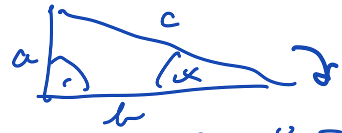
Trigonometric and cyclometric functions

6.6 The goniometric functions:

- sin (sine) ... see Fig. 21,
- cos (cosine) ... see Fig. 22,
- tan := $\frac{\sin}{\cos}$ (tangent) ... see Fig. 23,
- cot := $\frac{\cos}{\sin}$ (cotangent) ... see Fig. 24.

Definitions of trig function:
- from right angled △



$\cos x = \frac{b}{c}$
$\sin x =$
$\tan x =$
$\cot x =$

- from unit circle
see further in the lecture

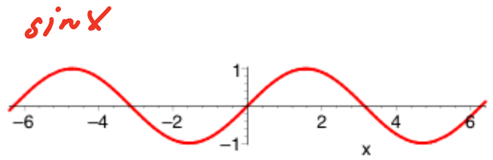


Fig. 21

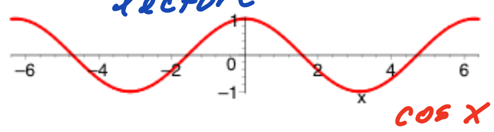


Fig. 22

We again find ourselves in a situation when we work with the functions whose definitions are beyond our comprehension in this moment. Similarly as in the case of the exponential function, let us mention that the functions sine and cosine are defined by the following sums of the infinite series:

$$\sin x := x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},$$

$$\cos x := 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

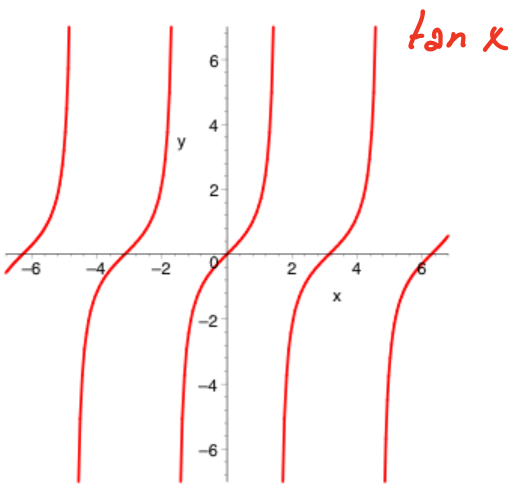


Fig. 23

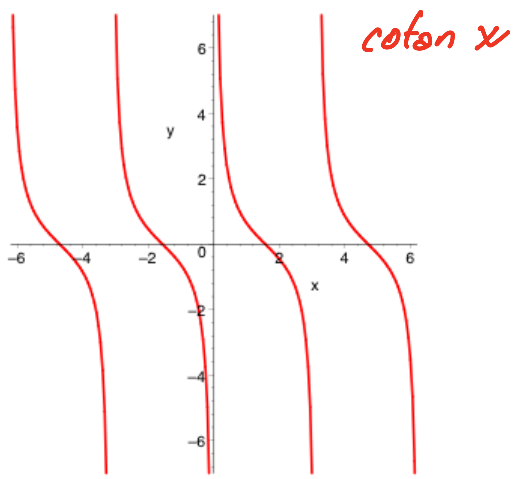


Fig. 24

$y = f(x) \iff f^{-1}(y) = x$

$y = f^{-1}(x)$

6.8 The <u>cyclometric functions</u>:

- $\arcsin := \left(\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]}\right)_{-1}$ (arcsine) ... see Fig. 25,
- $\arccos := \left(\cos|_{[0, \pi]}\right)_{-1}$ (arccosine) ... see Fig. 26,
- $\arctan := \left(\tan|_{(-\frac{\pi}{2}, \frac{\pi}{2})}\right)_{-1}$ (arctangent) ... see Fig. 27,
- $\text{arccot} := \left(\cot|_{(0, \pi)}\right)_{-1}$ (arccotangent) ... see Fig. 28.

arcsin x

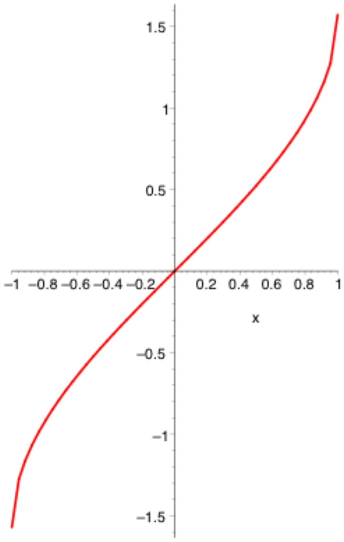
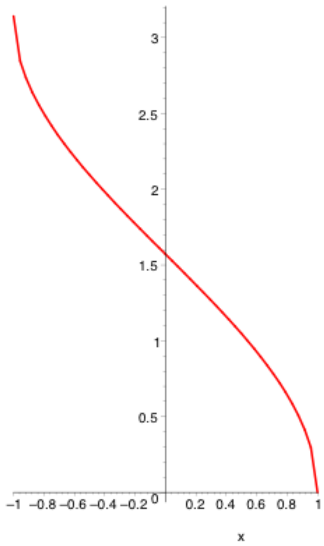


Fig. 25



arccos x

Fig. 26

arctan x

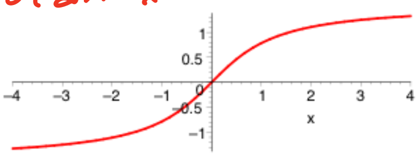
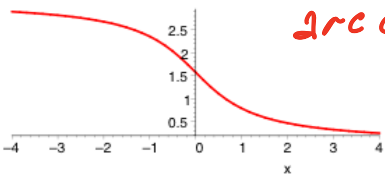


Fig. 27



arccot x

Fig. 28

in Stewart

Basic elementary func — essential func.

7 ELEMENTARY FUNCTIONS

algebraic functions in stewart

7.1 Definition. A function is said to be <u>elementary</u> if it is formed by the basic elementary functions using a finite number of algebraic operations ($+$, $-$, $\cdot$, $:$) and composition of functions.

7.2 Examples.

1) The function $f(x) = a^x := e^{x \log a}$ $(a \in \mathbb{R}^+)$ is elementary.

2) The inverse of a^x $(a \in \mathbb{R}^+ \setminus \{1\})$ is elementary $\left(\log_a x = \frac{\log x}{\log a}\right)$.

3) sgn is not elementary.

4) $|x| = \sqrt{x^2}$ is elementary. $|x| = \sqrt{x} \cdot \sqrt{x}$

5) Every real polynomial p, i.e., a function given by

$$p(x) := a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \quad (a_i \in \mathbb{R},\ i = 0, 1, \ldots, n),$$

is elementary.

Practicing properties of trigonometric and cyclometric functions.

