L–functions over Function Fields and Mean Values of L–functions associated with Half–Integral Weight forms

Julio Andrade
julio_andrade@brown.edu

ICERM–Brown University

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- The *zeta function* of A is defined by the infinite series

$$\zeta_A(s) := \sum_{\substack{f \in A \\ f \text{ monic}}} \frac{1}{|f|^s}, \quad \Re(s) > 1. \tag{1}$$

Mean Values of L–functions over Function Fields

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Problem

Establish asymptotic formulas for

$$\sum_{D \in \mathcal{H}_{2g+1,q}} L(\tfrac{1}{2}, \chi_D)^k, \quad (3)$$

as $g \to \infty$.

Some Results

Theorem (A. and Keating–2012)

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- $L(s,\chi_D)$ is the numerator of the zeta function associated with the hyperelliptic curve $C: y^2 = D(x)$ of genus g over $\mathbb{F}_q$.
- $L(s,\chi_D)$ is a polynomial of degree $2g$ and satisfies an analogue of the approximate functional equation.
- We can use the R.H. for curves to bound non–trivial character sums.

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$$L(w, f \otimes \chi) := \sum \frac{a(d)\chi_r(d)}{d^w}. \tag{7}$$

Problem

The recent project which I just started with J. Hoffstein is establish asymptotic formulas for

$$\sum_{r<X} L(\tfrac{1}{2}, f \otimes \chi)^k, \quad \text{for} \quad k = 1, 2. \tag{8}$$

making use of Multiple Dirichlet Series machinery.