Results on an Extended Torelli Map and Singularities of Degenerate Abelian Varieties

by

Joseph Anthony Tenini

(Under the direction of Valery Alexeev)

Abstract

The Torelli map associates to a smooth genus g projective curve a g-dimensional principally polarized abelian variety and is a map on the respective moduli spaces $M_g \to A_g$. There is a somewhat canonical compactification of M_g, the Deligne-Mumford compactification $\overline{M}_g$, but there are many natural compactifications of A_g. In this thesis we consider two toroidal compactifications, the central cone compactification $\overline{A}_g^{\text{cent}}$ and the second Voronoi compactification $\overline{A}_g^{\text{Vor}}$. Specifically, we answer the following two questions: Does the Torelli map extend to a regular map $\overline{M}_g \to \overline{A}_g^{\text{cent}}$ in genera 7 and 8? What singularities occur on pairs in the image of the extended Torelli map $\overline{M}_g \to \overline{A}_g^{\text{Vor}}$?

The first question is important because genera 7 and 8 represented the only remaining cases in which this particular extension question was unknown. The result, which is a product of joint work, is that the map does extend in these cases. The second question seeks to extend the 1995 result of Kollár which states that principally polarized abelian pairs (X, Θ) are log canonical. In this thesis we show that in fact all pairs (X, Θ) in the boundary of $\overline{A}_g^{\text{Vor}}$ are semi-log canonical, the analog of log canonical in the non-normal setting.

INDEX WORDS: Torelli Map, Singularities, Abelian Varieties, Degenerations, Moduli Spaces, Compactifications, Theta Divisors, Semiabelic Pairs, Semi-log Canonical, Vanishing.

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Contents

Chapter 1

Introduction

Throughout this thesis we work over an algebraically closed field of characteristic zero. The classical Torelli map associates to a smooth, projective curve C of genus g a g-dimensional principally polarized abelian variety (JC, Θ), known as the Jacobian variety of C. An abelian variety is a connected complete group variety, and a principal polarization can be viewed as a choice of ample Cartier divisor on it satisfying $h^0(JC, \mathcal{O}_{JC}(\Theta)) = 1$. More classically, over the complex numbers, JC may be constructed as the quotient

$$\frac{H^0(C, \Omega^1_C)^\vee}{H_1(C, \mathbb{Z})}$$

where $H^0(C, \Omega^1_C)^\vee$ is the vector space dual of the g-dimensional space of global holomorphic 1-forms on C and $H_1(C, \mathbb{Z})$ is the lattice of homology classes of 1-cycles in C. The principal polarization Θ may be described in terms of a polarizing form $\theta \in H^2(JC, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(\wedge^2 H_1(C, \mathbb{Z}), \mathbb{Z})$ given by intersecting cycles [12, p.307]. More generally, one can construct JC as the abelian variety $\mathrm{Pic}^0(C)$ of degree 0 invertible sheaves on C.

This association produces a map $\mathfrak{t} : M_g \to A_g$ from the moduli space of smooth, projective genus g curves to the moduli space of g-dimensional principally polarized abelian varieties.

In [8], Deligne and Mumford constructed a space $\overline{M}_g$ of genus g stable curves which gives a compactification of M_g. By the work of Ash, Mumford, Rapoport, and Tai [7], there is a compactification of A_g for every choice of a fan which is periodic with respect to $\mathrm{GL}(g, \mathbb{Z})$, supported on the space of quadratic forms in g variables which are positive semidefinite. In this thesis we will consider two compactifications arising in this way: the central cone compactification $\overline{A}_g^{\mathrm{cent}}$ and the second Voronoi compactification $\overline{A}_g^{\mathrm{Vor}}$.

One might ask, for a given compactification $\overline{A}_g$ of A_g, if the Torelli map extends to a regular map $\overline{M}_g \to \overline{A}_g$. Indeed this question has been asked and answered for different compactifications. A nice summary of the history of this question appears in [3].

The central cone compactification was defined by Igusa in 1967 in [16], and it may be obtained by blowing up the Satake-Baily-Borel compactification of A_g along the boundary and then normalizing the resulting space. In 1973, Namikawa in [23] showed that the map extended in low genus though no proof is given for the stated bound of $g \leq 6$. In 2012, Alexeev and Brunyate in [3] showed that the Torelli map does extend to $\overline{A}_g^{\mathrm{cent}}$ for $g \leq 6$ and does not extend for $g \geq 9$. Moreover, this paper provides a means for settling the extension question for $g = 7, 8$ by undertaking a large computation. The first result of this thesis represents joint work published in [4] which completes this computation and settles the remaining two cases via the following theorem:

Theorem 1.0.1. *The Torelli map extends to a regular morphism $\overline{M}_g \to \overline{A}_g^{\mathrm{cent}}$ for $g \leq 8$.*

The second part of the thesis deals with the second Voronoi compactification. The second Voronoi compactification was defined by G. Voronoi in [26] which appeared in 1908. By 1976, Mumford and Namikawa had shown in [24] that the Torelli map did indeed extend in this case to a map $\mathfrak{t}_{Vor} : \overline{M}_g \to \overline{A}_g^{\mathrm{Vor}}$. In 2002, Alexeev in [1] constructed the moduli space $\overline{AP}_g$ of stable semiabelic pairs. The space $\overline{A}_g^{\mathrm{Vor}}$ appears as the normalization of the main irreducible component of $\overline{AP}_g$, thus providing a modular interpretation to $\overline{A}_g^{\mathrm{Vor}}$. In particular, this allows us to ask questions such as: if we map a stable curve, which has at worst nodal singularities, under this extended Torelli map, what can we say about the singularities of the pair (X, Θ) corresponding to the image?

As a starting point, one might consider the following theorem of Kollár from 1995 [18, Thm.17.13]:

Theorem 1.0.2. *If (X, Θ) is a principally polarized abelian variety, then (X, Θ) is log canonical.*

It is easy to see that the pairs in $\overline{A}_g^{\mathrm{Vor}}$ arising as images under $\mathfrak{t}_{Vor}$, called Compactified Jacobians, are not even normal. So the most optimistic result one might hope for would be for such pairs to be semi-log canonical. In this thesis we prove that in fact all pairs in $\overline{A}_g^{\mathrm{Vor}}$ satisfy this condition.

Theorem 1.0.3. *Let $(\mathcal{X}_0, \Theta_0) \in \overline{A}_g^{\mathrm{Vor}}$ be a pair which is not a principally polarized abelian variety. Then $(\mathcal{X}_0, \Theta_0)$ is semi-log canonical.*

Chapter 2

Extended Torelli map to the Igusa blowup in genus 6, 7, and 8

This chapter describes the joint work found in [4].

2.1 Reducing to a question about graphs

Given a stable curve C, we can associate to it its dual graph Γ in the following way. For every irreducible component $C_i \subset C$ there is a vertex v_i. Given two vertices v_i and v_j, they are adjacent if and only if their corresponding irreducible components C_i and C_j intersect nontrivially. Let C be a stable curve and Γ its dual graph. In [3], Alexeev and Brunyate showed that the rational map $\overline{M}_g \to \overline{A}_g^{\text{cent}}$ is regular in a neighborhood of $[C] \in \overline{M}_g$ if and only if there is a positive definite integral valued quadratic form q on $H^1(\Gamma, \mathbb{Z})$ satisfying $q(e_i^*) = 1$ for every non-bridge edge e_i of G. Such a quadratic form is called an integral edge-minimizing metric or $\mathbb{Z}$-emm.

We now recall some terminology from graph theory that we will need for this chapter. An edge e of a graph Γ is said to be a bridge if $\Gamma - e$ has more connected components than

Γ. We say an edge is a loop if it incident to only one vertex. We call a graph planar if it embeds into $\mathbb{R}^2$ and projective planar if it embeds into $\mathbb{RP}^2$. After choosing an orientation on Γ, one constructs $H^1(\Gamma, \mathbb{Z})$ as

$$H^1(\Gamma, \mathbb{Z}) = C^1(\Gamma, \mathbb{Z})/dC^0(\Gamma, \mathbb{Z})$$

where $C^1(\Gamma, \mathbb{Z}) = \oplus_{\text{edges } e_i} \mathbb{Z}e_i^*$, $C^0(\Gamma, \mathbb{Z}) = \oplus_{\text{vertices } v_j} \mathbb{Z}v_j^*$ and the map $d : C^0(\Gamma, \mathbb{Z}) \to C^1(\Gamma, \mathbb{Z})$ is given by

$$dv_j^* = \sum_{e_i \text{ begins with } v_j} e_i^* - \sum_{e_i \text{ ends with } v_j} e_i^*$$

Given $e_i^* \in C^1(\Gamma, \mathbb{Z})$, we will also use e_i^* for its image in $H^1(\Gamma, \mathbb{Z})$ and call it a coedge.

Following [3], we define a graph Γ to be cohomology-irreducible if one cannot partition the edges into two non-empty set I_1 and I_2 such that $H^1(\Gamma, \mathbb{Z}) = \langle e_i^*, i \in I_1 \rangle \oplus \langle e_i^*, i \in I_2 \rangle$. A graph is cohomology-irreducible if and only if it contains no bridges and no loops. As pointed out in [3], for every graph one Γ, one has the decomposition

$$H^1(\Gamma, \mathbb{Z}) = \oplus H^1(\Gamma_k, \mathbb{Z})$$

where the Γ_k are cohomology-irreducible. Moreover, there exists a $\mathbb{Z}$-emm for Γ if and only if there exist $\mathbb{Z}$-emms for all Γ_k.

The final result from [3] we will need is a categorization of the possible $\mathbb{Z}$-emms. Specifically, if Γ is cohomology-irreducible and q is a $\mathbb{Z}$-emm, then $(H^1(\Gamma, \mathbb{Z}), 2q)$ is a root lattice of type A_g, D_g $(g \geq 4)$ or E_6, E_7 or E_8. Moreover, Γ has a $\mathbb{Z}$-emm of type A_g $(g \geq 4)$ if and only if Γ is planar and a $\mathbb{Z}$-emm of type D_g if and only if Γ is projective planar.

Thus, thanks to the work of Alexeev and Brunyate the extension question becomes a question of existence of some quadratic forms. Specifically, we would like to prove the following:

Theorem 2.1.1. *Let Γ be a cohomology-irreducible nonprojectively planar graph of genus $g = 6, 7$ or 8. Then Γ admits a $\mathbb{Z}$-emm of type E_g.*

In the following section we describe how to reduce this question to a finite computation.

2.2 Reduction to finitely many graphs

The Kuratowski theorem states that a graph is nonplanar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$. In [11] a list of 103 graphs was produced in an attempt to create a similar theorem for the projective plane. In [6, 5] it was confirmed that this list was indeed a complete list of minimal non-projectively planar graphs. Since the smallest graph on the list has genus 6, by the work of [3], there is a regular map $\overline{M}_g \to \overline{A}_g^{\text{cent}}$ for $g \leq 5$ and there is no hope for a regular extension in genera ≥ 9. Thus, we are left to consider the cases of $g = 6, 7, 8$.

As noted in [3, Sec.2], for the proof of 2.1.1 we may reduce to graphs which are trivalent. So let H be a cohomology-irreducible non-projectively planar trivalent graph of genus $g = 6, 7$ or 8. One says that H is *irreducible with respect to P* if H does not embed into P, but for any edge e in H, $H - e$ does embed into P. We now describe a process which will reduce H to a trivalent graph irreducible with respect to P. The operations (3a), (3b), (3c) are illustrated in Figures 2.2, 2.3, 2.4.

1. If the graph is irreducible with respect to P, stop and call this graph H'.

2. If not, choose an edge e so that $H - e$ does not embed into P and delete e from the graph.

(3a) If e was not a loop and did not have a parallel edge, then, denoting by v_1 and v_2 the distinct vertices to which e is incident, contract an edge incident to v_1 and an edge incident to v_2.

(3b) If e was not a loop but had a parallel edge f, then, denoting by v_1 and v_2 the distinct vertices to which e and f are incident, contract the edge incident to v_1 and different from f and the edge incident to v_2 and different from f.

(3c) If e was a loop incident to v, then delete the remaining edge f incident to v and, denoting by w the other vertex to which f is incident, contract one of the other two edges incident to w and different from f.

Notice that the above operations (3a)-(3b)-(3c) reduce the genus of the graph by 1 except for operation (3a) when e is a bridge. Repeating this process we get a graph H' irreducible with respect to P which is of the form $H' = \widetilde{H} \cup \{u_1, \dots, u_k\}$ where the u_i are isolated vertices and $\widetilde{H}$ is a trivalent graph irreducible with respect to P. By [10, 21] (see also [5, 6]), $\widetilde{H}$ is isomorphic to one of the following:

(i) The connected graph G of genus 6 shown in Figure 2.8.

(ii) The connected graphs $F_{11}, F_{12}, F_{13}, F_{14}$ of genus 7 shown in Figures 2.9-2.12.

(iii) The graph E_{42} shown in Figure 2.1.

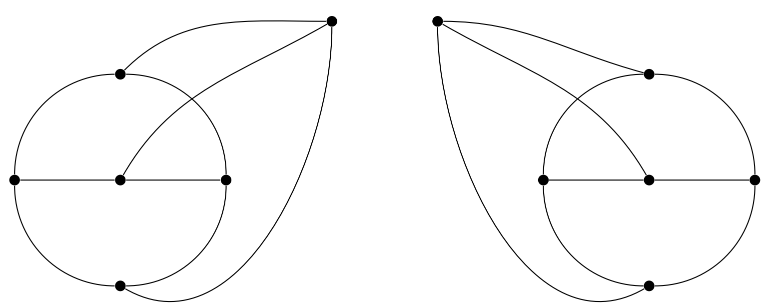

Figure 2.1: The Graph E_{42}.

Thus, we may construct H from $\widetilde{H}$ by reversing the algorithm above. We make this explicit for the relevant genera $6, 7$ and 8.

Consider the case when H has genus 6. Since H is cohomology-irreducible, it has no bridges and so operations (3a), (3b) and (3c) would all drop the genus. Thus H is already irreducible with respect to P and so $H = \widetilde{H} = G$. Thus, to show the existence of $\mathbb{Z}$-emms for graphs of genus 6, it suffices to produce one for G.