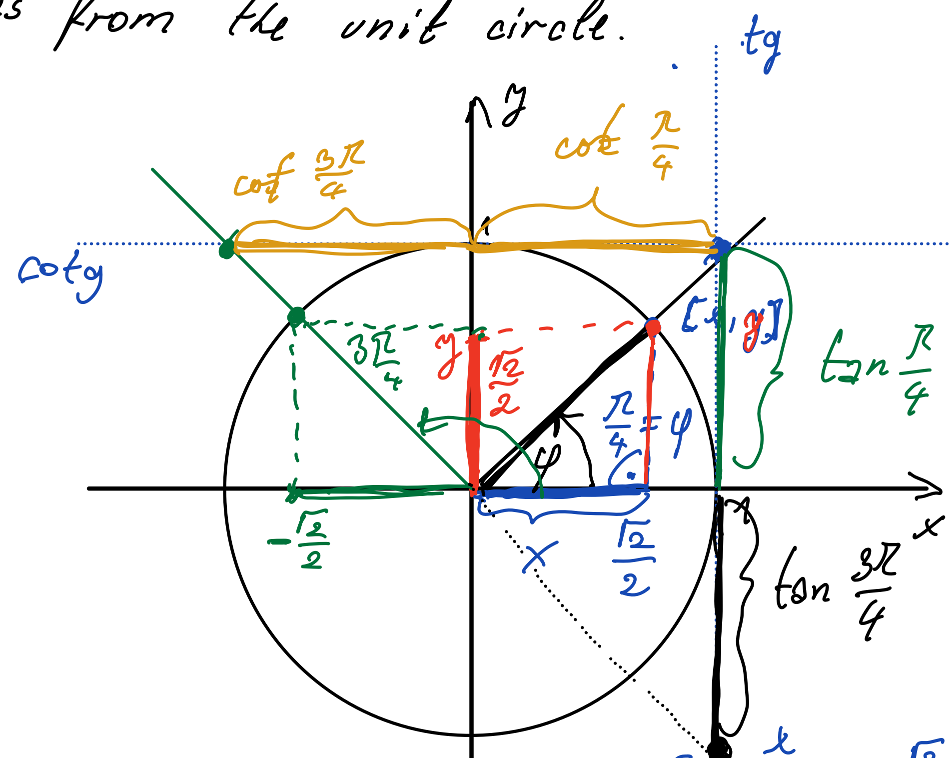
Ex: Determine the values from the unit circle.

a) $\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

b) $\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

c) $tg \frac{3\pi}{4} = -1$

d) $cotg \frac{3\pi}{4} = -1$



$\cos \varphi = \cos \frac{\pi}{4} = \frac{x}{1} = x = \frac{\sqrt{2}}{2}$

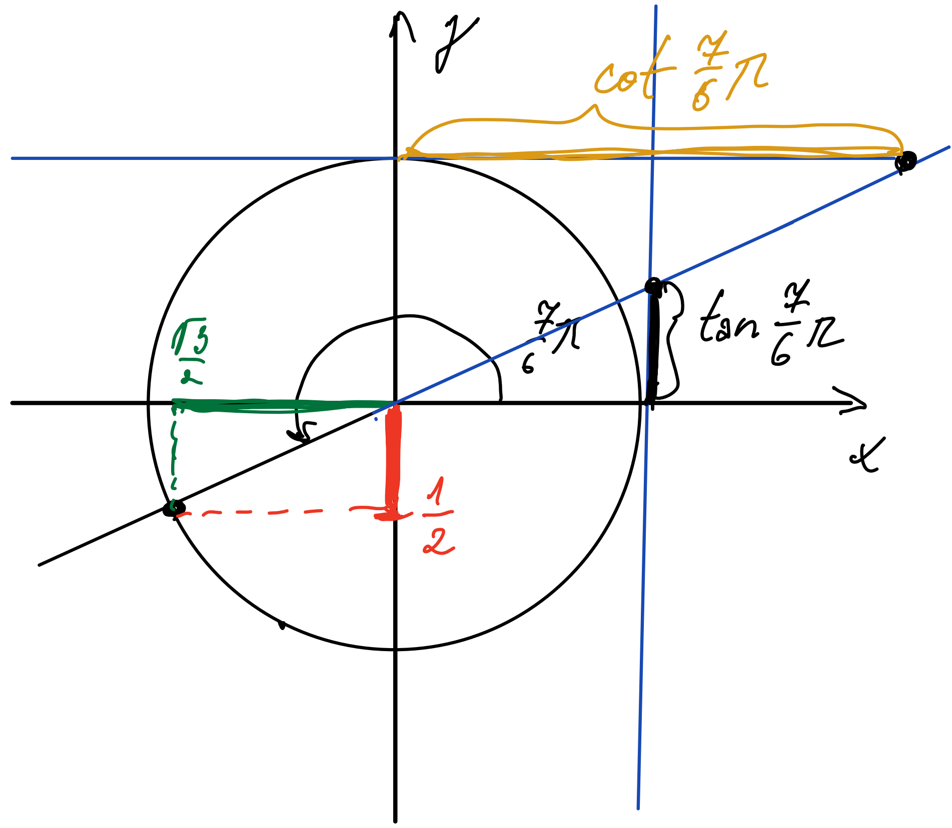
$\sin \varphi = \sin \frac{\pi}{4} = \frac{y}{1} = y = \frac{\sqrt{2}}{2}$

