Lecture 5-3: Applications polytrope models

Literature: MWW Chapter 19

g) Eddington's standard model

In this model, the energy equation and equation of diffusive radiative transfer are included (approximately) to arrive at simple solutions. In essence, this model boils down assuming a constant β throughout the star. We looked at this in section 5a. From a slightly different angle, with β is constant we have P_{rad}/P is constant:

$$\nabla = \frac{d\ln T}{d\ln P} = \frac{3}{64\pi\sigma}\frac{P\kappa}{T^4}\frac{L_r}{GM_r}$$

with $P_{rad} = 4\sigma T^4 / 3c$ we can write: $\nabla = \dfrac{P}{4P_{rad}}\dfrac{dP_{rad}}{dP}$

$$\frac{dP_{rad}}{dP} = \frac{\kappa L}{4\pi cGM}\frac{L_r/L}{M_r/M}$$

Introduce, $\varepsilon \equiv dL_r / dM_r$

$$\langle \varepsilon(r) \rangle = \int_0^r \varepsilon \, dM_r \Big/ M_r = L_r / M_r$$

$$\eta(r) \equiv \langle \varepsilon(r) \rangle / \langle \varepsilon(R) \rangle = \frac{L_r/L}{M_r/M}$$

$$\frac{dP_{rad}}{dP} = \frac{L}{4\pi cGM}\kappa(r)\eta(r)$$

$$P_{rad}(r) = \frac{L}{4\pi cGM} \langle \kappa(r)\eta(r) \rangle P(r) \text{ with}$$

$$\langle \kappa(r)\eta(r) \rangle = \int_0^{P(r)} \kappa\eta \, dP \Big/ P(r), \text{ and we have,}$$

$$1 - \beta = \frac{L}{4\pi cGM} \langle \kappa(r)\eta(r) \rangle$$

Now, realize that κ decreases rapidly inwards while η decreases rapidly outwards for main sequence stars. Eddington assumed that β is constant and that results in a simple T - P or T - ρ relation.

$$T(r) = \left(\frac{3ck}{4\sigma\mu m_u} \frac{1-\beta}{\beta} \right)^{1/3} \rho^{1/3}(r)$$

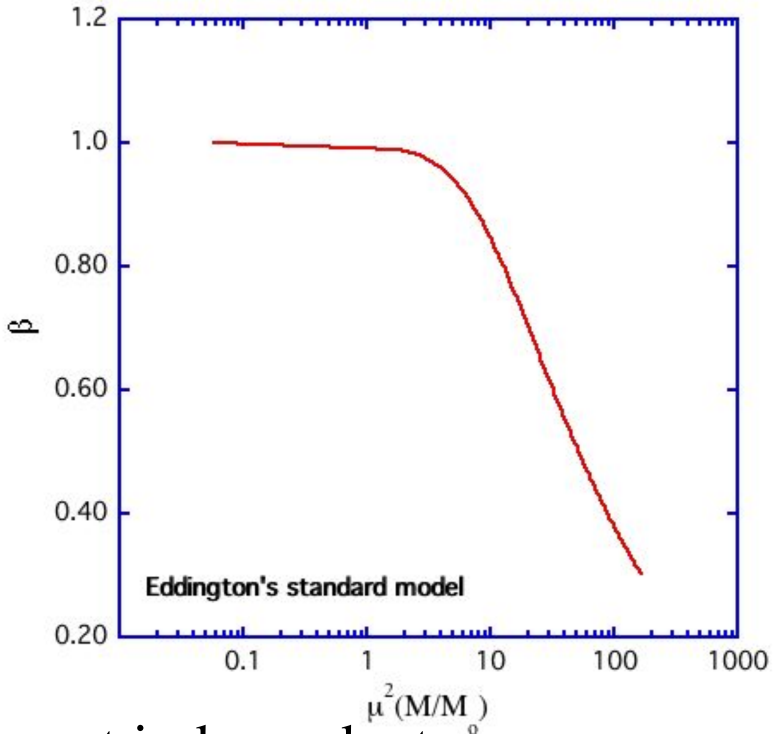
$$P(r) = \frac{P_{gas}}{\beta} = \frac{k}{\mu m_u}\frac{\rho T}{\beta} = \left[\left(\frac{k}{\mu m_u}\right)^4 \frac{3}{a}\frac{(1-\beta)}{\beta^4}\right]^{1/3} \rho^{4/3}(r),$$ see slide 4 lecture 5-1

The constant in brackets is also equal to the constant K in slide 8 lecture 5-1

$$K(n=3) = \frac{(4\pi)^{1/3}}{4}\frac{GM^{2/3}}{\left[\xi^4\left(-\dot{\theta}_3\right)^2\right]_{\xi_1}^{1/3}},$$ or

$$\frac{1-\beta}{\beta^4} = 0.002996\mu^4\left(\frac{M}{M_o}\right)^2,$$ and

$$T(r) = 4.62\times10^6\beta\mu\left(\frac{M}{M_o}\right)^{2/3}\rho^{1/3}(r)$$



but note β and M in this T - ρ relation are not independent

One-parameter family of models: β decreases with increasing M

Other properties: $\rho_c = 54.18\bar{\rho}$

$$P_c = 1.242\times10^{17}\left(\frac{M}{M_O}\right)^2\left(\frac{R_O}{R}\right)^4 \text{ dyne/cm}^2$$

$$T_c = 19.72\times10^{6}\beta\mu\left(\frac{M}{M_O}\right)\left(\frac{R_O}{R}\right) \text{ K}$$

$$\bar{T} = 0.5852T_c$$

Standard model provides the run of density, temperature, and pressure. For absolute values, we need to provide stellar mass and radius. i.e.,

$$M \;\&\; R \Rightarrow \bar{\rho} \Rightarrow \rho_c \text{ and hence } \rho(r), T(r), P(r) \text{ are set}$$

as the constant *K* in the pressure density relation is not specified aforehand in the standard model. E.g., the combination $\mu^2 M$ sets β and hence *K* but then there are still an infinite number of solutions corresponding to different *R*.

