (3) future research.

Discussion

This dissertation examined PSTs' understanding and reasoning regarding intensive quantities and functions involved in introductory mathematical models for global warming. For this reason, I divided the study into two parts: (1) exploring PSTs' conceptions of three intensive quantities commonly used to model global warming, and (2) examining PSTs' reasoning regarding three functions involved in introductory mathematical models for global warming. Each part of the study was guided by overarching research questions and supporting research questions. The study followed an exploratory, multiple-case research design. Three PSTs enrolled in a mathematics education program at a large Southeastern university participated in my study. Each PST completed six mathematical tasks designed for the study during four individual, task-based interviews. The analysis of PSTs' responses to the tasks provided information to answer the research questions. Next, I discuss the answer to the research questions guiding each part of the study.

PSTs' Reasoning Regarding Intensive Quantities for Modeling Global Warming

The first part of my study was guided by the general research question *what are PSTs' conceptions of intensive quantities commonly used to model global warming?* This question involved three supporting research questions:

- i. How do PSTs use division to define these intensive quantities?
- ii. What measurable attributes do PSTs identify for each intensive quantity?
- iii. What differences can be observed in PSTs' work with Type 1 intensive quantities (concentration) versus Type 2 intensive quantities (energy density and energy flux density)?

The intensive quantities involved in the study were *concentration*, *energy density*, and *energy flux density*. PSTs' conceptions of these intensive quantities were somewhat consistent for a particular quantity (with the exception of energy flux density) and significantly different across quantities. What follows is a discussion of the findings organized into three subsections: conceptions of intensive quantities (answer to supporting questions i and ii), type 1 versus type 2 intensive quantities (answer to supporting question iii), and the issue of temperature (an unexpected finding).

Conceptions of intensive quantities. PSTs' conceptions of concentration, energy density, and energy flux density were shaped by what and how PSTs made sense of: (a) *what is being measured (measurable attribute)* and (b) *how it is being measured (measurement process)*. PSTs identified the *intensity of a relationship between quantities* as the measurable attribute of each intensive quantity involved in this study. They identified the following attributes for each quantity:

- The *part* of a volume of air formed by the volume of a gas (concentration)

- The *crowdedness* of energy over a sheet's surface (energy density)
- The *rapidity* of increase of energy density (energy flux density)

PSTs also used division as the measurement process for each quantity. The ways they used division, however, varied depending on the intensive quantity. They made use of division in the following way for each quantity:

- *Measurement division* to quantify how much of a volume of air was formed by the volume of a gas (concentration)
- *Partitive division* to equally distribute the magnitude of a quantity among several units of another quantity (energy density and energy flux density)

These findings appeared consistent with Thompson's (2011) theoretical discussion regarding how a student constructs a quantity. Thompson has stated that a student's meaning for a quantity emerges in the interactions between identifying a measurable attribute (what is being measured) and developing a process to quantify such attribute (how it is being measured).

The PSTs did not demonstrate difficulties working on the tasks involving concentration. Although PSTs admitted they never worked with concentration before, it seemed that they were comfortable working with an intensive quantity expressing a part-whole relationship between two other quantities. In fact, all three PSTs demonstrated consistent conceptions regarding *concentration*. Their responses suggest that all three PSTs held a *Ratio as Measure* conception for this intensive quantity (Table 6.1). PSTs identified two similar measurable attributes of concentration: (a) the *part-whole relationship* between a gas and air (Pam and Kris) and (b) the *proportion* of a gas to air (Jodi and Kris). Notice that both attributes require a multiplicative comparison between the volume of a gas and the total volume of all gases (air). This suggests that PSTs used *measurement division* to quantify concentration; that is, they used measurement

division as the measurement process of concentration. More specifically, PSTs used division to measure the volume of the gas relative to the volume of air, or the proportion of air formed by that gas. PSTs interpreted particular values of concentration in terms of the percentage, fraction, or proportion of air volume formed by a particular gas. These interpretations further support the use of measurement division when defining concentration. Also, PSTs conceptualized concentration as a *constant multiplicative relationship* between co-varying quantities. They imagined the volume of a gas and the volume of air varying in constant ratio, which represented the value of concentration. This realization is central to develop a mature understanding of a ratio as a single intensive quantity because it symbolizes the structure of a ratio –dividend, divisor, and quotient– as a whole, but emphasizes the constant multiplicative relationship between the quantities forming the ratio as the defining aspect of the intensive quantity (Thompson, 1994a, 1994c; Simon & Placa, 2012).

Table 6.1

Summary of PSTs' Conceptions of Concentration, Energy Density, and Energy Flux Density

	Concentration	Energy Density	Energy Flux Density
PAM	Ratio as Measure	Ratio as Per-One	Ratio as Per-Composite-One
JODI	Ratio as Measure	Ratio as Measure	Ratio as Rate-Per-One
KRIS	Ratio as Measure	Ratio as Measure	Ratio as Rate-Measure

The PSTs demonstrated some difficulties working on the tasks involving energy density. They admitted never working with energy density before and appeared less comfortable working with an intensive quantity relating two quantities from different measure spaces (Joules and square meters). PSTs identified the relationship between radiation (energy) and the sheet's surface as the measurable attribute of energy density. To quantify such attribute, PSTs used *partitive division*; they equally distributed energy among several 1×1 squares. Their responses suggest that all three PSTs conceived energy density as a magnitude of energy corresponding to

one 1×1 square. This 1×1 square was not an area unit but a geometric object (a literal square). This suggests that they conceptualized surface area as a number of squares covering a sheet. This conception also downplayed the role of surface area in defining a magnitude of energy density. This resulted in PSTs occasionally treating energy and energy density as the same quantity. Additionally, PSTs demonstrated difficulties relating energy density and temperature. They initially conceptualized temperature as a magnitude of energy, ignoring the role of surface area. They overcame these difficulties by focusing on the multiplicative relationship defining energy density. It seemed that this helped them: (a) attend to the role of surface area in defining a magnitude of energy density and (b) relate energy density, as a multiplicative relationship, to temperature. This suggests that they no longer saw temperature as a magnitude of energy, but rather as a magnitude of energy density. There was one difference regarding PSTs' conceptions of energy density: whether they saw it as a constant multiplicative relationship between co-varying quantities. Jodi and Kris conceptualized energy density as a constant multiplicative relationship, which suggests that they held a *Ratio as Measure* of energy density. In contrast, Pam did not demonstrate conceptualizing energy density as a constant multiplicative relationship, which suggests that she held a *Ratio as Per-One* conception of energy density (see Table 6.1).

The PSTs demonstrated difficulties working on the tasks involving energy flux density. They admitted never working with energy flux density before and appeared less comfortable working with an intensive quantity relating three quantities from different measure spaces (Joules, square meters, and time). In fact, only Kris determined and used energy flux density to complete S1T4 without being directly prompted to do so. Pam and Jodi did not demonstrate acknowledging an intensive quantity relating three quantities until they were asked to calculate

it. Unlike concentration and energy density, PSTs' conceptions of energy flux density were qualitatively different. To define energy flux density, all three PSTs used partitive division when dividing radiation rate by surface area. They, however, applied partitive division in rather different ways. Pam used partitive division to equally distribute a magnitude of energy among several *time-area composite objects*. Since Pam demonstrated conceptualizing 1m^2 as an actual square, these composite objects can be thought of as *abstract boxes* with a 1×1 square base and a height of 1s. For Pam, energy flux density expressed the magnitude of energy per box, which is why I termed Pam's conception of energy flux density as *Ratio as Per-Composite-One* conception (see Table 6.1). Jodi used partitive division to equally distribute radiation rate among several 1×1 squares. Such application of partitive division would have defined a magnitude of radiation rate per square meter. Jodi, however, interpreted this ratio as the rate at which energy density corresponding to one 1×1 square was increasing over time. This suggests that she conceptualized energy flux density as rate expressing energy density corresponding to 1×1 square *per second*. Since Jodi's conception involved a rate quantity (radiation rate) corresponding to one unit of another quantity (one 1×1 square), I termed her conception of energy flux density as *Ratio as Rate-Per-One* conception (see Table 6.1). Finally, Kris used partitive division to define energy flux density as a magnitude of energy density *per second*. She decoupled radiation rate into a magnitude of energy and one second. Then, she used partitive division to equally distribute energy among several 1×1 squares in order to define energy density. Finally, she *re-coupled* energy density *and* one second to construct energy flux density. She conceptualized energy flux density as a rate measuring both how rapidly energy density increases and how rapidly temperature increases. Moreover, she was the only PSTs who conceptualized energy flux density as a constant multiplicative relationship between co-varying

quantities (energy density and time). Her conception was the most sophisticated and involved conceiving of energy flux density as a measure of the rate of change of a ratio. This is why I termed her conception *Ratio as Rate-Measure* conception (see Table 6.1).

Type 1 versus Type 2 intensive quantities. Concentration is an example of a *Type 1 Intensive Quantity* because its constituent quantities form a part-whole relationship. Intensive quantities of this type are often represented in the form of fractions, decimals, or percentages. Energy density and energy flux density are examples of *Type 2 Intensive Quantities* because they relate quantities whose measures belong to different measure spaces. Intensive quantities of this type are often represented in terms of rate and not as fractions. All three PSTs experienced more difficulties working with Type 2 intensive quantities than with Type 1 intensive quantities. Particularly, they had more difficulties identifying a measurable attribute and seeing division as the measurement process when working with energy density and energy flux density than when working with concentration. For instance, all three PSTs had no difficulties recognizing a proportion of air to gas or a part-whole relationship between a gas and air as measurable attributes of concentration. They also used division as a way to measure such attributes. For energy density, PSTs did use division to define such quantity; they, however, needed more time to realize that division was a measuring process to quantify the multiplicative relationship between energy and surface area. For energy flux density, Jodi and Kris needed more time to identify a measurable attribute than they needed for energy density, while Pam did not demonstrate identifying any measurable attribute. This finding contradicts findings from other research regarding intensive quantities. Nunes et al. (2003), for instance, found no significant differences in students' performance as they worked on tasks involving Type 1 intensive quantities versus Type 2 intensive quantities.

PSTs were consistent in their use of division when defining Type 1 Intensive Quantities versus Type 2 Intensive Quantities. All three PSTs used measurement division to define concentration. They interpreted values in terms of percentages or fractions indicating how much of a volume of air was formed by the volume of certain gas. Greer (1992) classified situations involving division of two quantities with like units (e.g., kg/kg or cm^3/cm^3) as cases in which measurement division is applied. Concentration, as presented in S1T1 and S1T2, emerged as a division of the volume of a gas and the volume of air, which were both measured in cm^3 . Thus, PSTs' consistent use of measurement division to define concentration appeared consistent with Greer's classification of situations involving division. Pam and Jodi also interpreted concentration in terms of rate indicating a volume of a gas per one volume unit of air. For instance, Pam and Jodi interpreted an oxygen concentration of 0.21 as indicating 0.21 cm^3 of oxygen per cm^3 of air. This suggests that PSTs could have used partitive division to define Type 1 Intensive Quantities as well. In other words, PSTs could have equally distributed a volume of a gas among several volumes units of air. This is different from Greer's classification since situations involving division of quantities with like units are not usually interpreted in terms of rate.