Now Consider the case when H has genus 7. Either $\widetilde{H}$ equals one of $F_{11}, F_{12}, F_{13}, F_{14}$ or $\widetilde{H} = G$. In the first case we have that H is equal to one of $F_{11}, F_{12}, F_{13}, F_{14}$ (again since H was cohomology-irreducible, thus bridgeless). The second case is slightly more complicated. First notice that H' has at most one isolated vertex v, because in the case of applying (3c), the genus drops by 1. Then H may be obtained from $\widetilde{H}$ by doing one of the following three operations. Notice that (a), (b) and (c) are the inverse operations of (3a), (3b) and (3c) (defined above) respectively.

(a) Choose two distinct edges e_1 and e_2 and add an edge from the midpoint of e_1 to the midpoint of e_2.

(b) Choose an edge and add a handle to it.

(c) Choose an edge e' and add an edge f from the midpoint of e' to the isolated vertex v. Then add a loop e to v.

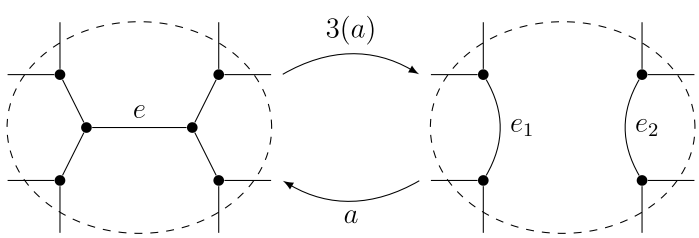

Figure 2.2: The procedures (3a) and (a).

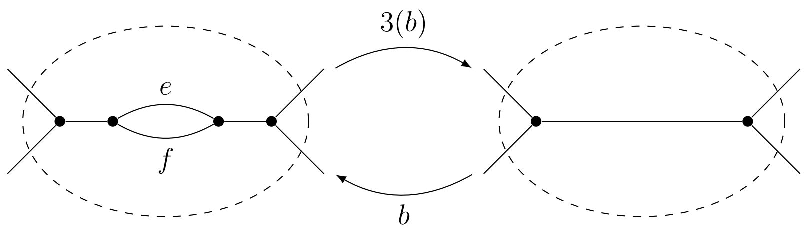

Figure 2.3: The procedures (3b) and (b).

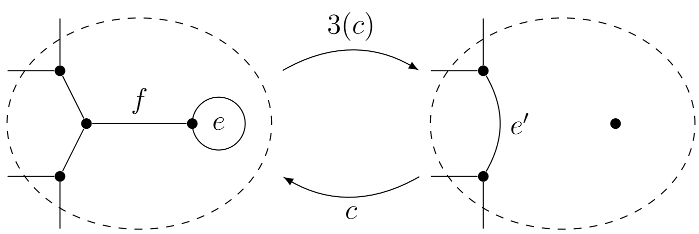

Figure 2.4: The procedures (3c) and (c).

In the case (c), f is a bridge and so we do not need to consider graphs acquired from $\widetilde{H}$ from operation (c).

A careful but elementary analysis shows that the cases (a) and (b), up to symmetries, produce ten possible graphs for H. We denote these graphs $G_1, \ldots G_{10}$, they appear in figures 2.13-2.22 below. Thus, to show the existence of $\mathbb{Z}$-emms for graphs of genus 7 it suffices to produce one for $F_{11}, F_{12}, F_{13}, F_{14}$ and G_i for $i \in \{1, \ldots, 10\}$. Below is the analysis which produces G_1 through G_{10}.

Below we have taken the graph G and created an equivalence relation on the edge set. Two edges e and e' are equivalent if there is an automorphism ϕ of G such that $\phi(e) = e'$.

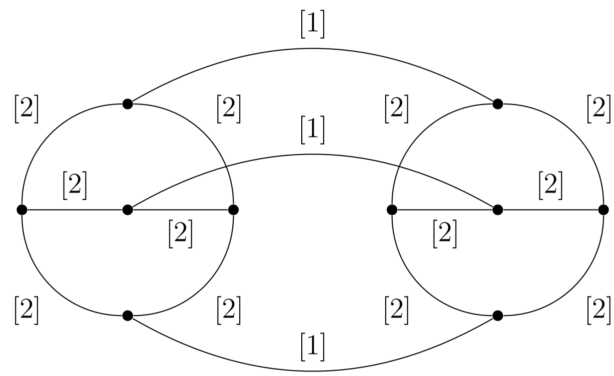

Figure 2.5: G with edges labeled by equivalence class.

Thus, we can get two non-isomorphic graphs from G by adding a handle. Let G_9 the graph obtained by adding a handle to edge [1] and let G_{10} be the graph obtained by adding a handle to edge [2]. The final step will be to list the isomorphism classes of graphs obtained from G by adding an edge joining the midpoints of two distinct edges. For this we proceed as follows: We must select an edge, create a vertex at its midpoint and relabel the equivalence classes of edges of this new graph (we will not label the edges incident to the midpoint). Selecting an edge in [1] we get $\widetilde{G}_{[1]}$ and selecting an edge in [2] we get $\widetilde{G}_{[2]}$. Now for each equivalence class $[i]$ of edges in $\widetilde{G}_{[j]}$ we can obtain a new graph by adding a vertex in the middle of an edge in class $[i]$ and adding an edge incident to the two midpoint vertices.

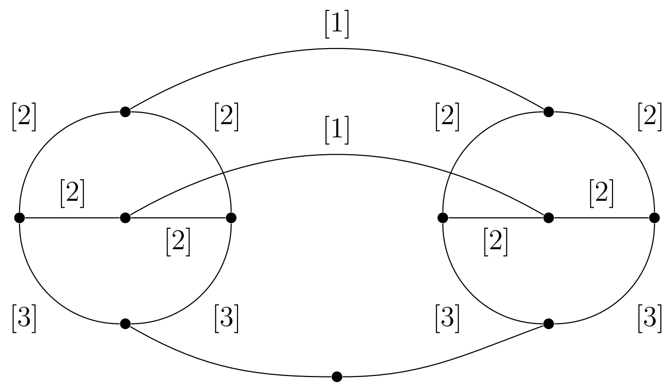

Figure 2.6: The graph $\widetilde{G}_{[1]}$

Since we have 3 equivalence classes of edges in $\widetilde{G}_{[1]}$, we may obtain 3 graphs from $\widetilde{G}_{[1]}$. The graph obtained by choosing [1] we call G_1. The graph obtained by choosing [2] we call G_8. Finally, the graph obtained by choosing [3] we call G_7. Suppose now that we choose an edge in the class [2], we get the graph $\widetilde{G}_{[2]}$ below, where edges have been relabeled according to their new equivalence classes.

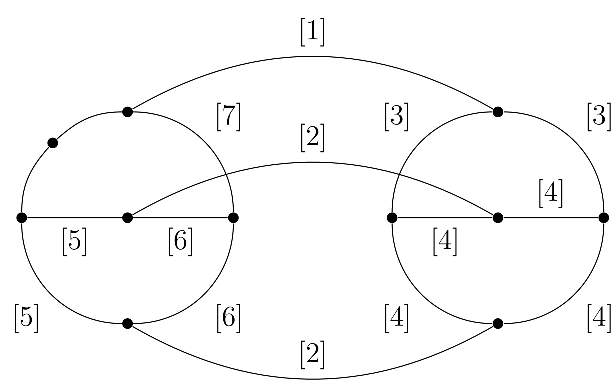

Figure 2.7: The graph $\widetilde{G}_{[2]}$.

Since we have 7 equivalence classes of edges, we may obtain 7 graphs from $\widetilde{G}_{[2]}$. The naming of the new graphs is as follows: choosing [1] we get G_7, choosing [2] we get G_8, choosing [3] we get G_5, choosing [4] we get G_6, choosing [5] we get G_2, choosing [6] we get

G_4, and choosing [7] we get G_3. Thus, there are a total of 14 genus 7 trivalent bridgeless graphs: $F_{11}, F_{12}, F_{13}, F_{14}$ and G_i for $i \in \{1, \ldots, 10\}$.

Now consider the case when H has genus 8. Since H is cohomology-irreducible, the graphs H and $\widetilde{H}$ cannot be isomorphic to E_{42}: otherwise, H would have genus ≥ 10. We may choose an edge e so that $H - e$ does not embed into P. Since e is not a bridge, we may construct a trivalent graph $\mathrm{Simp}(H - e)$ from $H - e$ by contracting edges which were incident to e, as in (3a) or (3b). So $\mathrm{Simp}(H - e)$ is a trivalent graph of genus 7 which does not embed into P. Hence by our above argument, $\mathrm{Simp}(H - e)$ is isomorphic to one of $F_{11}, F_{12}, F_{13}, F_{14}, G_i$ for $i \in \{1, \ldots, 10\}$, or a graph G' obtained from G by choosing an edge e', adding an edge f from the midpoint of e' to an isolated vertex v and then adding a loop e to v, as in (c).

In the latter case, H is obtained from the graph G by performing operation (c) and then (a). But, equivalently, this can be accomplished by the operations (a) and then (b). Thus, to prove 2.1.1 for $g = 8$, it is sufficient to find $\mathbb{Z}$-emms for the finitely many graphs obtained from one of the graphs F_{11}–F_{14}, G_1–G_{10} by performing one operation of type (a) or (b).

2.3 Genus 6

In this section, we explain the general method for finding a $\mathbb{Z}$-emm for any graph, and illustrate it in the case of the trivalent genus 6 graph G.

Let Γ be a directed graph of genus g with edge set $E = \{e_1, \ldots, e_n\}$. After renaming the edges, we may insist that the edges $\{e_{g+1}, \ldots, e_n\}$ induce a spanning tree T of Γ. Then for each e_i with $i \in \{1, \ldots, g\}$, we have a corresponding basis element f_i of the homology group $H_1(\Gamma, \mathbb{Z})$, given by:

$$f_i = e_i + \sum_{e_s \in T} b_{i,s} e_s, \quad b_{i,s} = 0, \pm 1, i \in \{1, \ldots, g\},$$

and the coedges $e_1^*, \ldots, e_g^*$ form a basis of the cohomology group $H^1(G, \mathbb{Z})$ (cf. [3, Lemma 2.3]).

Specifically, f_i is given by the unique simple cycle in Γ which uses only the edge e_i and edges of T. If we write the vectors f_i as the rows of a $g \times n$ matrix then the columns of this matrix are the coedges $e_i^* \in H^1(G, \mathbb{Z})$ written in the basis $\{e_1^*, \ldots, e_g^*\}$. In particular, the first g columns form an identity matrix.

Let q be a $\mathbb{Z}$-emm for Γ. Since q is a $\mathbb{Z}$-valued quadratic form, we may associate to q an even integral matrix $M_q = (a_{i,j})$ such that

$$q(x_1, \ldots, x_g) = (x_1, \ldots, x_g) \frac{1}{2} M_q (x_1, \ldots, x_g)^T.$$

Note here that $a_{i,j} = a_{j,i}$ is just the coefficient of the term $x_i x_j$ in $q(x_1, \ldots, x_g)$ if $i \neq j$ and $a_{i,i}$ is just twice the coefficient of the term x_i^2 in $q(x_1, \ldots, x_g)$.

We need to enforce the condition that $q(e_i^*) = 1$ for $i = 1, \ldots, n$. To ensure that $q(e_i^*) = 1$ for $i = 1, \ldots, g$ we must have $a_{i,i} = 2$. Now we must ensure that $q(e_i^*) = 1$ for $i = g+1, \ldots, n$. This is equivalent to $n - g$ linear equations on $a_{i,j}$:

$$1 = \sum_{i=1}^{g} c_i^2 + \sum_{1 \leq i < j \leq g} c_i c_j a_{i,j} \quad \text{for each column } (c_i).$$

Further, the condition that q is positive definite implies that each $a_{i,j} \in \{0, \pm 1\}$. Thus, for any given graph, we reduced the problem to a finite computation.

We now specialize to graph G. In Figure 2.8 it is shown as a labeled directed graph with a spanning tree denoted by solid edges.

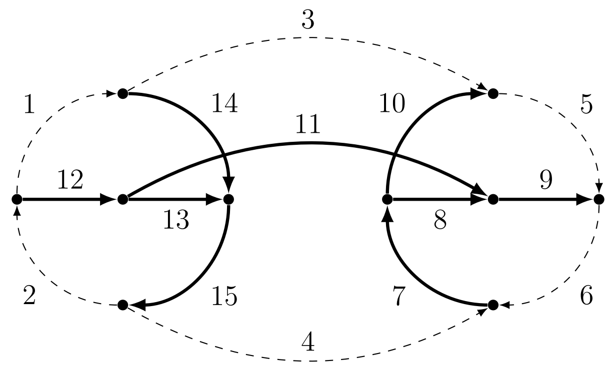

Figure 2.8: The Graph G.