e) $\sin\frac{7\pi}{6} = -\frac{1}{2}$

f) $\cos\frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$

g) $tg\frac{7\pi}{6} = \frac{\sqrt{3}}{3}$

h) $cotg\frac{7\pi}{6} = \sqrt{3}$

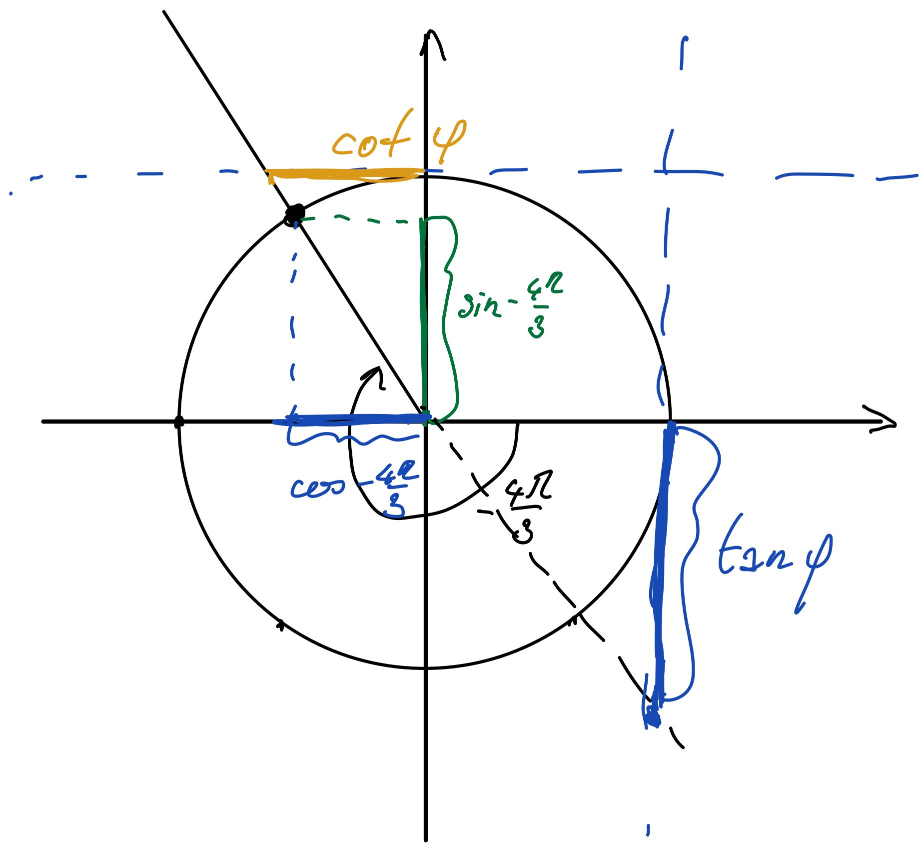


i) $\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$

j) $\cos\left(-\frac{4\pi}{3}\right) = -\frac{1}{2}$

k) tan $\cancel{tg}\left(-\frac{4\pi}{3}\right) = -\sqrt{3}$

l) $cot\cancel{g}\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{3}$
cot



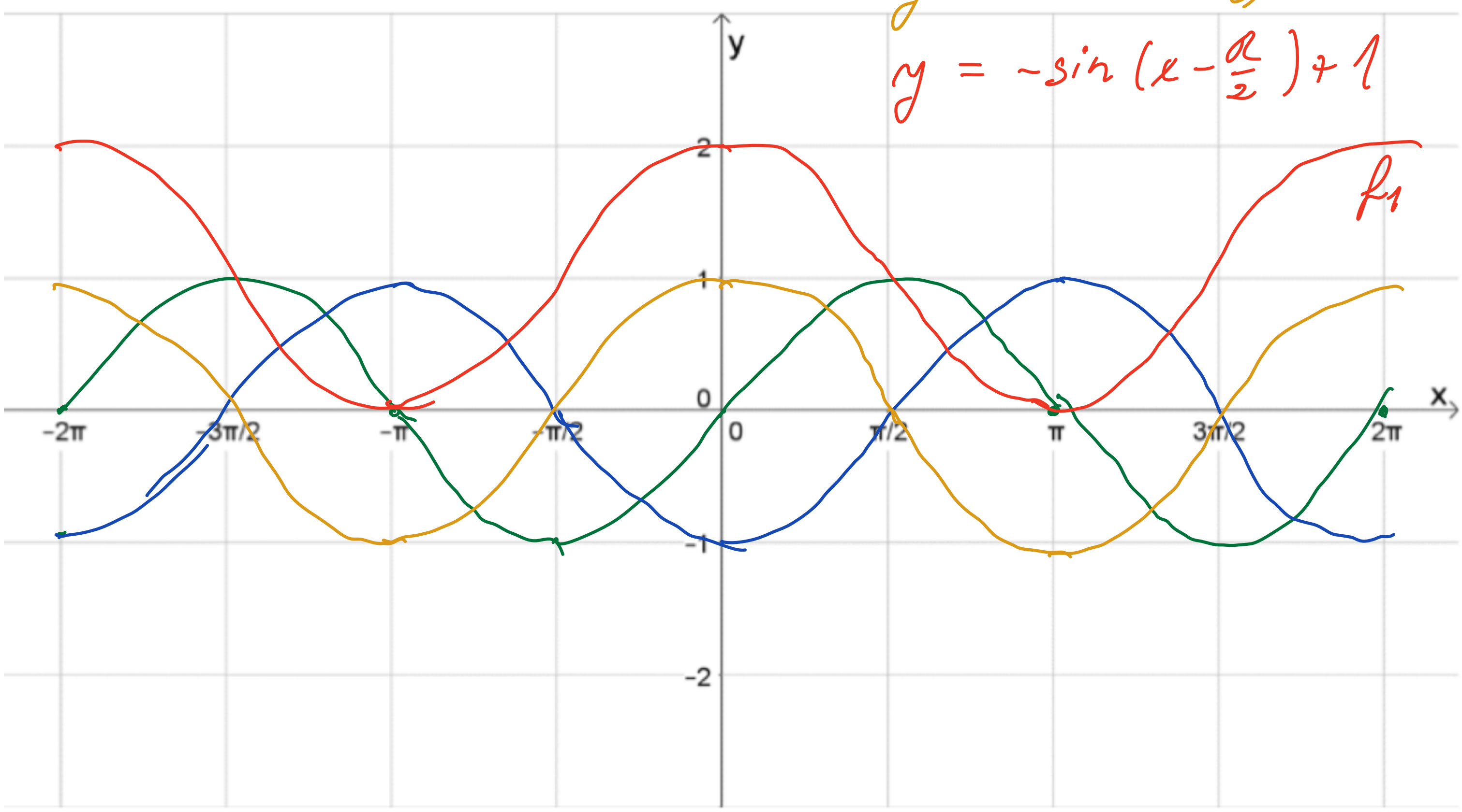
Sketch the graphs of given functions

$y = \sin x$ $\quad D(f) = \mathbb{R}$, $\quad H(f) = [-1, 1]$

$y = \sin\left(x - \frac{\pi}{2}\right)$

$y = -\sin\left(x - \frac{\pi}{2}\right)$

$y = -\sin\left(x - \frac{\pi}{2}\right) + 1$



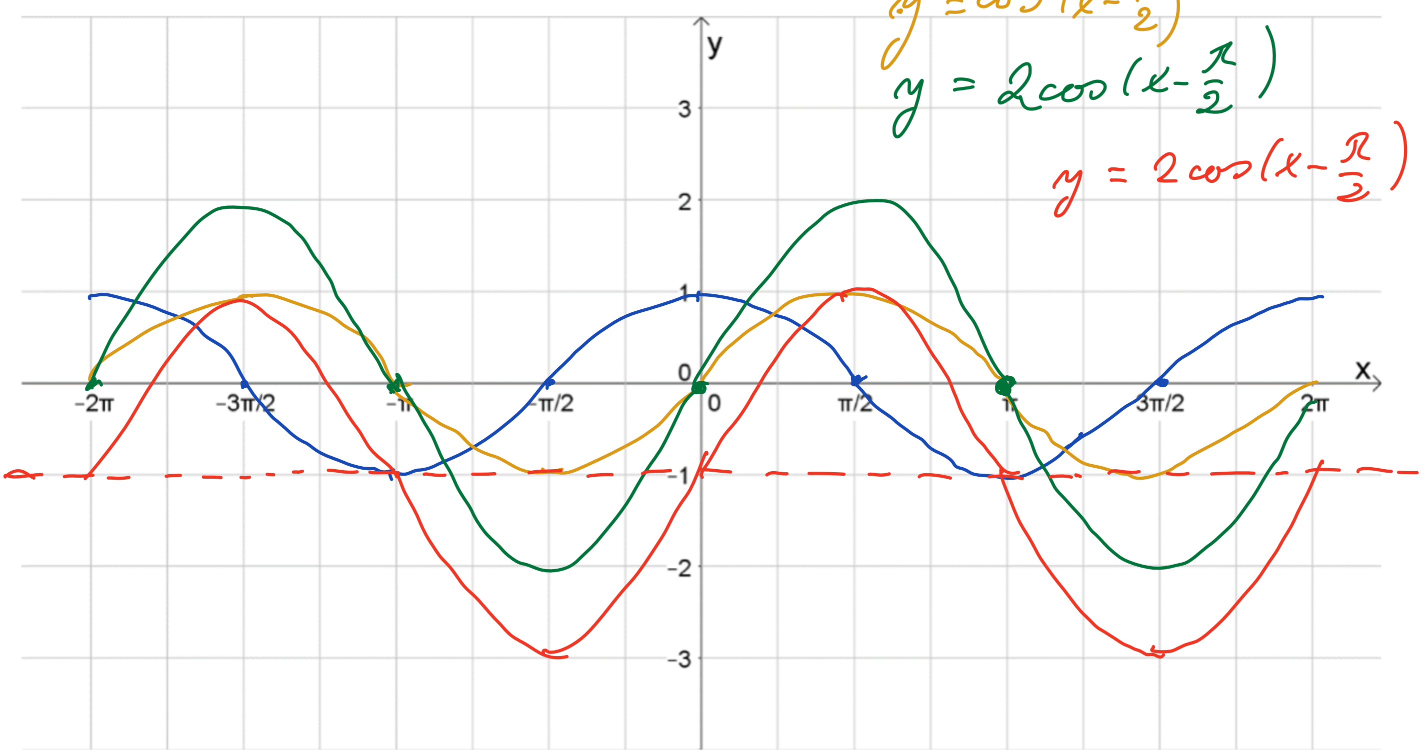
b) $f_2: y = 2\cos\left(x - \frac{\pi}{2}\right) - 1,$

$y = \cos x$ $\quad D(f) = \mathbb{R}$, $\quad H(f) = [-1, 1]$

$y = \cos\left(x - \frac{\pi}{2}\right)$

$y = 2\cos\left(x - \frac{\pi}{2}\right)$

$y = 2\cos\left(x - \frac{\pi}{2}\right) - 1$



d)