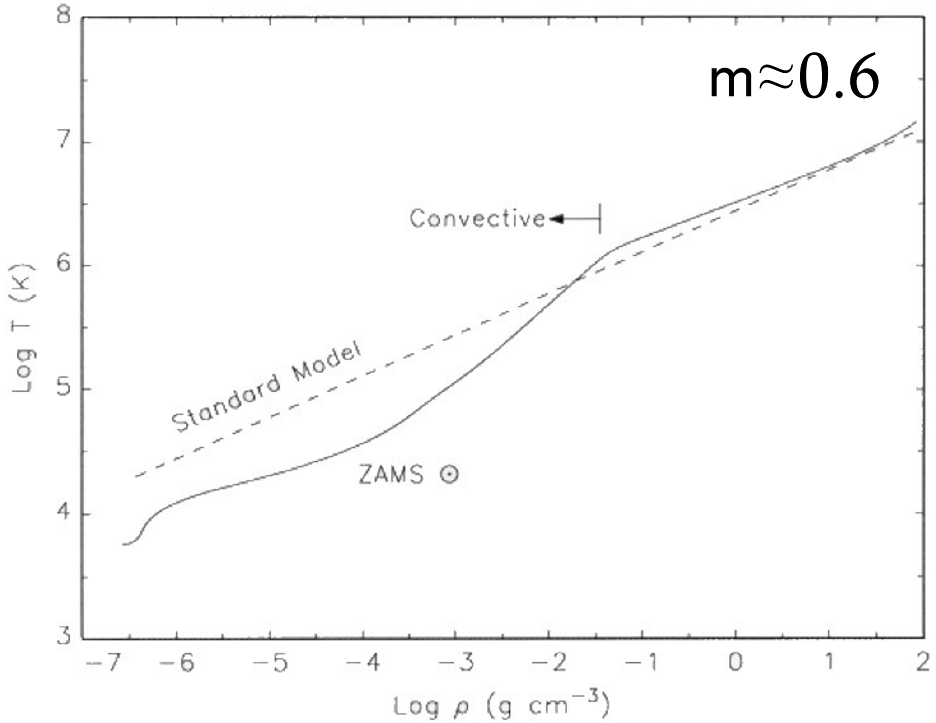
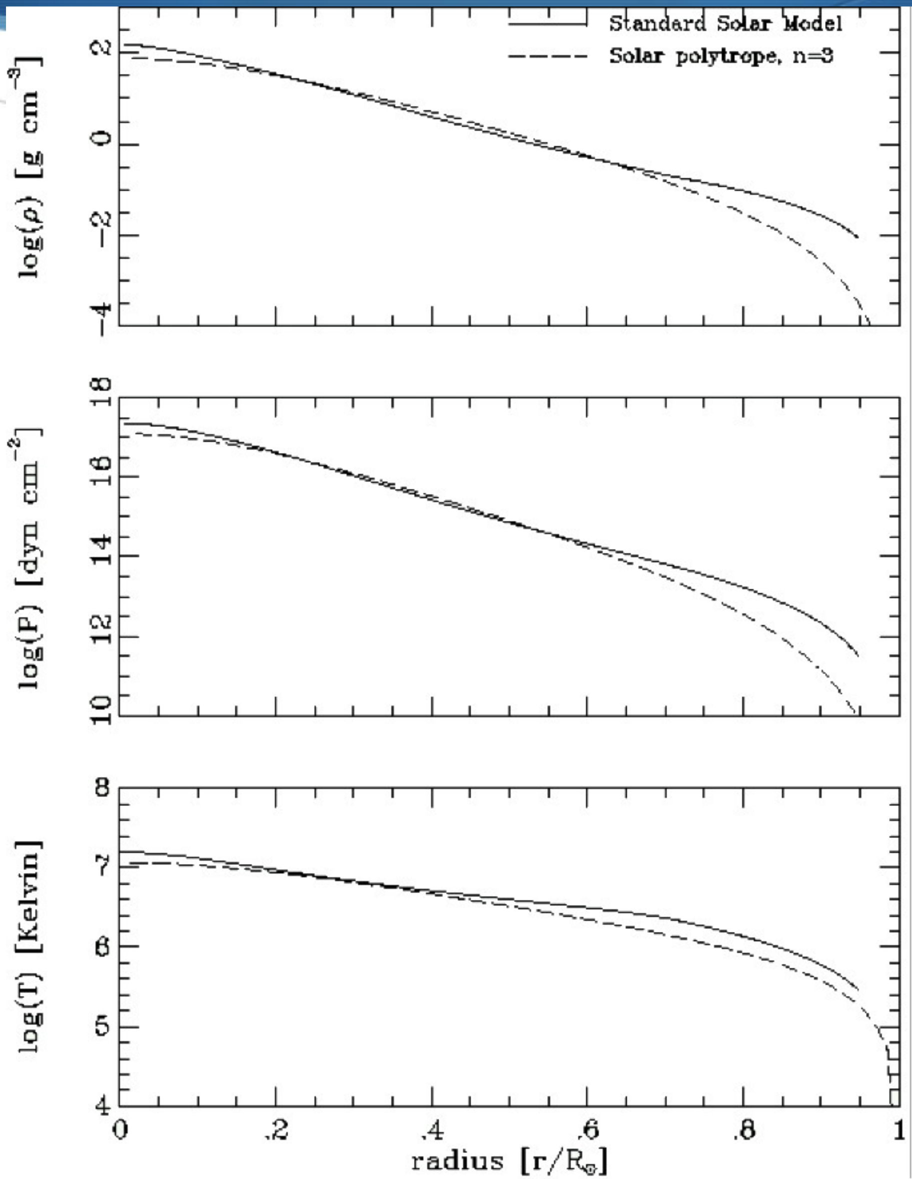


Fig. 7.4. The solid line is the temperature–density relation through a full-blown model of a ZAMS sun with total radius of 6.168×10^{10} cm or $0.886\ \mathcal{R}_\odot$. The dashed line shows the standard model result where the same total radius has been assumed. The centers of the models are at the right.