Using the spanning tree drawn and the process described above we get a basis for $H_1(G, \mathbb{Z})$

$$f_1 = e_1 - e_{12} - e_{13} + e_{14}$$

$$f_2 = e_2 + e_{12} + e_{13} + e_{15}$$

$$f_3 = e_3 + e_8 - e_{10} - e_{11} + e_{13} - e_{14}$$

$$f_4 = e_4 + e_7 + e_8 - e_{11} + e_{13} + e_{15}$$

$$f_5 = e_5 - e_8 - e_9 + e_{10}$$

$$f_6 = e_6 + e_7 + e_8 + e_9$$

which we may write in a matrix as

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}
f_1	1	0	0	0	0	0	0	0	0	0	0	-1	-1	1	0
f_2	0	1	0	0	0	0	0	0	0	0	0	1	1	0	1
f_3	0	0	1	0	0	0	0	1	0	-1	-1	0	1	-1	0
f_4	0	0	0	1	0	0	1	1	0	0	-1	0	1	0	1
f_5	0	0	0	0	1	0	0	-1	-1	1	0	0	0	0	0
f_6	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0

The linear equations become:

(1) $1 = 2 + a_{4,6}$

(2) $1 = 4 + a_{3,4} - a_{3,5} + a_{3,6} - a_{4,5} + a_{4,6} - a_{5,6}$

(3) $1 = 2 - a_{5,6}$

(4) $1 = 2 - a_{3,5}$

(5) $1 = 2 + a_{3,4}$

(6) $1 = 2 - a_{1,2}$

(7) $1 = 4 - a_{1,2} - a_{1,3} - a_{1,4} + a_{2,3} + a_{2,4} + a_{3,4}$

(8) $1 = 2 - a_{1,3}$

(9) $1 = 2 + a_{2,4}$

So, equations (1),(3),(4),(5),(6),(8),(9) immediately imply that $1 = a_{5,6} = a_{3,5} = a_{1,2} = a_{1,3}$ and $-1 = a_{4,6} = a_{3,4} = a_{2,4}$. Applying this information to (2) and (7) we get $1 = a_{3,6} - a_{4,5}$ and $1 = a_{2,3} - a_{1,4}$ respectively. Let us arbitrarily choose $a_{3,6} = a_{2,3} = 1$ and $a_{4,5} = a_{1,4} = 0$. Hence, we will get a $\mathbb{Z}$-emm if we can choose the remaining terms of the below matrix in such a way that it is positive definite.

$$\begin{pmatrix} 2 & 1 & 1 & 0 & a_{1,5} & a_{1,6} \\ 1 & 2 & 1 & -1 & a_{2,5} & a_{2,6} \\ 1 & 1 & 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 2 & 0 & -1 \\ a_{5,1} & a_{5,2} & 1 & 0 & 2 & 1 \\ a_{6,1} & a_{6,2} & 1 & -1 & 1 & 2 \end{pmatrix}$$

One such choice is to set all the unknowns to 0.

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & -1 & 1 & 1 \\ 0 & -1 & -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{pmatrix}$$

Then the quadratic form corresponding to this matrix is:

$$\begin{aligned} q(x_1, x_2, x_3, x_4, x_5, x_6) = \ & x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 - \\ & x_2x_4 + x_3^2 - x_3x_4 + x_3x_5 + x_3x_6 + \\ & x_4^2 - x_4x_6 + x_5^2 + x_5x_6 + x_6^2 \end{aligned}$$

One can easily check by diagonalizing that this quadratic form is indeed positive definite. Moreover, in an appropriately chosen basis, it is isomorphic to the standard quadratic form E_6.

2.4 Genus 7

We repeat the general procedure of the previous section for the graphs F_{11}–F_{14} and G_1–G_{10}. Below, we list one explicit $\mathbb{Z}$-emm for each of these graphs.

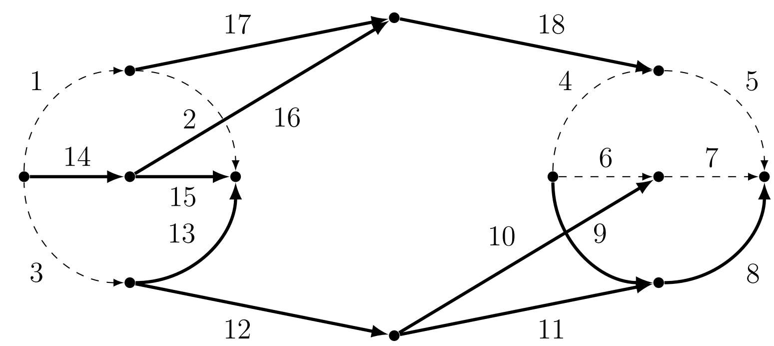

Figure 2.9: The Graph F_{11}.

The basis for F_{11} is given by:

$$f_1 = e_1 + e_{17} - e_{16} - e_{14}$$

$$f_2 = e_2 - e_{15} + e_{16} - e_{17}$$

$$f_3 = e_3 + e_{13} - e_{15} - e_{14}$$

$$f_4 = e_4 - e_{18} - e_{16} + e_{15} - e_{13} + e_{12} + e_{11} - e_9$$

$$f_5 = e_5 - e_8 - e_{11} - e_{12} + e_{13} - e_{15} + e_{16} + e_{18}$$

$$f_6 = e_6 - e_{10} + e_{11} - e_9$$

$$f_7 = e_7 - e_8 - e_{11} + e_{10}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	0	0	0	0	0	0	−1	0	−1	1	0
f_2	0	0	0	0	0	0	0	−1	1	−1	0
f_3	0	0	0	0	0	1	−1	−1	0	0	0
f_4	0	−1	0	1	1	−1	0	1	−1	0	−1
f_5	−1	0	0	−1	−1	1	0	−1	1	0	1
f_6	0	−1	−1	1	0	0	0	0	0	0	0
f_7	−1	0	1	−1	0	0	0	0	0	0	0

which gives equations:

(1) $1 = 2 + a_{5,7}$

(2) $1 = 2 + a_{4,6}$

(3) $1 = 2 - a_{6,7}$

(4) $1 = 4 - a_{4,5} + a_{4,6} - a_{4,7} - a_{5,6} + a_{5,7} - a_{6,7}$

(5) $1 = 2 - a_{4,5}$

(6) $1 = 3 - a_{3,4} + a_{3,5} - a_{4,5}$

(7) $1 = 2 + a_{1,3}$

(8) $1 = 4 + a_{2,3} - a_{2,4} + a_{2,5} - a_{3,4} + a_{3,5} - a_{4,5}$

(9) $1 = 4 - a_{1,2} + a_{1,4} - a_{1,5} - a_{2,4} + a_{2,5} - a_{4,5}$

(10) $1 = 2 - a_{1,2}$

(11) $1 = 2 - a_{4,5}$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & 1 & -1 & 0 & 0 & -1 & -1 \\ 1 & 2 & 0 & 1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 2 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 & 2 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{F_{11}}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \quad & x_1^2 + x_1x_2 - x_1x_3 - x_1x_6 - x_1x_7 + x_2^2 + \\ & x_2x_4 - x_2x_6 + x_3^2 + x_3x_4 + x_3x_7 + x_4^2 + \\ & x_4x_5 - x_4x_6 + x_5^2 - x_5x_6 - x_5x_7 + \\ & x_6^2 + x_6x_7 + x_7^2 \end{aligned}$$

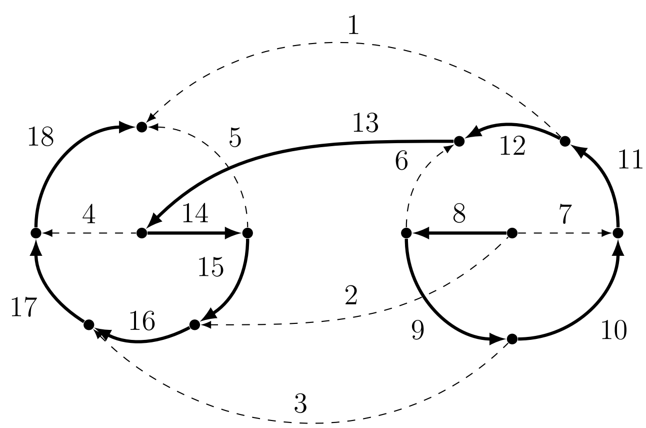

Figure 2.10: The graph F_{12}.

The basis for F_{12} is given by:

$$f_1 = e_1 - e_{12} - e_{13} - e_{14} - e_{15} - e_{16} - e_{17} - e_{18}$$

$$f_2 = e_2 - e_8 - e_9 - e_{10} - e_{11} - e_{12} - e_{13} - e_{14} - e_{15}$$

$$f_3 = e_3 - e_{10} - e_{11} - e_{12} - e_{13} - e_{14} - e_{15} - e_{16}$$

$$f_4 = e_4 - e_{14} - e_{15} - e_{16} - e_{17}$$

$$f_5 = e_5 - e_{15} - e_{16} - e_{17} - e_{18}$$

$$f_6 = e_6 - e_9 - e_{10} - e_{11} - e_{12}$$

$$f_7 = e_7 - e_8 - e_9 - e_{10}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	0	0	0	0	-1	-1	-1	-1	-1	-1	-1
f_2	-1	-1	-1	-1	-1	-1	-1	-1	0	0	0
f_3	0	0	-1	-1	-1	-1	-1	-1	-1	0	0
f_4	0	0	0	0	0	0	-1	-1	-1	-1	0
f_5	0	0	0	0	0	0	0	-1	-1	-1	-1
f_6	0	-1	-1	-1	-1	0	0	0	0	0	0
f_7	-1	-1	-1	0	0	0	0	0	0	0	0

which gives equations:

(1) $1 = 2 + a_{2,7}$

(2) $1 = 3 + a_{2,6} + a_{2,7} + a_{6,7}$

(3) $1 = 4 + a_{2,3} + a_{2,6} + a_{2,7} + a_{3,6} + a_{3,7} + a_{6,7}$

(4) $1 = 3 + a_{2,3} + a_{2,6} + a_{3,6}$

(5) $1 = 4 + a_{1,2} + a_{1,3} + a_{1,6} + a_{2,3} + a_{2,6} + a_{3,6}$

(6) $1 = 3 + a_{1,2} + a_{1,3} + a_{2,3}$

(7) $1 = 4 + a_{1,2} + a_{1,3} + a_{1,4} + a_{2,3} + a_{2,4} + a_{3,4}$

(8) $1 = 5 + a_{1,2} + a_{1,3} + a_{1,4} + a_{1,5} + a_{2,3} + a_{2,4} + a_{2,5} + a_{3,4} + a_{3,5} + a_{4,5}$

(9) $1 = 4 + a_{1,3} + a_{1,4} + a_{1,5} + a_{3,4} + a_{3,5} + a_{4,5}$

(10) $1 = 3 + a_{1,4} + a_{1,5} + a_{4,5}$

(11) $1 = 2 + a_{1,5}$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & -1 & 1 & 0 & -1 \\ -1 & -1 & 2 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & -1 & 1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{F_{12}}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \; & x_1^2 - x_1x_3 - x_1x_5 + x_2^2 - x_2x_3 \\ & -x_2x_4 + x_2x_5 - x_2x_7 + x_3^2 - x_3x_6 + x_3x_7 + \\ & x_4^2 - x_4x_5 + x_5^2 + x_6^2 - x_6x_7 + x_7^2 \end{aligned}$$

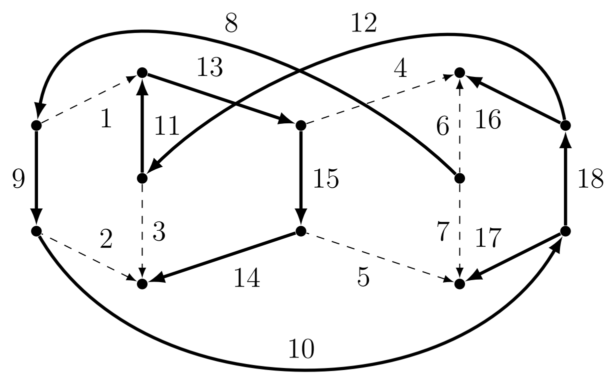

Figure 2.11: The Graph F_{13}

The basis for F_{13} is given by:

$$f_1 = e_1 - e_9 - e_{10} - e_{11} - e_{12} - e_{18}$$

$$f_2 = e_2 - e_{10} - e_{11} - e_{12} - e_{13} - e_{14} - e_{15} - e_{18}$$

$$f_3 = e_3 - e_{11} - e_{13} - e_{14} - e_{15}$$

$$f_4 = e_4 + e_{11} + e_{12} + e_{13} - e_{16}$$

$$f_5 = e_5 + e_{11} + e_{12} + e_{13} + e_{15} - e_{17} + e_{18}$$

$$f_6 = e_6 - e_8 - e_9 - e_{10} - e_{16} - e_{18}$$

$$f_7 = e_7 - e_8 - e_9 - e_{10} - e_{17}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	0	-1	-1	-1	-1	0	0	0	0	0	-1
f_2	0	0	-1	-1	-1	-1	-1	-1	0	0	-1
f_3	0	0	0	-1	0	-1	-1	-1	0	0	0
f_4	0	0	0	1	1	1	0	0	-1	0	0
f_5	0	0	0	1	1	1	0	1	0	-1	1
f_6	-1	-1	-1	0	0	0	0	0	-1	0	-1
f_7	-1	-1	-1	0	0	0	0	0	0	-1	0

which gives equations:

(1) $1 = 2 + a_{6,7}$

(2) $1 = 3 + a_{1,6} + a_{1,7} + a_{6,7}$

(3) $1 = 4 + a_{1,2} + a_{1,6} + a_{1,7} + a_{2,6} + a_{2,7} + a_{6,7}$

(4) $1 = 5 + a_{1,2} + a_{1,3} - a_{1,4} - a_{1,5} + a_{2,3} - a_{2,4} - a_{2,5} - a_{3,4} - a_{3,5} + a_{4,5}$

(5) $1 = 4 + a_{1,2} - a_{1,4} - a_{1,5} - a_{2,4} - a_{2,5} + a_{4,5}$

(6) $1 = 4 + a_{2,3} - a_{2,4} - a_{2,5} - a_{3,4} - a_{3,5} + a_{4,5}$

(7) $1 = 2 + a_{2,3}$

(8) $1 = 3 + a_{2,3} - a_{2,5} - a_{3,5}$

(9) $1 = 2 + a_{4,6}$

(10) $1 = 2 + a_{5,7}$

(11) $1 = 4 + a_{1,2} - a_{1,5} + a_{1,6} - a_{2,5} + a_{2,6} - a_{5,6}$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 2 & -1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 2 & -1 \\ -1 & 0 & 0 & 1 & -1 & -1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{F_{13}}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \quad & x_1^2 + x_1x_5 - x_1x_7 + x_2^2 - x_2x_3 + \\ & x_2x_5 - x_2x_6 + x_3^2 + x_4^2 - x_4x_5 - \\ & x_4x_6 + x_4x_7 + x_5^2 - x_5x_7 + x_6^2 - \\ & x_6x_7 + x_7^2 \end{aligned}$$