$f_1: \ y = \sin\frac{x}{2}$

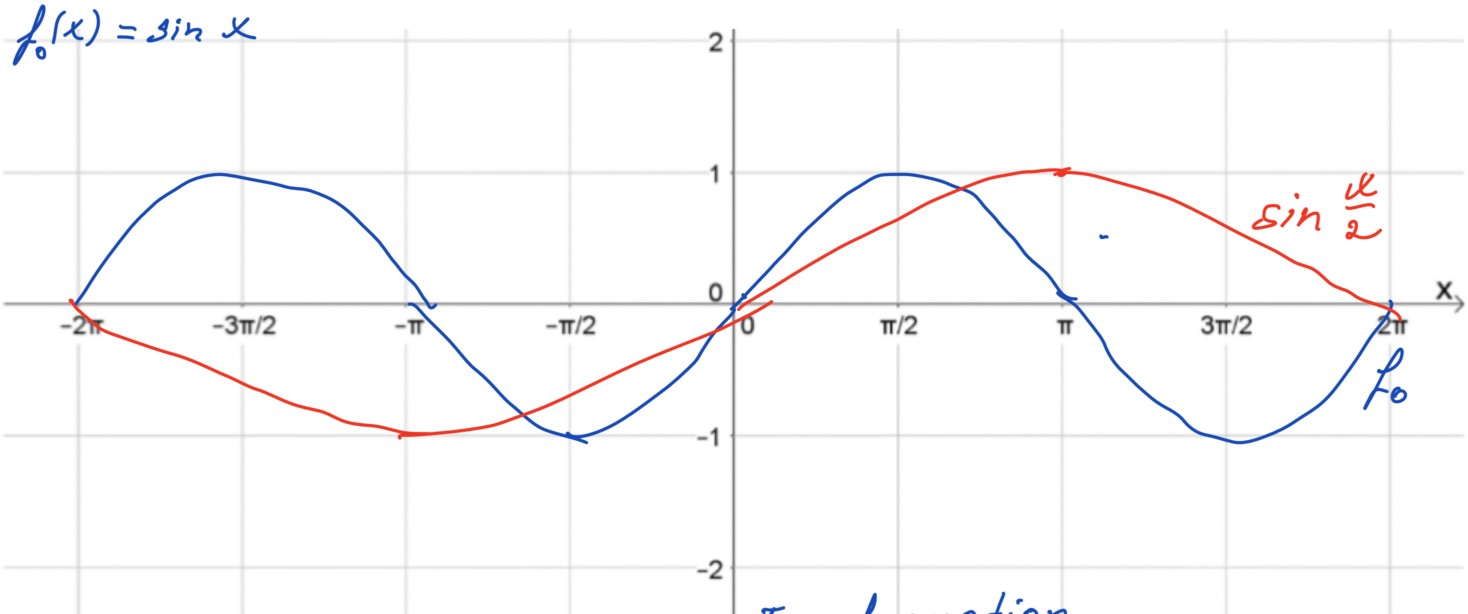
$f_2: \ y = \cos 2x$

Transformation

$f_1(x) = f\left(\frac{x}{2}\right)$ graph of f is stretched in x-axis by 2

$f_1(x) = \sin\frac{x}{2}$

$f_0(x) = \sin x$

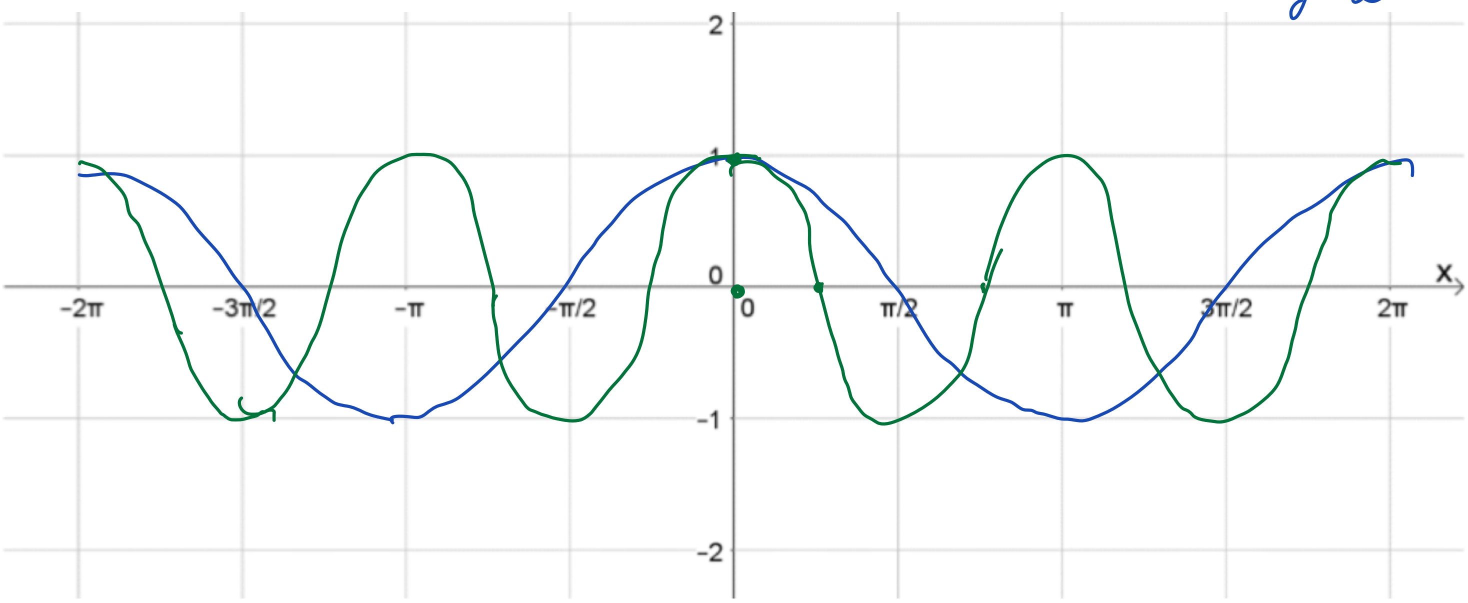


$f_2(x) = \cos 2x$

$f_0(x) = \cos x$

Transformation

$f_2(x) = f(2x)$ graph of f is shrinked in x-axis by $\frac{1}{2}$



Determine the period of a given function

- $f_1(x) = \sin\frac{x}{2} = \sin\left(\frac{x}{2} + 2\pi k\right) = \sin\left[\frac{1}{2}\left(x + \underbrace{4\pi k}_{\text{period}}\right)\right] \quad T = 4\pi$

  $k \in \mathbb{Z}$