Inner region well approximated by a polytrope of index 3. Outer region is convective and better approximated by a polytrope of index 1.5 (but this is only 0.6% of the mass)

The Eddington Solar model



h) Completely convective stars

Examples: late M dwarfs
cool, gravitationally contracting stars

Internal Structure

Consider case with $\nabla = \nabla_{ad} = (\Gamma_2 - 1)/\Gamma_2$ everywhere in the star

Then: $P = K\rho^{1+\frac{1}{n}}$ or $P = K'T^{n+1}$ with $n = \frac{1}{\Gamma_2 - 1}$ (effective polytropic index)

If Γ_2 constant ⇒ completely convective star: polytrope of index n

Useful specific case: ideal gas ⇒ n = 3/2 (§5-2, slides 8-10) ⇒

$$K = 2.476 \times 10^{14} \left(\frac{M}{M_O}\right)^{1/3} \left(\frac{R}{R_O}\right)$$

$$K' = \frac{0.0796}{\mu^{5/2}} \left(\frac{M_O}{M}\right)^{1/2} \left(\frac{R_O}{R}\right)^{3/2}$$

$$\rho_c = 5.991\bar{\rho}$$

$$P_c = 8.658 \times 10^{15} \left(\frac{M}{M_O}\right)^2 \left(\frac{R_O}{R}\right)^4 \quad \text{dyne/cm}^2$$

$$T_c = 1.234 \times 10^7 \mu \left(\frac{M}{M_O}\right)\left(\frac{R_O}{R}\right) \quad \text{K}$$

$$\bar{T} = 0.5306 T_c$$

Radial behavior of ρ, P, T

follows from Lane-Emden function: $\theta_{3/2}$

i) Hayahsi track

Hayashi track: Location of low-mass (<3 M_o), fully convective stars in the Hertzsprung-Russell diagram. Relevant for low-mass protostars, red giants, and asymptotic red giants.

Match fully convective stellar structure models with boundary conditions for the photosphere. Assume the star is fully convective (polytrope with n=3/2) with a thin radiative outer layer where the opacity is dominated by H^-.

$$\frac{dP}{dr} = -g\rho \text{ and } \frac{d\tau}{dr} = -\kappa\rho \text{ or } \frac{dP}{d\tau} = \frac{g}{\kappa}$$

Assume κ is constant,

$$P\left(\tau = 2/3\right) = \frac{2g}{3\kappa} \text{ with } g = \frac{GM}{R^2}$$

$$\text{with } P = \frac{k}{\mu m_u}\rho T,\ \kappa = \kappa_o \rho^{0.5} T^{7.7}, \text{ and } \kappa_o = 10^{-25} Z^{0.5} \text{ we have,}$$

$$\rho^{1.5} T^{8.7} = \frac{2\mu m_u}{3k}\frac{G}{\kappa_o}\frac{M}{R^2}$$

For a polytrope with $n = 3/2$ we have,

$$\frac{\rho}{\rho_c} = \left(\frac{T}{T_c}\right)^{3/2}, \ \rho_c = 5.99\bar{\rho} = 5.99\frac{3M}{4\pi R^2}, \text{ and } T_c = 0.539\frac{\mu m_u}{k}\frac{GM}{R}$$

$$T^{10.95} = \frac{T_c^{2.25}}{\rho_c^{1.5}}\frac{2\mu m_u}{3k}\frac{G}{\kappa_o}\frac{M}{R^2} \approx \frac{0.1}{\kappa_o}\left(\frac{\mu m_u G}{k}\right)^{3.25} M^{1.75}R^{0.25}$$

We assume that convection starts at the photosphere ($T = T_{eff}$)

and use $L = 4\pi R^2 \sigma T_{eff}^4$ to replace R

$$T^{11.45} \approx \frac{0.07}{\kappa_o \sigma^{1/8}}\left(\frac{\mu m_u G}{k}\right)^{3.25} M^{1.75}L^{1/8}$$

$$T_{eff} \approx 2000\left(\frac{M}{M_o}\right)^{0.15}\left(\frac{L}{L_o}\right)^{0.01}\left(\frac{0.02}{Z}\right)^{0.04} \text{ K; actually, } \approx 3000 \text{ K}$$

Summary

Fully convective stars (with a radiative atmosphere) of a given mass and composition in hydrostatic equilibrium lie at a constant (low) effective temperature independent of luminosity. Conversely, the effective temperature is nearly independent of how the luminosity is generated.

Objects to the right of these Hayashi tracks have too steep a temperature gradient and convection will set in. Convection quickly sets the adiabatic temperature gradient (lecture 3-3, slide 14). That quickly rearranges the stellar structure and the star will settle on the Hayashi track.

j) protostars & the Hayashi track

Premain sequence evolution: Molecular cloud core collapses on a slow timescale ($t \sim 1/(Gr)^{1/2}$). When the core becomes optically thick, the internal temperature rises (virial theorem) and molecules (e.g., H_2) will dissociate, and then H and He will ionize.

$$\frac{R_{ps}}{R_o} \approx 50 \frac{M}{M_o}, \text{ and } \overline{\mathrm{T}} \approx 10^5 \text{ K (virial theorem)}$$

No nuclear reactions. High luminosity and high opacity. Protostar is fully convective except for an H^- dominated atmosphere. Star will start high up on the Hayashi track. As it contracts, the radius will decrease but the effective temperature stays constant.