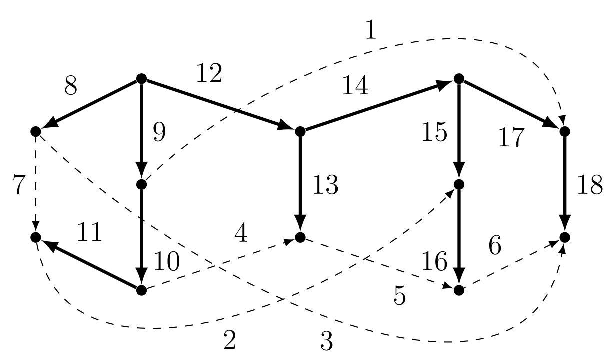

Figure 2.12: The Graph F_{14}

The basis for F_{14} is given by:

$$f_1 = e_1 + e_9 - e_{12} - e_{14} - e_{17}$$

$$f_2 = e_2 + e_9 + e_{10} + e_{11} - e_{12} - e_{14} - e_{15}$$

$$f_3 = e_3 + e_8 - e_{12} - e_{14} - e_{17} - e_{18}$$

$$f_4 = e_4 + e_9 + e_{10} - e_{12} - e_{13}$$

$$f_5 = e_5 + e_{13} - e_{14} - e_{15} - e_{16}$$

$$f_6 = e_6 + e_{15} + e_{16} - e_{17} - e_{18}$$

$$f_7 = e_7 + e_8 - e_9 - e_{10} - e_{11}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	0	1	0	0	-1	0	-1	0	0	-1	0
f_2	0	1	1	1	-1	0	-1	-1	0	0	0
f_3	1	0	0	0	-1	0	-1	0	0	-1	-1
f_4	0	1	1	0	-1	-1	0	0	0	0	0
f_5	0	0	0	0	0	1	-1	-1	-1	0	0
f_6	0	0	0	0	0	0	0	1	1	-1	-1
f_7	1	-1	-1	-1	0	0	0	0	0	0	0

which gives equations:

$$
\begin{aligned}
&(1) \quad 1 = 2 + a_{3,7} \\
&(2) \quad 1 = 4 + a_{1,2} + a_{1,4} - a_{1,7} + a_{2,4} - a_{2,7} - a_{4,7} \\
&(3) \quad 1 = 3 + a_{2,4} - a_{2,7} - a_{4,7} \\
&(4) \quad 1 = 2 - a_{2,7} \\
&(5) \quad 1 = 4 + a_{1,2} + a_{1,3} + a_{1,4} + a_{2,3} + a_{2,4} + a_{3,4} \\
&(6) \quad 1 = 2 - a_{4,5} \\
&(7) \quad 1 = 4 + a_{1,2} + a_{1,3} + a_{1,5} + a_{2,3} + a_{2,5} + a_{3,5} \\
&(8) \quad 1 = 3 + a_{2,5} - a_{2,6} - a_{5,6} \\
&(9) \quad 1 = 2 - a_{5,6} \\
&(10) \quad 1 = 3 + a_{1,3} + a_{1,6} + a_{3,6} \\
&(11) \quad 1 = 2 + a_{3,6}
\end{aligned}
$$

The following matrix satisfies these equations and is positive definite:

$$
\begin{pmatrix}
2 & -1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & -1 & -1 & 0 & 1 \\
-1 & 0 & 2 & 0 & 0 & -1 & -1 \\
0 & -1 & 0 & 2 & 1 & 0 & 0 \\
0 & -1 & 0 & 1 & 2 & 1 & -1 \\
0 & 0 & -1 & 0 & 1 & 2 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 & 2
\end{pmatrix}
$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{F_{14}}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \; & x_1^2 - x_1x_2 - x_1x_3 + x_2^2 - x_2x_4 - \\ & x_2x_5 + x_2x_7 + x_3^2 - x_3x_6 - x_3x_7 + \\ & x_4^2 + x_4x_5 + x_5^2 + x_5x_6 - x_5x_7 + \\ & x_6^2 + x_7^2 \end{aligned}$$

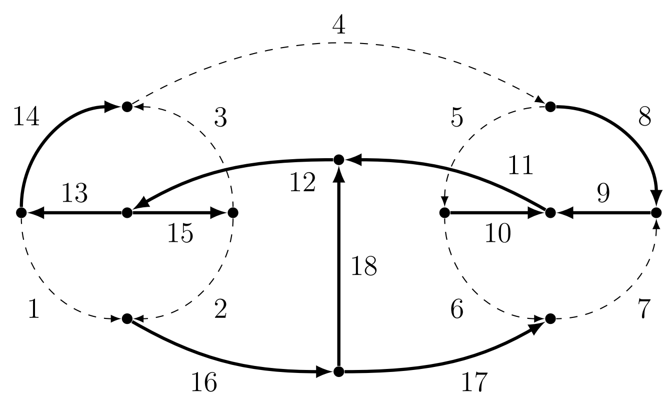

Figure 2.13: The graph G_1.

The basis for G_1 is given by:

$$f_1 = e_1 + e_{12} + e_{13} + e_{16} + e_{18}$$

$$f_2 = e_2 + e_{12} + e_{15} + e_{16} + e_{18}$$

$$f_3 = e_3 - e_{13} - e_{14} + e_{15}$$

$$f_4 = e_4 + e_8 + e_9 + e_{11} + e_{12} + e_{13} + e_{14}$$

$$f_5 = e_5 - e_8 - e_9 + e_{10}$$

$$f_6 = e_6 - e_{10} - e_{11} - e_{17} + e_{18}$$

$$f_7 = e_7 + e_9 + e_{11} + e_{17} - e_{18}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	0	0	0	0	1	1	0	0	1	0	1
f_2	0	0	0	0	1	0	0	1	1	0	1
f_3	0	0	0	0	0	-1	-1	1	0	0	0
f_4	1	1	0	1	1	1	1	0	0	0	0
f_5	-1	-1	1	0	0	0	0	0	0	0	0
f_6	0	0	-1	-1	0	0	0	0	0	-1	1
f_7	0	1	0	1	0	0	0	0	0	1	-1

which gives equations:

$$
\begin{aligned}
&(1) \quad 1 = 2 - a_{4,5} \\
&(2) \quad 1 = 3 - a_{4,5} + a_{4,7} - a_{5,7} \\
&(3) \quad 1 = 2 - a_{5,6} \\
&(4) \quad 1 = 3 - a_{4,6} + a_{4,7} - a_{6,7} \\
&(5) \quad 1 = 3 + a_{1,2} + a_{1,4} + a_{2,4} \\
&(6) \quad 1 = 3 - a_{1,3} + a_{1,4} - a_{3,4} \\
&(7) \quad 1 = 2 - a_{3,4} \\
&(8) \quad 1 = 2 + a_{2,3} \\
&(9) \quad 1 = 2 + a_{1,2} \\
&(10) \quad 1 = 2 - a_{6,7} \\
&(11) \quad 1 = 4 + a_{1,2} + a_{1,6} - a_{1,7} + a_{2,6} - a_{2,7} - a_{6,7}
\end{aligned}
$$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & -1 & 1 & 0 & 1 & 1 & 1 \\ -1 & 2 & -1 & -1 & -1 & -1 & 0 \\ 1 & -1 & 2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 2 & 1 & 0 & -1 \\ 1 & -1 & 1 & 1 & 2 & 1 & 0 \\ 1 & -1 & 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & -1 & 0 & 1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{G_1}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \; & x_1^2 - x_1x_2 + x_1x_3 + x_1x_5 + x_1x_6 + \\ & x_1x_7 + x_2^2 - x_2x_3 - x_2x_4 - x_2x_5 - \\ & x_2x_6 + x_3^2 + x_3x_4 + x_3x_5 + x_3x_6 + \\ & x_4^2 + x_4x_5 - x_4x_7 + x_5^2 + x_5x_6 + \\ & x_6^2 + x_6x_7 + x_7^2 \end{aligned}$$

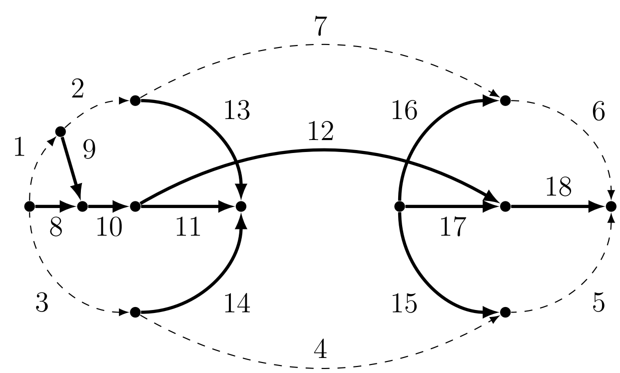

Figure 2.14: The graph G_2.

The basis for G_2 is given by:

$$f_1 = e_1 - e_8 + e_9$$

$$f_2 = e_2 - e_9 - e_{10} - e_{11} + e_{13}$$

$$f_3 = e_3 - e_8 - e_{10} - e_{11} + e_{14}$$

$$f_4 = e_4 + e_{11} - e_{12} - e_{14} - e_{15} + e_{17}$$

$$f_5 = e_5 + e_{15} - e_{17} - e_{18}$$

$$f_6 = e_6 + e_{16} - e_{17} - e_{18}$$

$$f_7 = e_7 + e_{11} - e_{12} - e_{13} - e_{16} + e_{17}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	-1	1	0	0	0	0	0	0	0	0	0
f_2	0	-1	-1	-1	0	1	0	0	0	0	0
f_3	-1	0	-1	-1	0	0	1	0	0	0	0
f_4	0	0	0	1	-1	0	-1	-1	0	1	0
f_5	0	0	0	0	0	0	0	1	0	-1	-1
f_6	0	0	0	0	0	0	0	0	1	-1	-1
f_7	0	0	0	1	-1	-1	0	0	-1	1	0

which gives equations:

$$
\begin{array}{ll}
(1) & 1 = 2 + a_{1,3} \\
(2) & 1 = 2 - a_{1,2} \\
(3) & 1 = 2 + a_{2,3} \\
(4) & 1 = 4 + a_{2,3} - a_{2,4} - a_{2,7} - a_{3,4} - a_{3,7} + a_{4,7} \\
(5) & 1 = 2 + a_{4,7} \\
(6) & 1 = 2 - a_{2,7} \\
(7) & 1 = 2 - a_{3,4} \\
(8) & 1 = 2 - a_{4,5} \\
(9) & 1 = 2 - a_{6,7} \\
(10) & 1 = 4 - a_{4,5} - a_{4,6} + a_{4,7} + a_{5,6} - a_{5,7} - a_{6,7} \\
(11) & 1 = 2 + a_{5,6}
\end{array}
$$

The following matrix satisfies these equations and is positive definite:

$$
\begin{pmatrix}
2 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 2 & -1 & -1 & 0 & 0 & 1 \\
-1 & -1 & 2 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 2 & 1 & -1 & -1 \\
0 & 0 & 0 & 1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & 2 & 1 \\
0 & 1 & 0 & -1 & 0 & 1 & 2
\end{pmatrix}
$$

And the quadratic form given by this matrix is:

$$
\begin{aligned}
q_{G_2}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \quad & x_1^2 + x_1x_2 - x_1x_3 + x_2^2 - x_2x_3 - \\
& x_2x_4 + x_2x_7 + x_3^2 + x_3x_4 + x_4^2 + \\
& x_4x_5 - x_4x_6 - x_4x_7 + x_5^2 - x_5x_6 + \\
& x_6^2 + x_6x_7 + x_7^2
\end{aligned}
$$

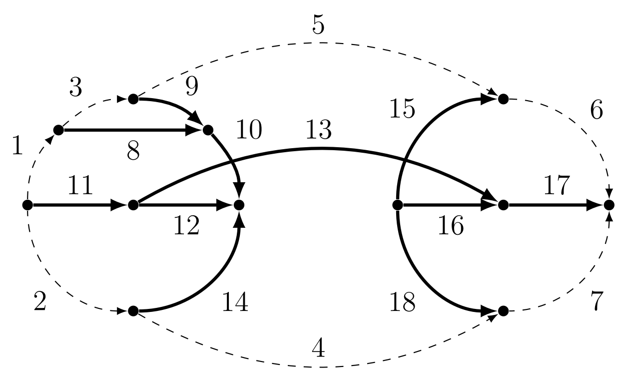

Figure 2.15: The Graph G_3

The basis for G_3 is given by:

$$f_1 = e_1 + e_8 + e_{10} - e_{11} - e_{12}$$

$$f_2 = e_2 - e_{11} - e_{12} + e_{14}$$

$$f_3 = e_3 - e_8 + e_9$$

$$f_4 = e_4 + e_{12} - e_{13} - e_{14} + e_{16} - e_{18}$$

$$f_5 = e_5 - e_9 - e_{10} + e_{12} - e_{13} - e_{15} + e_{16}$$

$$f_6 = e_6 + e_{15} - e_{16} - e_{17}$$

$$f_7 = e_7 - e_{16} - e_{17} + e_{18}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	1	0	1	-1	-1	0	0	0	0	0	0
f_2	0	0	0	-1	-1	0	1	0	0	0	0
f_3	-1	1	0	0	0	0	0	0	0	0	0
f_4	0	0	0	0	1	-1	-1	0	1	0	-1
f_5	0	-1	-1	0	1	-1	0	-1	1	0	0
f_6	0	0	0	0	0	0	0	1	-1	-1	0
f_7	0	0	0	0	0	0	0	0	-1	-1	1

which gives equations:

(1) $1 = 2 - a_{1,3}$

(2) $1 = 2 - a_{3,5}$

(3) $1 = 2 - a_{1,5}$

(4) $1 = 2 + a_{1,2}$

(5) $1 = 4 + a_{1,2} - a_{1,4} - a_{1,5} - a_{2,4} - a_{2,5} + a_{4,5}$

(6) $1 = 2 + a_{4,5}$

(7) $1 = 2 - a_{2,4}$

(8) $1 = 2 - a_{5,6}$

(9) $1 = 4 + a_{4,5} - a_{4,6} - a_{4,7} - a_{5,6} - a_{5,7} + a_{6,7}$

(10) $1 = 2 + a_{6,7}$

(11) $1 = 2 - a_{4,7}$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & -1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{G_3}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \quad & x_1^2 - x_1x_2 + x_1x_3 - x_1x_4 + x_1x_5 + \\ & x_2^2 + x_2x_4 + x_3^2 + x_3x_5 + x_4^2 - \\ & x_4x_5 - x_4x_6 + x_4x_7 + x_5^2 + x_5x_6 + \\ & x_6^2 - x_6x_7 + x_7^2 \end{aligned}$$

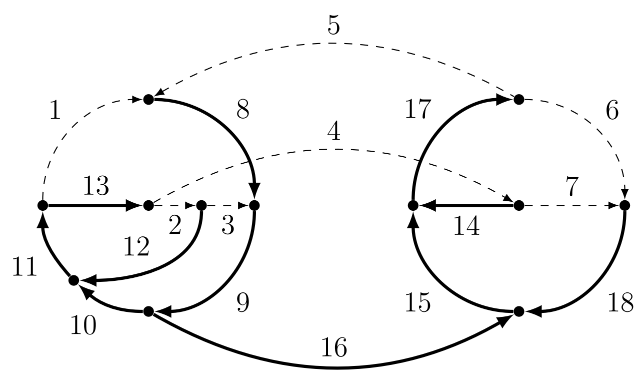

Figure 2.16: The Graph G_4

The basis for G_4 is given by:

$$f_1 = e_1 + e_8 + e_9 + e_{10} + e_{11}$$

$$f_2 = e_2 + e_{11} + e_{12} + e_{13}$$

$$f_3 = e_3 + e_9 + e_{10} - e_{12}$$

$$f_4 = e_4 + e_{10} + e_{11} + e_{13} + e_{14} - e_{15} - e_{16}$$

$$f_5 = e_5 + e_8 + e_9 + e_{15} + e_{16} + e_{17}$$

$$f_6 = e_6 + e_{15} + e_{17} + e_{18}$$

$$f_7 = e_7 - e_{14} + e_{15} + e_{18}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	1	1	1	1	0	0	0	0	0	0	0
f_2	0	0	0	1	1	1	0	0	0	0	0
f_3	0	1	1	0	−1	0	0	0	0	0	0
f_4	0	0	1	1	0	1	1	−1	−1	0	0
f_5	1	1	0	0	0	0	0	1	1	1	0
f_6	0	0	0	0	0	0	0	1	0	1	1
f_7	0	0	0	0	0	0	−1	1	0	0	1

which gives equations:

$$
\begin{aligned}
&(1) \quad 1 = 2 + a_{1,5} \\
&(2) \quad 1 = 3 + a_{1,3} + a_{1,5} + a_{3,5} \\
&(3) \quad 1 = 3 + a_{1,3} + a_{1,4} + a_{3,4} \\
&(4) \quad 1 = 3 + a_{1,2} + a_{1,4} + a_{2,4} \\
&(5) \quad 1 = 2 - a_{2,3} \\
&(6) \quad 1 = 2 + a_{2,4} \\
&(7) \quad 1 = 2 - a_{4,7} \\
&(8) \quad 1 = 4 - a_{4,5} - a_{4,6} - a_{4,7} + a_{5,6} + a_{5,7} + a_{6,7} \\
&(9) \quad 1 = 2 - a_{4,5} \\
&(10) \quad 1 = 2 + a_{5,6} \\
&(11) \quad 1 = 2 + a_{6,7}
\end{aligned}
$$

The following matrix satisfies these equations and is positive definite:

$$
\begin{pmatrix}
2 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & 2 & 1 & -1 & -1 & 0 & 0 \\
0 & 1 & 2 & -1 & -1 & 0 & 0 \\
-1 & -1 & -1 & 2 & 1 & 0 & 1 \\
-1 & -1 & -1 & 1 & 2 & -1 & 1 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
-1 & 0 & 0 & 1 & 1 & -1 & 2
\end{pmatrix}
$$

And the quadratic form given by this matrix is:

$$
\begin{aligned}
q_{G_4}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \;& x_1^2 - x_1x_4 - x_1x_5 - x_1x_7 + x_2^2 + \\
& x_2x_3 - x_2x_4 - x_2x_5 + x_3^2 - x_3x_4 - \\
& x_3x_5 + x_4^2 + x_4x_5 + x_4x_7 + x_5^2 - \\
& x_5x_6 + x_5x_7 + x_6^2 - x_6x_7 + x_7^2
\end{aligned}
$$

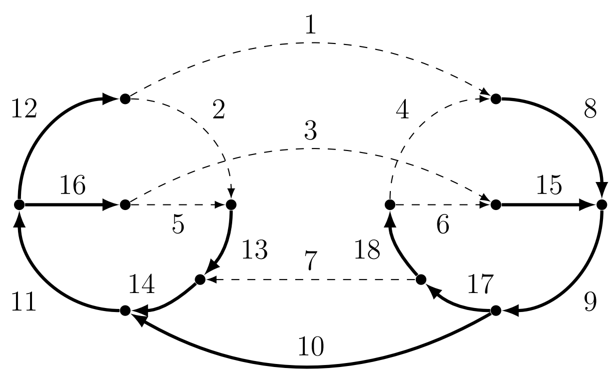

Figure 2.17: The Graph G_5

The basis for G_5 is given by:

$$f_1 = e_1 + e_8 + e_9 + e_{10} + e_{11} + e_{12}$$

$$f_2 = e_2 + e_{11} + e_{12} + e_{13} + e_{14}$$

$$f_3 = e_3 + e_9 + e_{10} + e_{11} + e_{15} + e_{16}$$

$$f_4 = e_4 + e_8 + e_9 + e_{17} + e_{18}$$

$$f_5 = e_5 + e_{11} + e_{13} + e_{14} + e_{16}$$

$$f_6 = e_6 + e_9 + e_{15} + e_{17} + e_{18}$$

$$f_7 = e_7 - e_{10} + e_{14} + e_{17}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	1	1	1	1	1	0	0	0	0	0	0
f_2	0	0	0	1	1	1	1	0	0	0	0
f_3	0	1	1	1	0	0	0	1	1	0	0
f_4	1	1	0	0	0	0	0	0	0	1	1
f_5	0	0	0	1	0	1	1	0	1	0	0
f_6	0	1	0	0	0	0	0	1	0	1	1
f_7	0	0	-1	0	0	0	1	0	0	1	0

which gives equations:

(1) $1 = 2 + a_{1,4}$

(2) $1 = 4 + a_{1,3} + a_{1,4} + a_{1,6} + a_{3,4} + a_{3,6} + a_{4,6}$

(3) $1 = 3 + a_{1,3} - a_{1,7} - a_{3,7}$

(4) $1 = 4 + a_{1,2} + a_{1,3} + a_{1,5} + a_{2,3} + a_{2,5} + a_{3,5}$

(5) $1 = 2 + a_{1,2}$

(6) $1 = 2 + a_{2,5}$

(7) $1 = 3 + a_{2,5} + a_{2,7} + a_{5,7}$

(8) $1 = 2 + a_{3,6}$

(9) $1 = 2 + a_{3,5}$

(10) $1 = 3 + a_{4,6} + a_{4,7} + a_{6,7}$

(11) $1 = 2 + a_{4,6}$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 & -1 & 1 \\ -1 & 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 2 & 1 & -1 \\ 0 & 0 & -1 & -1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 0 & -1 & -1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{G_5}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \quad & x_1^2 - x_1x_2 - x_1x_4 + x_1x_7 + x_2^2 - \\ & x_2x_5 + x_3^2 - x_3x_5 - x_3x_6 + x_3x_7 + \\ & x_4^2 - x_4x_6 + x_5^2 + x_5x_6 - x_5x_7 + \\ & x_6^2 - x_6x_7 + x_7^2 \end{aligned}$$

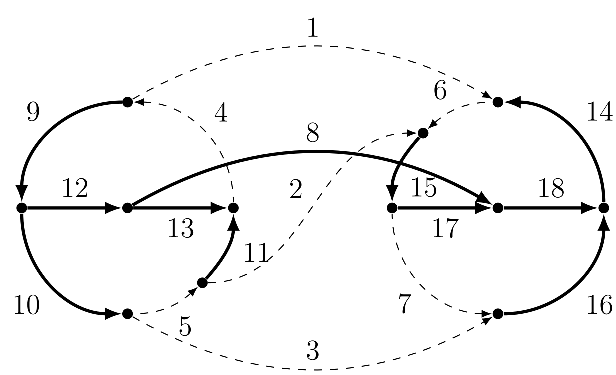

Figure 2.18: The Graph G_6

The basis for G_6 is given by:

$$f_1 = e_1 - e_8 - e_9 - e_{12} - e_{14} - e_{18}$$

$$f_2 = e_2 - e_8 - e_{11} + e_{13} + e_{15} + e_{17}$$

$$f_3 = e_3 - e_8 + e_{10} - e_{12} + e_{16} - e_{18}$$

$$f_4 = e_4 + e_9 + e_{12} + e_{13}$$

$$f_5 = e_5 + e_{10} + e_{11} - e_{12} - e_{13}$$

$$f_6 = e_6 + e_{14} + e_{15} + e_{17} + e_{18}$$

$$f_7 = e_7 + e_{16} - e_{17} - e_{18}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	-1	-1	0	0	-1	0	-1	0	0	0	-1
f_2	-1	0	0	-1	0	1	0	1	0	1	0
f_3	-1	0	1	0	-1	0	0	0	1	0	-1
f_4	0	1	0	0	1	1	0	0	0	0	0
f_5	0	0	1	1	-1	-1	0	0	0	0	0
f_6	0	0	0	0	0	0	1	1	0	1	1
f_7	0	0	0	0	0	0	0	0	1	-1	-1

which gives equations:

$$
\begin{array}{ll}
(1) & 1 = 3 + a_{1,2} + a_{1,3} + a_{2,3} \\
(2) & 1 = 2 - a_{1,4} \\
(3) & 1 = 2 + a_{3,5} \\
(4) & 1 = 2 - a_{2,5} \\
(5) & 1 = 4 + a_{1,3} - a_{1,4} + a_{1,5} - a_{3,4} + a_{3,5} - a_{4,5} \\
(6) & 1 = 3 + a_{2,4} - a_{2,5} - a_{4,5} \\
(7) & 1 = 2 - a_{1,6} \\
(8) & 1 = 2 + a_{2,6} \\
(9) & 1 = 2 + a_{3,7} \\
(10) & 1 = 3 + a_{2,6} - a_{2,7} - a_{6,7} \\
(11) & 1 = 4 + a_{1,3} - a_{1,6} + a_{1,7} - a_{3,6} + a_{3,7} - a_{6,7}
\end{array}
$$

The following matrix satisfies these equations and is positive definite:

$$
\begin{pmatrix}
2 & -1 & -1 & 1 & 0 & 1 & 0 \\
-1 & 2 & 0 & -1 & 1 & -1 & 0 \\
-1 & 0 & 2 & 0 & -1 & -1 & -1 \\
1 & -1 & 0 & 2 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 2 & 0 & 0 \\
1 & -1 & -1 & 0 & 0 & 2 & 1 \\
0 & 0 & -1 & -1 & 0 & 1 & 2
\end{pmatrix}
$$

And the quadratic form given by this matrix is:

$$
\begin{aligned}
q_{G_6}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \quad & x_1^2 - x_1x_2 - x_1x_3 + x_1x_4 + x_1x_6 + \\
& x_2^2 - x_2x_4 + x_2x_5 - x_2x_6 + x_3^2 - \\
& x_3x_5 - x_3x_6 - x_3x_7 + x_4^2 - x_4x_7 + \\
& x_5^2 + x_6^2 + x_6x_7 + x_7^2
\end{aligned}
$$

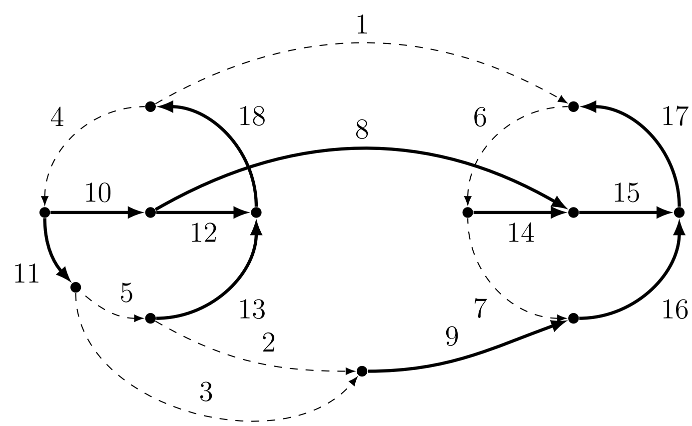

Figure 2.19: The Graph G_7

The basis for G_7 is given by:

$$f_1 = e_1 - e_8 + e_{12} - e_{15} - e_{17} + e_{18}$$

$$f_2 = e_2 - e_8 + e_9 + e_{12} - e_{13} - e_{15} + e_{16}$$

$$f_3 = e_3 - e_8 + e_9 - e_{10} + e_{11} - e_{15} + e_{16}$$

$$f_4 = e_4 + e_{10} + e_{12} + e_{18}$$

$$f_5 = e_5 - e_{10} + e_{11} - e_{12} + e_{13}$$

$$f_6 = e_6 + e_{14} + e_{15} + e_{17}$$

$$f_7 = e_7 - e_{14} - e_{15} + e_{16}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	-1	0	0	0	1	0	0	-1	0	-1	1
f_2	-1	1	0	0	1	-1	0	-1	1	0	0
f_3	-1	1	-1	1	0	0	0	-1	1	0	0
f_4	0	0	1	0	1	0	0	0	0	0	1
f_5	0	0	-1	1	-1	1	0	0	0	0	0
f_6	0	0	0	0	0	0	1	1	0	1	0
f_7	0	0	0	0	0	0	-1	-1	1	0	0

which gives equations:

(1) $1 = 3 + a_{1,2} + a_{1,3} + a_{2,3}$

(2) $1 = 2 + a_{2,3}$

(3) $1 = 3 - a_{3,4} + a_{3,5} - a_{4,5}$

(4) $1 = 2 + a_{3,5}$

(5) $1 = 4 + a_{1,2} + a_{1,4} - a_{1,5} + a_{2,4} - a_{2,5} - a_{4,5}$

(6) $1 = 2 - a_{2,5}$

(7) $1 = 2 - a_{6,7}$

(8) $1 = 5 + a_{1,2} + a_{1,3} - a_{1,6} + a_{1,7} + a_{2,3} - a_{2,6} + a_{2,7} - a_{3,6} + a_{3,7} - a_{6,7}$

(9) $1 = 3 + a_{2,3} + a_{2,7} + a_{3,7}$

(10) $1 = 2 - a_{1,6}$

(11) $1 = 2 + a_{1,4}$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & -1 & 0 & -1 & -1 & 1 & 0 \\ -1 & 2 & -1 & 0 & 1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 2 & 1 & -1 & 0 \\ -1 & 1 & -1 & 1 & 2 & -1 & 0 \\ 1 & -1 & 0 & -1 & -1 & 2 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{G_7}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \quad & x_1^2 - x_1x_2 - x_1x_4 - x_1x_5 + x_1x_6 + \\ & x_2^2 - x_2x_3 + x_2x_5 - x_2x_6 - x_2x_7 + \\ & x_3^2 - x_3x_5 + x_4^2 + x_4x_5 - x_4x_6 + \\ & x_5^2 - x_5x_6 + x_6^2 + x_6x_7 + x_7^2 \end{aligned}$$

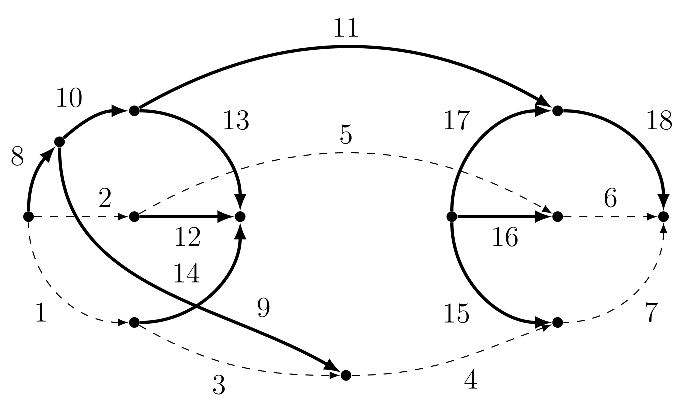

Figure 2.20: The Graph G_8

The basis for G_8 is given by:

$$f_1 = e_1 - e_8 - e_{10} - e_{13} + e_{14}$$

$$f_2 = e_2 - e_8 - e_{10} + e_{12} - e_{13}$$

$$f_3 = e_3 - e_9 + e_{10} + e_{13} - e_{14}$$

$$f_4 = e_4 + e_9 - e_{10} - e_{11} - e_{15} + e_{17}$$

$$f_5 = e_5 - e_{11} - e_{12} + e_{13} - e_{16} + e_{17}$$

$$f_6 = e_6 + e_{16} - e_{17} - e_{18}$$

$$f_7 = e_7 + e_{15} - e_{17} - e_{18}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	-1	0	-1	0	0	-1	1	0	0	0	0
f_2	-1	0	-1	0	1	-1	0	0	0	0	0
f_3	0	-1	1	0	0	1	-1	0	0	0	0
f_4	0	1	-1	-1	0	0	0	-1	0	1	0
f_5	0	0	0	-1	-1	1	0	0	-1	1	0
f_6	0	0	0	0	0	0	0	0	1	-1	-1
f_7	0	0	0	0	0	0	0	1	0	-1	-1

which gives equations:

$$
\begin{aligned}
&(1) \quad 1 = 2 + a_{1,2} \\
&(2) \quad 1 = 2 - a_{3,4} \\
&(3) \quad 1 = 4 + a_{1,2} - a_{1,3} + a_{1,4} - a_{2,3} + a_{2,4} - a_{3,4} \\
&(4) \quad 1 = 2 + a_{4,5} \\
&(5) \quad 1 = 2 - a_{2,5} \\
&(6) \quad 1 = 4 + a_{1,2} - a_{1,3} - a_{1,5} - a_{2,3} - a_{2,5} + a_{3,5} \\
&(7) \quad 1 = 2 - a_{1,3} \\
&(8) \quad 1 = 2 - a_{4,7} \\
&(9) \quad 1 = 2 - a_{5,6} \\
&(10) \quad 1 = 4 + a_{4,5} - a_{4,6} - a_{4,7} - a_{5,6} - a_{5,7} + a_{6,7} \\
&(11) \quad 1 = 2 + a_{6,7}
\end{aligned}
$$

The following matrix satisfies these equations and is positive definite:

$$
\begin{pmatrix}
2 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 2 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 2 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & -1 & -1 & 1 \\
0 & 1 & 0 & -1 & 2 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 2 & -1 \\
0 & 0 & 0 & 1 & 0 & -1 & 2
\end{pmatrix}
$$

And the quadratic form given by this matrix is:

$$
\begin{aligned}
q_{G_8}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \; & x_1^2 - x_1x_2 + x_1x_3 + x_2^2 + x_2x_5 + \\
& x_3^2 + x_3x_4 + x_4^2 - x_4x_5 - x_4x_6 + \\
& x_4x_7 + x_5^2 + x_5x_6 + x_6^2 - x_6x_7 + \\
& x_7^2
\end{aligned}
$$

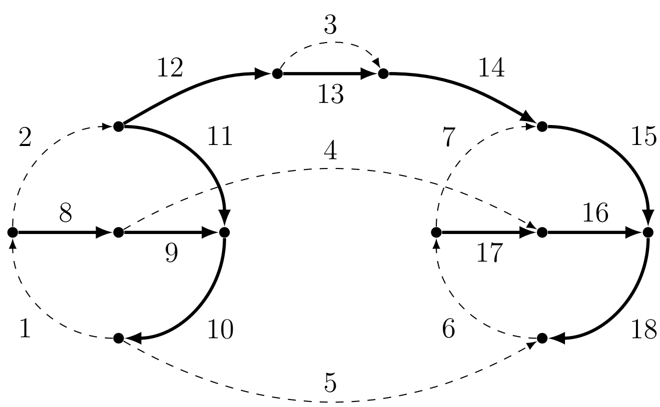

Figure 2.21: The Graph G_9

The basis for G_9 is given by:

$$f_1 = e_1 + e_8 + e_9 + e_{10}$$

$$f_2 = e_2 - e_8 - e_9 + e_{11}$$

$$f_3 = e_3 - e_{13}$$

$$f_4 = e_4 - e_9 + e_{11} - e_{12} - e_{13} - e_{14} - e_{15} + e_{16}$$

$$f_5 = e_5 + e_{10} + e_{11} - e_{12} - e_{13} - e_{14} - e_{15} - e_{18}$$

$$f_6 = e_6 + e_{16} + e_{17} + e_{18}$$

$$f_7 = e_7 + e_{15} - e_{16} - e_{17}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	1	1	1	0	0	0	0	0	0	0	0
f_2	-1	-1	0	1	0	0	0	0	0	0	0
f_3	0	0	0	0	0	-1	0	0	0	0	0
f_4	0	-1	0	1	-1	-1	-1	-1	1	0	0
f_5	0	0	1	1	-1	-1	-1	-1	0	0	-1
f_6	0	0	0	0	0	0	0	0	1	1	1
f_7	0	0	0	0	0	0	0	1	-1	-1	0

which gives equations:

$$(1) \quad 1 = 2 - a_{1,2}$$

$$(2) \quad 1 = 3 - a_{1,2} - a_{1,4} + a_{2,4}$$

$$(3) \quad 1 = 2 + a_{1,5}$$

$$(4) \quad 1 = 3 + a_{2,4} + a_{2,5} + a_{4,5}$$

$$(5) \quad 1 = 2 + a_{4,5}$$

$$(6) \quad 1 = 3 + a_{3,4} + a_{3,5} + a_{4,5}$$

$$(7) \quad 1 = 2 + a_{4,5}$$

$$(8) \quad 1 = 3 + a_{4,5} - a_{4,7} - a_{5,7}$$

$$(9) \quad 1 = 3 + a_{4,6} - a_{4,7} - a_{6,7}$$

$$(10) \quad 1 = 2 - a_{6,7}$$

$$(11) \quad 1 = 2 - a_{5,6}$$

The following matrix satisfies these equations and is positive definite:

$$\begin{pmatrix} 2 & 1 & 0 & 1 & -1 & 0 & 0 \\ 1 & 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 2 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{G_9}(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = \quad & x_1^2 + x_1x_2 + x_1x_4 - x_1x_5 + x_2^2 + \\ & x_2x_3 - x_2x_5 + x_3^2 - x_3x_4 + x_3x_6 + \\ & x_4^2 - x_4x_5 - x_4x_6 + x_5^2 + x_5x_6 + \\ & x_5x_7 + x_6^2 + x_6x_7 + x_7^2 \end{aligned}$$

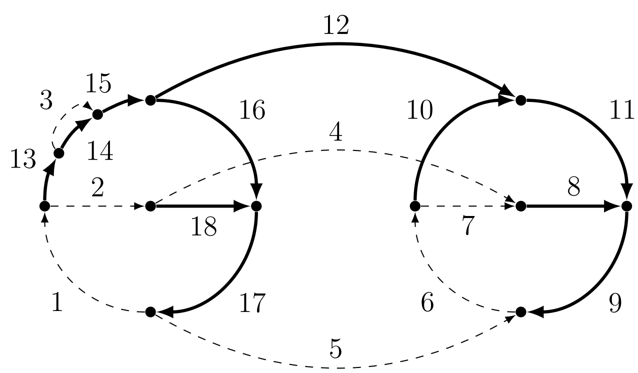

Figure 2.22: The Graph G_{10}

The basis for G_{10} is given by:

$$f_1 = e_1 + e_{13} + e_{14} + e_{15} + e_{16} + e_{17}$$

$$f_2 = e_2 - e_{13} - e_{14} - e_{15} - e_{16} + e_{18}$$

$$f_3 = e_3 - e_{14}$$

$$f_4 = e_4 + e_8 - e_{11} - e_{12} + e_{16} - e_{18}$$

$$f_5 = e_5 - e_9 - e_{11} - e_{12} + e_{16} + e_{17}$$

$$f_6 = e_6 + e_9 + e_{10} + e_{11}$$

$$f_7 = e_7 + e_8 - e_{10} - e_{11}$$

Which gives us the following array:

	e_8	e_9	e_{10}	e_{11}	e_{12}	e_{13}	e_{14}	e_{15}	e_{16}	e_{17}	e_{18}
f_1	0	0	0	0	0	1	1	1	1	1	0
f_2	0	0	0	0	0	-1	-1	-1	-1	0	1
f_3	0	0	0	0	0	0	-1	0	0	0	0
f_4	1	0	0	-1	-1	0	0	0	1	0	-1
f_5	0	-1	0	-1	-1	0	0	0	1	1	0
f_6	0	1	1	1	0	0	0	0	0	0	0
f_7	1	0	-1	-1	0	0	0	0	0	0	0

which gives equations:

$$
\begin{aligned}
&(1) \quad 1 = 2 + a_{4,7}\\
&(2) \quad 1 = 2 - a_{5,6}\\
&(3) \quad 1 = 2 - a_{6,7}\\
&(4) \quad 1 = 4 + a_{4,5} - a_{4,6} + a_{4,7} - a_{5,6} + a_{5,7} - a_{6,7}\\
&(5) \quad 1 = 2 + a_{4,5}\\
&(6) \quad 1 = 2 - a_{1,2}\\
&(7) \quad 1 = 3 - a_{1,2} - a_{1,3} + a_{2,3}\\
&(8) \quad 1 = 2 - a_{1,2}\\
&(9) \quad 1 = 4 - a_{1,2} + a_{1,4} + a_{1,5} - a_{2,4} - a_{2,5} + a_{4,5}\\
&(10) \quad 1 = 2 + a_{1,5}\\
&(11) \quad 1 = 2 - a_{2,4}
\end{aligned}
$$

The following matrix satisfies these equations and is positive definite:

$$
\begin{pmatrix}
2 & 1 & 1 & 1 & -1 & 0 & 0\\
1 & 2 & 0 & 1 & 0 & 0 & 0\\
1 & 0 & 2 & 0 & 0 & 0 & 0\\
1 & 1 & 0 & 2 & -1 & 0 & -1\\
-1 & 0 & 0 & -1 & 2 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 2 & 1\\
0 & 0 & 0 & -1 & 1 & 1 & 2
\end{pmatrix}
$$

And the quadratic form given by this matrix is:

$$\begin{aligned} q_{G_{10}}(x_1,x_2,x_3,x_4,x_5,x_6,x_7) = \quad & x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 - x_1x_5 + \\ & x_2^2 + x_2x_4 + x_3^2 + x_4^2 - x_4x_5 - \\ & x_4x_7 + x_5^2 + x_5x_6 + x_5x_7 + x_6^2 + \\ & x_6x_7 + x_7^2 \end{aligned}$$

This concludes the calculations.

2.5 Genus 8

As we have explained, it is sufficient to find a $\mathbb{Z}$-emm for each of the finitely many graphs obtained from F_{11}–F_{14} and G_1-G_{10} by applying procedure (a) or (b). This gives $14 \cdot \left(\binom{18}{2} + 18\right) = 2394$ graphs.

We have written a Mathematica program for computing the 8×13 matrices for these graphs, and a Fortran program which uses integer arithmetic for finding the $\mathbb{Z}$-emms. We confirmed that they exist for all of these graphs. The lists of the matrices and the $\mathbb{Z}$-emms are available at `http://www.math.uga.edu/~valery/vigre2010`.