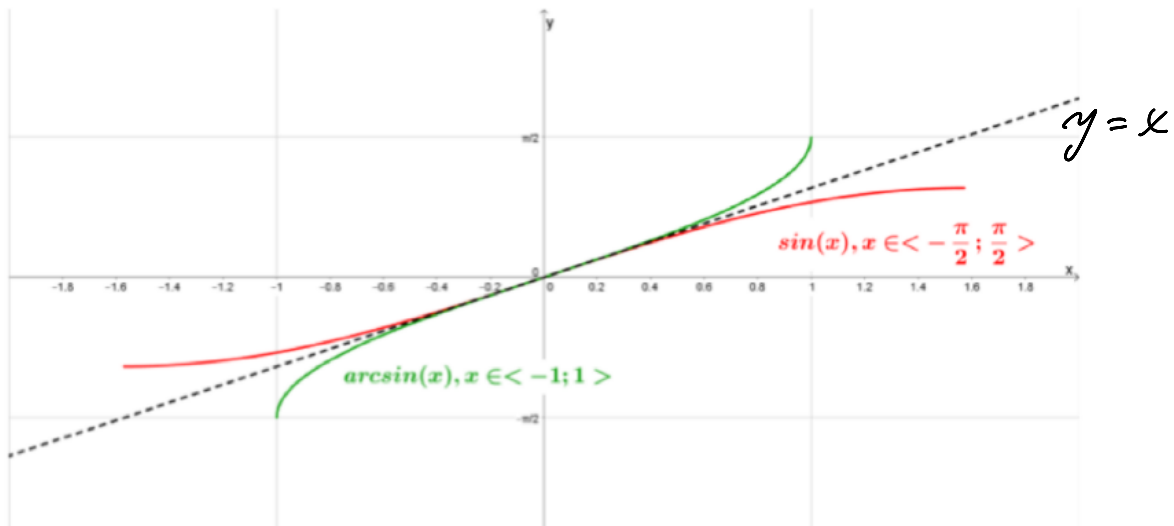
- $f_2(x) = \cos\left(3x + \frac{\pi}{2}\right) = \cos\left(3x + \frac{\pi}{2} + 2\pi k\right) =$

  $k \in \mathbb{Z}$

$= \cos\left[3\left(x + \frac{\pi}{6} + \underbrace{\frac{2}{3}\pi k}_{\frac{2}{3}\pi \text{ period}}\right)\right]$

Cyclometric functions as inverse functions to restrictions of trigonometric functions.

Arkussinus

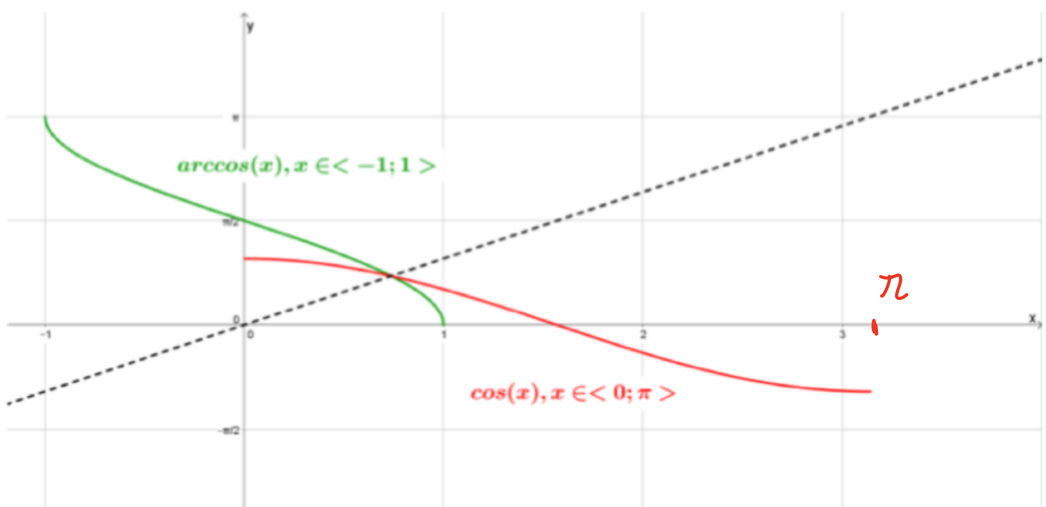


$H(f) = [-1, 1]$ $\quad H(f^{-1}) = [-\frac{\pi}{2}, \frac{\pi}{2}]$

$f: y = \sin x, x \in \langle -\frac{\pi}{2}; \frac{\pi}{2} \rangle, \quad f^{-1}: \boldsymbol{y = \arcsin x, x \in \langle -1; 1 \rangle}$