In contrast, all PSTs used a form of partitive division to define energy density and energy flux density. They interpreted values in terms of rates indicating a magnitude of a quantity per one unit of another quantity. For instance, energy density was defined by equally distributing a magnitude of energy (total number of elements) over several square meters (number of groups), thus defining a magnitude of energy per square meter (number of elements per group). For energy flux density, PSTs used qualitatively different forms of partitive division in terms of what was being equally distributed and what represented the number of groups. Pam equally

distributed energy among several composite time-area units and defined a rate indicating energy per composite unit, or $J/(s \times m^2)$. Jodi equally distributed radiation rate over several square meters and defined a rate indicating radiation rate per square meter, or $(J/s)/m^2$. Kris decoupled radiation rate into energy and one time unit, and then equally distributed that energy over several square meters to finally define a rate indicating energy density per second, or $(J/m^2)/s$. Greer (1992) classified situations involving division of two quantities with unlike units (e.g., m/s , kg/cm^3 or J/m^2) as cases in which partitive division is applied. Energy density and energy flux density, as presented in S1T3 and S1T4, respectively, emerged as equally sharing or distributing the magnitude of a quantity (energy or radiation rate) among several units of another quantity (surface area). Thus, PSTs' consistent use of partitive division to define energy density and energy flux density appeared consistent with Greer's classification of situations involving division.

The issue of temperature. The analysis of PSTs' responses to tasks S1T3 and S1T4 produced an unexpected finding concerning the concept of temperature and its relationship to energy density and energy flux density. All three PSTs showed difficulties making sense of these relationships. Specifically, PSTs' difficulties were rooted in the notion that temperature and energy (or heat) were equivalent quantities. Although energy in the form of heat is closely related to temperature, these two quantities are different: temperature is a measure of the average internal energy of an object, while heat is a measure of the magnitude of energy flowing from one object to another due to their temperature difference. In the context of the tasks, the sheets' surface temperature was a measure of the energy incident to the sheets averaged over their surface area. PSTs' inability to distinguish between temperature and heat became a conceptual obstacle when working with energy density and energy flux density. For instance, energy

density, as a ratio between energy and surface area, represented a measure of the sheets' surface temperature. PSTs, however, had difficulties conceptualizing such ratio as a measure of temperature for they saw the magnitude of energy alone as a measure of temperature. This notion that temperature and heat are equivalent quantities seemed to be a common misconception among students as reported in the research in science education (Harrison, Grayson, & Treagust, 1999; Prince, Vigeant, & Nottis, 2011; Thomaz, Malaquias, Vanente, & Antunes, 1995). This suggests an interesting interaction between conceptualizing a mathematical object, such as a ratio, and conceptualizing a scientific object, such as temperature. Pam's and Jodi's conceptions of energy density and energy flux density were less sophisticated than Kris's conceptions of those quantities. This coincided with Pam and Jodi having more difficulties than Kris in seeing energy density as a measure of temperature and energy flux density as a measure of how rapidly temperature was increasing.

PSTs' Reasoning Regarding Functions Involved in Modeling Global Warming

The second part of my study was guided by the general research question *how do PSTs reason about co-variation regarding three functions involved in introductory mathematical models for global warming?* This question involved two supporting research questions:

- i. What level of covariational reasoning do PSTs demonstrate when working with the functions involved in this study?
- ii. How do PSTs' understanding of the Earth's energy budget and radiative equilibrium relate to PSTs' understanding of the functions involved in this study?

The functions involved in the study were *the forcing by CO₂ function $F(C)$* , *the planetary energy imbalance function $N(t)$* , and *the mean surface temperature function $T(t)$* . In general, PSTs did not demonstrate reasoning about co-variation at the rate of change levels (average rate of change

level L4 and instantaneous rate of change level L5). Although Pam and Kris made references to the rate of change, their arguments were centered around attending to the variation in *the amounts of change* (e.g., $N(t)$ increases by smaller and smaller amounts over time), rather than attending to variation in *the rate of change* (e.g., $N(t)$ increases more rapidly over time). This finding is consistent with previous research documenting students' difficulties reasoning (or inability to reason) at the rate of change levels (Carlson et al., 2002; Johnson, 2012, 2015). What follows is a discussion of the findings organized into three subsections: level of covariational reasoning (answer to supporting question i), conceptions of the Earth's energy budget and radiative equilibrium (answer to supporting question ii), and monotonically asymptotic variation (unexpected finding).

Level of covariational reasoning. All three PSTs drew a line as a graph of *the forcing by CO₂ function* $F(C)$. They, however, were unable to develop a quantitative argument supporting such decision. This suggests that PSTs did not demonstrate consistent reasoning about the covariation represented by $F(C)$ beyond the *direction of change level* (L2) (Table 6.2). In other words, PSTs only presented quantitative arguments to justify why $F(C)$ was an increasing function, but they did not present quantitative arguments to explain, for instance, why $F(C)$ was a linear function. They attempted to construct a *non-ratio-based object* associating change in F with change in C . Such object may not involve reasoning about rate of change since it was not defined by a ratio and was based on coordinating amounts of change. Although PSTs could not construct that object based on the information available, they wanted to use it for different purposes, which provided information about their reasoning. Pam and Jodi attempted to construct the object to determine the slope of their lines. When constructing the object, Pam and Jodi were not quantifying the *variation in the intensity of change of F with respect to C* (Johnson, 2012) to

confirm a line as the graph of $F(C)$. This suggests that the line became their object of reasoning when making sense of the co-variation represented by $F(C)$. Johnson (2012) stated that conceptualizing the variation in the intensity of change is necessary to move from the direction of change level to the amounts of change level, and finally to the rate of change levels. This may explain why Pam and Jodi did not demonstrate reasoning beyond the direction of change level. They assumed that the increments in F remained constant for equal changes in C because the graph was a line. For them, the object was meant to determine the value of such constant increment. In contrast, Kris attempted to construct an object relating change in F and change in C in order to confirm (or discard) a line as the graph of $F(C)$. Although Kris ended up drawing a line, she did not demonstrate confidence doing it because, as she stated, that implied that the rate of change of $F(C)$ was constant. Kris was attempting to quantify the *variation in the intensity of change of F with respect to C* in order to determine the shape of the graph of $F(C)$. This suggests that Kris was reasoning about the situation (i.e., the energy budget and forcing by CO_2) in order to make sense of the co-variation represented by $F(C)$. This suggests that Kris was reasoning about co-variation at a higher level than L2. Since she attempted to construct an object associating amounts of change in F with changes in C , she was beginning to reason at the *amounts of change level (L3)* (Table 6.2).

Table 6.2

PSTs' Level of Covariational Reasoning per Function

	Forcing by CO_2 $F(C)$	Energy Imbalance $N(t)$	Surface Temperature $T(t)$
<i>Pam</i>	Direction of Change (L2)	Amounts of Change (L3)	Direction of Change (L2)
<i>Jodi</i>	Direction of Change (L2)	Direction of Change (L2)	Direction of Change (L2)
<i>Kris</i>	Amounts of Change (L3)	Amounts of Change (L3)	Amounts of Change (L3)

All three PSTs drew a concave-up, decreasing curve as a graph of *the planetary energy imbalance function* $N(t)$. They made conceptual connections between the decreasing magnitude of $N(t)$ and the concept of radiative equilibrium (the differences regarding their conceptions about such radiative equilibrium are discussed later in this section). Such connections allowed them to make sense of the direction of change of $N(t)$ (i.e., N decreases as time passes). Based on the concept of radiative equilibrium, Pam and Kris anticipated that $N(t)$ should be a decreasing function. By watching an introductory video, PSTs learned that the energy budget reaches a radiative equilibrium after a positive forcing occurs. This meant that $N(t)$, the measure of energy imbalance, approaches to zero over time. In other words, Pam and Kris reasoned about the situation (i.e., the energy budget reaching equilibrium) *in order to establish* the direction of change of $N(t)$. In contrast, Jodi anticipated that $N(t)$ should increase over time because she noticed that the flows R and A were increasing over time. When she determined values of $N(t)$ for particular time values, Jodi noticed that $N(t)$ was decreasing over time. Jodi found an explanation for the decreasing $N(t)$ in the energy budget approaching to a radiative equilibrium. In other words, Jodi reasoned about the situation *in order to make sense of* an already established direction of change.

Although all three PSTs drew a concave-up curve for $N(t)$, they demonstrated different reasoning regarding their concavity. Pam and Kris demonstrated reasoning about the co-variation represented by $N(t)$ at the *amounts of change level* (L3) (see Table 6.2). Pam determined values of $N(t)$ for some equally-spaced time values. She noticed that $N(t)$ was decreasing by smaller and smaller amounts between time values. She constructed a concave-up, decreasing curve to represent such pattern. This suggests that Pam reasoned about the situation (i.e., the energy budget reaching equilibrium) *in order to establish* the variation in the intensity of change of $N(t)$.

To construct the graph of $N(t)$, Kris attended to the variation of the energy flow B after a positive forcing. She noticed that B was increasing by smaller and smaller amounts over time. She interpreted that pattern as the way by which the budget approaches to a new equilibrium, which she translated into $N(t)$ decreasing to zero by smaller and smaller amounts. Kris drew a concave-up, decreasing curve to represent such pattern, which suggests that she reasoned about the situation *in order to establish* the variation in the intensity of change of $N(t)$. Unlike Pam and Kris, Jodi did not demonstrate reasoning about the co-variation represented by $N(t)$ beyond the *direction of change level* (L2) (see Table 6.2). Jodi defined the concavity of her curve by *following* particular points on a coordinated plane. She determined values of $N(t)$ for three equally-spaced time values and plotted three points representing the pairs (t, N) . The concavity of her graph of $N(t)$ was the result of joining all such points by a curve. Notice that she need not reason beyond L2 to draw the curve in this way. Although she interpreted the concavity as representing $N(t)$ decreasing by smaller and smaller amounts, her interpretation emerged from reasoning about the graph and not about the situation being represented by it. In other words, Jodi reasoned about the already established curve *in order to make sense of* the variation in the intensity of change of $N(t)$.