$$E_t = -E_i = E_g/2 \text{ and } L = \dot{E}_t = -\dot{E}_g\big/2 \approx \frac{3GM^2}{7R^2}\frac{dR}{dt}$$

with $L = 4\pi R^2 \sigma T_{eff}^4$, we then have:

$$\frac{1}{R^4}\frac{dR}{dt} = \frac{28\pi\sigma T_{eff}^4}{3GM^2},$$ which integrates to ($T_{eff} \approx cst$):

$$t = \frac{GM^2}{28\pi\sigma T_{eff}^4 R^3} \text{ and } L = L_0\left(\frac{\tau_{KH}}{3t}\right)^{2/3}$$

where $R_o = R(t = \tau_{KH}/3)$ with $\tau_{KH} = 3GM^2\big/7R_o L_0$ and $L_0 = 4\pi R_0^2 \sigma T_{eff}^4$

Radiative (Henyey) protostellar tracks

Convection will stop when $\nabla_{rad} = \dfrac{d\ln T}{d\ln P} = \dfrac{3\kappa_R P}{64 G\pi\sigma T^4}\dfrac{L}{M} < 0.4$

As the star contracts, the luminosity will drop and radiative energy transport will take over in the core

Note that P_c/T_c^4 does not decrease (slide 10 of section 5.2).

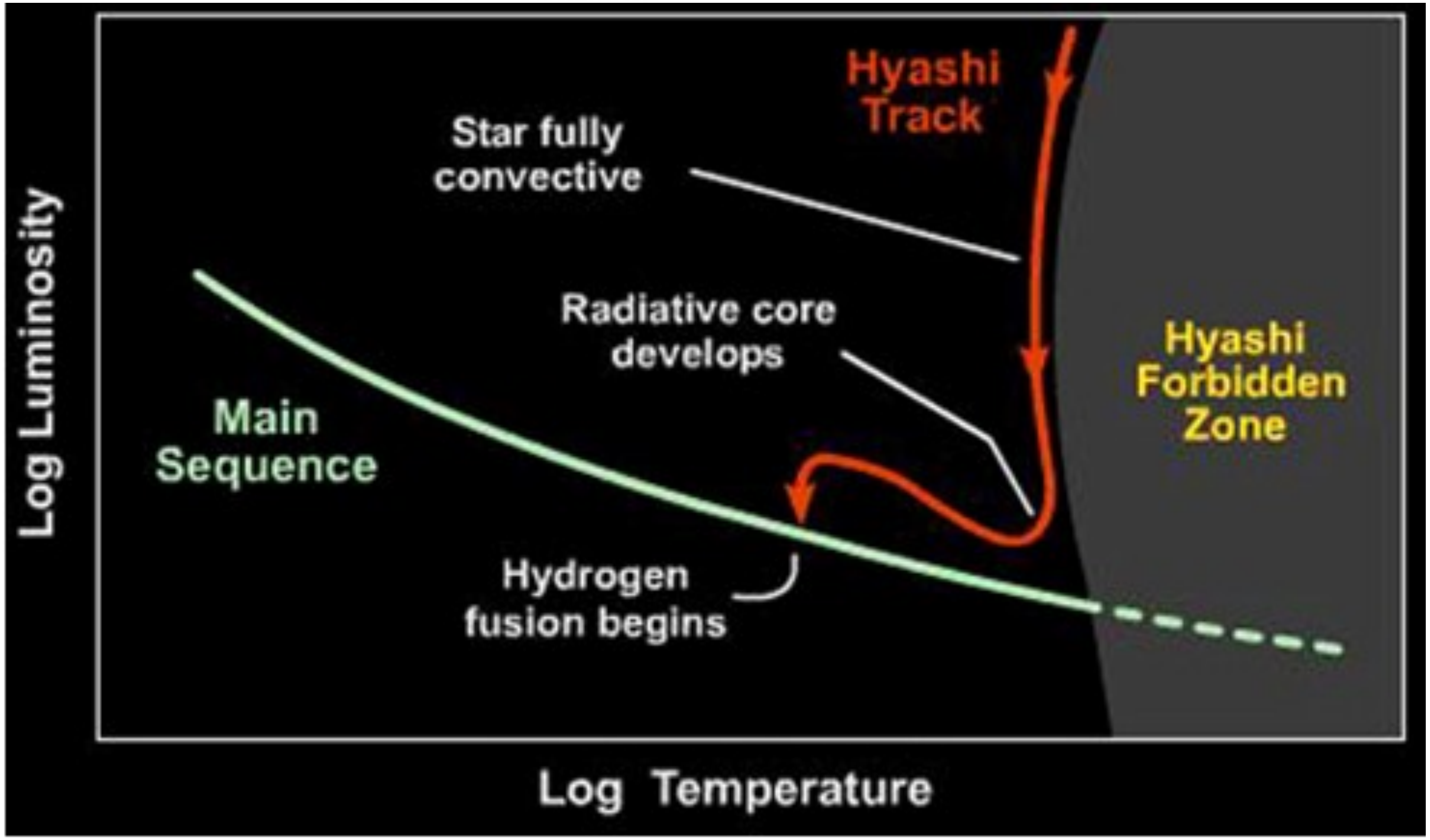
$T \propto M/R$ and $\rho \propto M/R^3$ and $\kappa \propto \rho T^{-3.5}$ and adopt $dT/dr = T_c/R$