Chapter 3

Singularities and Kollár's Result on PPAV's

The second part of this thesis seeks to extend in a natural way the result of Kollár [18, Thm.17.13] which states that principally polarized abelian pairs (X, Θ) are log canonical. We would like to extend this result to degenerations of such pairs. Though there is more than one compactification of A_g, the moduli space of principally polarized abelian varieties, in [1] Alexeev provided a modular interpretation for the toroidal compactification of A_g for the second Voronoi fan. We will show that the pairs in the boundary of this compactification are in fact semi-log canonical.

To do so, we will first discuss the definition of being log canonical. Next we will review the proof given by Kollár showing that principally polarized abelian varieties are log canonical. Finally we will discuss the definition of being semi-log canonical as an extension of log canonical in the non-normal setting.

3.1 Log Canonical Pairs

The class of singularities termed "log canonical" is a class that arises in the study of the minimal model program. Indeed, there are several such classes, all closely related. Intuitively, if one considers a pair (X, Θ) and a birational morphism $f : Y \to X$, then under the right conditions one can compare $f^*(K_X + \Theta)$ and $K_Y + f_*^{-1}(\Theta)$. Terminal, canonical, log terminal and log canonical classes of singularities represent the degrees by which these divisors may differ.

More precisely we consider the following setup as described in [20, p.50-56]. Let X be a normal variety and $\Theta = \sum a_i D_i$ a $\mathbb{Q}$-divisor on X such that $K_X + \Theta$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a birational morphism where Y is normal. Since $m(K_X + \Theta)$ is Cartier for some $m \in \mathbb{Z}_{>0}$, we can compare the two divisors $m(K_Y + f_*^{-1}(\Theta))$ and $f^*(m(K_x + \Theta))$. The difference will be exceptional and so there exist rational numbers $a(E_i, X, \Theta)$ so that $m(a(E_i, X, \Theta))$ are integers and

$$m(K_Y + f_*^{-1}\Theta) = f^*(m(K_x + \Theta)) + \sum ma(E_i, X, \Theta)E_i$$

for exceptional divisors E_i. Passing to numerical equivalence, we can write

$$K_Y + f_*^{-1}\Theta \equiv f^*(K_X + \Theta) + \sum a(E_i, X, \Theta)E_i$$

We are now in a position to make the following important definition.

Definition 3.1.1. *The discrepancy of* (X, Θ) *is defined as*

$$\mathrm{Discrep}(X, \Theta) = \inf_E \{a(E, X, \Theta)\}$$

where the infimum is taken over all exceptional divisors E of all birational morphisms $f : Y \to X$.

This allows us to define different singularity classes based on the discrepancy of the pair.

Definition 3.1.2. *We say the pair (X, Θ) is*

terminal		> 0
canonical		≥ 0
klt	*if* Discrep(X, Θ) *is*	> -1 *and* $\lfloor \Theta \rfloor \leq 0$
plt		> -1
log canonical		≥ -1

As stated there may seem to be little hope of actually computing the discrepancy of a pair since, a priori, we must consider all birational maps to the pair. However, it suffices to consider a log resolution. A log resolution of (X, Θ) is a proper, birational morphism $f : Y \to X$ satisfying the conditions that Y is smooth, the exceptional locus $\text{Ex}(f)$ is a divisor, and $ex(f) \cup f^{-1}(\text{Supp}(\Theta))$ is a simple normal crossing divisor. By Hironaka's result in [15], log resolutions always exist over an algebraically closed field of characteristic 0.

We are then able to compute the information needed for our purposes by the following lemma [19, Corollary 2.13]:

Lemma 3.1.1. *Let (X, Θ) be a pair as above and let $f : Y \to X$ be a log resolution. Write*

$$K_Y \sim_Q f^*(K_X + \Theta)) + \sum a_i E_i$$

where $\sim_{\mathbb{Q}}$ denotes $\mathbb{Q}$-linear equivalence. Then (X, Θ) is log canonical if and only if $a_i \geq -1$ for every i.

We end this section with an example.

Example 3.1.1. *Let X be a smooth surface and let $\Theta = \sum D_i$. If $P \in X$ is a point, we might consider the blowup map $f : \mathrm{BL}_P(X) \to X$. Writing Y for $\mathrm{BL}_P(X)$ and E for the exceptional divisor of f we have $K_Y = f^*K_X + (\operatorname{codim}_P X - 1)E$ and $f_*^{-1}(\Theta) = f^*\Theta - \operatorname{mult}_P \Theta E$. Adding these two equations we get that $a(E, X, \Theta) = \operatorname{codim}_P X - 1 - \operatorname{mult}_P \Theta = 2 - 1 - \operatorname{mult}_P \Theta$. Moreover, if Θ is nodal at P then f is a log resolution of (X, Θ). So we see that in this case (X, Θ) is log canonical if the singularities of Θ are at worst nodal.*

3.2 Kollár's Result on PPAV's

In this section we include Kollár's result which will serve as a model for the main result of this chapter. The main tool of the proof is the vanishing theorem of Kawamata and Viehweg from 1982 [17, 25] which we state here.

Theorem 3.2.1. *Let X be a smooth projective variety. If $L \equiv M + \Delta$ is a $\mathbb{Z}$-divisor with M a nef and big $\mathbb{Q}$ divisor and (X, Δ) klt, then $H^i(X, K_X \otimes L) = 0$ for all $i > 0$.*

The following appears in [18, Thm.17.13] and the presentation of the proof is credited to Lazarsfeld. We will, in the proof of 4.3.1, try to use a similar strategy: first create an opportunity to make use of vanishing and then use the group action on X to arrive at a contradiction.

Theorem 3.2.2. *Let (A, Θ) be a principally polarized abelian variety. Then (A, Θ) is log canonical.*

Proof: Let $f : A' \to A$ be a log resolution. Set $K_{A'} = \sum e_i E_i$, and $f^*\Theta = \sum b_i E_i$. Let $c = \min\{(e_i + 1)/b_i\}$. This is what we will later call the log canonical threshold of (A, Θ). That is, c is the largest number so that $(A, c\Theta)$ is log canonical. Thus, we would like to show that $c \geq 1$. Formally we have

$$f^*\Theta \equiv K_{A'} + (1-c)f^*\Theta + \sum (cb_i - e_i)E_i \tag{3.1}$$

Let $\sum \lfloor cb_i - e_i \rfloor E_i = P - N$ where P and N are effective and share no irreducible components. If E_j is a component of N, then $e_j > 0$, and so N is exceptional. P is reduced and non-empty by the definition of c. This allows us to rewrite (3.1) as

$$f^*\Theta + N - P \equiv K_{A'} + (1-c)f^*\Theta + \sum \{cb_i - e_i\}E_i$$

where $\{a\}$ is the fractional part of a. Set $Z = f(P)$. Letting $Z = f(P)$ we get a commutative diagram:

$$\begin{array}{ccc} H^0(A', \mathcal{O}(f^*\Theta + N)) & \rightarrow & H^0(P, \mathcal{O}(f^*\Theta + N)|_P) \\ \uparrow \cong & & \uparrow \\ H^0(A, \mathcal{O}(\Theta)) & \xrightarrow{r} & H^0(Z, \mathcal{O}(\Theta)|_Z) \end{array}$$

The left vertical arrow is an isomorphism since N is exceptional and the right vertical arrow is injective. Assume that $c < 1$. By Kawamata-Vieweg vanishing, $H^1(A', \mathcal{O}(f^*\Theta + N - P)) = 0$, which implies that the top horizontal arrow is surjective. Therefore r is surjective.

On the other hand, r is zero since $Z \subset \Theta$ and the unique section of $\mathcal{O}_A(\Theta)$ vansishes along Θ. If we can show that $h^0(Z, \mathcal{O}_A(\Theta)|_Z) > 0$ then we will arrive at a contradiction. Let $\tau : A \to A$ be a general translation. Then $Z \not\subset \tau^*\Theta$, so $h^0(Z, \tau^*\mathcal{O}_A(\Theta)|_Z) > 0$. By semicontinuity we get that $h^0(Z, \mathcal{O}_A(\Theta)|_Z) > 0$ as well. Thus, we arrive at our contradicition and so $c \geq 1$, as desired.

□

In the proof of 4.3.1, we will use the same basic strategy while considering a family of pairs and using a more general vanishing theorem.

3.3 Semi-Log Canonical Pairs

The concept of a pair (X, Δ) being semi-log canonical (or slc) comes from generalizing to the non-normal setting. This can be done either by working with the normalization of the pair or by working with semiresolutions which somehow preserve the "double locus" of X. However, some care must be taken in discussing the canonical class on singular varieties. A nice discussion appears in [13, p.35], which we summarize here. Suppose that X is $S2$ and all codimension 1 points of X are regular or ordinary nodes. Thus, we have that X is Gorenstein away from a codimension 2 locus. This allows us to define a dualizing sheaf ω_X° away from this locus, which we may then extend to all of X by the $S2$ property. We take this coherent sheaf as our canonical sheaf with corresponding Weil divisor K_X.

We can now give the definition of a semi-log canonical pair from [19, p.193-194].

Definition 3.3.1. *Let X be a scheme that is $S2$ and whose codimension 1 points are either regular or ordinary nodes (a so called demi-normal scheme). Let X have normalization $\nu : X^\nu \to X$ and conductors $D \subset X$ and $D^\nu \subset X^\nu$. Let Δ be an effective $\mathbb{Q}$-divisor whose support does not contain any irreducible component of D and Δ^ν the divisorial part of $\nu^{-1}(\Delta)$.*

The pair (X, Δ) is called semi-log canonical if

1. *$K_X + \Delta$ is $\mathbb{Q}$-Cartier, and*
2. *One of the following equivalent conditions holds:*
   - *$(X^\nu, D^\nu + \Delta^\nu)$ is log canonical*
   - *$a(E, X, \Delta) \geq -1$ for every exceptional divisor E of a birational morphism $f : Y \to X$ such that Y is regular at the generic point of E.*

where $a(E, X, \Delta)$ is the coefficient appearing in the formula

$$K_Y + f_*^{-1}\Delta \sim_{\mathbb{Q}} f^*(K_X + \Delta) + \sum_{E_i \text{ exceptional}} a(E_i, X, \Delta)E_i$$

In the proof of 4.3.1, we will use the first of the equivalent conditions above as our characterization of being semi-log canonical. We conclude this section with two examples. The following are compactified jacobians and should be viewed as motivation for 4.3.1. For details on computing the compactified jacobian of a stable curve, see [2].

Example 3.3.1. *The compactified jacobian of a rational curve C with one node is again C with Θ a smooth point on C. This is described below in the simple toric picture where we identify the two endpoints of the polytope. Θ appears as a point in the interior.*

Figure 3.1: Compactified jacobian of $\mathbb{P}^1$ with node.

Thus the pair $(C^\nu, D^\nu + \Theta^\nu)$ is just a smooth $\mathbb{P}^1$ with three points which is log canonical since the identity map is a log resolution.

Example 3.3.2. *The compactified jacobian of a rational curve C with two nodes is given by a $\mathbb{P}^1 \times \mathbb{P}^1$ with two pairs of lines identified. This is described below in the toric picture where we identify the opposite edges of the polytope. This will again be semi-log canonical since $\Theta^\nu + D^\nu$ is normal crossing.*

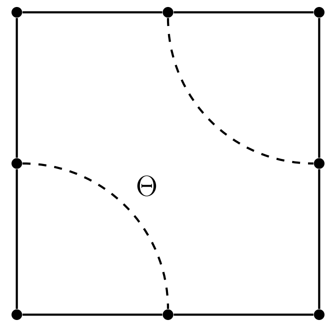

Figure 3.2: Compactified jacobian of $\mathbb{P}^1$ with 2 nodes.

Chapter 4

The Singularities of Degenerate Abelian Varieties

4.1 Stable Semiabelic Pairs

In [1] Alexeev discusses the moduli of semiabelic pairs. Here we will be concerned with principally polarized stable semiabelic pairs, objects introduced in the compactification of abelic pairs. More specifically we will consider the so-called "smoothable" pairs which appear as limits of one-parameter families of principally polarized abelian varieties. A smoothable pair $(\mathcal{X}_0, \Theta_0)$ is one such that there is a family $\mathcal{X} \to S$ where S is a smooth curve with generic fiber a principally polarized abelian variety and special fiber $(\mathcal{X}_0, \Theta_0)$. What follows is a summary of the pertinent definitions and results from [1].

A semiabelian variety is a group variety G which is an extension

$$1 \to T \to G \to A \to 0$$

of an abelian variety A by a torus $T \cong \mathbb{G}_m^r$ (where $\mathbb{G}_m$ is the multiplicative group of the ground field). A semiabelic variety is a variety P together with an action of a semiabelian variety G such that:

1. P is normal.

2. The action has only finitely many orbits.

3. The stabilizer of every point is connected, reduced and lies in the toric part T of G.

In the special case that G is abelian, we say that P is an abelic variety.

Just like in the story of curves, one would like to generalize these definitions to make "stable" versions of these objects. A stable semiabelic variety is a variety P along with an action of a semiabelian variety G of the same dimension satisfying the same conditions except condition (1) is weakened to require only seminormality. We say that a reduced scheme P is seminormal if every proper bijective morphism $f : P' \to P$ with reduced P' inducing isomorphisms on the residue fields $\kappa(p') \supset \kappa(p)$ for each $p \in P$ is an isomorphism.

A stable semiabelic variety is polarized if it is projective and comes with an ample invertible sheaf L. The degree of the polarization is $h^0(L)$. A stable semiabelic pair (P, Θ) is a stable projective semiabelic variety P and an effective ample Cartier divisor Θ which does not contain any G-orbits. P is polarized by $L = \mathcal{O}_P(\Theta)$. We define an abelic pair analogously as a projective abelic variety (also called an abelian torsor) with an effective ample Cartier divisor.

In [1] we see that such pairs correspond to certain complexes of lattice polytopes referenced by $\overline{\Lambda}/Y$ where $\Lambda \cong \mathbb{Z}^g$ is a lattice, $Y \subset \Lambda$ is a subgroup and $\overline{\Lambda}$ is a trivial Λ-torsor. Analysis of such pairs is categorized by type which is determined by $\overline{\Lambda}$ and Y. For this thesis we will be concerned with the principally polarized case, which corresponds to the case when $|\Lambda/Y| = 1$.