All three PSTs constructed a graph of *the mean surface temperature function* $T(t)$ by attending to the variation in the energy flows R and A with respect to time. This was unexpected since they were asked to construct the graph of $T(t)$ based on their graphs of $N(t)$. This suggests that PSTs did not develop a clear or strong connection between these two functions while working on S2T2. In particular, PSTs were expected to conceptualize $N(t)$ as a measure of the rate of change of $T(t)$. Although they all attended to R and A to construct the graph of $T(t)$, their graphs were qualitatively different. Pam conceptualized R as a magnitude of energy leaving (or

being released by) the planet's surface. She noticed that R was increasing over time, which she interpreted as the planet trying to "cool itself off." Pam concluded that $T(t)$ had its maximum value at $t = 0$ (when the positive forcing occurs), and then it decreased back to the *original equilibrium temperature* (before the positive forcing) as the surface cooled down. She drew a decreasing curve with a concavity shift at some $t = t_c > 0$: the curve was concave-down for $0 < t < t_c$ and concave-up for $t > t_c$. Pam neither justified the concavity shift nor the order in which each concavity appeared. This suggests that Pam did not reason about the co-variation represented by $T(t)$ beyond the *direction of change level* (L2) (see Table 6.2). Jodi conceptualized R and A as magnitudes of energy leaving and coming into the surface, respectively. She coordinated values of R and A with discrete time values she called *cycles* or *revolutions*. She did not demonstrate conceiving of fractional cycles (e.g., half a cycle, a third of a cycle, or 0.23 cycles) or variation within cycles, which suggests Jodi used *chunky thinking* when conceiving co-variation. Castillo-Garsow et al. (2010) characterized chunky thinking as coordinating values of the dependent variable with discrete units of the independent variable or *chunks*. A student using chunky thinking does not conceive of variation within a chunk so that the dependent variable only varies between chunks. Jodi noticed that $\Delta A(\text{cycle } i) = \Delta R(\text{cycle } [i + 1])$, which she interpreted as the surface absorbing and releasing the same energy between cycles. Based on this, she concluded that $T(t)$ was *oscillating* between cycles since the surface energy was *oscillating* between cycles. Then, Jodi attended to the variation in the energy flow B (i.e., B increased by smaller and smaller amounts). Jodi interpreted the decreasing increments in B as an indication that $T(t)$ was returning to the *original equilibrium temperature*, which she represented by a curve with quasi-periodic oscillations stabilizing around a horizontal asymptote (original equilibrium temperature). Although Jodi attended to amounts of change in B , she interpreted the pattern

exclusively in terms of direction of change (i.e., $T(t)$ approaches to a value as t increases).

Additionally, Jodi did not justify the concavities of her curve. This further supports that Jodi remained reasoning about the co-variation represented by $T(t)$ at the *direction of change level* (L2) (see Table 6.2).

Kris demonstrated reasoning about the co-variation represented by $T(t)$ at the *amounts of change level* (L3) (see Table 6.2). She drew a concave-down, increasing curve to represent $T(t)$. She noticed that R and A continued to increase over time after a positive forcing. She interpreted that the continuous exchange of radiation between the surface and the atmosphere as an indication that the surface was absorbing more and more radiation. Based on that, Kris concluded that the surface was warming up, which meant that $T(t)$ was increasing over time. To construct the graph of $T(t)$, Kris attended to the variation of the energy flow A after a positive forcing. She noticed that A was increasing by smaller and smaller amounts over time. Since A represented the radiation reabsorbed by the surface, she interpreted the variation in A as $T(t)$ increasing by smaller and smaller amounts, which she represented by a concave-down, increasing curve. This led her to conclude that there existed a new higher equilibrium temperature to which $T(t)$ was approaching as time increased. This also suggests that she reasoned about the situation *in order to establish* the variation in the intensity of change of $T(t)$.

Conceptions of the Earth's energy budget and radiative equilibrium. As discussed above, all three PSTs made connections between $N(t)$ decreasing to zero and the energy budget reaching a radiative equilibrium. PSTs, however, demonstrated two different conceptions regarding the concept of radiative equilibrium. Pam and Jodi conceptualized the process of reaching radiative equilibrium as returning to the *original radiative equilibrium* (before the positive forcing occurred at $t = 0$). Pam and Jodi conceptualized $N(t)$ as the magnitude of energy

in the planet's surface. This conception led them to interpret their decreasing graphs of $N(t)$ as representing an energy budget losing energy. Because the budget was losing energy, they concluded that the Earth's surface must have been cooling down after the temperature increased at $t = 0$. Notice that both Pam and Jodi drew a graph of $T(t)$ approaching to the *original equilibrium temperature* after a positive forcing. In this frame of reasoning, the idea of reaching a radiative equilibrium means returning to the *original radiative equilibrium* since the budget was losing energy (and the Earth's surface was cooling down). Unlike Pam and Jodi, Kris conceptualized reaching a radiative equilibrium as approaching to a *new radiative equilibrium*. This realization allowed her to understand that, after a positive forcing, the temperature of the Earth's surface continues to increase until a new radiative equilibrium is reached. She noticed that R and A continued to increase after a positive forcing. Also, R and A represented the continuous exchange of radiation between the surface and the atmosphere, which for Kris indicated that the surface was absorbing more and more radiation. This represents a central understanding concerning the greenhouse effect, which refers to the exchange of radiation between atmosphere and surface that maintains the planet's temperature higher than it would be without an atmosphere. Kris finally concluded that the temperature of the Earth's surface was increasing and drew an increasing curve for $T(t)$. Kris's graph showed $T(t)$ approaching to a *new higher equilibrium temperature*. This is consistent with Kris's realization that the energy budget reaches a *new radiative equilibrium* after a positive forcing. This is an important realization in order to understand the long term effect of carbon dioxide pollution over the planet's surface temperature: the effects of CO_2 emissions manifest for years after they were released.

In summary, PSTs' meaning of the three functions involved in this study were influenced by their conceptions of two important scientific concepts: (a) *the Earth's energy budget* and (b)

the radiative equilibrium. This is consistent with previous research in science education and climate change education which have found these two concepts to be central for understanding global warming and the role of human activity in driving it (Lambert et al., 2013; Pruneau et al., 2010). The present study documented the relevance of these two scientific concepts to make sense of the quantities and functions used to model global warming. Particularly, it documents how PSTs explored the link between carbon dioxide pollution and global warming by reasoning quantitatively about the Earth's energy budget and the radiative equilibrium concept.

Monotonically asymptotic variation. The analysis of PSTs' responses to tasks S2T1 and S2T2 produced an unexpected finding concerning the concept of *monotonically asymptotic variation*. The three functions involved in S2T1 and S2T2 (ideally) exhibited such variation, which I characterized in terms of a function $f(x)$ such that $\lim_{x \rightarrow \infty} f(x) = c$ and either $f(x) \leq f(y) < c$ or $f(x) \geq f(y) > c$ for all x and y in an interval (a, ∞) such that $x \leq y$. Particularly, these functions had horizontal asymptotes governing their *long-term* variation: the forcing by CO₂ function $F(C)$ had an asymptote at $F = 45$, the planetary energy imbalance function $N(t)$ had an asymptote at $N = 0$, and the mean surface temperature function $T(t)$ had an asymptote at $T = T_{eq}$, where T_{eq} represents the new equilibrium temperature after a positive forcing. PSTs' responses to S2T1 and S2T2 suggested three central understandings for conceiving monotonically asymptotic variation for these functions. First, PSTs needed to conceptualize a local extreme value for the function $f(x)$. This allowed them to imagine $f(x)$ as taking values smaller than (or larger than) the local extreme within a *sufficiently small vicinity*⁵ around such extreme value. Second, PSTs needed to imagine the independent variable to continue to increase as the dependent variable approaches to

⁵ I am aware that the term "*sufficiently small vicinity*" does not represent a formal definition for the situation. I do not consider such formal definition necessary here since I am only interested in describing PSTs' understandings of $f(x)$ having a local extreme value. In other words, I am interested in examining if PSTs can imagine $f(x)$ increases toward a maximum value without exceeding it or decreases toward a minimum value without taking a smaller value.

its extreme value. This allowed PSTs to realize that there is no finite value $x = a$ such that $f(a)$ equals its extreme value. PSTs imagined that there is always a value of x such that $f(x)$ is *closer* to its extreme value than it was before. Finally, PSTs needed to reason about the co-variation represented by $f(x)$ (at least) at the *amounts of change level* (L3). Conceiving monotonically asymptotic variation requires PSTs to imagine $f(x)$ as changing by smaller and smaller amounts for equal changes in x , as it approaches to its extreme value.

PSTs did not demonstrate conceiving of *monotonically asymptotic variation* when constructing their graphs of $F(C)$. In order to conceive of monotonically asymptotic variation for $F(C)$, PSTs needed to: conceptualize $45 \text{ J/m}^2/\text{s}$ as the maximum forcing, imagine C to continue to increase, and reason about $F(C)$ in terms of amounts of change. Pam and Jodi initially drew lines to represent $F(C)$, but these lines showed $F(C)$ taking values larger than $45 \text{ J/m}^2/\text{s}$. They scratched that section of their lines after conceptualizing $45 \text{ J/m}^2/\text{s}$ as the maximum possible forcing. This realization involved the understanding that a graph of $F(C)$ could not contain points with ordinate values larger than $45 \text{ J/m}^2/\text{s}$. All three PSTs imagined a finite value $C = C_{\text{opa}}$ such that $F(C_{\text{opa}}) = 45 \text{ J/m}^2/\text{s}$. Jodi and Kris decided that their lines should stop at the point $(C_{\text{opa}}, 45)$. This suggests that they conceptualized $C = C_{\text{opa}}$ as the maximum possible concentration of CO_2 in the atmosphere (i.e., an atmosphere completely filled with CO_2), which meant that C could not exceed C_{opa} . Unlike Jodi and Kris, Pam drew the horizontal line $F = 45$ for $C > C_{\text{opa}}$, which suggests that she imagined C to continue to increase, exceeding $C = C_{\text{opa}}$. She, however, did not conceive of monotonically asymptotic variation for $F(C)$ either. Pam did not demonstrate reasoning about the co-variation represented by $F(C)$ at the amounts of change level, which would explain why she did not conceive of monotonically asymptotic variation.

While constructing their graphs of $N(t)$, PSTs needed to reason about whether $N(t)$ either took the value zero or asymptotically approached zero. This provided further evidence concerning the three understandings needed for conceiving monotonically asymptotic variation. For the case of $N(t)$, PSTs needed to: conceptualize zero as the minimum energy imbalance, imagine t to continue to increase, and reason about $N(t)$ in terms of amounts of change. All three PSTs demonstrated conceptualizing zero as the minimum energy imbalance. Jodi and Kris, however, did not demonstrate conceiving monotonically asymptotic variation for $N(t)$. They stated that $N(t)$ must eventually take the value zero because the energy budget actually reaches a radiative equilibrium at some point in time. They drew concave-up, decreasing curves incident to the horizontal axis at some finite $t = a$ or, mathematically, $N(a) = 0$. This suggests that Jodi and Kris did not imagine t to continue to increase such that $N(t)$ can always take values *closer* to zero than before. Unlike them, Pam demonstrated conceiving monotonically asymptotic variation for $N(t)$. Pam determined values of $N(t)$ for particular time values $t = 0, h,$ and $2h$ (for some $h > 0$). She noticed that the values of $N(t)$ decreased by smaller and smaller amounts between time values. For her, such pattern suggested that $N(t)$ was approaching zero without ever taking such value for a finite time. This suggests that Pam imagined t to continue to increase. Finally, Pam demonstrated reasoning about the co-variation represented by $N(t)$ at L3. This allowed her to imagine $N(t)$ approaching zero by smaller and smaller amounts, hence producing a concave-up, decreasing curve with an asymptote at $N = 0$.