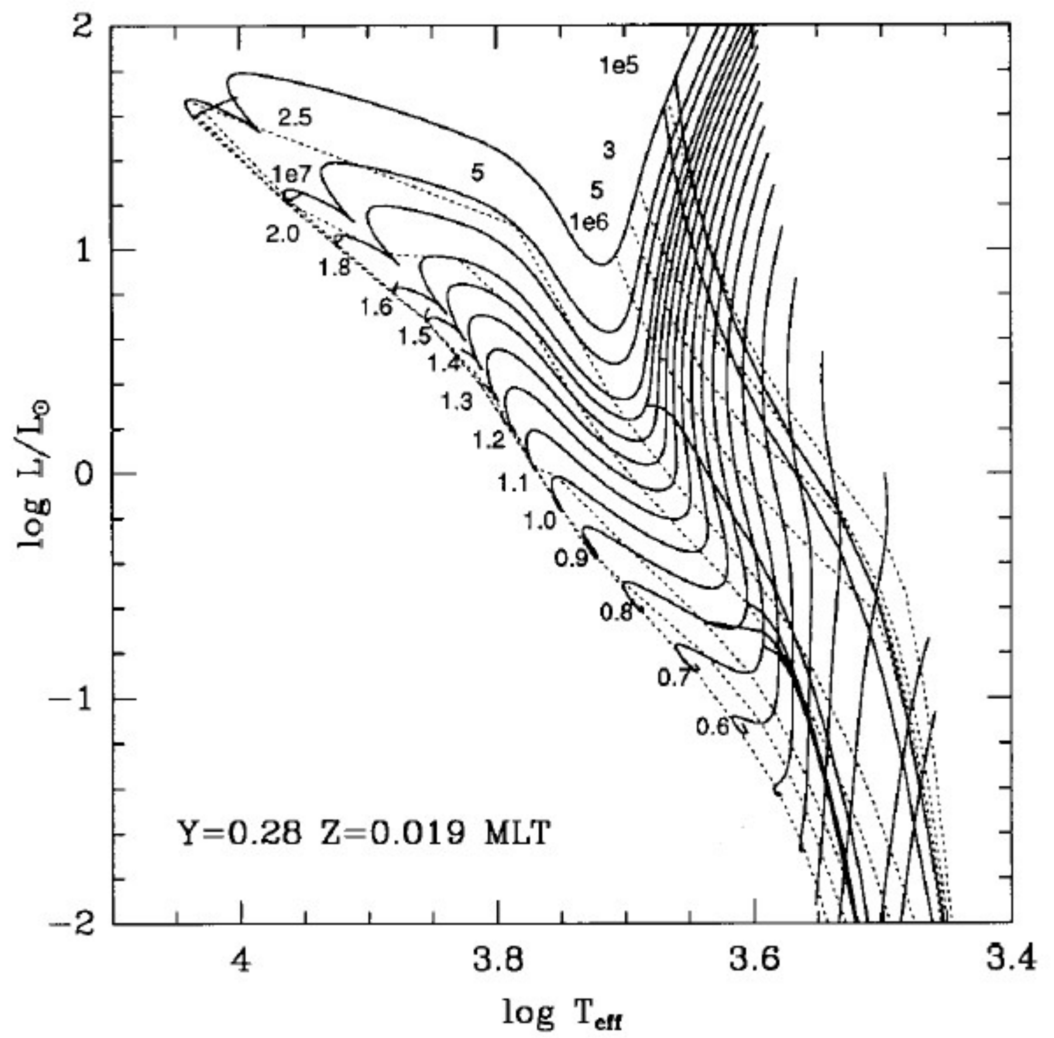
$L \propto M^{5.5} R^{-0.5}$

The star evolves now towards the left in the HR diagram with a slowly increasing luminosity.

Summary

A protostar will initially be fully convective and contract on a Hayashi track of (almost) constant effective temperature. Eventually, the core will become radiative, the star will switch to a Henyey track and the luminosity will (moderately) increase. As the star contracts, the central temperature will increase (Virial theorem). Eventually, H-burning will be ignited in the core and the star will settle on the main sequence. The timescale for this is set by the Kelvin Helmholtz timescale and this is set for when the star evolves the slowest (eg., near the main sequence).


Hyashi
Track
Star fully
convective
Radiative core
develops
Hyashi
Forbidden
Zone
Main
Sequence
Hydrogen
fusion begins
Log Luminosity
Log Temperature



Protostellar tracks (solid lines) for different protostellar stellar masses. Isochrones are indicated by dashed lines. The solid lines crossing the tracks indicate the region of D-burning and Li-burning.

k) White dwarfs

A white dwarf is the stellar remnant of a low mass star with a mass typically half that of the Sun but a radius comparable to the Earth, corresponding to a density of 10^6 g/cm^3. The star is pressure-supported by a degenerate electron gas with (cf., Lecture 3-2 slide 14):

$$P = K_{NR}\left(\rho/\mu_e\right)^{5/3} \quad \text{non-relativistic}$$

$$P = K_{ER}\left(\rho/\mu_e\right)^{4/3} \quad \text{extremely relativistic}$$

where the constants K_i only depend on atomic physics

The mechanical properties are controlled by the electrons while the thermal energy is stored in the ions. We can use polytropes of index 3/2 and 3 to describe them.

Mass-radius relation

For the non-relativistic equation of state, we find (Lecture 5-2, slide 8):

$$M^{1/3}R = cst \quad \text{and} \quad \bar{\rho} \propto M/R^3 \propto M^2$$

Thus, a more massive white dwarf has to be smaller and will be denser than a less massive white dwarf. Eventually when the density is high enough, the electron gas will become relativistic (cf., lecture 3-2 slide 25) and we have:

$$M \approx \left(\frac{K_{ER}}{0.3639\mu_e^{4/3}G}\right)^{3/2} \approx 0.197\left(\frac{hc}{Gm_u^{4/3}}\right)^{3/2}\frac{1}{\mu_e^2}$$

Chandrasekhar mass

$$M_{ch} = \frac{5.86}{\mu_e^2}M_o \approx 1.46M_o \text{ for C, O, ... composition}$$

Heuristically: Mass-radius relation

When analyzing stars using the virial theorem, we discovered that stars are unstable when g<4/3 (lecture 2 slide 11). Evaluate hydrostatic equilibrium