$D(f) = [-\frac{\pi}{2}, \frac{\pi}{2}]$ $\quad D(f) = [-1, 1]$

Arkuskosinus



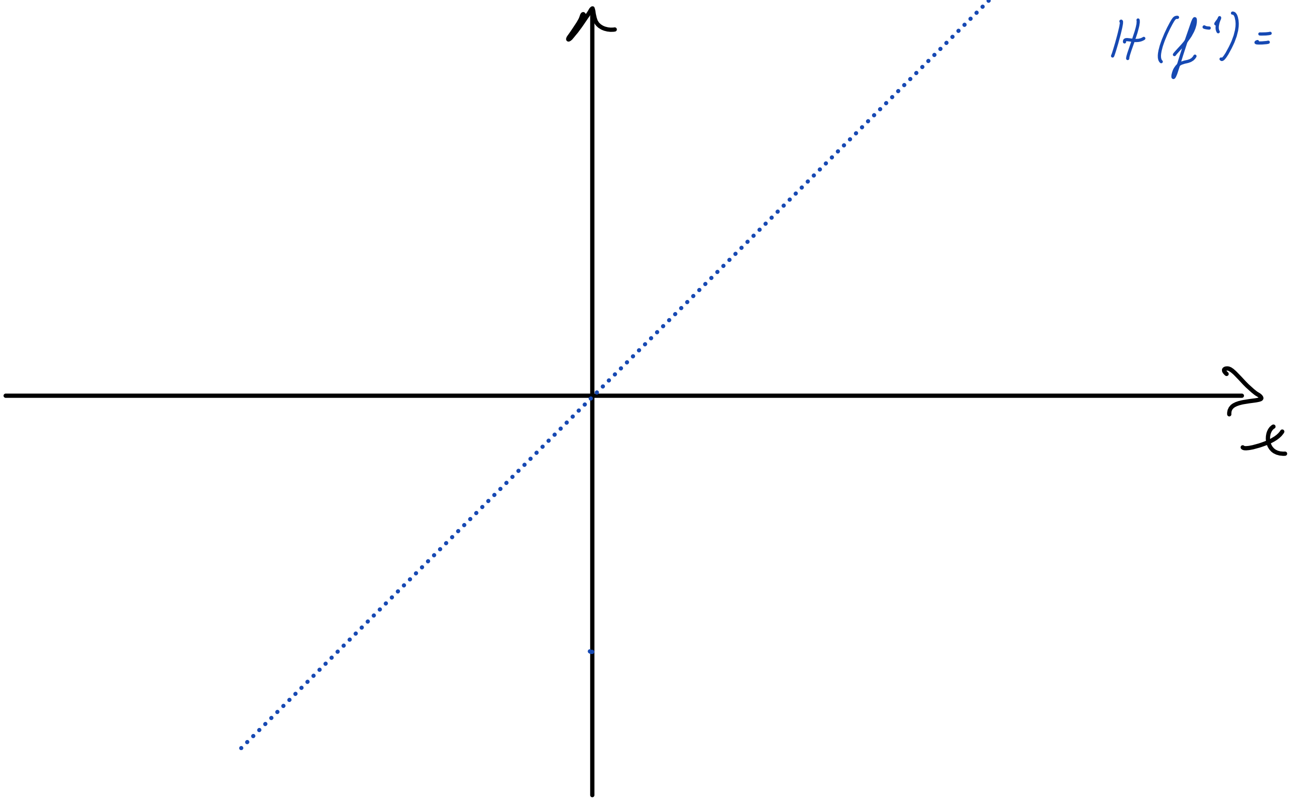
$H(f^{-1}) = [0, \pi]$

$f: y = \cos x, x \in \langle 0; \pi \rangle, \quad f^{-1}: \boldsymbol{y = \arccos x, x \in \langle -1; 1 \rangle}$

Sketch graphs of functions:

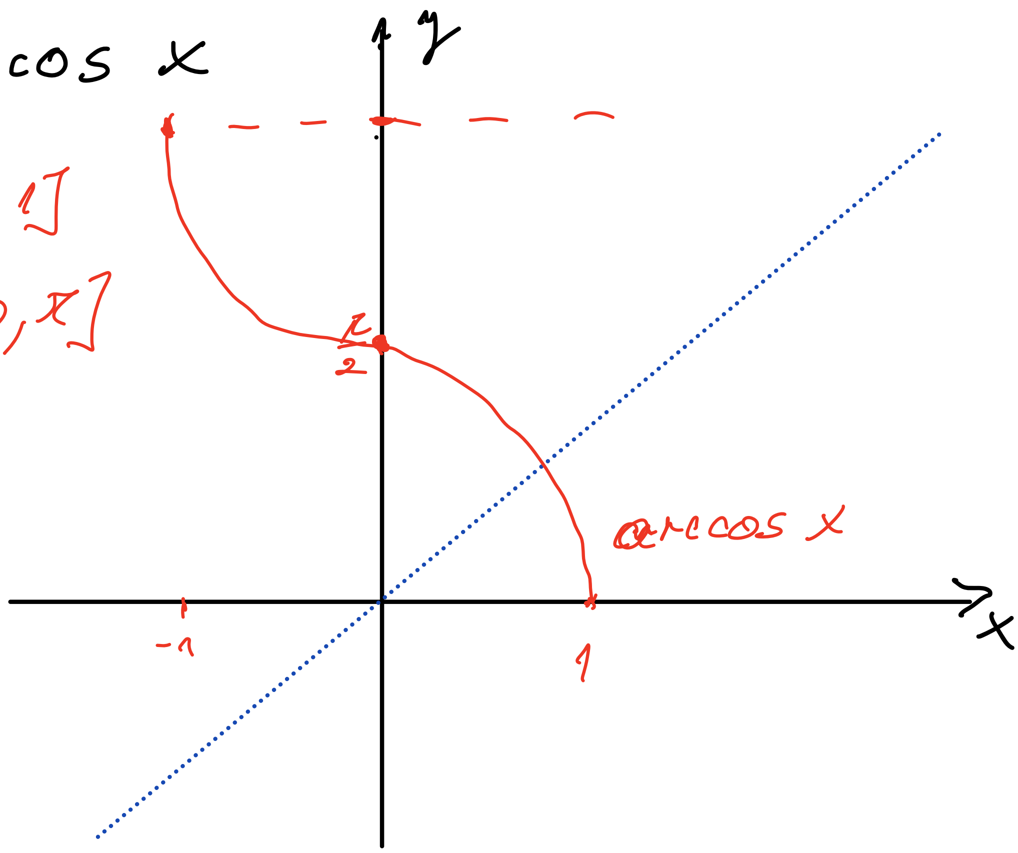
1.) $f(x) = \arcsin x$

$D(f) =$
$H(f) =$
$D(f^{-1}) =$
$H(f^{-1}) =$

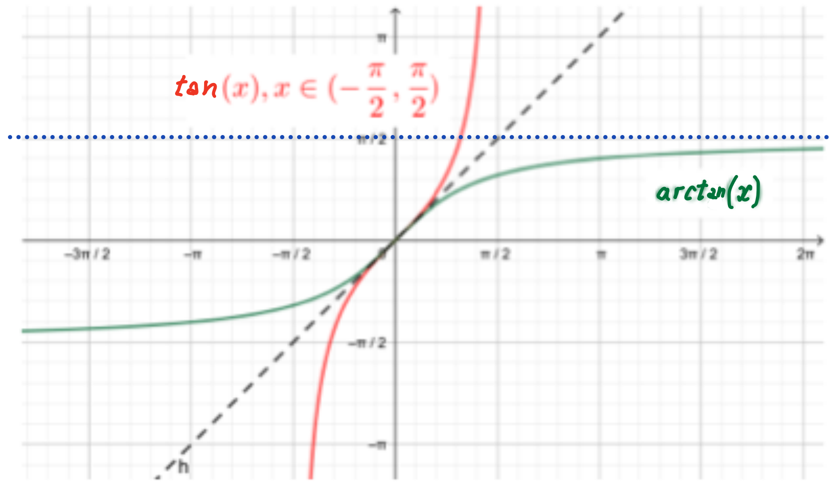


2.) $f(x)$ $y = \arccos x$

$D(f^{-1}) = [-1, 1]$
$H(f^{-1}) = [0, \pi]$

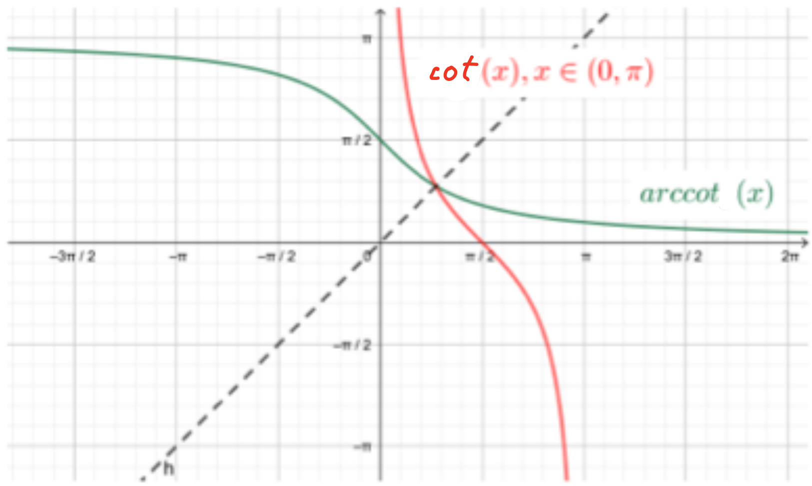


Arctangent



$$f: y = \tan x, x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right), \quad f^{-1}: \boldsymbol{y} = \mathbf{arctan}\, \boldsymbol{x}, \boldsymbol{x} \in \mathbb{R}$$

Arccotangent



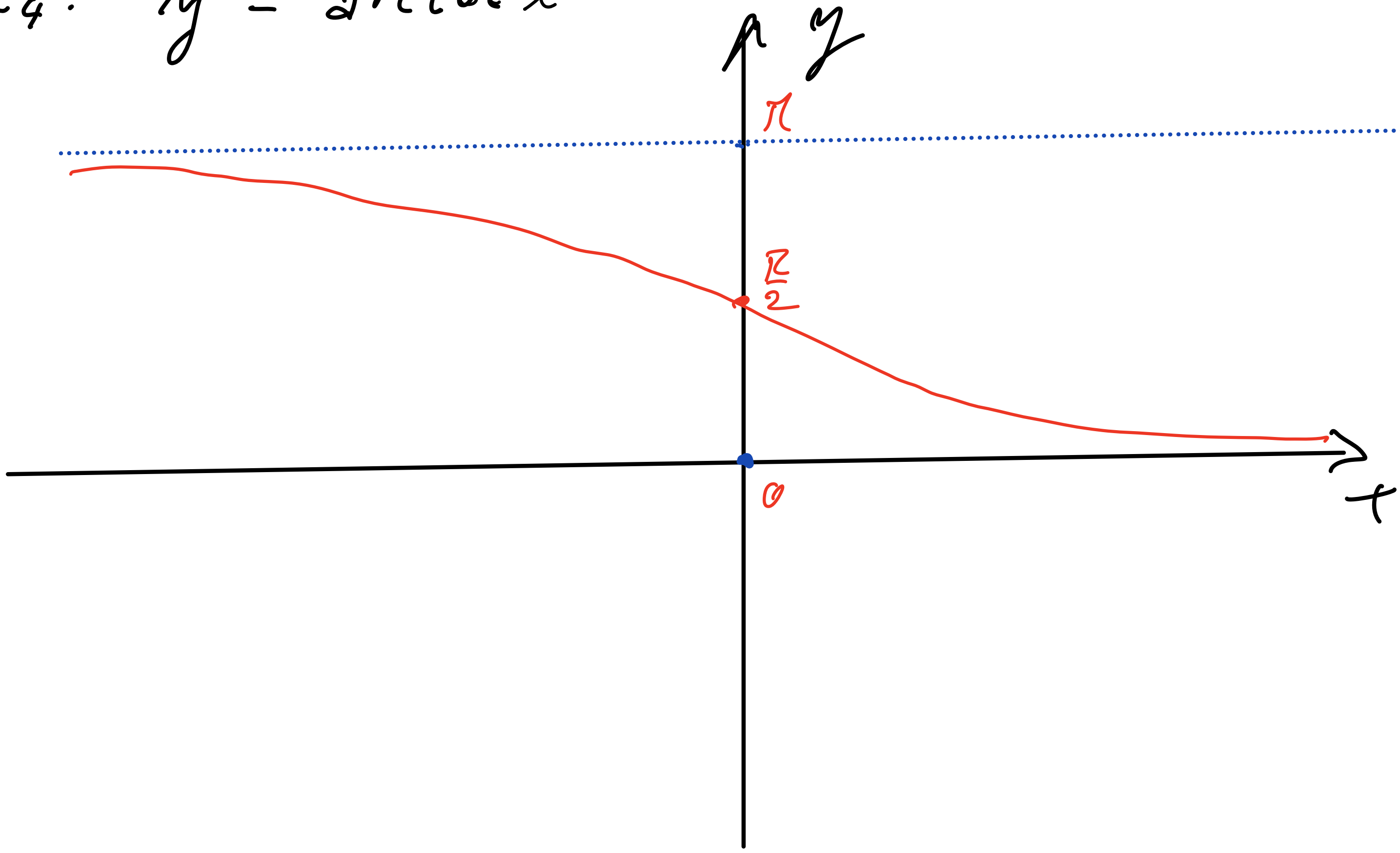
$$f: y = \cot x, x \in (0; \pi), \quad f^{-1}: \boldsymbol{y} = \mathbf{arccot}\ \boldsymbol{x}, \boldsymbol{x} \in \mathbb{R}$$