When constructing the graph of $T(t)$, all three PSTs drew curves suggesting a horizontal asymptote. Kris, however, was the only PST who demonstrated clear evidence of conceiving monotonically asymptotic variation. All three PSTs demonstrated imagining $T(t)$ approaching to a particular value over time. While Pam and Jodi imagined $T(t)$ returning to the original

equilibrium temperature, or the value of $T(t)$ before the positive forcing (a minimum value), Kris imagined $T(t)$ increasing toward a new higher equilibrium temperature (a maximum value). Additionally, all three PSTs demonstrated imagining t to continue to increase. This time variation appeared to be consistent with our *ordinary* perception of time, which may explain why all three PSTs did not have difficulties imagining time to continue to increase. Pam and Jodi did not demonstrate reasoning about $T(t)$ in terms of amounts of change (their reasoning remained at L2). This suggests that the concavity of their curves was not supported by imagining $T(t)$ decreasing toward a minimum value by smaller and smaller amounts. In other words, their curves did not represent monotonically asymptotic variation albeit their curve appeared to exhibit a horizontal asymptote. In contrast, Kris demonstrated reasoning about $T(t)$ in terms of amounts of change. She concluded that $T(t)$ was increasing by smaller and smaller amounts and drew a concave-down, increasing curve to represent it. Kris concluded that $T(t)$ was increasing toward a new higher equilibrium temperature and drew a horizontal asymptote to represent this. This suggests that Kris conceived monotonically asymptotic variation when constructing the graph of $T(t)$.

Relation between Conceptions of Intensive Quantities and Reasoning about Functions

The study also examined relationships between its two parts guided by the research question *in what ways do PSTs' conceptions of intensive quantities relate to PSTs' understanding of the functions involved in this study?* In particular, the functions involved in modeling global warming make use of concentration measured in ppmv and energy flux density. Thus, I was interested in exploring the extent to which PSTs' conceptions of these two intensive quantities relate to their reasoning regarding the functions involved in this study. The analysis of PSTs' responses unveiled that their conceptions of concentration and energy flux density were

not necessarily transferred into them making sense of functions involved in modeling global warming. For the case of the forcing by CO₂ function $F(C)$, all three PSTs did not demonstrate difficulties conceptualizing variation in the independent variable C (the atmospheric concentration of CO₂) for changes in the volume of CO₂ in the atmosphere. This may be explained by the direct proportional relationship between C and the volume of CO₂. Students have significantly less difficulties reasoning about the relationship between an intensive quantity and one its constituent quantities when they are directly proportional (Nunes et al., 2003). PSTs, however, did not demonstrate being aware of the maximum value of C when learning about $F(C)$. When PSTs reasoned about the minimum and maximum values of $F(C)$, they needed to imagine two scenarios: an atmosphere containing zero CO₂ (completely transparent) and an atmosphere completely formed by CO₂ (completely opaque). PSTs demonstrated no difficulties stating that the first case meant $C = 0$ ppmv. They, however, could not determine a value of C corresponding to the second case. All PSTs conceived of a finite value $C = C_{\text{opa}}$ such that $F(C_{\text{opa}}) = 45 \text{ J/m}^2/\text{s}$, where C_{opa} represent the concentration of CO₂ for a completely opaque atmosphere. Jodi's and Kris's graphs of $F(C)$ suggests that they conceptualized $C = C_{\text{opa}}$ as the maximum possible concentration of CO₂ in the atmosphere (or an atmosphere full of CO₂). They, however, did not state that $C_{\text{opa}} = 1,000,000 \text{ ppmv}$ ⁶, which suggests that they might not have been thinking about C in terms of concentration measured in ppmv. Pam did not demonstrate conceptualizing C_{opa} as the maximum concentration of CO₂. Her graph suggests that C continues to increase indefinitely. This suggests that Pam was not aware that C had a maximum value of 1,000,000 ppmv. Although all PSTs demonstrated a *Ratio as Measure* conception of concentration while

⁶ If the atmospheric concentration of CO₂ is 1,000,000 ppmv, then the atmosphere is completely formed by CO₂ since there are 1,000,000 parts of CO₂ per every 1,000,000 parts of air.

working on S1T1 and S1 T2, they did not apply such conception when learning working on S2T1.

The planetary energy imbalance $N(t)$ is of particular interest because of two reasons: (1) $N(t)$ is connected to the mean surface temperature of the planet $T(t)$ since $N(t)$ is a measure of the rate of change of $T(t)$, and (2) $N(t)$ as a difference between downward radiation and upward radiation is the rate at which the surface energy is changing. These two notions rely on conceptualizing $N(t)$ as a magnitude of energy flux density (Joules per square meter per second), which suggests that PSTs' conceptions of such intensive quantity could have influenced their interpretation of $N(t)$. All three PSTs, however, did not draw connections between $N(t)$ and $T(t)$. When they drew their graphs of $T(t)$, PSTs relied on their understanding of changes in the energy flows in the energy budget, particularly the flows R and A . None of the PSTs interpreted their graphs of $N(t)$ in terms of the properties of the graphs of $T(t)$. Pam's *Ratio as Per-Composite-One* of energy density did not support conceptualizing such intensive quantity as a measure of how rapidly the sheets' temperature was increasing. This may explain why she did not construct her graph of $T(t)$ based on her graph of $N(t)$. The cases of Jodi and Kris, however, were intriguing. Although their conceptions were different, Jodi's *Ratio as Rate-Per-One* and Kris's *Ratio as Rate-Measure* did involve identifying energy flux density as a measure of how rapidly the sheets' temperature was increasing. It seemed, however, that such connection was not sufficiently stable since neither Jodi nor Kris applied their conceptions of energy flux density to make sense of the connection between $N(t)$ and $T(t)$. This finding seems consistent with Johnson's (2015) characterization of a relationship between students' conceptions of ratios and their understanding of the rate of change of a function. In her study, she found that students who did not see ratios as intensive quantities had difficulties reasoning in terms of rate of change of a

function. In my study, PSTs' conceptions of energy flux density impacted their ability to conceive of $N(t)$ as a rate of change of $T(t)$.

When working with the function $N(t)$, PSTs drew the same graph: a concave-up, decreasing curve. Their interpretation of $N(t)$ as a quantity, however, were qualitatively different. Pam and Jodi demonstrated conceptualizing $N(t)$ as the magnitude of energy *within* the planet's surface. This conception of $N(t)$ appears consistent with Pam's *Ratio as Per-Composite-One* conception of energy flux density (energy *per* time-area composite box). The Ratio as Per-Composite-One seems to emphasizing energy over the composite object, probably because this object may not be conceptualized as a quantity. Additionally, a Ratio as Per-Composite-One did not involve conceptualizing energy flux density as a measure of the *rapidity of change of energy density*. This appears consistent with Pam not seeing $N(t)$ as a rate of change of the surface energy; for her, $N(t)$ was energy and not an intensive quantity measuring a rate of change. Jodi's *Ratio as Rate-Per-One* conception (energy density in a 1×1 square *per* second) appeared, at least in theory, consistent with conceptualizing $N(t)$ as a rate. She, however, did not demonstrate applying her conception of energy flux density in order to make sense of $N(t)$. It is possible that Jodi's conception of energy flux density was dependent on performing the division between radiation rate and surface area. When working with $N(t)$, Jodi did not have a chance to do that, which may explain why she did not see a magnitude of $N(t)$ as a rate.

PSTs' conceptions of $N(t)$ shaped their understanding of radiative equilibrium. Pam and Jodi interpreted $N(t)$ as a magnitude of energy within the surface. When they saw that $N(t)$ was decreasing after the positive forcing, they concluded that the energy content in the surface was decreasing after the forcing made it *jump up*. This led them to think that the energy budget was returning to the *original radiative equilibrium*; that is, the radiative equilibrium before the

positive forcing occurred. In contrast, Kris demonstrated understating that the concept of radiative equilibrium was not unique; that is, the budget can reach different radiative equilibriums. She interpreted the decreasing magnitude of $N(t)$ as an indication that the budget was reaching a radiative equilibrium. Also, she demonstrated attending to the sign of $N(t)$ as a difference of downward radiation and upward radiation. Kris interpreted the positive value of $N(t)$ for all $t > 0$ as an indication that the energy in the planet's surface was increasing. These two realizations seemed to have led Kris to conclude that the energy budget was reaching a *new radiative equilibrium*, different from the original radiative equilibrium before the positive forcing. Kris's attention to the sign of $N(t)$ appeared consistent with her *Ratio as Rate-Measure* conception of energy flux density (energy density *per second*). A positive $N(t)$ may indicate an increasing surface energy content when $N(t)$ is conceptualized as the rate of change of energy density.

Conclusions and Implications

By examining PSTs' understandings regarding intensive quantities and functions involved in introductory mathematical models for global warming, my dissertation study makes two fundamental contributions to mathematics education research. First, it generates new knowledge about ways that mathematics and science can be studied together, which addresses recent calls to focus on integrated STEM education (Baldwin, 2009; McCright, O'Shea, Sweeder, Urquhart, & Zeleke, 2013). The study shows that the PSTs' conceptions of mathematical concepts interacted with their conceptions of scientific concepts as they were making sense of introductory mathematical models for global warming. A sound understanding of temperature is needed to make sense of what quantities such as energy density and energy flux density were measuring in the context of the situations. PSTs needed to conceptualizing energy

density as *a measure of the surface temperature* and energy flux density as *a measure of the rate of change of surface temperature* in order to make sense of introductory mathematical models for global warming. To develop these conceptions, PSTs needed to clearly understand the relationship and the difference between the concepts of temperature and heat (or energy). Additionally, the study's findings suggest that a *Ratio as Measure* conception of energy density and a *Ratio as Rate-Measure* conception of energy flux density supported conceiving these quantities as measures of temperature and the rate of change of temperature, respectively.

The study's findings showed that reasoning about co-variation regarding three important functions was central to understand the link between carbon dioxide pollution and global warming. Particularly, the findings showed that reasoning at *the amounts of change level (L3)* was necessary to make sense of *the forcing by CO₂ function F(C)*, *the planetary energy imbalance function N(t)*, and *the mean surface temperature function T(t)*. To make sense of each function, PSTs needed to identify *what quantities were co-varying* (dependency of change) and *in what direction these quantities were co-varying* (direction of change) in the Earth's energy budget. For instance, energy flows B and A increases when the atmospheric CO₂ concentration C increases, or R, B, and A increase over time t after a positive forcing. By attending to the direction of change, it is possible to conclude that F(C) is an increasing function and N(t) is a decreasing function. Since F(C) connects CO₂ concentration with rate of heat transfer (energy flux density), reasoning at *the direction of change level (L2)* is necessary to establish a connection between carbon dioxide pollution and global warming.

PSTs made sense of the relationship between N(t) and the concept of radiative equilibrium by reasoning about co-variation in terms of amounts of change. The energy flows R and A increase over time after a positive forcing, but they do so at a decreasing rate. Pam and

Kris noticed that R and A were increasing by smaller and smaller amounts for equal changes in time. Since the values appeared to be *stabilizing*, they concluded that the Earth's energy budget was approaching a radiative equilibrium after a forcing. By attending to variation in the intensity of change (Johnson, 2012), it is possible to conclude that $N(t)$ is a function decreasing less and less rapidly (formally, this is a negative and increasing rate of change), which is represented by a concave-up, decreasing curve asymptotically approaching zero. Since $N(t)$ measures the energy imbalance in the Earth's energy budget, such curve indicates that the budget tends to radiative equilibrium ($N(t) = 0$) after an energy imbalance caused by CO_2 pollution (positive forcing). Kris also interpreted R and A increasing by smaller and smaller amounts as an indication that the energy budget was gaining energy over time after a positive forcing. This energy gain translates into an increase of the planet's surface temperature. This means that, after an energy imbalance caused by CO_2 pollution, the surface temperature increases in order for the budget to reach a *new radiative equilibrium*. This indicates that $T(t)$ is a function increasing less and less rapidly (a positive and decreasing rate of change), which is represented by a concave-down, increasing curve. Thus, reasoning at *the amounts of change level* (L3) is necessary to understand the concept of radiative equilibrium, its implications for global warming, and long-term effects of CO_2 pollution over the planet's surface temperature.