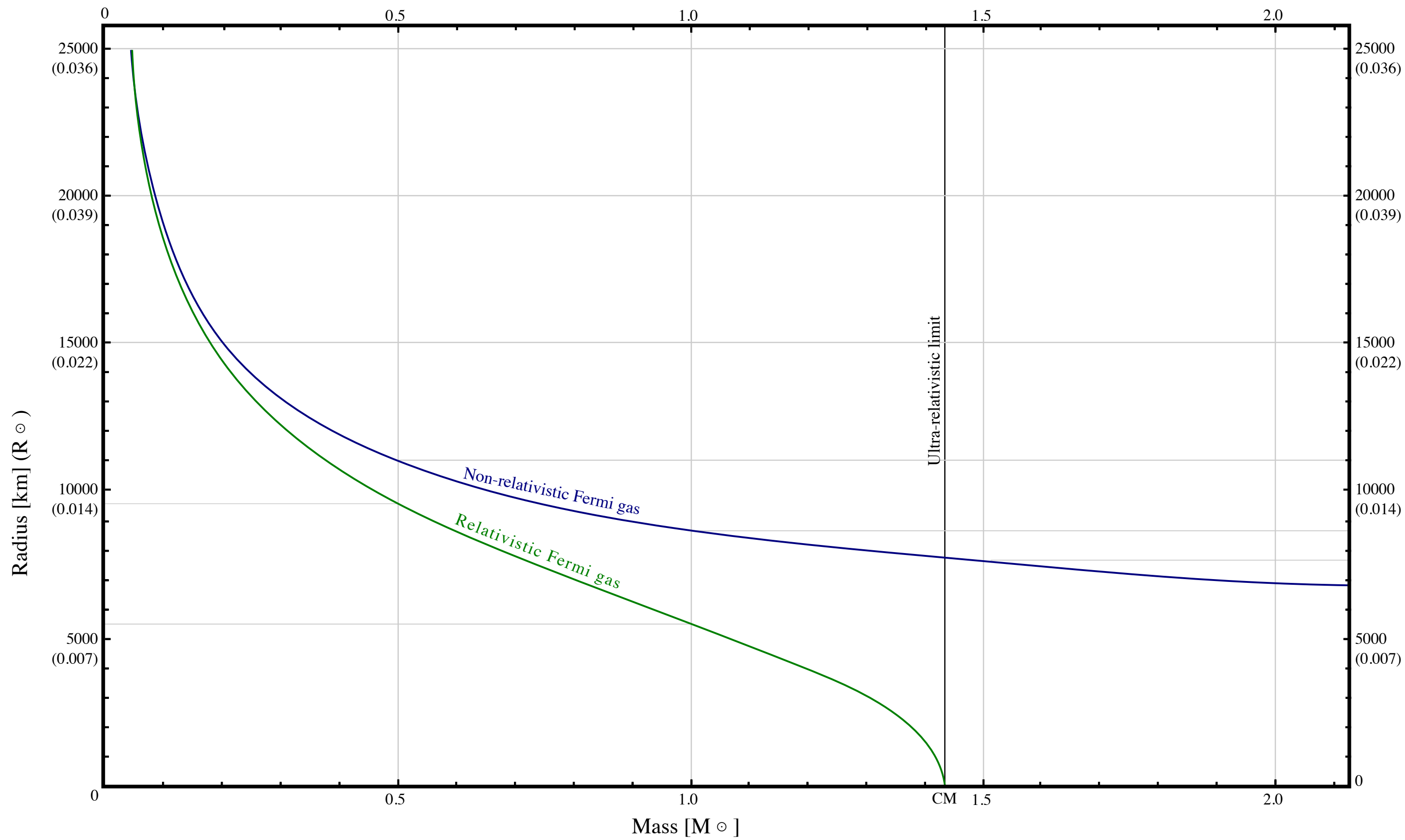
$$\frac{dP}{dr} = -q\frac{GM\rho}{R^2} \propto \frac{M^2}{R^5}$$

$$\frac{dP}{dr} \approx \frac{P_c}{R} \propto \frac{M^{5/3}}{R^6} \quad \text{(non-relativistic)}$$

$$\frac{dP}{dr} \approx \frac{P_c}{R} \propto \frac{M^{4/3}}{R^5} \quad \text{(extreme-relativistic)}$$

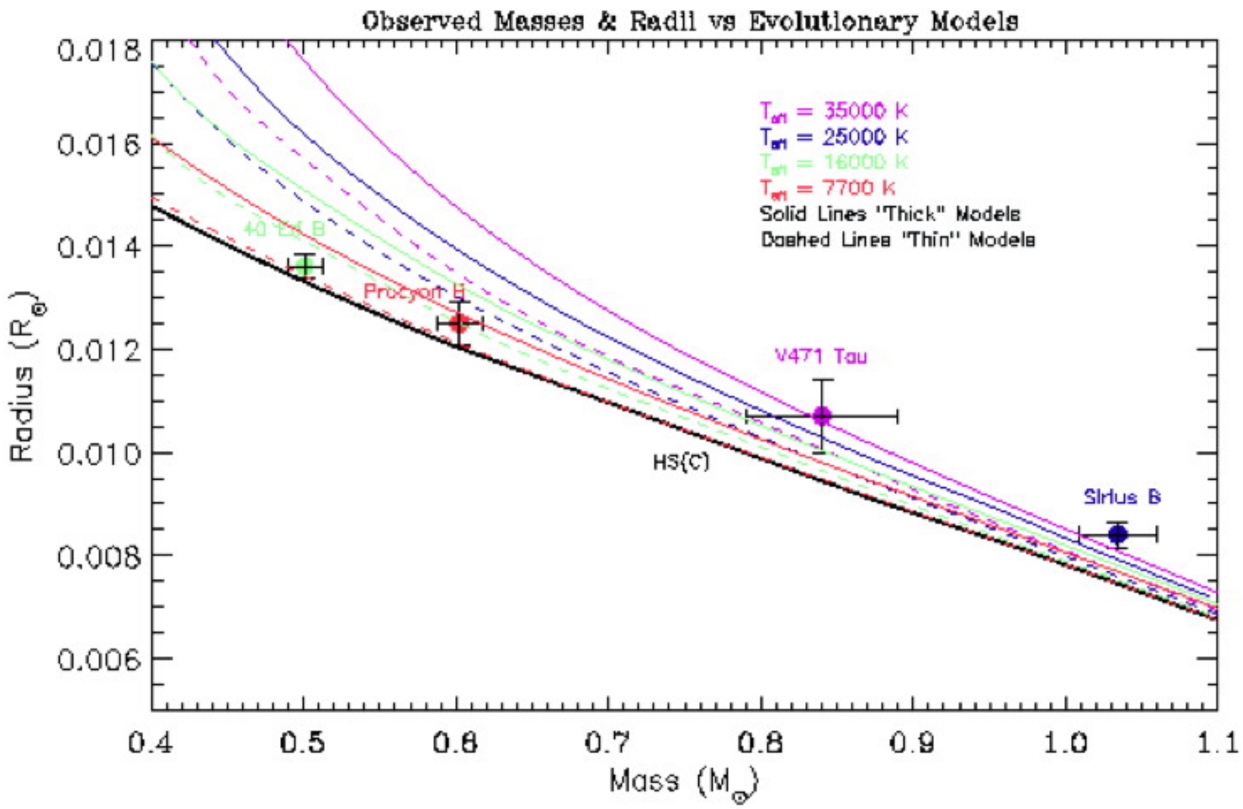
Non-relativistic: For any given mass, star can adjust its radius to reach equilibrium.