Sketch graphs of given functions

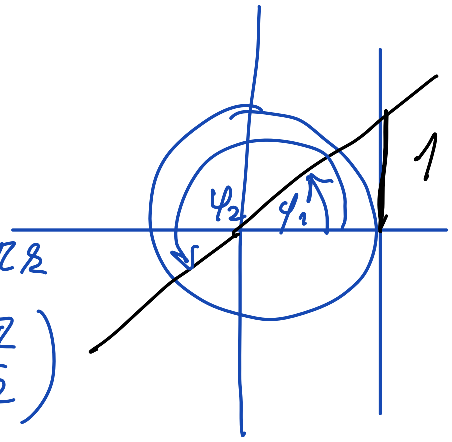
$f_3: \ y = \arctan x$

$f_4: \ y = \operatorname{arccot} x$

$D(f_4) = \mathbb{R} \quad H(f_4) = (0, \pi)$



Determine the values of the functions at given x.



a) $\operatorname{arctg} 1 = x \iff \operatorname{tg} x = 1$

$= x = \frac{\pi}{4}$

$x = \frac{\pi}{4} + \pi k$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

b) $\operatorname{arctg} \sqrt{3} = x \iff \operatorname{tg} x = \sqrt{3}$

$= \frac{\pi}{3}$

$x = \frac{\pi}{3} + \pi k$

$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

c) $\operatorname{arccotg} 1 = x \iff \cot x = 1$ $\quad x \in (0, \pi)$

$= \frac{\pi}{4}$

$x = \frac{\pi}{4} + k\pi$

d) $\arccos \frac{\sqrt{2}}{2} = y \iff \cos y = \frac{\sqrt{2}}{2}$

$= \frac{\pi}{4}$

$y_1 = \frac{\pi}{4} + 2k\pi$

$y_2 = -\frac{\pi}{4} + 2k\pi$

$y \in [0, \pi]$

e) $\arcsin \frac{\sqrt{3}}{2} =$ do it yorself

f) $\sin(\underbrace{\operatorname{arccotg} \frac{\sqrt{3}}{3}}_{u}) = y$

$\sin u = y$ $\qquad u = \operatorname{arccot} \frac{\sqrt{3}}{3} \iff \cot u = \frac{\sqrt{3}}{3}$

$u = \frac{\pi}{3} + \pi k$

$\sin \frac{\pi}{3} = y$

$u \in (0, \pi)$

$u = \frac{\pi}{3}$

$y = \frac{\sqrt{3}}{2}$

Determin domains of the given functions

b) $g(x) = 2\arccos\left(\frac{1}{x+3}\right)$

$x+3 \neq 0$, $-1 \leq \frac{1}{x+3} \leq 1$ $\arccos(20)$ meaningless

$x \neq -3$

$$\text{I.}\quad -1 \leq \frac{1}{x+3} \quad \wedge \quad \text{II.}\quad \frac{1}{x+3} \leq 1$$

$$0 \leq \frac{1}{x+3} + 1 \qquad\qquad \frac{1}{x+3} - 1 \leq 0$$

$$0 \leq \frac{1+x+3}{x+3} \qquad\qquad \frac{1-x-3}{x+3} \leq 0$$

$$0 \leq \frac{x+4}{x+3} \qquad\qquad \frac{-2-x}{x+3} \leq 0 \quad / \cdot (-1)$$

$$\frac{x+2}{x+3} \geq 0$$

$+ \quad - \quad +$ (-4, -3) $\qquad + \quad - \quad +$ (-3, -2)

I. $(-\infty, -4] \cup (-3, \infty)$ $\qquad$ II. $(-\infty, -3) \cup [-2, \infty)$

intersection

$$D(g) = \text{I.} \cap \text{II.} = (-\infty, -4] \cup [-2, \infty)$$

3.

a) $f(x) = \frac{1}{\sqrt[3]{x+2}} - \sin(x^2+4) + \ln(1-|x-3|)$

1) $\sqrt[3]{x+2} \neq 0$

$x \neq -2$

2) No condition

3.) $1-|x-3| > 0$

$|x-3| < 1$

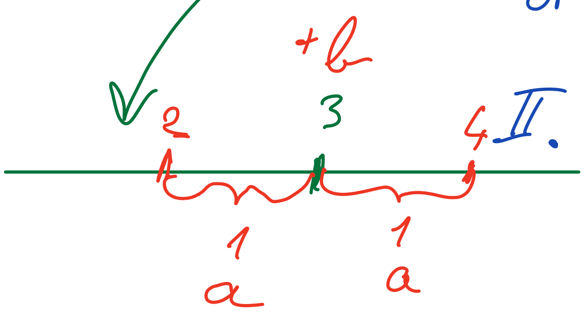
$|x-b| < a$

I. $x-3 \geq 0 \wedge x-3 < 1$

$x \geq 3 \qquad x < 4$

$x \in [3,4)$

or



II. $x-3 < 0 \wedge -x+3 < 1$

$x < 3 \wedge x > 2$

$x \in (2,3)$

solution I. ∪ II. $= (2,4)$

$D(f) = (2,4)$

c) $h(x) = e^{\sin(x)+x} - \log_2(x+3) + \frac{4x-7}{\sqrt{x^2+x-2}}$

1.) No condition

2.) $(x+3) > 0$

$x > -3$

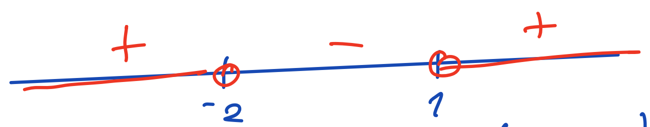
3.) $\sqrt{x^2+x-2} \neq 0$

$x^2+x-2 \neq 0$

4.) $x^2+x-2 \geq 0$

$x^2+x-2 > 0$

$(x+2)(x-1) > 0$



$x \in (-\infty,-2) \cup (1,\infty)$

$D(h) = (-3,-2) \cup (1,\infty)$

Find the inverse function

$$f(x) = -3 + \cos\frac{4x-5}{2}$$

Supplementary problems to solve.

Solve the given inequalities

a) $|\tan x| > 0$

b) $2\cos x - \sqrt{3} > 0$