In summary, the study's findings emphasized the importance of combining research on mathematics education and science education in order to examine how PSTs can make sense of mathematical and scientific concepts behind complex phenomenon such as global warming and its environmental implications. These topics are consistent with both science and mathematics education standards. Particularly, it relates to the Next Generation Science Standards HS-ESS 2 through 4 (NRC, 2013) and to the Common Core State Standard for Mathematics standards HS-

NQ 1 through 3, HS-F-IF 2, HS-F-BF 1, and modeling with mathematics in general (CCSSI, 2010). By working on the tasks in this study, PSTs had opportunities to explore scientific concepts such as temperature, the Earth's energy budget, radiative equilibrium, and the greenhouse effect, in order to make sense of global warming. They also had opportunities to develop their understanding of intensive quantities, ratio and rates, functions and co-variation, properties of graphs, and rate of change, in order to make sense of global warming. Thus, this study provides a set of tasks to answer Barnwell's (2013a) call to help teachers and students learn about these issues. This has implications for teacher education programs since teacher educators can use these tasks to include global warming into their mathematics teacher preparation courses. The tasks not only help PSTs explore mathematics related to global warming, but also represent material that can be used with high school students. By using these tasks, teacher educators can provide their PSTs with material they could use with their future students.

The second contribution to educational research is that this study fills an important gap in the literature regarding research connecting mathematics learning and global warming. Specifically, the study discussed ways in which *quantitative reasoning* and *covariational reasoning* can support or constrain PSTs' ability to make sense of mathematical models for real-world situations. To make sense of the introductory mathematical models for global warming, PSTs needed to make sense of intensive quantities –especially concentration and energy flux density– and three important functions involved in such models. As a result, this dissertation combines distinct strands of mathematics education research that have not been combined within a single study: intensive quantities, co-variation, mathematical modeling, and global warming.

This study adds two intrinsic characteristics of intensive quantities that may influence a person's meaning for them. First, in order to make sense of *what was being measured* by a particular intensive quantity, PSTs needed to conceptualize such quantity as a *constant multiplicative relationship between co-varying quantities*. In other words, PSTs needed to conceive that the ratio remains constant for different values of the constituent quantities. This helped PSTs identify what was an invariant attribute of the situation that could be measured. Second, the way that division is used to measure such attribute (measurement process) was dependent on the type of intensive quantity (Type 1 versus Type 2) and conceptions of the constituent quantities. PSTs made use of *measurement division* when working with a Type 1 intensive quantity (concentration) and *partitive division* when working with a Type 2 intensive quantity (energy density and energy flux density). Their conceptions regarding the constituent quantities also affected the measurement process. PSTs seemed to conceive surface area as the number of 1×1 squares covering a sheet's surface. For energy density, PSTs used partitive division to equally distribute energy among several 1×1 squares, thus interpreting this intensive quantity as a magnitude of energy per 1×1 square. This conception appeared to hinder PSTs' ability to conceptualize an intensive quantity as a rate. For the particular case of energy density, PSTs appeared to emphasize energy over surface area probably because they saw the area unit as a geometric object, which meant that energy was the only quantity involved in the ratio. This may explain why Pam and Jodi conceptualized the planetary energy imbalance $N(t)$ as a magnitude of energy instead of the rate at which the surface's energy content was changing.

Mathematical models for global warming commonly known in climate sciences as Energy Balance Models (EBMs) "rest on the idea that energy received by the planet is balanced by the energy it emits. Any imbalance in the system will result in a temperature change"

(Widiasih, 2013, p. 2068). In other words, EBMs rest on the idea that $\frac{dT}{dt} = \alpha N(t)$ for some constant $\alpha > 0$. In this study, PSTs were asked to draw the graph of the mean surface temperature function $T(t)$ by reflecting on their graphs of the planetary energy imbalance function $N(t)$. They, however, did not demonstrate understanding of the relationship between these two functions since none made use of the graph of $N(t)$ to construct a graph of $T(t)$. One possible explanation is that PSTs did not conceptualize such relationship between $N(t)$ and $T(t)$ because they did not demonstrate reasoning about co-variation beyond the amounts of change level (L3). At such level, PSTs could not think about the rate of change of $T(t)$, and thus they did not see a connection between it and $N(t)$. This suggests that that PSTs need to reason at the rate of change levels (*the average rate of change level LA or the instantaneous rate of change level L5*) in order to make sense of the EBMs for global warming. Thus, the study's findings have implications for researchers, teacher educators and teacher education programs in both mathematics and science education. The results provide examples about how PSTs think about intensive quantities and functions related to mathematical models of global warming. Researchers who study these areas can build on the knowledge learned from this study. Teacher educators can use the results from this study to structure teacher preparation courses that help PSTs develop understanding or correct misconceptions regarding the quantities and functions used to model global warming. The study's findings also emphasized the importance of incorporating scientific concepts in mathematics teacher education courses in order to promote PSTs' ability to model real-world situations. Mathematics teacher educators can use the results of this study to structure their teacher preparation courses so that they include scientific concepts or to incorporate scientific concepts into existing courses.

Future Research

This study investigated PSTs' understanding of intensive quantities and functions used in introductory mathematical models for global warming. One extension of this work is to develop and test effective instructional sequences for teaching the mathematics of global warming. This work might take the form of teaching experiments (Steffe & Thompson, 2000) centered around instructional strategies that can help PSTs: (a) develop a *Ratio as Measure* conception of concentration and energy density, and a *Ratio as Rate-Measure* conception of energy flux density; (b) reason about co-variation at *the instantaneous rate of change level* (L5) when working with $F(C)$, $N(t)$, and $T(t)$; and (c) make sense of these functions when the atmospheric CO_2 concentration is a function of time $C(t)$. By examining these sequences, researchers can identify conditions under which PSTs can develop *Ratio as Measure* and *Ratio as Rate-Measure* conceptions for intensive quantities used in modeling global warming. Also, researchers can identify which instructional strategy may promote PSTs' ability to reason about co-variation at the rate levels while modeling real-world situations such as global warming.

Although this study did not explicitly examine PSTs learning of mathematical and scientific concepts related to global warming, future studies can explore what PSTs learn from working on tasks like the ones designed for this study. These researchers can assess what PSTs learn about intensive quantities such as concentration and energy flux density or functions as they work on such tasks. This research can also assess what PSTs learn about scientific concepts such as heat, temperature, radiation, the greenhouse effect, or the Earth's energy budget. Additionally, it is important to examine the impact of such tasks on PSTs' beliefs regarding: (a) the reality of global warming, (b) the role human activity has in driving it, and (c) global warming's social, economic, and environmental consequences. These studies might include pre-

and post-measures of PSTs' understandings about intensive quantities and co-variation to target the ways they deepen their understandings and connect these understandings to issues of global warming. By extension, examining these issues can assist researchers and educators identify how PSTs (and their future students) can make informed decisions regarding the impact of global warming in their lives and in the lives of others.

Another potential area of investigation is to study how PSTs learn about the mathematics behind other concepts related to global warming. One of these concepts is the *global warming potential*, which represents a complex intensive quantity created to measure the warming potential of greenhouse gases emissions. By learning about such quantity, PSTs can discuss how to quantify the impact of global warming in the planet's temperature and climate. Another concept involves the family of mathematical models known as Energy Balance Models (EBMs). These models can be described by the differential equation

$$c \frac{dT}{dt} = F(t) - \lambda^{-1}[T(t) - T_0]$$

where $F(t)$ and $T(t)$ are the forcing by CO_2 and the mean surface temperature in function of time, c and λ are two positive constants, and T_0 represents an arbitrary initial surface temperature.

Notice that the present study is a particular case of such differential equation, a case when $F(t) = F_0$ constant. After the positive forcing F_0 occurs, $T(t)$ increases since its rate of change is positive. As $T(t)$ increases, its rate of change decreases in magnitude, which results in $T(t)$ increasing less and less rapidly as it approaches to a new equilibrium temperature. That was the case presented in this study. By studying such equations, PSTs can learn how the rate of change of a quantity connects to the magnitude of such quantity, and how by knowing the magnitude of the quantity at certain time and its rate of change, predictions are possible. This research can add to the literature on mathematics education and global warming education by providing

information regarding concepts, conceptions, and tasks involved in exploring and learning the mathematics and science behind global warming. Mathematics teacher educators can incorporate concepts and create tasks regarding the mathematics and science of global warming as they design teacher preparation courses. All in all, I believe that, by studying global warming, PSTs and their future students can experience how mathematics and science, together, can help them understand environmental changes that are reaching all corners of our planet (Intergovernmental Panel on Global warming [IPCC], 2013).

REFERENCES

- Baldwin, R. (2009). Climate change: Creating conditions conducive to quality STEM undergraduate education. *New Directions for Teaching and Learning*, 117, 97-101.
- Barwell, R. (2013a). Climate change. In A. Coles, R. Barwell, T. Cotton, J. Winter & L. Brown (Eds.), *Teaching mathematics as if the planet matters* (pp. 31-49). New York, NY: Routledge.
- Barwell, R. (2013b). The mathematical formatting of climate change: Critical mathematics education and post-normal science. *Research in Mathematics Education*, 15(1), 1-16.
- Beckmann, S. (2014). *Mathematics for elementary teachers* (4th ed.). Boston, MA: Pearson.
- Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 33(5), 352-378.
- Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, 33(3), 31-37.
- Common Core State Standards Initiative (CCSSI). (2010). *Common Core State Standards for mathematics*. Retrieved from http://www.corestandards.org/assets/CCSSi_Math%20Standards.pdf
- Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. Kelly & R. A. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education* (pp. 517–545). Mahwah, NJ: Lawrence Erlbaum Associates.

- Greer, B. (1992). Multiplication and division as models of situations. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 243-275). New York, NY: Macmillan.
- Gregory, J. M., & Forster, P. M. (2008). Transient climate response estimated from radiative forcing and observed temperature change. *Journal of Geophysical Research*, *113*, 1-15
- Harrison, A., Grayson, D., & Treagust, D. (1999). Investigating a grade 11 student's evolving conceptions of heat and temperature. *Journal of Research in Science Teaching*, *36*(1), 55-87.
- Howe, C., Nunes, T., & Bryant, P. (2010). Intensive quantities: Why they matter to developmental research. *British Journal of Developmental Psychology*, *28*, 307-329.
- Intergovernmental Panel on Climate Change (IPCC). (2013). *Climate Change 2013: The Physical Science Basis. Contribution of Working Group I to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change*. In T. F. Stocker, D. Qin, G. K. Plattner, M. Tignor, S. K. Allen, J. Boschung, A. Nauels, Y. Xia, V. Bex & P. M. Midgley (Eds.). New York: Cambridge University Press.
- Johnson, H. L. (2015). Secondary students' quantification of ratio and rate: a framework for reasoning about change in covarying quantities. *Mathematical Thinking and Learning*, *17*, 64-90.
- Johnson, H. L. (2012). Reasoning about variation in the intensity of change in covarying quantities involved in rate of change. *The Journal of Mathematical Behavior*, *31*, 313-330.
- Lambert, J. L., & Bleicher, R. E. (2013). Climate change in the preservice teacher's mind. *Journal of Science Teacher Education*, *24*, 999-1022.