Extreme-relativistic: Internal energy and gravity have same dependence on radius but different dependence on mass. So, if mass is too large, gravity will win.



White dwarfs that reach the Chandrasekhar mass – either through accretion from a companion or through merging of binary WDs – will explode as SN Ia.

Observed mass-size relation for White Dwarfs



Radiative cooling

The mechanical and thermal structure are decoupled. Hence, we have to relate the temperature to the luminosity and mass to calculate the cooling rate and timescale. Or equivalently, we will use the luminosity to calculate the internal temperature. Conduction keeps the interior at (almost) uniform temperature. We will assume that the photosphere is radiative (only true for white dwarfs with SpT earlier than A) and is described by the ideal gas law. Use the radiative zero conditions (lecture 5-1, slide 5/6) with Kramers opacity, $\kappa = \kappa_o \rho T^{-3.5}$

$$P = \left(\frac{2}{8.5} \frac{16\sigma}{3} \frac{k}{\mu m_u} \frac{4\pi GM}{\kappa_0 L} \right)^{1/2} T^{4.25}$$

Ideal gas: $\rho = \left(\frac{64}{51} \frac{\mu m_u}{k} \frac{4\pi\sigma GM}{\kappa_0 L} \right)^{1/2} T^{3.25}$ (i)

Transition from non-degenerate to degenerate gas at $\rho = \rho_{tr}$

with: $P(\text{degenerate,NR}) = P(\text{ideal}) \Leftrightarrow 9.91 \times 10^{12} \left(\frac{\rho_{tr}}{\mu_e} \right)^{5/3} = \frac{k}{\mu_e m_u} \rho_{tr} T_{tr}$

$$\Rightarrow \rho_{tr} = 2.40 \times 10^{-8} T_{tr}^{3/2} \quad \text{(ii)}$$

Use (i) and (ii), and κ_0 for bound-free transitions $\Rightarrow$

$$L = 5.7 \times 10^5 \frac{\mu}{\mu_e^2} \frac{T_{tr}^{7/2}}{Z(1+X)} \frac{M}{M_O} \quad \text{(iii)}$$

Since interior is isothermal: T_{tr} is interior temperature

Integrate radiative gradient between T=0 and T=T_{tr} $\Rightarrow$

$$T_{tr} = \frac{\mu m_u GM}{4.25k} \left(\frac{1}{r_{tr}} - \frac{1}{R} \right) \qquad r_{tr} = r(\rho = \rho_{tr})$$

Cooling time scale can be evaluated from:

$$L = -\frac{dE}{dt} = -c_V M \frac{dT_{tr}}{dt} \ \& \ c_V = \frac{3}{2}\frac{k}{\mu_A m_u} \qquad \text{(iv)}$$

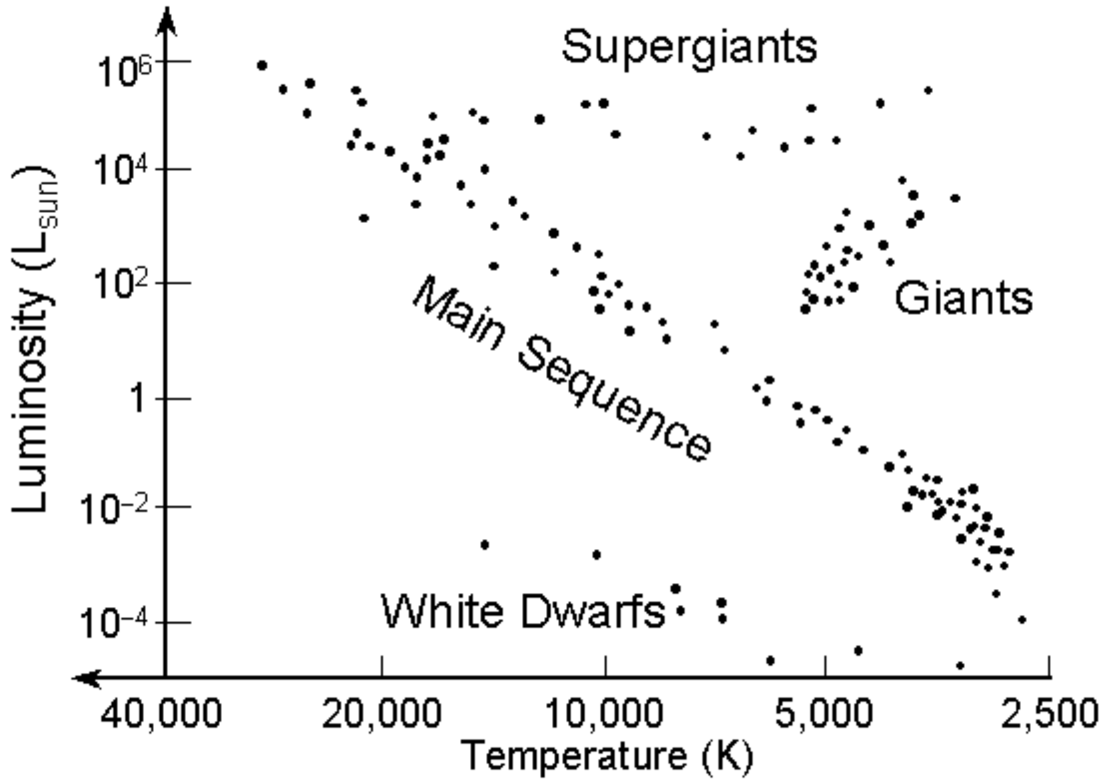
Differentiate eqn (iii), substitute in eqn (iv) and use eqn (iii) to eliminate T_{tr}:

$$\frac{dL}{dt} = \frac{7}{2}\frac{L}{T_{tr}}\frac{dT_{tr}}{dt} \ \& \ T_{tr} = \left(\frac{L}{C_2 M}\right)^{2/7}$$

$$\frac{1}{L^{12/7}}\frac{dL}{dt} = -\frac{7}{2}\frac{C_2^{2/7}}{C_V M^{5/7}}$$

$$t_{cool} = \frac{2}{5}\frac{c_V M^{5/7}}{C_2}\left(L^{-5/7} - L_0^{5/7}\right) = 8\times10^6\left(\frac{2}{\mu}\right)^{2/7}\left(\frac{M/M_\odot}{L/L_\odot}\right)^{5/7} \ \text{yr}$$

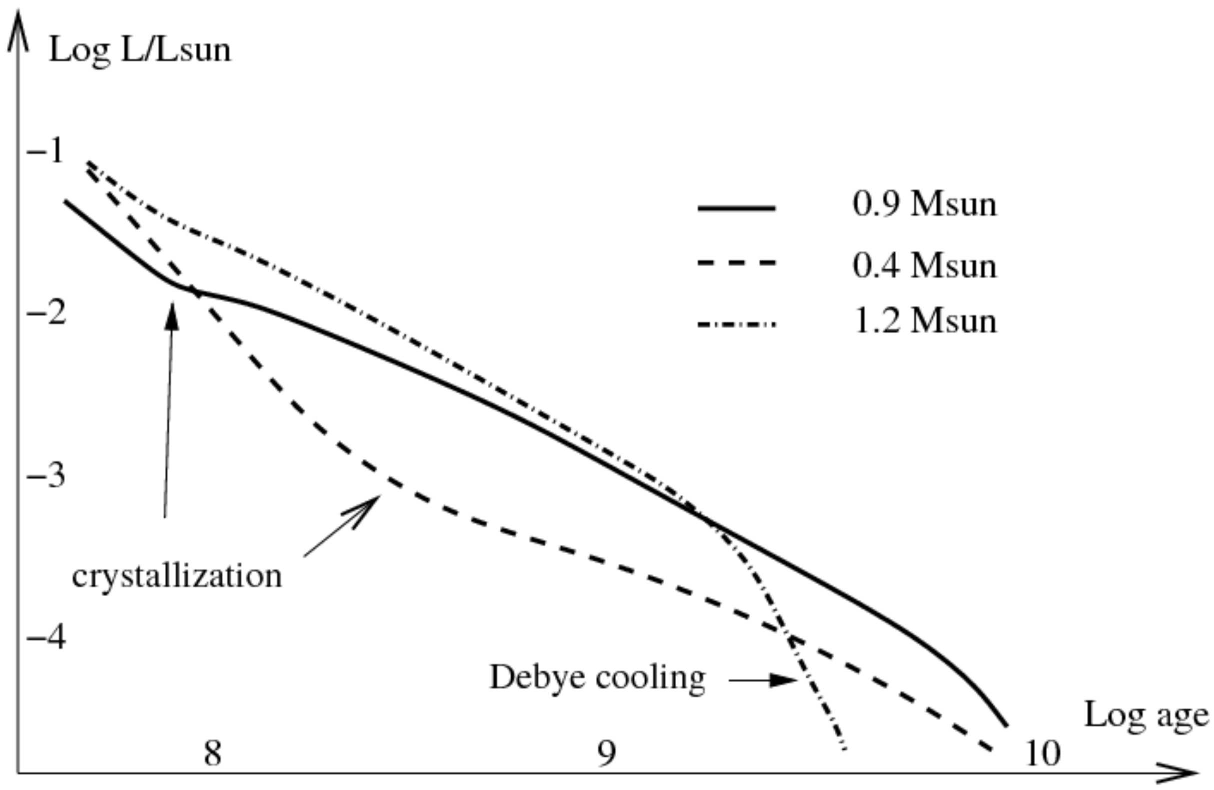
L_0 is luminosity at start of cooling



(μ_A mean molecular weight per nucleus; 4.4 for Y=0.9, Z=0.1, 12 for C white dwarf)

Cooling time: t_{cool}>10^{10} yr when L<$10^{-4}L_\odot$

Evolution along line of constant radius in HRD



All energies sources have to be taken into account: neutrino losses & nuclear energy left-overs (early on), gravitational contraction (early on and in the last stages) crystallization energy, followed by Debye cooling.

I) Neutron stars

Neutron stars pack the mass of the Sun in a radius of some 10 km (e.g., the size of Amsterdam), corresponding to a density of ~10^{14} g/cm^3.

Nuclear energy generation stops after Fe-core formation. The core will collapse and neutronization occurs (Lecture 4-2 slide 20): $e^- + p^+ \rightarrow n^0 + \nu_e$

If the core is not too massive, collapse will be halted by the degeneracy pressure due to neutrons. The equation of state is complex but let's just scale the Chandrasekhar mass:

$$M_{ch,n} = 1.46\left(\frac{2}{\mu_n}\right)^2 \approx 5.8\ \mathrm{M}_o$$

Actually, have to take into account that neutrons can couple and form a boson. Limiting mass ~3-4 M_o. Cores more massive than this will collapse and form a black hole.