In [1], Alexeev establishes the following facts concerning the moduli of abelic and semi-abelic pairs:

Theorem 4.1.1. *1. The moduli stack $\mathcal{AP}_{g,d}$ of abelic pairs (P, Θ) of degree d is a separated Artin stack with finite stabilizers and comes with a natural map of relative dimension $d-1$ to the stack $\mathcal{A}_{g,d}$ of polarized abelian varieties [1, Thm.1.2.2].*

2. *$\mathcal{AP}_{g,d}$ has a coarse moduli space $AP_{g,d}$ which is a separated scheme and comes with a natural projective map of relative dimension $d-1$ to $A_{g,d}$ [1, Thm.1.2.2].*

3. *In the principally polarized case (that is, when $d = 1$), we have $\mathcal{AP}_g = \mathcal{A}_g$ and $AP_g = A_g$ [1, Thm.1.2.3].*

4. *The component $\overline{\mathcal{AP}}_{g,d}$ of the moduli stack of semiabelic pairs containing $\mathcal{AP}_{g,d}$ and pairs of the same numerical type is a proper Artin stack with finite stabilizers [1, Thm.1.2.16].*

5. *It has a coarse moduli space $\overline{AP}_{g,d}$ as a proper algebraic space [1, Thm.1.2.16].*

6. *In the principally polarized case (when $Y = \Lambda$) the toroidal compactification of A_g for the second Voronoi decomposition is isomorphic to the normalization of the main irreducible component of $\overline{AP}_{g,1}$, the one containing $A_g = AP_g$ [1, Thm.1.2.17].*

In the above paper there are also many facts about principally polarized stable semiabelic pairs, some of which are implicit in the arguments. Below we summarize and make explicit the ones that we will use in our analysis.

Theorem 4.1.2. *Let $(\mathcal{X}_0, \Theta_0)$ be a stable semi-abelic pair that appears as the special fiber of a flat family $\pi : (\mathcal{X}, \Theta) \to S$ where S is a smooth curve and a general fiber of π is a principally polarized abelian variety.*

1. *$K_{X_0} \sim 0$, where $\sim$ denotes linear equivalence.*

2. There is an $\epsilon > 0$ such that $(X_0, \epsilon\Theta_0)$ is semi-log canonical.

3. $h^0(X_0, \mathcal{O}_{X_0}(\Theta_0)) = |\Lambda/Y| = 1$.

4. $H^i(X_0, \mathcal{O}_{X_0}(\Theta_0)) = 0$ for all $i > 0$ [1, Thm.5.4.1].

These statements require some remarks.

(1) By the work of [22], there is a toric model of (X, X_0). That is, we may write $(X, X_0) = (Y, Y_0)/\mathbb{Z}^r$ where Y is toric and Y_0 is the toric boundary. Thus, by adjunction $K_{Y_0} \sim 0$ and so, since the action is $\mathbb{Z}^r$-equivariant, we also have $K_{X_0} \sim 0$.

(2) By [1, Thm.1.2.14], X_0 is Cohen-Macaulay and hence S2 and so the existence of an $\epsilon > 0$ such that $(X_0, \epsilon\Theta_0)$ is semi-log canonical amounts to showing that (in the notation of definition 3.3.1) $K_{X_0} + \Theta_0$ is $\mathbb{Q}$-Cartier, (X_0^ν, D^ν) is log canonical and Θ_0^ν does not contain any log canonical centers of (X_0^ν, D^ν). $K_{X_0} + \Theta_0$ is $\mathbb{Q}$-Cartier since Θ_0 is Cartier and $K_{X_0} \sim 0$. (X_0^ν, D^ν) is log canonical because $K_{X_0^\nu} + D^\nu \sim 0$ and (X_0^ν, D^ν) is a toric pair. Finally, by definition, Θ_0 does not contain any G-orbits, and the log canonical centers of (X_0^ν, D^ν) are precisely the closures of the codimension ≥ 1 orbits. This implies in particular that X_0 is a demi-normal scheme such that Θ_0 does not contain any irreducible component of the conductor of normalization. By continuity of discrepancies, the pair $(X_0, \epsilon\Theta_0)$ is semi-log canonical for $0 < \epsilon \ll 1$.

(3) Since we work with principally polarized pairs, the condition $h^0(X, \mathcal{O}_X(\Theta)) = |\Lambda/Y| = 1$ is satisfied by definition.

(4) This vanishing result is the most important property that we will need in our proof of the main result. A complete proof of this fact is given in [1, Thm.5.4.1].

4.2 Vanishing

In this part of the thesis we aim to prove that principally polarized stable semiabelic pairs are semi-log canonical, and it has been mentioned that we will proceed in analogy with the proof of Theorem 3.2.2. However, the vanishing theorem we will need to apply is due to Fujino [9, Thm.6.3], which we use only with $\mathbb{Q}$-divisors.

Theorem 4.2.1. *Let Y be a smooth variety and let B be a boundary $\mathbb{R}$-divisor such that* $\operatorname{Supp} B$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and let L be a Cartier divisor on Y such that $L - (K_Y + B)$ is $f-semi-ample$. Let $\pi : X \to S$ be a projective morphism. Assume that $L - (K_X + B) \sim_{\mathbb{R}} f^*H$ for some π-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor H on X. Then $R^p\pi_* R^q f_* \mathcal{O}_Y(L) = 0$ for every $p > 0$ and $q \geq 0$.*

We are now ready to state and prove the main result of the paper.

4.3 Statement and proof of main result

We work now over an algebraically closed field of characteristic zero.

Theorem 4.3.1. *Let $(\mathcal{X}_0, \Theta_0)$ be a stable semi-abelic pair that is a one-parameter degeneration. That is, we have a flat family $\pi : (\mathcal{X}, \Theta) \to S$ where S is a smooth curve, a general fiber of π is a principally polarized abelian variety, and the special fiber of π is $(\mathcal{X}_0, \Theta_0)$. Then $(\mathcal{X}_0, \Theta_0)$ is semi-log canonical.*

Proof: As we have seen in Theorem 4.1.2, we already have that $\mathcal{X}_0$ is demi-normal, $K_{\mathcal{X}_0} + \Theta_0$ is Cartier, and Θ_0 does not contain any irreducible component of the conductor of normalization. Thus, by adjunction $(\mathcal{X}_0, \Theta_0)$ is semi-log canonical if $(\mathcal{X}, \Theta + \mathcal{X}_0)$ is log canonical in a neighborhood of $\mathcal{X}_0$. So, it would suffice to show that $(\mathcal{X}, \Theta + (1-\epsilon)\mathcal{X}_0)$ is log canonical for every $0 < \epsilon \ll 1$.

Let $f : \mathcal{Y} \to \mathcal{X}$ be a resolution of singularities so that $\mathcal{Y}$ is smooth and $\widehat{\mathcal{X}_0} \cup \widehat{\Theta} \cup E$ is a simple normal crossing divisor where $\widehat{\mathcal{X}_0} = f_*^{-1}(\mathcal{X}_0)$, $\widehat{\Theta} = f_*^{-1}(\Theta)$, and E is the exceptional locus of f. Such a resolution is guaranteed by Hironaka [15].

Fix $0 < \epsilon \ll 1$ and put $d = 1 - \epsilon$. Let c be the log canonical threshold so that $(\mathcal{X}, c(\Theta + d\mathcal{X}_0))$ is log canonical but $(\mathcal{X}, c'(\Theta + d\mathcal{X}_0))$ is not log canonical for any $c' > c$.

Seeking a contradiction, suppose that $c < 1$ and put $cd = d'$. This means that we may write the following equation:

$$K_{\mathcal{Y}} + c\widehat{\Theta} + d'\widehat{\mathcal{X}_0} = f^*(K_{\mathcal{X}} + c\Theta + d'\mathcal{X}_0) + D$$

where D is exceptional. Write D as $D = B - A - \Delta$ where A, B, Δ are effective, A, B are integral and nonzero, $\lfloor \Delta \rfloor = 0$, and A, B have no irreducible components in common.

Now the equation becomes:

$$K_{\mathcal{Y}} + c\widehat{\Theta} + d'\widehat{\mathcal{X}_0} = f^*(K_{\mathcal{X}} + c\Theta + d'\mathcal{X}_0) + B - A - \Delta$$

Which we can rewrite as:

$$-f^*(K_{\mathcal{X}} + c\Theta + d'\mathcal{X}_0) = B - A - (K_{\mathcal{Y}} + c\widehat{\Theta} + d'\widehat{\mathcal{X}_0} + \Delta)$$

Adding $f^*(K_{\mathcal{X}} + \Theta + K_{\mathcal{X}_0})$ to each side:

$$f^*((1-c)\Theta + (1-d')\mathcal{X}_0) = (f^*(K_{\mathcal{X}} + \Theta + \mathcal{X}_0) + B - A) - (K_{\mathcal{Y}} + c\widehat{\Theta} + d'\widehat{\mathcal{X}_0} + \Delta)$$

Now we would like to apply Theorem 4.2.1 to the above situation. The conclusion is:

$$R^p\pi_* R^q f_* \mathcal{O}_{\mathcal{Y}}(f^*(K_{\mathcal{X}} + \Theta + \mathcal{X}_0) + B - A) = 0 \tag{4.1}$$

for $q \geq 0$ and $p > 0$.

Let $\mathcal{I}$ be the ideal given by $f_*\mathcal{O}_{\mathcal{Y}}(B - A) \subset \mathcal{O}_{\mathcal{X}}$ and let $Z \subset \mathcal{X}$ be the subscheme determined by $\mathcal{I}$. First let's see that $\mathcal{I}$ is in fact a non-trivial ideal sheaf. Since B and A are exceptional, $f(B) = Z_B$ and $f(A) = Z_A$ in $\mathcal{X}$ are of codimension ≥ 2. Let $s \in f_*\mathcal{O}_{\mathcal{Y}}(B-A)(U)$ for some open set $U \subset \mathcal{X}$. So s is a function on $f^{-1}(U)$ which may have poles of various orders on the different components of B and must have zeros on all components of A. Since $f_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$, s descends to a unique regular function s' on $\mathcal{X} - Z_B$. Moreover, since the codimension of Z_B in $\mathcal{X}$ is at least 2 and $\mathcal{X}$ is $S2$, we may extend s' uniquely to a regular function on all of $\mathcal{X}$. Moreover, if we had two functions $s_1, s_2 \in f_*\mathcal{O}_{\mathcal{Y}}(B - A)(U)$, which agreed away from B, then by continuity they would agree on B. Hence, we see that $f_*\mathcal{O}_{\mathcal{Y}}(B - A)$ is an ideal of $\mathcal{O}_{\mathcal{X}}$.

Consider the short exact sequence:

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_Z \to 0$$

Tensoring by $\mathcal{O}_{\mathcal{X}}(\Theta)$ we get:

$$0 \to \mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I} \to \mathcal{O}_{\mathcal{X}}(\Theta) \to \mathcal{O}_Z(\Theta) \to 0$$

Applying π_*, we arrive at the long exact sequence:

$$0 \to \pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I}) \to \pi_*(\mathcal{O}_{\mathcal{X}}(\Theta)) \xrightarrow{\rho} \pi_*(\mathcal{O}_Z(\Theta)) \to R^1\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I}) \to \dots$$

By (4.1), we have $R^1\pi_*(\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}} + \Theta + \mathcal{X}_0) \otimes \mathcal{I}) = 0$, and we would like to conclude that $R^1\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I}) = 0$. Indeed, we may write $\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}} + \mathcal{X}_0) = \pi^*L$ for some line bundle L.

Using the projection formula we have:

$$\begin{aligned} 0 = \quad & R^1\pi_*(\mathcal{O}_{\mathcal{X}}(K_{\mathcal{X}} + \Theta + \mathcal{X}_0) \otimes \mathcal{I}) \\ = \quad & R^1\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I} \otimes \pi^* L) \\ = \quad & R^1\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I}) \otimes L \end{aligned}$$

Since L is a line bundle, $R^1\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I}) \otimes L = 0$ implies that $R^1\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta) \otimes \mathcal{I}) = 0$. Thus, we have that $\rho : \pi_*(\mathcal{O}_{\mathcal{X}}(\Theta)) \to \pi_*(\mathcal{O}_Z(\Theta))$ is surjective.

Let us now consider the sheaf $\pi_*\mathcal{O}_{\mathcal{X}}(\Theta)$. From Theorem 4.1.2 Part (4), we have that $H^i(\mathcal{X}_0, \mathcal{O}_{\mathcal{X}_0}(\Theta)) = 0$ for all $i > 0$. Thus, by the cohomology and base change formula (see for example [14, III.12.11]) $\pi_*\mathcal{O}_{\mathcal{X}}(\Theta)$ is locally free of rank 1. Moreover, all sections of $\pi_*(\mathcal{O}_{\mathcal{X}}(\Theta))$ vanish on Z (since $Z \subset \Theta$). Thus, ρ is in fact the zero map since it is merely the restriction to Z. We will reach a contradiction if we can show that $\pi_*(\mathcal{O}_Z(\Theta))$ is not the zero sheaf.

Indeed, if τ is a general translation of $\mathcal{X}$, we will have $\tau(\Theta) \not\subset Z$. Specifically, τ is a section of a semiabelian scheme G over S. That is, G is a smooth separated group scheme over S such that every geometric fiber G_p is a semiabelian variety over $\kappa(p)$. There is a question as to the existence of such a section, and indeed, such a section may not exist. However, a section does exist after an extension $G \times_S S' \to S'$. Taking τ to be such a section we have $\tau(\Theta) \not\subset Z$ as desired. Hence, $H^0(S, \pi_*(\mathcal{O}_Z(\tau\Theta))) \neq 0$ and so by semi-continuity $H^0(S, \pi_*(\mathcal{O}_Z(\Theta))) \neq 0$ as well. We have thus arrived at a contradiction and the proof is complete.

$\square$

Chapter 5

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