Leiserowitz, A., Smith, N. & Marlon, J.R. (2010) *Americans' Knowledge of Climate Change*.

Yale University. New Haven, CT: Yale Project on Climate Change Communication.

<http://environment.yale.edu/climate/files/ClimateChangeKnowledge2010.pdf>

Lenton, T. (2000). Land and ocean carbon cycle feedback effects on global warming in a simple Earth system model. *Tellus*, 52B, 1159-1188.

McCright, A., O'Shea, B., Sweeder, R., Urquhart, G., & Zeleke, A. (2013). Promoting interdisciplinarity through climate change education. *Nature Climate Change*, 3, 713-716.

McKeown, R., & Hopkins, C. (2010). Rethinking climate change education: Everyone wants it, but what is it? *Green Teacher*, 89, 17-21.

National Research Council (NRC). (2013). *The next generation science standards*. Washington, D.C.: The National Academies Press.

Nunes, T., Desli, D., & Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research*, 39, 651-675.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

Prince, M., Vigeant, M., & Nottis, K. (2011). Using inquiry-based activities to repair student misconceptions related to heat, energy, and temperature. *Proceedings of the ASEE Annual Conference, Vancouver, BC, Canada*.

Pruneau, D., Khattabi, A., & Demers, M. (2010). Challenges and possibilities in climate change education. *US-China Education Review*, 7(9), 15-24.

- Saldanha, L., & Thompson, P. W. (1998). Re-thinking co-variation from a quantitative perspective: Simultaneous continuous variation. In S. B. Berensah & W. N. Coulombe (Eds.), *Proceedings of the Annual Meeting of the Psychology of Mathematics Education - North America*. Raleigh, NC: North Carolina State University.
- Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale, NJ: Lawrence Erlbaum.
- Simon, M. (2006). Key developmental understanding in mathematics: A direction for investigating and establishing learning goals. *Mathematical Thinking and Learning*, 8(4), 359-371
- Simon, M., & Placa, N. (2012). Reasoning about intensive quantities in whole-number multiplication? A possible basis for ratio understanding. *For the Learning of Mathematics*, 32, 35-41
- Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In R. Lesh & A. E. Kelly (Eds.), *Research design in mathematics and science education* (pp. 267-307). Hillsdale, NJ: Erlbaum.
- Stephens, G., Li, J., Wild, M., Clayson, C. A., Loeb, N., Kato, S., ... Andrews, T. (2012). An update on Earth's energy balance in light of the latest global observation. *Nature Geoscience*, 5, 691-696
- Thomaz, M. F., I. M. Malaquias, M. C. Vanente, & M. J. Antunes. (1995). An attempt to overcome alternative conceptions related to heat and temperature. *Physics Education* 30, 19-26.

- Thompson, P. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education* (Vol. 1, pp. 33-57). Laramie, WY: University of Wyoming.
- Thompson, P. (1994a). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics*, 26(2-3), 229-274.
- Thompson, P. W. (1994b). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in Collegiate Mathematics Education, 1* (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.
- Thompson, P. (1994c). The development of the concept of speed and its relationship to concepts of rate. In Harel, G., & Confrey, J. (Eds.) *The Development of Multiplicative Reasoning in the Learning of Mathematics*, pp. 181-234. Albany, NY: SUNY press.
- UNESCO. (2012). Climate change education as an integral part of the United Nations Framework Convention on Climate Change. *Journal of Education for Sustainable Development*, 6(2), 237-239.
- Wachholz, S., Artz, N., & Chene, D. (2014). Warming to the idea: University students' knowledge and attitudes about climate change. *International Journal of Sustainability in Higher Education*, 15(2), 128-141.
- Ward, D. J., Furber, C., Tierney, S., & Swallow, V. (2013). Using Framework Analysis in nursing research: a worked example. *Journal of Advanced Nursing*, 69(11), 2423–2431.
- Widiasih, E. R. (2013). Dynamics of the Budyko energy balance model. *SIAM Journal of Applied Dynamical Systems*, 12(4), 2068-2092.

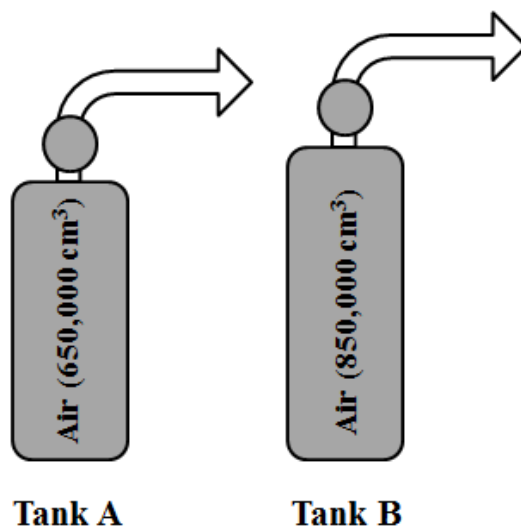
Yin, R. K. (2013). *Case Study research: Design and methods* (Fifth Edition ed.). Thousand Oaks, CA: Sage Publications Inc.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, 8, 103–122

APPENDIX A

Mathematical Tasks & Interview Protocol (Set 1)**Set 1 – Task 1**

The air you breathe is a mixture of gases such as nitrogen (N), oxygen (O₂), and other trace gases. Diving tanks contain air and come in different sizes depending on how much air you would need for diving. Let's consider two diving tanks of different size (see Figure below): Tank A contains a total air volume of 650,000 cubic centimeters (cm³), while Tank B contains a total air volume of 850,000 cubic centimeters (cm³).



The composition of each tank's air is:

Tank	Nitrogen (cm ³)	Oxygen (cm ³)	Others gases (cm ³)
A	503,000	136,500	10,500
B	667,000	161,500	21,500

Assume both tanks have the same pressure. If you take one breath of air from each tank (assume you breathe in the same amount of air from each tank), then:

Which breath of air contained more oxygen? Can you define a quantity that measures which tank provides breaths of air with more oxygen?

Protocol Set 1 – Task1

Questions if participants provide an answer

- How did you find the tank that provides breaths of air with more oxygen?
- Could you explain why you decide to divide the O₂ volume by the total air volume? What is the meaning of division in this situation?
- What is the meaning of the ratio's value in this situation?

Questions if participant struggles to provide an answer

- What is meant by a “breath of air with more oxygen” in this situation?
- Tank B has more oxygen than Tank A. Does that mean a breath of air from Tank B would contain more oxygen than a breath of air from Tank A? Why?
- If you could breathe in 500 cm³ of air in one inhalation, then how much O₂ would you inhale from each tank? Why?
- Dividing the O₂ volume by the air volume gives you the O₂ concentration. Can this quantity help you decide which tank gives breath of air with more oxygen?
- What is the meaning of the division in this situation?
- What is the meaning of the ratio's value in this situation?

Follow-up questions

- What is the difference between O₂ volume and O₂ concentration? What exactly is each of these quantities measuring about oxygen?
- Let V_x , V_a , and Q_x be the volume of a gas X in an air mixture, the volume of air mixture, and the concentration of the gas X in the mixture, respectively. How would you define gas X concentration mathematically? Can you give me a “formula”?
- How does O₂ concentration relate to O₂ volume (holding air volume constant)? What about O₂ concentration and air volume (holding O₂ volume constant)?
- In what unit is O₂ (or any gas) concentration measured? For instance, what is one unit of O₂ concentration?

Set 1 – Task 2

The concentration of a gas X (denoted by Q_x) in an air mixture is the ratio:

$$Q_x = \frac{\text{Volume of gas X (cm}^3\text{)}}{\text{Volume of air mixture (cm}^3\text{)}}$$

The ratio's value represents the average volume of gas X (in cm^3) contained in 1 cm^3 of air.

Diving tanks also contain a small volume of carbon dioxide (CO_2) in their air. The table below shows two diving tanks' total air volume and the CO_2 volume contained in their air.

Tank	Air volume (cm^3)	CO_2 volume (cm^3)
A	4,000,000	1,448
B	800,000	316

Solve the following:

- Using the definition of gas concentration above, calculate each tank's CO_2 concentration (denoted by Q_{CO_2}). Interpret your result in the context of this situation.
- When concentrations are small, they are often measured in *ppmv* units (parts per million by volume units). This unit represents the volume of a gas (in cm^3) contained in $1,000,000 \text{ cm}^3$ of air. Diving tanks must have a CO_2 concentration between 350 ppmv and 400 ppmv in order to have safe air for a diver. Calculate each tank's CO_2 concentration in *ppmv* and check whether these tanks have safe air for a diver.

Protocol Set 1 – Task 2

Questions if participants provide an answer

- How did you find the CO₂ concentration in *ppmv* for Tank A? How did you find the CO₂ concentration in *ppmv* for Tank B? Did you use the same method? Why?
- Could you explain the meaning of the division in this situation?
- What is the meaning of the ratio's value in this situation?
- How did you figure out you could use the calculation in 1) to find CO₂ concentration in *ppmv*? (assuming they connect with the concept of concentration in 1))
- Which tank has a safe air for the diver?

Questions if participants struggle to provide an answer

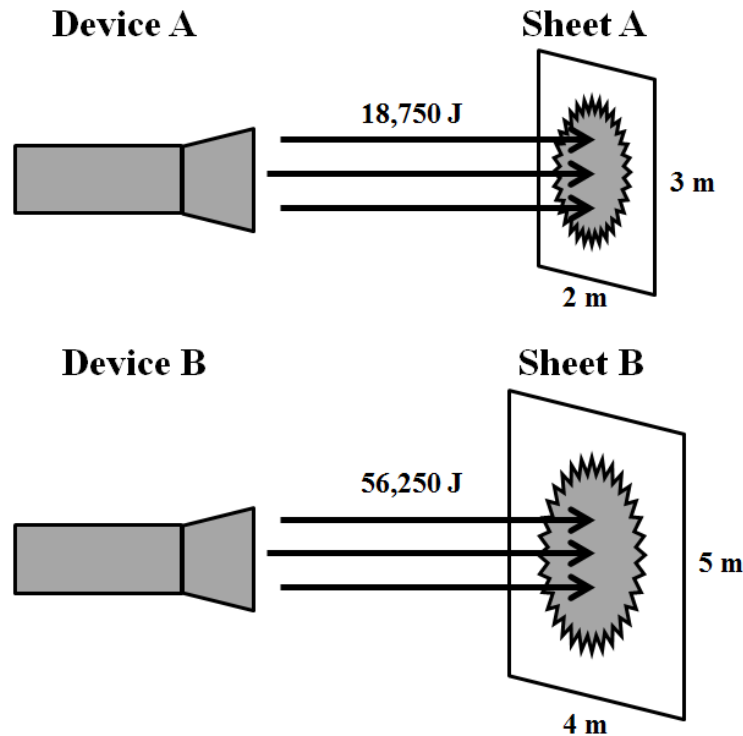
- What is meant by “parts per million by volume”? “Parts” of what? “Per million” of what?
- In Tank A, there is a CO₂ volume of 1,448 cm³ contained in 4,000,000 cm³ of air. How much CO₂ volume would there be in only 1,000,000 cm³ of air? How did you figure it out?
- Is the answer (above) the CO₂ concentration in *ppmv* for Tank A? Why?
- Can you repeat the process to obtain the CO₂ concentration in *ppmv* for Tank B? How much CO₂ would there be in 1,000,000 cm³ of air?
- Remember the ratio you calculated 1). This ratio represents the CO₂ volume (in cm³) contained in 1 cm³ of air. Can you use that to obtain the concentration in *ppmv* for each tank?
- Which tank has a safe air for the diver?

Follow-up questions

- Let V_x , V_a , and P_x be the volume of a gas X in an air mixture, the volume of air mixture, and the concentration of the gas X in *ppmv*, respectively. How would you define gas X concentration in *ppmv* mathematically? Can you give me a “formula”?
- In 1), Tank A's CO₂ concentration was 0.000362. In 2), Tank A's CO₂ concentration is 362 *ppmv*. What are these quantities measuring about CO₂ in the air mixture? Why?
- How would you convert from concentration in 1) to concentration in *ppmv*?

Set 1 – Task 3

Radiation is the transmission of energy from one object (source) to another (receiver). Radiation can rise the receiver's temperature since it increases the receiver's internal energy. In an experiment, two (very thin) rectangular sheets of a new metal known as *daridium* were positioned at the same distance from two devices that produce radiation (see Figure).



The radiation from Device A added 18,750 Joules (J) of energy to Sheet A's internal energy, while the radiation from Device B added 56,250 J of energy to Sheet B's internal energy. An object's temperature involves a comparison between the object's internal energy and the object's size. If both sheets were at room temperature (around 15 °C) at the beginning of the experiment, then:

Can you determine which sheet had a higher temperature by the end of the experiment? Can you define a quantity that measures which sheet will have a higher temperature by the end of the experiment?

Protocol Set 1 – Task 3

Questions if participants provide an answer

- Why did you decide to divide the energy added to each sheet by the surface area of the corresponding sheet?
- What is the meaning of the division in this situation?
- What is the meaning of this ratio's value in this situation?

Questions if participants struggle to provide an answer

- Why is it important to compare each sheet's internal energy to their size in order to anticipate any difference in temperature? Can two sheets with the same internal energy but different size have the same temperature? Why?
- If we change the sheets' dimensions so they are of the same size and we maintain the energy added by each device, then would both sheets have the same temperature by the end of the experiment? Why?
- If both devices radiate the same total energy and we maintain the sheets' dimensions, then do you think that the sheets would have the same average temperature? Why?
- Is there a way to relate the total energy radiated and each sheet's surface area that could help you explain differences in temperatures? How would you relate them?
- Divide the energy added by each device by the corresponding sheet's surface area. What is the meaning of the division in this situation?
- What is the meaning of the ratio's value in this situation?
- Can you use the ratio's value to anticipate which sheet will have a higher average temperature?

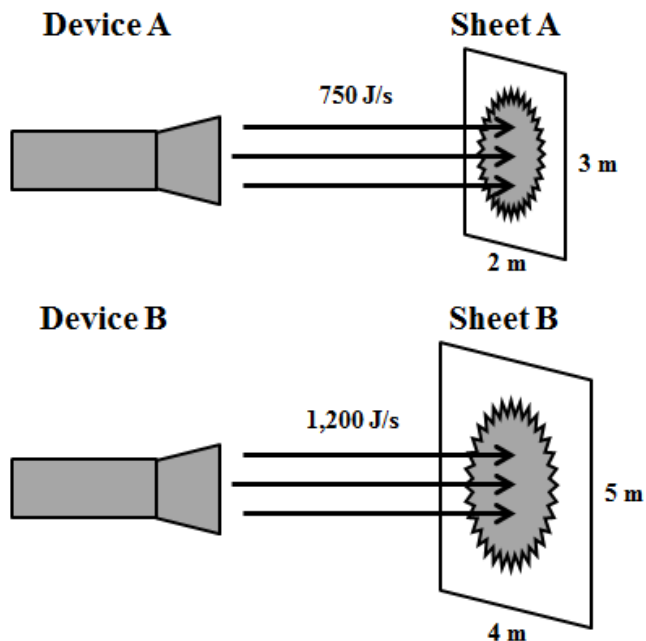
Follow-up questions

- The ratio you calculated is known as *energy density*. Let E , A , and D be the energy added (J), the surface area (m^2), and the energy density, respectively. How would you define energy density mathematically? Can you give me a "formula"?
- How does energy density relate to the energy added (holding surface area constant)? What about energy density and surface area (holding energy added constant)?

- Does the energy density represent the exact energy added to each 1m^2 of the sheet's surface area? Is this an average? Why?
- What relationship is there between temperature and energy density? Will two sheets with same energy density have the same temperature? Why?
- In what unit is energy density measured? For instance, what is one unit of energy density?

Set 1 – Task 4

Energy density and temperature are directly related. For instance, when the energy density of a sheet of *daridium* increases by $2,500 \text{ J/m}^2$ (Joules per square meter), the sheet's temperature rises by $4 \text{ }^\circ\text{C}$ (degree Celsius). In an experiment, two sheets of *daridium* were positioned at the same distance from two devices that produce radiation (see Figure).



Device A adds energy to Sheet A's internal energy at a rate of 750 J/s (Joules per second), while Device B adds energy to Sheet B's internal energy at a rate of $1,200 \text{ J/s}$. If both sheets were at room temperature (around $15 \text{ }^\circ\text{C}$) at the beginning of the experiment and both devices started producing radiation at the same time, then:

Which sheet will first reach a temperature of $25 \text{ }^\circ\text{C}$? Can you define a quantity that shows this?

Protocol Set 1 – Task 4

Questions if participants provide an answer

- Why did you decide to divide the rate at which energy is added to each sheet by the surface area of the corresponding sheet?
- What is the meaning of that division in this situation?
- What is the meaning of this ratio's value in this situation?

Questions if participants struggle to provide an answer

- If both sheets had a temperature of 15 °C at the beginning of the experiment, then their temperatures need to increase 10 °C in order to reach 25 °C. What does that mean in terms of energy density? How much would the sheets' energy density increase?
- Device B is radiating energy at a higher rate than Device A. Does that mean Sheet B's temperature will reach 25 °C before Sheet A's does? Why?
- If both sheets were of the same size and the devices' rates of radiation are the original rates, would both sheets' temperatures rise up to 25 °C at the same time? Why?
- If both devices were radiating energy at the same rate and sheets' dimensions are the original dimensions, would both sheets' temperatures rise up to 25 °C at the same time? Why?
- Is there a way to relate each device's rate of radiation to the corresponding sheet's surface area that could show which sheet's temperature will first rise up to 25 °C?
- Divide each device's rate of radiation by the corresponding sheet's surface area. What is the meaning of that division in this situation?
- What is the meaning of the ratio's value in this situation?
- Can you use the ratio's value to anticipate which sheet's temperature will rise up to 25 °C first?

Follow-up questions

- A device's rate of radiation is known as *energy flux* and the ratio between *energy flux* and the sheet's surface area is known as *energy flux density*. What is the meaning of each quantity in this situation? Are they providing the same information?

- In this task, another important quantity appeared: *energy density*. How does this quantity differ from *energy flux density* in this situation? Are they providing the same information?
- Let E , A , t , and S be amount of energy added (Joules), surface area (square meters), elapsed time (seconds), and the energy flux density, respectively. How would you define energy flux density mathematically? Can you give me a “formula”?
- In what unit is energy flux density measured? What is one unit of energy flux density?
- In relation to energy flux density, How does this quantity relate to the energy added (holding surface area and time constant)? What about energy flux density and surface area (holding energy added and time constant)? What about energy flux density and time (holding energy added and surface area constant)?

APPENDIX B

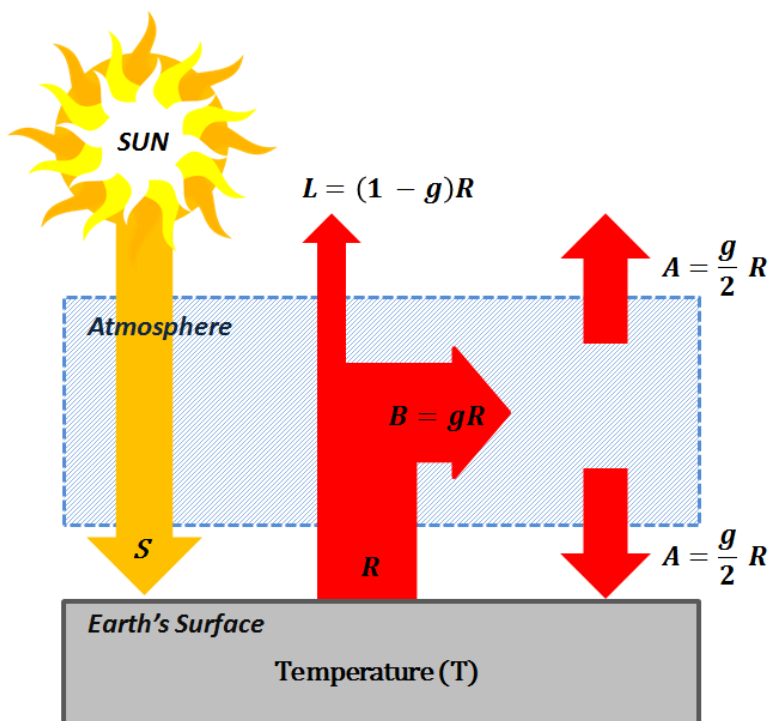
Mathematical Tasks & Interview Protocol (Set 2)**Introductory Videos**

Before participants start working on Set 2, I will show them two short Youtube videos that explain the concepts of *Earth's energy budget* and the *greenhouse effect*. The information in these videos will provide participants with necessary background to work on Set 2 tasks (see Table below)

Concept	Video URL
Earth's energy budget	https://www.youtube.com/watch?v=DOAqECd70Ww
Greenhouse effect	https://www.youtube.com/watch?v=ZzCA60WnoMk

The Planetary Energy Budget

The diagram below represents the Earth's energy budget. Participants will use this diagram to draw conclusions about how quantities relate to each other and change over time, such as: the concentration of CO₂ in the atmosphere, the planetary energy imbalance, and the planet's average surface temperature.



The planetary energy imbalance N (J/s/m^2) is a quantity defined to measure the magnitude of an imbalance of energy in the Earth's energy budget.

Protocol for the introduction of the planetary energy imbalance

Warm-up Questions

- If N measures the magnitude of an imbalance of energy, how would you define N mathematically? Can you give me a formula for N ?

Define PLANETARY ENERGY IMBALANCE

The planetary energy imbalance N is defined as the difference between energy inflows and energy outflows in the budget. Mathematically,

$$N = (\text{Energy Inflows}) - (\text{Energy Outflows})$$



$$N = \underbrace{(S + A)}_{\text{At the planet's surface}} - \underbrace{R}_{\text{At the top of atmosphere}} = S - (L + A)$$

At the planet's
surface

At the top of
atmosphere

- Why do you think N is defined as a difference between energy inflows and energy outflows?
- In what unit is N measured? Why?

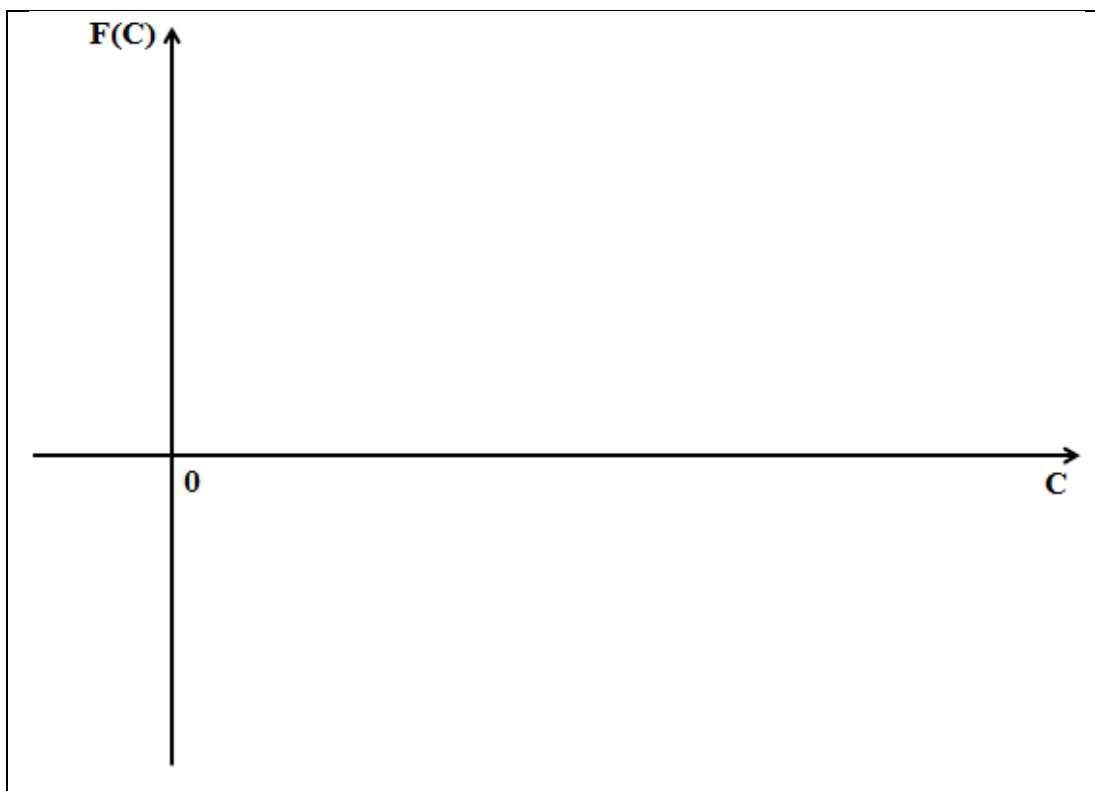
Define the idea of BALANCE

- If we assume the initial values $S = 240 \text{ J/s/m}^2$, $R = 390 \text{ J/s/m}^2$, and $A = 150 \text{ J/s/m}^2$, then what would the value of N be? What does this value mean in terms of the budget?
- Let's assume N is not zero, then:
 - What does $N > 0$ mean in terms of the budget?
 - Is the budget gaining or losing energy? Why?
 - What does $N < 0$ mean in terms of the budget?
 - Is the budget gaining or losing energy? Why?

Set 2 – Task 1

The atmosphere's capacity to absorb infrared energy depends on the concentration of greenhouse gases in the atmosphere. Thus, a change in the concentration of CO_2 can result in an imbalance in the Earth's energy budget. An imbalance created by a change in the concentration of CO_2 is known as *forcing by CO_2* .

Let F and C be the forcing by CO_2 in J/s/m^2 and the concentration of CO_2 in ppmv, respectively. Based on the concepts of *Earth's energy budget* and *greenhouse effect*, determine how F varies with respect to C and sketch a graph for the relationship $F(C)$.



Protocol Set 2 – Task 1

Questions if participants recognize a relationship and provide a graph

- How did you figure out the way F and C are related? Can you describe the relationship between F and C?
- Why did you draw a straight line/concave-down curve/concave-up curve? What makes you think the relationship is linear/non-linear?
- Why is your graph of F(C) discrete/ smooth?

Questions if participant struggles to recognize a relationship and provide a graph

- Do you think there is a linear relationship between F and C? How can you check whether there is a linear relationship between F and C?
- Is there a limit for the growth of F? Why?
- How does F change when C increases? What about if C decreases?
- Assume the atmosphere absorbs all of R, what would the value of F be?
- What is the meaning of that value?
- Assume the atmosphere absorbs none of R, what would the value of F be?
- What is the meaning of that value?

Follow-up questions

- How does F change for equal increments of C? For example, does F increase by the same amount?
- If you wanted F to increase by the same amount again, what value should C reach?
- The rate of change of F with respect to C is known as the *sensitivity of F to C*. How does the sensitivity of F to C change as C increases? Can you sketch the graph of it?

Set 2 – Task 2

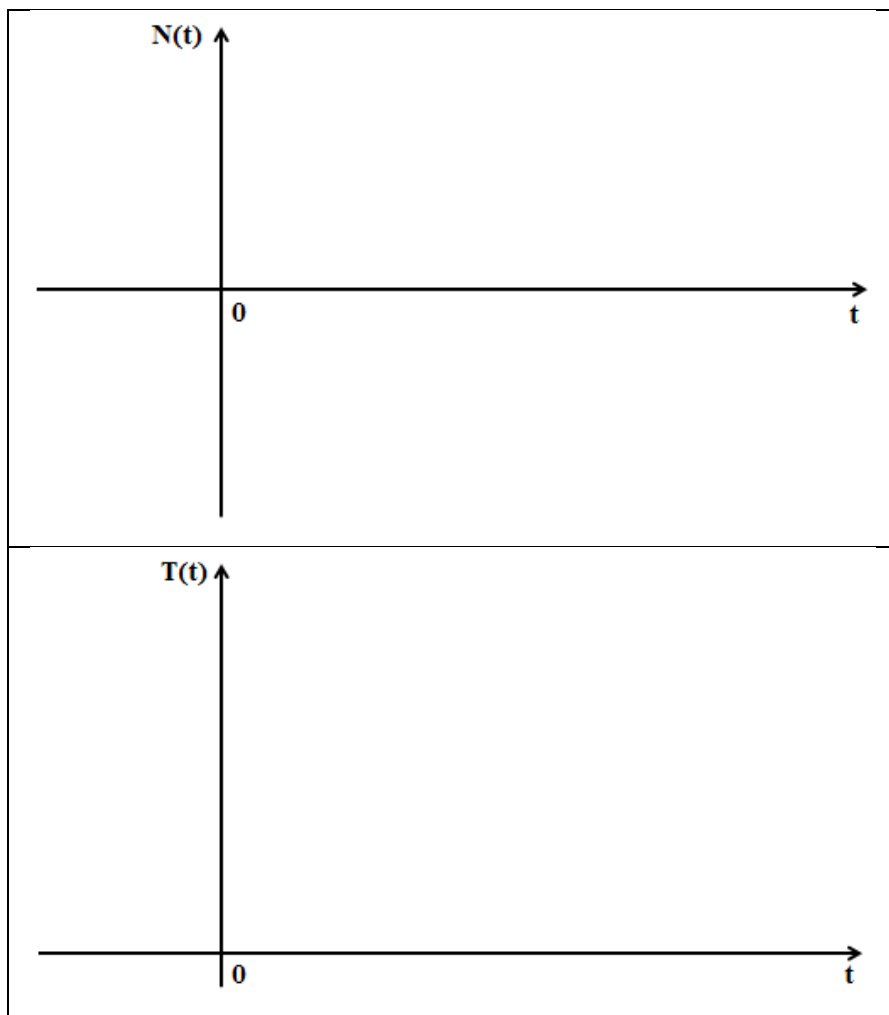
A change in the concentration of CO_2 in the atmosphere can result in an imbalance in the Earth's energy budget; this imbalance is known as *forcing by CO_2* . We want to examine how the budget evolves after a positive forcing has occurred.

Let:

- N be the planetary energy imbalance in J/s/m^2 ,
- C be the atmospheric concentration of CO_2 in ppmv, and
- T be the average temperature of the planet's surface in $^\circ\text{C}$

Based on the concepts of *Earth's energy budget* and *greenhouse effect*, determine how N and T vary with respect to time t (in years after forcing occurs) and sketch graphs of the relationships $N(t)$ and $T(t)$.

C suddenly increases at $t = 0$ (positive forcing)



Protocol Set 2 – Task 2

Questions if participants recognize a relationship and sketch graphs

- How did you figure out the way N and T vary over time? Can you describe the relationships $N(t)$ and $T(t)$ after a forcing occurs?
- Why did you draw a straight line/concave-down curve/concave-up curve? What makes you think the relationship is linear/non-linear?
- Why is your graph of $N(t)$ discrete/smooth? Why does it show a “jump” at $t = 0$?
- Why is your graph of $T(t)$ broken/smooth? Why does it “break” at $t = 0$?

Questions if participants struggle to recognize a relationship and sketch graphs

- Do you think $N(t)$ and $T(t)$ are linear relationships? Why?
- After a forcing occurs, the energy budget will reach a new balance over time. What does that mean in terms $N(t)$? For instance, how does $N(t)$ vary after a positive forcing?
- Can $N(t)$ increase/decrease indefinitely? Why?
- If N measures the magnitude of the energy imbalance, what should its value be at $t = 0$?
- After a forcing occurs, will the planet’s surface heat up or cool down? What does this mean in terms of $T(t)$? For instance, how does $T(t)$ vary after a positive forcing?
- Can $T(t)$ increase/decrease indefinitely? Why?
- How should the graph of $N(t)/T(t)$ look like after a positive forcing occurs?

Follow-up questions

- How should the graphs of $N(t)$ and $T(t)$ look like for $t < 0$? Why?
- How do you interpret the graph of $N(t)$ in relation to the Earth’s energy budget? When is the budget in balance? Why does $N(t)$ increase/decrease asymptotically towards zero?
- When does N decrease most rapidly? How do you see that in the graph of $N(t)$?
- Examine the connection between the graphs of $N(t)$ and $T(t)$. What does the graph of $T(t)$ look like for $N > 0$, $N < 0$, and $N = 0$? Why does the graph look like this?
- Sketch the graph of the rate of change of $T(t)$. What does this graph look like (compare to $N(t)$)?
- What can be said about the relationship between $N(t)$ and $T(t)$?